Precautionary Saving

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1 Overview

- Precautionary saving arises when the expected marginal utility in future is lower than the marginal utility of expected consumption. Euler equation requires the marginal utility of today versus tomrrow is equal to certain constant ratio. Therefore, a lower expected marginal utility in future requires a higher marginal utility, therefore a lower consumption today. Mathematically speaking, a positive third derivative of utility function $u''' > 0$ is necessary for precautionary saving. Or sometimes, equivalently, the prudence, defined as the negative ratio of third to second derivative $-\frac{u'''}{u''} > 0$, when $u'' < 0$. CRRA is the most typical utility function of such kind. In contrast, quadratic utility does not have such property.

2 Precautionary Saving in a Two-Period Model

Consumer's problem:

$$\begin{aligned} Max \quad & U(C_1) + E(U(\tilde{C}_2)) \\ s.t. \quad & \tilde{C}_2 = (1-\tau_1)Y_1 - C_1 + (1-\tau_2)\tilde{Y}_2 \end{aligned} \tag{1}$$

Generically we use ˜ supscript over Y_2 and C_2 as they may not be certain. Optimal condition F.O.C.

$$\begin{aligned} U'(C_1) &= E(U'(\tilde{C}_2)) \\ U'(C_1) &= E(U'((1-\tau_1)Y_1 + (1-\tau_2)\tilde{Y}_2) - C_1)) \end{aligned} \tag{2}$$

Scenario 1: Quadratic Utility

$$C_1 = E(\tilde{C}_2) \tag{3}$$

This is the case of random-walk.

Scenario 2: Income Y_2 is Certain

Expectation sign can be dropped. And the supscript is also dropped.

$$U'(C_1) = U'(C_2) \tag{4}$$

U' is not a constant and typically monotone. Therefore, the only solution to the equality is

$$C_1 = C_2$$

This naturally gives

$$C_1 = E(C_2)$$

Scenario 3: Precautionary Saving
The key condition for precautionary motives to arise is

$$U''' > 0 \tag{5}$$

We consider the case when the expected life-long income does not change. Tax rate τ_2 is adjusted by government in response to different realization of $\tilde{Y}_2$.

$$(1-\tau_1)Y_1 + E[(1-\tau_2)\tilde{Y}_2] = \bar{Y} \tag{6}$$

From $U'''() > 0$, we know marginal utility U' is concex. By Jensen Inequality, we have the right half inequality below

$$U'(C_1) = E[U'(\tilde{C}_2)] > U'(C_2) \equiv U'(E(\tilde{C}_2)) \tag{7}$$

Here $\tilde{C}_2$ is a random variable. $C_2 = E(C_2)$ is its expectation. The marginal utility from a mean-spread consumption is higher than the case of certainty.

This leads to the inequality that reflects precautionary motives.

$$C_1 < E(\tilde{C}_2)$$

The simple 2-period model has important implications about the Recardian Equivalence. As we have shown, the Reicardian Equivalence holds conditional on perfect foresight, namely absense of certainty. If this is not the case, and the consumer has precautionary motives, the classical result does not hold anymore.

3 With Uncertainty of Future Income

Now we shift gear to a slightly different model with specified assumption on the distribution of income in future. Assume income Y_{t+1} or beyond follows log-normal distribution. There is a risk-free return factor R. Time discount factor $\beta \equiv \frac{1}{1+\theta}$. CRRA utility function with ρ being the coefficient of relative risk aversion. Euler equation:

$$C_t^{-\rho} = \beta R E_t[C_{t+1}^{-\rho}] \tag{8}$$

It turns out that we can express this inter-temporal condition in terms of $E(ln(C_{t+1}))$ and its variance σ^2.

Rewrite Equation 8

$$C_t^{-\rho} = \beta R E_t[e^{ln(C_{t+1}^{-\rho})}] \tag{9}$$

$$C_t^{-\rho} = \beta R E_t[e^{-\rho ln(C_{t+1})}] \tag{10}$$

We know

$$ln(C_{t+1}) \sim N(E(ln(C_{t+1})), \sigma^2) \tag{11}$$

$$-\rho ln(C_{t+1}) \sim N(-\rho E(ln(C_{t+1})), \rho^2\sigma^2) \tag{12}$$

By the following mathematic fact

$$x \sim N(\mu, \sigma^2) \quad \Rightarrow E(e^x) = e^{\mu + \sigma^2/2}$$

We have the following:

$$C_t^{-\rho} \approx \beta R e^{-\rho E(ln(C_{t+1})) + \frac{\rho^2\sigma^2}{2}} \tag{13}$$

Taking the log on both sides

$$-\rho ln(C_t) \approx ln(\beta) + ln(R) - \rho E[ln(C_{t+1})] + \frac{\rho^2\sigma^2}{2} \tag{14}$$

Rearranging and approximating.

$$\rho(E_t(ln(C_{t+1})) - ln(C_t)) \approx ln(\beta) + ln(R) + \frac{\rho^2\sigma^2}{2} \tag{15}$$

$$\rho(E_t(ln(C_{t+1})) - ln(C_t)) \approx ln(\frac{1}{1+\theta}) + ln(1+r) + \frac{\rho^2\sigma^2}{2} \tag{16}$$

$$E_t[\Delta ln(C_{t+1})] \equiv E_t(ln(C_{t+1})) - ln(C_t) \approx \frac{1}{\rho}(r-\theta) + \frac{\rho\sigma^2}{2} \tag{17}$$

The second term of Equation 17 is zero if there is no uncertainty regarding future income thus consumption. With uncertainty, $\sigma^2 > 0$, the term is positive. It reflects the precautionary saving motives again. The higher the uncertainty, the greater the consumption growth. The first term shows up in the same form under perfect sight.

4 With Uncertainty of Capital Income

The income risks could come from capital income instead of labor income. Consider a consumer with CRRA utility with wealth W_t at time t. No labor income. The asset earns a stochastical return $\tilde{R}_{t+1}$ in next period. No portfolio choice can be made. It is the only asset the agent can hold.

Consumer's problem remains to pick the optimal consumption subject the the dynamic budget constraint.

$$(W_t - C_t)\tilde{R}_{t+1} = W_{t+1} \tag{18}$$

Then the F.O.C. with respect to C_t:

$$1 = \beta E_t[\tilde{R}_{t+1}(\frac{C_{t+1}}{C_t})^{-\rho}] \tag{19}$$

We assume $\tilde{R}_t = R\epsilon_t \forall t$, where ϵ is log-normally distributed with mean $-\frac{\sigma^2}{2}$ and variance σ^2. Formally:

$$log(\epsilon_t) \sim N(-\frac{\sigma^2}{2}, \sigma^2) \quad \forall t \tag{20}$$

Again, in order to see clearly the precautionary motives that arise from this model, we need to either find an expression for consumption growth or an explicit consumption function for a certain level of wealth. We adopt the later approach in this context. Using guess and verify approach, we start from assuming consumption is a fraction of the wealth. This is legimate as there is no labor income. The current wealth is the life-long wealth the consumer.

$$C_t(W_t) = \gamma W_t \tag{21}$$

By the dynamic budget constraint, we have

$$C_{t+1} = \gamma W_{t+1} = \gamma(1-\gamma)W_t\tilde{R}_{t+1} \tag{22}$$

This can be plugged in the Euler equation 19

$$\begin{aligned} 1 &= \beta E_t[\tilde{R}_{t+1}(\frac{\gamma(1-\gamma)W_t\tilde{R}_{t+1}}{\gamma W_t})^{-\rho}] \\ 1 &= \beta E_t[\tilde{R}_{t+1}^{1-\rho}(1-\gamma)^{-\rho}] \\ (1-\gamma)^{\rho} &= \beta E_t[\tilde{R}_{t+1}^{1-\rho}] \\ \gamma &= 1 - \beta^{1/\rho}(E_t[\tilde{R}_{t+1}^{1-\rho}])^{1/\rho} \end{aligned} \tag{23}$$

It turns out further approximation can be made to the expression γ. The key fact we use is again $E(e^x) \approx e^{\mu}e^{\frac{\sigma^2}{2}}$ if $x \sim N(\mu, \sigma^2)$.

$$\begin{aligned}\gamma &= 1-\beta^{1/\rho}(E_t[R^{1-\rho}\epsilon^{1-\rho}])^{1/\rho}\\ \gamma &= 1-\beta^{1/\rho}R^{\frac{1-\rho}{\rho}}(E_t[\epsilon^{1-\rho}])^{1/\rho}\\ \gamma &= 1-\beta^{1/\rho}R^{\frac{1-\rho}{\rho}}(E_t[e^{ln(\epsilon^{1-\rho})}])^{1/\rho}\\ \gamma &= 1-\beta^{1/\rho}R^{\frac{1-\rho}{\rho}}(E_t[e^{(1-\rho)ln(\epsilon)}])^{1/\rho}\end{aligned} \tag{24}$$

For Equation 20, we have

$$(1-\rho)ln(\epsilon) \sim N(-\frac{(1-\rho)\sigma^2}{2}, \frac{(1-\rho)^2\sigma^2}{2}) \tag{25}$$

Therefore

$$E[e^{(1-\rho)ln(\epsilon)}] \approx e^{-\frac{(1-\rho)\sigma^2}{2}} e^{\frac{(1-\rho)^2\sigma^2}{2}} \tag{26}$$

$$\begin{aligned}\gamma &= 1-\beta^{1/\rho}R^{\frac{1-\rho}{\rho}}(e^{-\frac{(1-\rho)\sigma^2}{2}} e^{\frac{(1-\rho)^2\sigma^2}{2}})^{1/\rho}\\ \gamma &= 1-\beta^{1/\rho}R^{\frac{1-\rho}{\rho}} e^{-\frac{(1-\rho)\sigma^2}{2\rho}} e^{\frac{(1-\rho)^2\sigma^2}{2\rho}}\\ 1-\gamma &= \beta^{1/\rho}R^{\frac{1-\rho}{\rho}} e^{\frac{(\rho-1)\sigma^2}{2}}\end{aligned} \tag{27}$$

$$\begin{aligned}-\gamma \approx ln(1-\gamma) &= \frac{1}{\rho}ln(\beta) + \frac{1-\rho}{\rho}ln(R) + \frac{(\rho-1)\sigma^2}{2}\\ -\gamma &\approx -\frac{1}{\rho}\theta + \frac{1-\rho}{\rho}r + \frac{(\rho-1)\sigma^2}{2}\\ \gamma &\approx \frac{1}{\rho}\theta - \frac{1-\rho}{\rho}r - \frac{(\rho-1)\sigma^2}{2}\\ &...\\ \gamma &\approx \underbrace{r-\frac{1}{\rho}(r-\theta)}_{\kappa=1-p_R} - \underbrace{\frac{(\rho-1)\sigma^2}{2}}_{Precautionary}\end{aligned} \tag{28}$$

It is clear that when $\rho > 1$, namely the prudence of consumer is high enough, a higher uncertainty of income leads to lower marginal propensity to consume outof wealth. This is another example of precautionary motives.

By the same token, we can also show the following.

$$\begin{aligned}\Delta(ln(C_{t+1})) &= ln(\frac{C_{t+1}}{C_t})\\ \Delta(ln(C_{t+1})) &= ln(\frac{\gamma W_{t+1}}{\gamma W_t})\\ \Delta(ln(C_{t+1})) &= ln(\frac{\gamma(1-\gamma)W_t\tilde{R}_{t+1}}{\gamma W_t})\\ \Delta(ln(C_{t+1})) &= ln(1-\gamma)+ln(R)+ln(\epsilon)\\ Var(\Delta ln(C_{t+1})) &= Var(ln(\epsilon)) = \sigma^2\end{aligned} \tag{29}$$