A Note on GMM Estimation of Theories of Expectation Formation

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1 A Generic Framework

For a given process of inflation and a particular theory of expectation formation, the GMM estimate of the vector of parameters Ω is defined as the following.

$$\widehat{\Omega} = \underset{\Omega}{argmin}(M_{\text{data}} - F(\Omega, Y))W(M_{\text{data}} - F(\Omega, Y))' \tag{1}$$

- Ω is a vector of size of k, depending on the number of parameters to be estimated.
- M_{data} is the moments computed from data, i.e. mean forecasts, average forecast errors, the cross-sectional variance of forecasts (disagreement), average uncertainty, etc. Also the autocovariance of all the abovementioned.
- F is the moments that are generated from a certain theory of expectation formation and inflation process. It is a function of parameters Ω as well as the Y, the real-time data(including history) that is available to forecasters at each point of the time t.
- For instance, for T periods, Y includes T sequences of real-time inflation data of different lengths that terminate at each point of the forecasting: $t = 0$, $t = 1...$ $t = T$.
- Both M and F is of the size m, depending on the number of moments used for estimation. For instance, if we only estimate expectation formation using mean forecasts, disagreements and forecast errors while taking the inflation process as given, there are three moments, thus $m = 3$. If we include autocovariance of forecast errors, then $m = 4$.
- W is the weighting matrix. For now, I stick to the identity matrix.

The above procedure is specific to a pair of assumed inflation process and a theory of expectation. It can be estimated only for the theory of expectation formation with exogenous fed parameters of the inflation process using the entire history of inflation data, or we could jointly estimate the parameters of expectation formation and the inflation process.

2 Estimation

2.1 Real-time Data

When agents form their expectations of inflation at any point of the time, what is potentially available to them is the real-time inflation data, namely those released from the most update-to-date vintage of the inflation. In order to match as close as possible the information set available to the forecasting agents, we need to use real-time data for each particular point of time.

These real-time vintage inflation data since 1998 were obtained from the Real-Time Data Research Center hosted by the Federal Reserve Bank of Philadelphia.

To get an idea of how much of the differences there are between the real-time inflation and the most recent vintage in 2018, Figure 1 and 2 plot, respectively, the distributions of the revisions as well as the time series of the inflation. Although overall, there is no obvious skewness of the revisions, there have been indeed sizable revisions made for the real-time data.

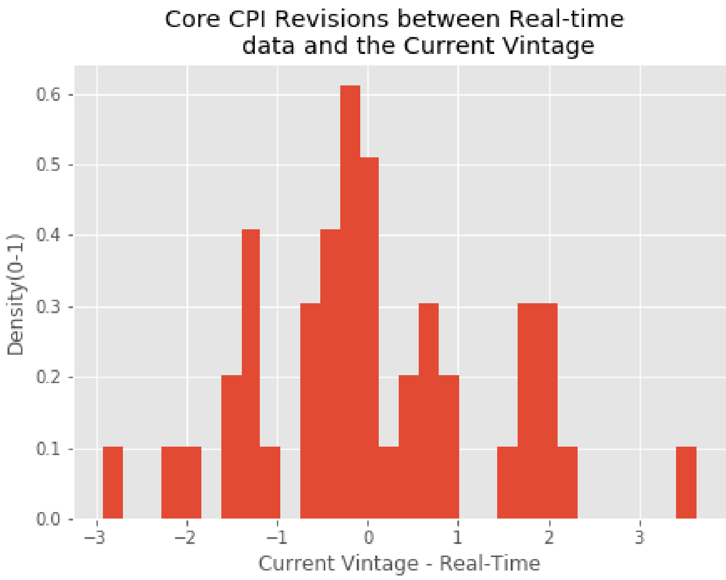


Figure 1: Revisions of Current-vintage from Real-time Core CPI

Note: real-time data at time t is defined the inflation from $t-1$ to t according to the most recent vintage of CPI inflation at time t. The period is between 2000 M1-2018 M3.

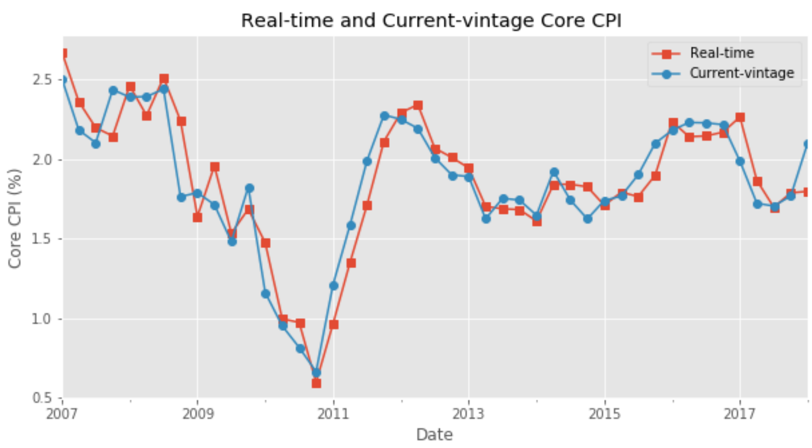


Figure 2: Current-vintage and Real-time Core CPI

2.2 Results

2.2.1 Professional Forecasters

Figure 3a and 3b plot the estimated moments for SE model together with the data moments professional forecasts of SPF.

From left to the right, the four columns of the figures are based on estimation using different choices of moments.

Figure 4a and 4b plot the estimated moments for NI model together with the data moments professional forecasts of SPF.

2.2.2 Households

Figure 5a and 5b plot the estimated moments for SE model together with the data moments of households inflation forecasts from SCE.

From left to the right, the four columns of the figures are based on estimation using different choices of moments.

Figure 6a and 6b plot the estimated moments for NI model together with the data moments of households inflation forecasts from SCE.

3 Stochastic volatility model of inflation

3.1 Process of inflation

Assume that the inflation follows a process of unobserved components model with stochastical volatility.

$$\begin{aligned} y_t &= \tau_t + \eta_t, \quad \text{where } \eta_t = \sigma_{\eta,t}\xi_{\eta,t} \\ \tau_t &= \tau_{t-1} + \epsilon_t, \quad \text{where } \epsilon_t = \sigma_{\epsilon,t}\xi_{\epsilon,t} \\ \log\sigma^2_{\eta,t} &= \log\sigma^2_{\eta,t-1} + \mu_{\eta,t} \\ \log\sigma^2_{\epsilon,t} &= \log\sigma^2_{\epsilon,t-1} + \mu_{\epsilon,t} \end{aligned} \tag{2}$$

The distributions of shocks to levels of the components and their volatilities are, respectively, the following.

$$\begin{aligned} \xi_t &= [\xi_{\eta,t}, \xi_{\epsilon,t}] \sim N(0, I_2) \\ \mu_t &= [\mu_{\eta,t}, \mu_{\epsilon,t}]' \sim N(0, \gamma I_2) \end{aligned} \tag{3}$$

The only parameter of the model is γ, which determines the time-varying volatilities.

3.2 Rational expectation

At the point of time t, the RE agent sees the realization of all stochastic variables above with subscript t, $t-1$, etc, including y_t, τ_t, η_t, $\sigma_{\eta,t}$, $\sigma_{\epsilon,t}$ and their realizations in the whose past. Again, $*$ stands for FIRE benchmark.

$$\overline{y}^*_{t+h|t} \equiv y^*_{t+h|i,t} = E^*_{i,t}(y_{t+h}|I_{i,t}) = \theta_t \tag{4}$$

Forecast error is simply the cumulated sum of unrealized shocks from t to $t+h$, which is

$$\overline{FE}^*_{t+h|t} \equiv FE^*_{t+h|i,t} = \sum_{s=1}^{h}(\eta_{t+s} + \epsilon_{t+s}) \tag{5}$$

Conditional h-step-ahead variance, uncertainty is

$$
\begin{aligned}
\overline{Var}^*_{t+h|t} \equiv Var^*_{t+h|i,t} &= \sum_{k=1}^{h} E_{i,t}(\sigma^2_{\eta,t+k}) + E_{i,t}(\sigma^2_{\epsilon,t+k}) \\
&= \sum_{k=1}^{h} E_{i,t}(exp^{\log\sigma^2_{\eta,t}+\sum_{k=1}^{h}\mu_{\eta,t+k}}) + E_{i,t}(exp^{\log\sigma^2_{\epsilon,t}+\sum_{f=1}^{h}\mu_{\epsilon,t+f}}) \\
&= \sum_{k=1}^{h} \sigma^2_{\eta,t} E_{i,t}(exp^{\sum_{k=1}^{h}\mu_{t+h,\eta}}) + \sigma^2_{\epsilon,t} E_{i,t}(exp^{\sum_{f=1}^{h}\mu_{\epsilon,t+f}}) \\
&= \sum_{k=1}^{h} \sigma^2_{\eta,t} exp^{E_{i,t}(\sum_{k=1}^{h}\mu_{t+k,\eta})-0.5Var_{i,t}(\sum_{k=1}^{h}\mu_{t+k,\eta})} + \sigma^2_{\epsilon,t} E_{i,t}(exp^{\sum_{f=1}^{h}\mu_{\epsilon,t+f}}) \\
&= \sigma^2_{\eta,t} \sum_{k=1}^{h} exp^{-0.5k\gamma_\eta} + \sigma^2_{\epsilon,t} exp^{-0.5h\gamma_\epsilon}
\end{aligned} \tag{6}
$$

One immediately see that now the volatility is stochastic at any point of the time.
For instance, set $h = 1$, the conditional volatility for the 1-step-ahead inflation is

$$
Var^*_{t+1|i,t} = exp^{-0.5\gamma_\eta}\sigma^2_{\eta,t} + exp^{-0.5\gamma_\epsilon}\sigma^2_{\epsilon,t} \tag{7}
$$

Disgreement is zero across agents in RE.

$$
\overline{Disg}^*_{t+h|t} = 0 \tag{8}
$$

3.3 Sticky Expectation

An agent whose most recent up-do-date update happened at $t-\tau$, thus she sees all the realizations of stochastic variables up to $t-\tau$, including $y_{t-\tau}$, $\tau_{t-\tau}$, $\eta_{t-\tau}$, $\sigma_{\eta,t-\tau}$, $\sigma_{\epsilon,t-\tau}$.

Her forecast is the permanent component that realized at time $t-\tau$.

$$
y_{t+h|i,t-\tau} = \theta_{t-\tau} \tag{9}
$$

Her forecast uncertainty is

$$
Var_{t+h|i,t-\tau} = \sigma^2_{\eta,t-\tau} \sum_{k=1}^{h+\tau} exp^{-0.5k\gamma_\eta} + \sigma^2_{\epsilon,t-\tau} exp^{-0.5(h+\tau)\gamma_\epsilon} \tag{10}
$$

The population average of the two are, respectively, a weighted average of people whose the most update was in $t, t-1...t-\tau, t-\infty$, respectively.

$$
\begin{aligned}
\bar{y}^{se}_{t+h|t} &= \sum_{\tau=0}^{\infty}(1-\lambda)^\tau \lambda y_{t+h|t-\tau} \\
&= \sum_{\tau=0}^{\infty}(1-\lambda)^\tau \lambda \theta_{t-\tau}
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
\overline{Var}^{se}_{t+h|t} &= \sum_{\tau=0}^{\infty}(1-\lambda)^\tau \lambda Var_{t+h|t-\tau} \\
&= \sum_{\tau=0}^{\infty}(1-\lambda)^\tau \lambda [\sigma^2_{\eta,t-\tau} \sum_{k=1}^{h+\tau} exp^{-0.5k\gamma_\eta} + \sigma^2_{\epsilon,t-\tau} exp^{-0.5(h+\tau)\gamma_\epsilon}]
\end{aligned} \tag{12}
$$

Both forecast errors $\overline{FE}_{t+h|t}$ and disagreements takes similar form to that in AR process with time-invariant volatility.

$$\overline{FE}^{se}_{t+h|t} = \sum_{\tau=0}^{\infty}(1-\lambda)^{\tau}\lambda FE^{*}_{t+h|t-\tau} \quad = \sum_{\tau=0}^{\infty}(1-\lambda)^{\tau}\lambda\sum_{s=1}^{\tau+h}(\theta_{t+s}+\epsilon_{t+s}) \tag{13}$$

The disagreement is the following.

$$\begin{aligned}\overline{Disg}^{se}_{t+h|t} &= \sum_{\tau=0}^{\infty}(1-\lambda)^{2\tau}\lambda^2(y_{t+h|t-\tau}-\overline{y}^{se}_{t+h|t})^2 \\ &= \sum_{\tau=0}^{\infty}(1-\lambda)^{2\tau}\lambda^2(\theta_{t-\tau}-\overline{y}^{se}_{t+h|t})^2 \\ &= \sum_{\tau=0}^{\infty}(1-\lambda)^{2\tau}\lambda^2\{\theta_{t-\tau}-\sum_{\tau=0}^{\infty}(1-\lambda)^{\tau}\lambda\theta_{t-\tau}\}^2\end{aligned} \tag{14}$$

3.4 Noisy Information

Now, the agent at time t needs to recover the real-time permanent component θ_t to make the best forecast for future y_{t+h} using nosiy signals.

$$y^{ni}_{t+h|t} \equiv y^{ni}_{t+h|i,t} = \bar{\theta}_{t|t} \tag{15}$$

where $\bar{\theta}_{t|t}$ is generated through Kalman filtering.

Assume that the nosiy signals of θ_t consists of a public signals s^{pb}_t and the private signals $s^{pr}_{i,t}$. For simplicity, let us assume the public signal is basically the y_t. A more general case would be an independently drawn public signal sequence. The two signals can be again stacked into a vector of 2×1 to $s^{\theta}_{i,t}$.

Then the filterred θ_t by agent i is

$$\bar{\theta}_{t|t} = (1-\tilde{P}_t H)\bar{\theta}_{t|t-1} + \tilde{P}s^{\theta}_{i,t} \tag{16}$$

where $\theta_{t|t-k}$ is the filterred forecast of θ_t using all the information up to $t-k$., and $\tilde{P}t$ is the time-specific Kalman gain that is dependent on the noisy ratios of signals.

$$\tilde{P}_t = \Sigma^{\theta}_{i,t|t-1}H(H'\Sigma^{\theta}_{i,t|t-1}H+\Sigma^{\theta}_t)^{-1} \tag{17}$$

Now the noisiness of signals are time varying as well.

$$\Sigma^{\theta}_t = \begin{bmatrix}\sigma^2_{\eta,t} & 0 \\ 0 & \sigma^2_{\xi}\end{bmatrix} \tag{18}$$

where the variance of public signal is the time-varying $\sigma^2_{\eta,t}$ and let us assume the private signals have constant nosiness σ^2_{ξ}.

The uncertainty matrix evolves in the following manner.

$$\Sigma^{\theta}_{i,t|t} = \Sigma^{\theta}_{i,t|t-1} - \Sigma^{\theta}_{i,t|t-1}H'(H\Sigma^{\theta}_{i,t-1}H'+\Sigma^{\theta}_t)H\Sigma^{\theta}_{i,t|t-1} \tag{19}$$

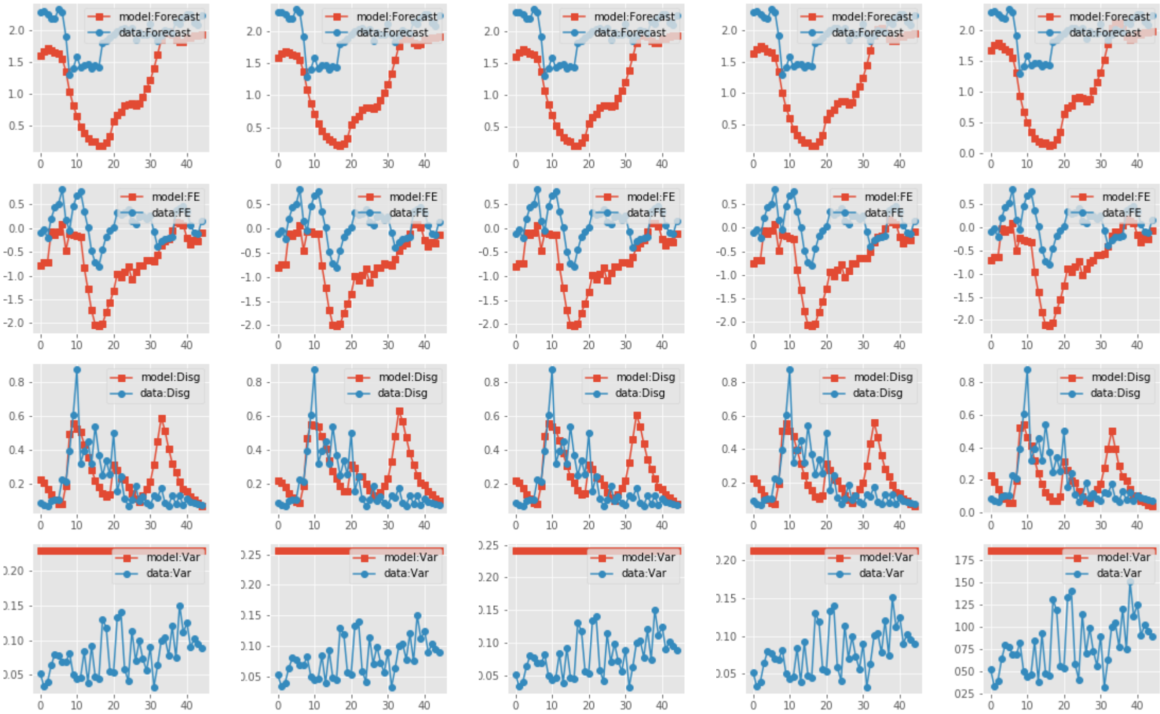
Notice now that since prior of θ_t before time period t is a weighted average of previous realizations of y and past signals, current forecast depends on the past realizations even though the rational forecast is up-to-date θ_t.

Due to the time-varying volatility $\sigma^2_{\eta,t}$, the noisyness of the public signal is also time-varying, which governs the degree of rigidity. For instance, if volatility to the transitory shock is high, the Kalman gain is low, the forecast responds less to the new realizations.

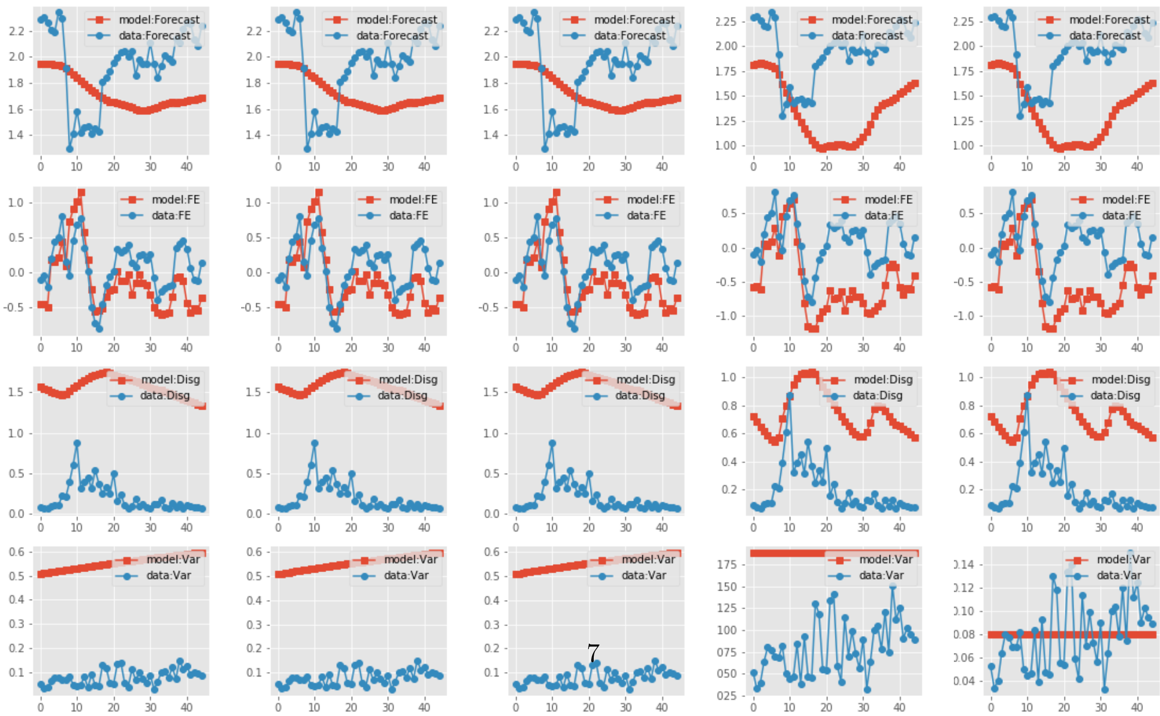
It is also worth thinking about what is the one-step ahead uncertainty in the context of stochastic valotility world.

$$\begin{aligned}\Sigma^{\theta}_{i,t|t-1} &= \Sigma^{\theta}_{i,t-1|t-1} + Var^{*}_{t|t-1}(y_t) \\ &= \Sigma^{\theta}_{i,t-1|t-1} + exp^{-0.5\gamma_{\eta}}\sigma^2_{\eta,t-1} + exp^{-0.5\gamma_{\epsilon}}\sigma^2_{\epsilon,t-1}\end{aligned} \tag{20}$$

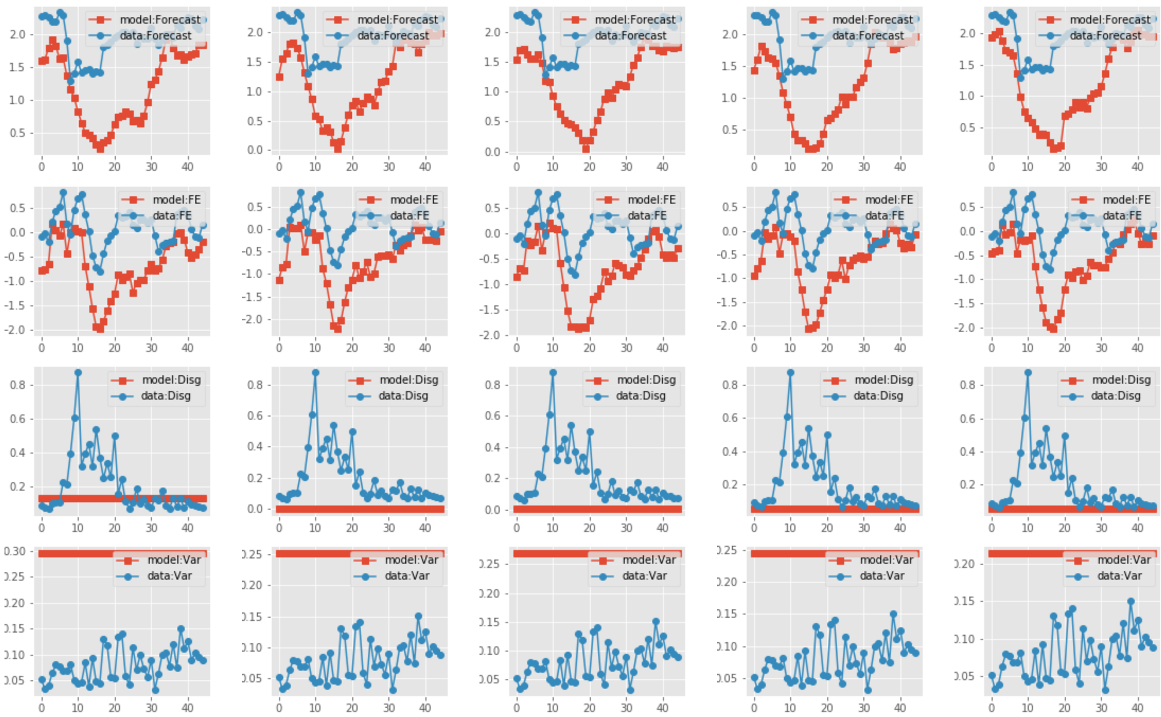
(a) Estimation of SE for Professionals



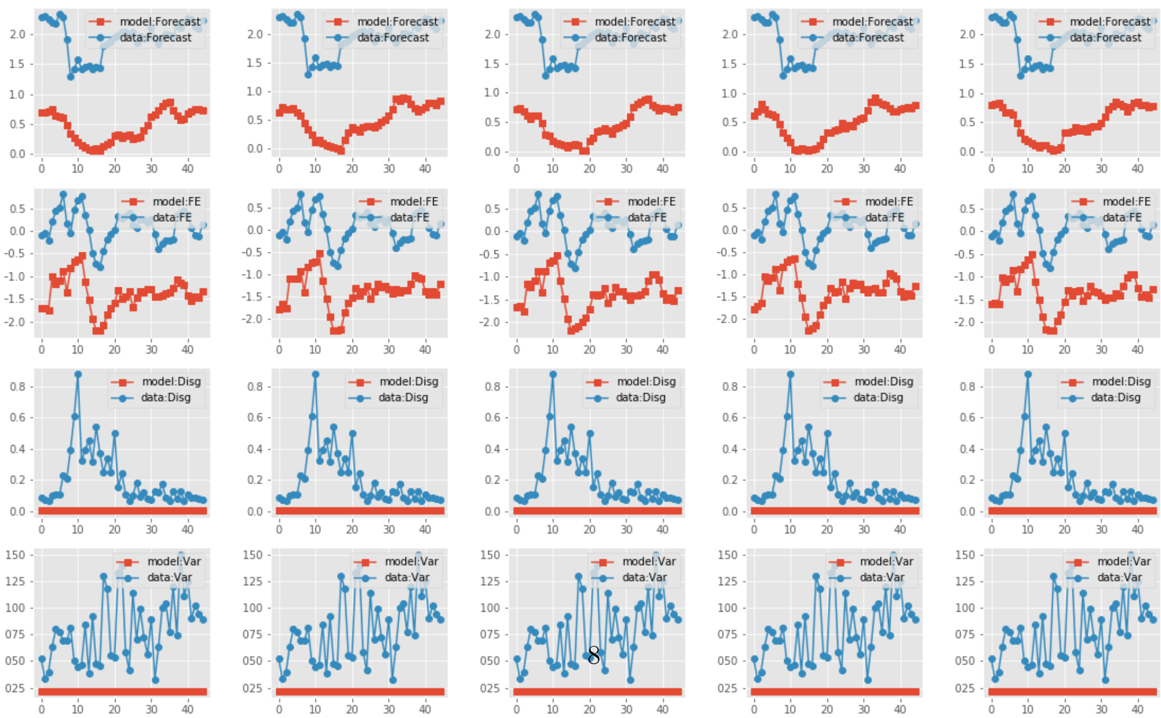
(b) Joint Estimation of SE and Inflation Process



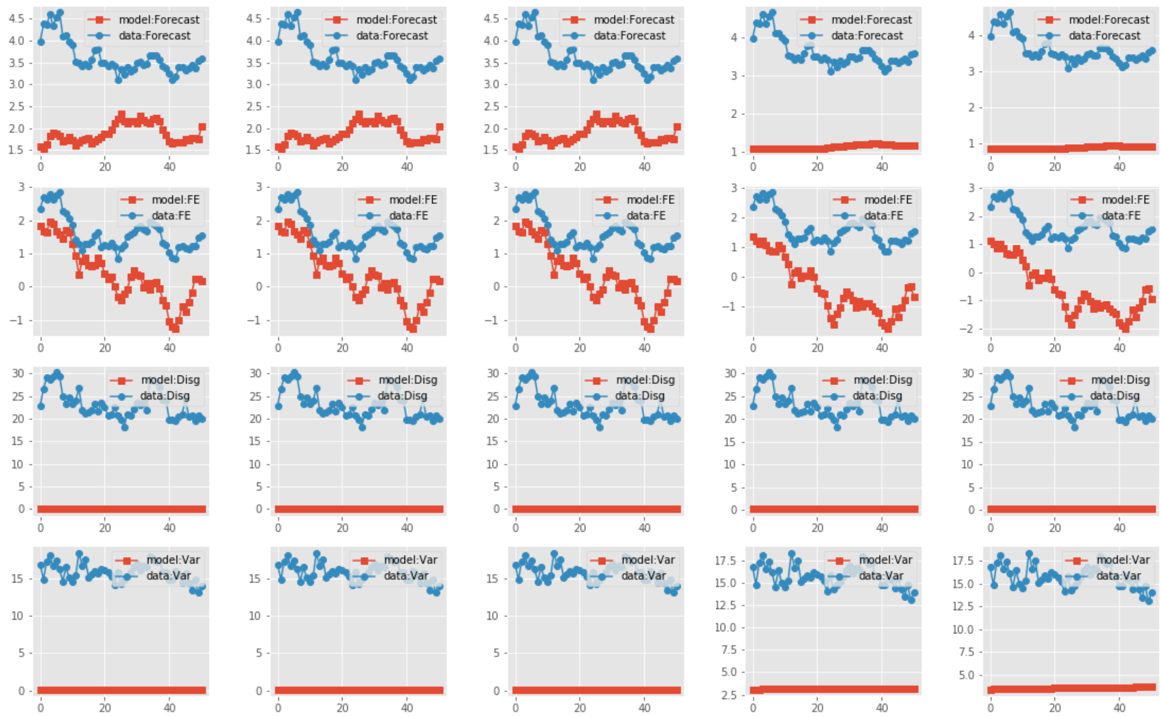
(a) Estimation of NI for Professionals



(b) Joint Estimation of NI and Inflation Process

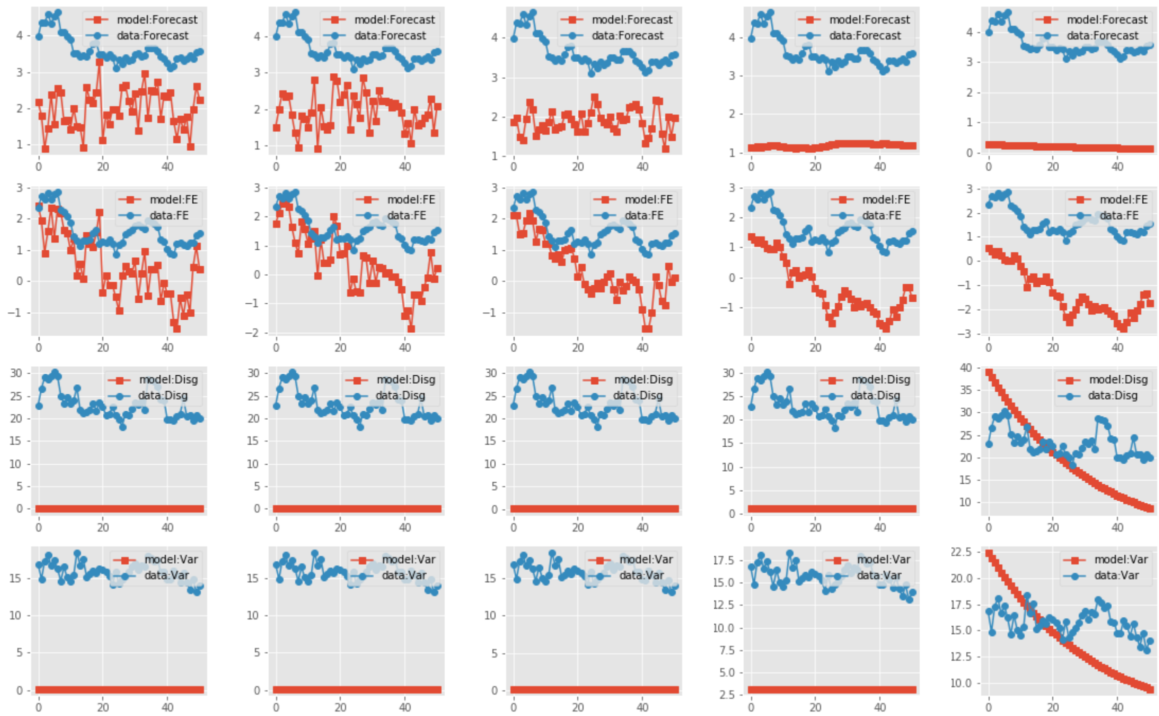


(a) Estimation of SE for Households



(b) Joint Estimation of SE and Inflation Process

(a) Estimation of NI for Households



(b) Joint Estimation of SE and Inflation Process

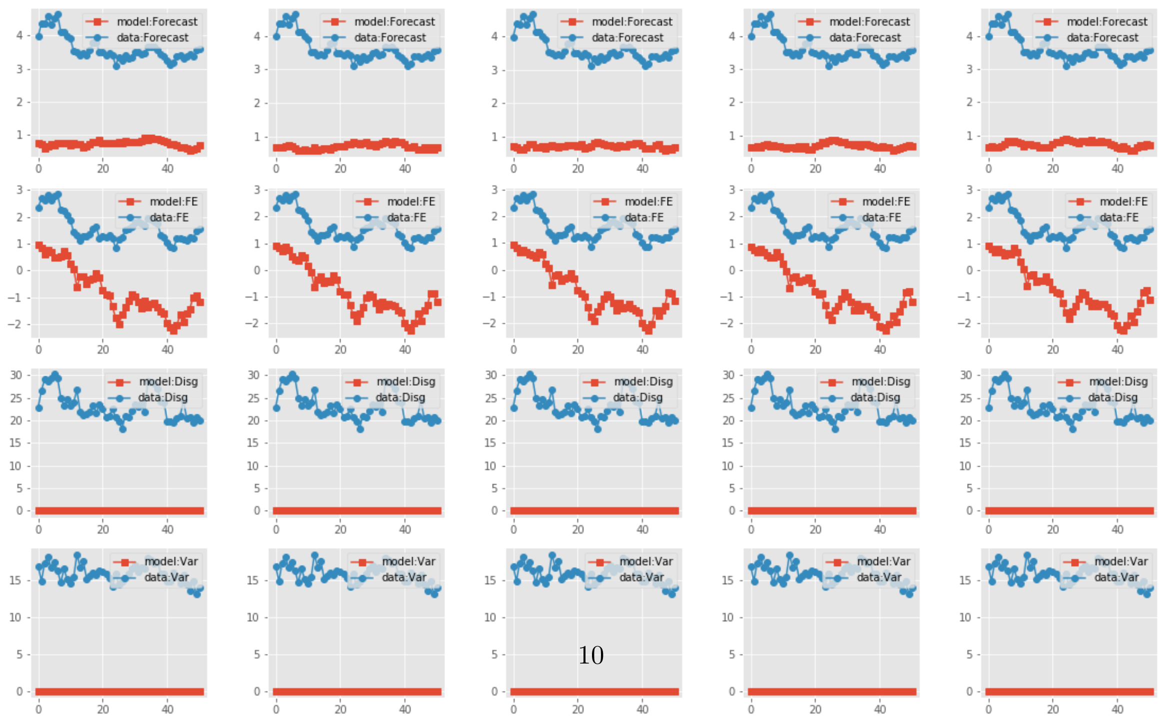


Table 1: GMM Estimates of Parameters of SE and Inflation Process

0	1	2	3	SE: $\hat{\lambda}_{SPF}$(Q)	SE: $\hat{\lambda}_{SPF}$(Q)	SE: ρ	SE: σ	SE: $\hat{\lambda}_{SCE}$(M)	SE: $\hat{\lambda}_{SCE}$(M)	SE: ρ	SE: σ
Forecast				0.21	0.01	1	0.1	1	1	1	0.1
FE				0.18	0.01	1	0.1	1	1	1	0.1
Forecast	FE			0.2	0.01	1	0.1	1	1	1	0.1
Forecast	FE	Disg		0.23	0.05	1	0.1	0.02	0.03	0.98	0.1
Forecast	FE	Disg	Var	0.26	0.05	1	0.07	0.01	0.03	0.98	1.11

Table 2: GMM Estimates of Parameters of NI and Inflation Process

0	1	2	3	NI: $\hat{\sigma}_{pb,SPF}$	$\hat{\sigma}_{pr,SPF}$	NI: $\hat{\sigma}_{pb,SPF}$	$\hat{\sigma}_{pr,SPF}$	NI: ρ	NI: σ	NI: $\hat{\sigma}_{pb,SCE}$	$\hat{\sigma}_{pr,SCE}$	NI: $\hat{\sigma}_{pb,SCE}$	$\hat{\sigma}_{pr,SCE}$	NI: ρ	NI: σ
Forecast				376.66	1.26	0.5	0.5	0.8	0.1	0	1.13	0.5	0.5	0.8	0.1
FE				1.04	159.74	0.5	0.5	0.8	0.1	0	1.26	0.5	0.5	0.8	0.1
Forecast	FE			1.14	6.74	0.5	0.5	0.8	0.1	0.59	0.36	0.5	0.5	0.8	0.1
Forecast	FE	Disg		1.39	1.46	0.5	0.5	0.8	0.1	2354.37	10.93	0.5	0.5	0.8	0.1
Forecast	FE	Disg	Var	1.37	1.1	0.5	0.5	0.8	0.1	35263.31	332.04	0.5	0.5	0.8	0.1