Intermediate Microeconomics Useful to Macroeconomics

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1 Consumer's Theory

1.1 Marshallian Demand

X is the vector of consumption.

$$Max \quad u(x) \quad s.t. \sum p_i x_i \leq w \tag{1}$$

Marshalian demand is defined as

$$\bar{x}(w,p) = \underset{\{x\}}{argmax} \quad u(x) \quad s.t. \sum p_i x_i \leq w \tag{2}$$

Correspondingly, define indirect utility as the maximum value attained by the Marshallian demand.

$$v^*(w,p) = u(\bar{x}(w,p)) \tag{3}$$

Roy's Identity

$$\bar{x}_i(p,w) = -\frac{\partial v^*(p,w)/\partial p_i}{\partial v^*(p,w)/\partial w} \tag{4}$$

1.2 Hicksian Demand

Minimize cost to attain utility level non-below a reference point. The problem is then.

$$min \quad px \quad s.t. \quad u(x) \geq \bar{u} \tag{5}$$

Hicksian demand is defined.

$$h(p,\bar{u}) = \underset{\{x\}}{argmin} \quad px \quad s.t. \quad u(x) \geq \bar{u} \tag{6}$$

Expenditure function:

$$e(p, \bar{u}) = ph(x, \bar{u}) \tag{7}$$

Shephard's Lemma:

$$h_i(p, \bar{u}) = \frac{\partial e(p, \bar{u})}{\partial p_i} \tag{8}$$

1.3 Connection

Slutsky decomposition.

$$\frac{\partial \bar{x}_i(p, w)}{\partial p_j} = \underbrace{\frac{\partial h_i(p, v^*(p, w))}{\partial p_j}}_{S:nsd} - \underbrace{\frac{\partial \bar{x}_i(p, w)}{\partial w} \bar{x}_j(p, w)}_{I:ambiguous} \tag{9}$$

S captures substitution effect. It is the change in demand for i following a price change in j holding the utility fixed. Specifically, it is the indirect utility associated with the current Marshallian demand. I captures income effect. It is the change in demand for i in response to change in income multiplied by the change in demand for j. In general, there is no guarantee $\frac{\partial \bar{x}_i(p,w)}{\partial w} > 0$, namely the demand increases with income. Therefore, the income effect is ambiguis. We impose the assumption that good i is normal good, Therefore, I is positive semi definite. As a result, the whole equation is negative, an price increase in j decreases the demand for i.

The following two establish the duality of Hicksian and Marshallian demand. It is the most intuitive to understand in a graph.

Think of Marshallian demand as pushing outward indifference curve holding a budget line fixed to where the indifference curve is tangent with budget line. Think of Hicksian demand as pushing inward budget line holding indifference curve fixed to where the two are tengent. For a well-behaving preference, that implies the two optimal bundle is the same and uniquely determined.

$$\bar{x}(p, e(p, \bar{u})) = h(p, \bar{u}) \tag{10}$$

$$h(p, v^*(p, w)) = \bar{x}(p, w) \tag{11}$$

For the same reason above, we also have following two equality relations.

Expenditure that attains utility equal to an indirect utility function, is equal to income. Indirect utility associated with the minimum expenditure for a certain utility level is exactly equal to the reference utility.

$$e(p, v^*(p, w)) = w \tag{12}$$

$$v(p, e(p, \bar{u})) = \bar{u} \tag{13}$$

Equation 10 is also what we use to prove the Slutsky Decomposition. Take its i-th element

$$\bar{x}_i(p, e(p, \bar{u})) = h_i(p, \bar{u}) \tag{14}$$

Taking total derivative with respect to p_j and utilizing Shepard's Lemman and duality results at optimum 12, 13.

$$\underbrace{\frac{\partial h_i(p, \bar{u})}{\partial p_j}}_{\frac{\partial \bar{x}_i(p,w)}{\partial p_j}} = \underbrace{\frac{\partial \bar{x}_i(p, e(p, \bar{u}))}{\partial p_j}}_{\frac{\partial h_i(p,\bar{u})}{\partial p_j}} + \underbrace{\frac{\partial \bar{x}_i(p, e(p, \bar{u}))}{\partial e(p, \bar{u})}}_{\frac{\partial x_i(p,w)}{\partial w}} \underbrace{\frac{\partial e(p, \bar{u})}{\partial p_j}}_{h_j(p,\bar{u})=x_j(p,w)} \tag{15}$$

1.4 Properties

Here are some properties.

- $v(p, w)$ is quasiconvex in p. That is $v(tp+(1-t)p', w) \geq max v(p, w), v(p', w)$
- $\bar{x}(\lambda p, \lambda w) = \bar{x}(p, w)$. Marshallian demand is homogenous of degree zero.
- $v(\lambda p, \lambda w) = v(p, w)$. Indirect utility is homogenous of degree zero.
- $h(p, \bar{u})$ is concave in p. That is $h(tp+(1-t)p', \bar{u}) < th(p, \bar{u})+(1-t)h(p, \bar{u})$
- $e(\lambda p, \bar{u}) = \lambda(p, \bar{u})$. Expenditure function is homogeneous of degree 1.
- $\bar{x}(p, w)$ obeys single law of demand if normal good.
- $\bar{x}(p, w)$ obeys multi-good law of demand if $\frac{-x\partial^2 u(x)x}{x\partial u(x)} < 4$
- $h(p, \bar{u})$ obeys law of demand for sure, i.e. $(p - p')(h(p, \bar{u}) - h(p', \bar{u})) \leq 0$.
- $\bar{x}(p, w)$ obeys budget identity if $p\bar{x}(p, w) = w$. A continuous, quasiconcave and monotone preference $\succeq$ guarantees budget identity.
- $\bar{x}(p, w)$ obeys boundary property if $x_i(p, w) \to \infty$ for $p_t = 0$ and $w > 0$. Strictly monotone, continuous and strictly quasiconcave preference $\succeq$ guarantees boundary property.

2 Producer's Theory

2.1 Cost-Minimizing Problem

Cost function is defined as below. w is the factor price, x is factor demand, y is the production target, $F()$ is the production function.

$$c(w, y) = \underset{x}{min} \quad wx \quad s.t. F(x) = y \tag{16}$$

The solution to the minimization problem is the conditonal factor demand defined as below.

$$f(w,y) = \underset{x}{argmin} \quad wx \quad s.t. F(x) \geq y \tag{17}$$

The solution to the problem is *non-empty* if $\{x : F(x) \geq y\} \neq \emptyset$ and the set is also convex and compact, which follow from the production functing being quasi-concave and continuous.

The solution to the problem is *unique* if the production function is also strictly quasi-concave and monotone.

Here are some properties of the cost function. c is eactly the same as expenditure fucntion in consumer's problem.

- $c(\lambda w, y) = c(w, y)$, homogeneous of degree 0.
- $c(w, y)$ increaes with w and y.
- $c(w, y)$ is concave in w.
- Shepard's Lemma. Conditional factor factor $f_i(x, y) = \frac{\partial c(w,y)}{\partial w_i}$.
- $c(w, y)$ is convex in y if $f(w, y)$ is concave in y.

2.2 Profits Maximization Problem

The general form of firm's problem can be written as below. Y is production possiblity set. y is a $l + 1$ vector with the first l entries being the factor inputs and the last being the output. p is now also a $l + 1$ vector of input and output price.

$$\underset{y}{max} \quad py \quad s.t. \quad y \in Y \tag{18}$$

Specifically, we define Y as below.

$$Y = \{(-x, y) | y \leq F(x), x \in R^l_+\}$$

The firm's profits maximization can be seen as a two-step problem. First, determining the profits-maximizing output. Second, minimizing the cost for the production target. We define profits function being the following.

$$\underset{y}{max} \quad R(y) - c(w, y) s.t. y \geq 0 \tag{19}$$

The solution to the above problem is supply correspondence.

$$s(p) = \underset{y}{argmax} \quad py \quad s.t. \quad y \in Y \tag{20}$$

And profits function is below.

$$\pi(p) = s(p) * p$$

Here are some properties.

- Hotelling's Lemma. $y_i^* = \frac{\partial \pi(p)}{\partial p_i}$
- $\pi(p)$ is convex in p.