SIR model as a Dynamic Macroeconomic Model

Tao Wang

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1 Introduction

When I was a first-year graduate student in economics, I found many advanced macro models we were supposed to learn essentially boil down to how individuals' choices interact with others in a constantly evolving, dynamic environment. But it is often via self-taught in the context of the specific models that the generic structure and common principles gradually emerge.

Taking separate classes devoted to mathematical tools such as dynamics programming helps, but it does not substitute a generic structure in the context of economic problems, that summarizes the intrinsic nature of these models stripping off the model specifics and technicalities, especially in heterogeneous-agent macro.

To me, all of these economic models inherently share the same nature: how the situation facing an individual agent with certain primitives such as what they like (the preference) and what they think (expectations), changes in a constantly evolving environment which itself changes due to the interaction between the agent's situations in this environment. The change in the individual's situations also typically depends on how agents act in their respective situations. But it does not need to be so, and the component of individual choice is not entirely indispensable in making these models. In some cases where no choice can be made. For instance, one cannot change how time moves forward. Such a change in the situation may not necessarily depend on how the environment and others' situations change, either. Neither of these makes a model less of a macroeconomic model.

Economists typically like to think the purposive behaviors where agents can make choices and take actions in facing different situations as questions economics should study, to me this narrow focus somehow creates a conceptual hurdle for understating the nature of the problem. Even if no choice is made, the individual situations may change due to aggregation/ interaction of all the agents which changes the environment, and it does generate outcomes and affect the well-being/payoffs to every agent in the environment. What's more, oftentimes, even in many dynamic economic models where individual choices are made optimally contingent on their situations, there are exogenous stochastic and endogenous processes that affect the situation of each individual.

2 SIR model

The classical SIR model since the seminal work by Kermack and McKendrick (1927) belongs what epidemiologist broadly refer to as compartmental models. The essence of the model encapsulates ex-ante homogeneous agents transiting between three states associated with the disease infection: susceptible, infected and recovered, and the transitions depending on the aggregate state and the distributions of the infection states across the entire population.

It is through the wide popularity of the epidemiological models, and working in quantitative heterogeneous-agent models, that I realized that actually the classical SIR model in epidemiology presents a great starting example and a conceptual building block of various macroeconomic models, especially those that involve heterogeneity. In this note, I try to recast an SIR model into a macroeconomic model.

3 SIR model through the lens of an economic model

Epidemiologists are concerned about the dynamics of an infectious disease in the population. Economists' concerns typically go at least one step further to the welfare/utility outcome bore by each individual from the disease. We summarize the utility consequence in the following value function.

3.1 Payoff function

$$V(z,Z) = u(z,Z) + \beta E(V'(z',Z')) \tag{1}$$

z is the micro state, and indicates the individual's status of infection, which fall into one of the three possible states $\{s,i,r\}$, representing susceptible, infected and recovered, respectively. Z is the macro state, containing information about the epidemic status of the entire population.

V is the payoff of the individual, which unsurprisingly depends on her own status. But it is worth asking why it does so on the aggregate state Z. In the narrowest sense, how many people are infected and recovered now affect the probability of this individual getting infected in future, according to the SIR model. Of course, in addition to this mechanism, we can also assume there is direct effects of the aggregate epidemic status on individual welfare. [1]

The payoff V is the summation of payoff today u(s,S) and the discounted expected payoff from the future, $\beta E(V'(z',Z'))$. The future utility is within

[1] For instance, the aggregate epidemic affects the economic resources available to the individual agent, or affects her subjective well being via her altruism concerns.

expectation operator because the payoff tomorrow depends on the realized states z' and Z'. Such expectations are evaluated by the perceived law of motions of z and Z.

3.2 Law of motion of idiosyncratic and aggregate state

We first think through the transition of the idiosyncratic state z.

For anyone who is susceptible today, the probabilities of transitioning to different outcomes tomorrow is given by the following.

$$\begin{aligned} p(z' = i|z = s) &= \chi \frac{I}{N} \equiv \beta \\ p(z' = s|z = s) &= 1 - \beta \\ p(z' = r|z = s) &= 0 \end{aligned} \tag{2}$$

where the term χ is the transmission probability given a contact between an infected and a susceptible. Via random mixing assumption, each susceptible person will be likely to gain contact with I/N, which is the population proportion of the infected people in the current period. Notice it is time varying as the fraction of the infected in the population changes each period.

For any individual whose current state is infected ($z = i$). The transition matrix is the following.

$$\begin{aligned} p(z' = i|z = i) &= 1 - \gamma \\ p(z' = s|z = i) &= 0 \\ p(z' = r|z = i) &= \gamma \end{aligned} \tag{3}$$

The recovery rate γ, typically implicitly depending on biological forces, is assumed to be a constant and state-dependent.

For any individual whose current state is recovered ($z = r$), the transition matrix is the simplest.

$$\begin{aligned} p(z' = i|z = r) &= 0 \\ p(z' = s|z = r) &= 0 \\ p(z' = r|z = r) &= 1 \end{aligned} \tag{4}$$

The only possible transition for a recovered person is to stay in the recovered state.

We can summarize the entire transition matrix of the individual state among $\{s, i, r\}$ as the following.

$$\pi = \begin{bmatrix} 1-\beta & \beta & 0 \\ 1-\gamma & 0 & \gamma \\ 0 & 0 & 1 \end{bmatrix} \tag{5}$$

The transition matrix is state-dependent because the infection rate β depends on the contemporary fraction of the population that has been infected, which itself changes over time.

The key assumption underlying the classical SIR model is **random-mixing**. The assumption states that every individual in the population is equally likely to encounter anyone else in each period, regardless their own and the others' status of infection. This makes the aforementioned transition matrix of each individual easily collapse to an aggregate transition matrix of the aggregate state of infection Z.

To see this through clearly, let's start by pointing out that, generally in an environment where the distribution across agents matter for the aggregate dynamics, Z could be a high-dimension object. In the extreme case, it shall a size equal to the number of agents in the economy, i.e. every one's states matters for the aggregate dynamics.

But to the extent that in the simple SIR model, it is only the fraction of agents in each of the three compartments/states that affect the aggregate and individual dynamics, Z's dimension can substantially reduce to just three and all it needs to contain is the fraction of agents in the population currently in each state.

Furthermore, the transition function of the aggregate state Z, we can call Π, turns out to be exactly the same as the individual transition matrix π.

$$\Pi = \pi \tag{6}$$

The transition between aggregate state Z and Z' tomorrow follows the following dynamics.

$$Z' = \Pi \times Z \tag{7}$$

This is equivalent to the standard representation via a system of three differential equations as written below.

$$\begin{aligned}
\frac{\Delta S}{N} &= -\beta \frac{S}{N} \\
\frac{\Delta I}{N} &= \beta S - \gamma \frac{I}{N} \\
\frac{\Delta R}{N} &= \gamma \frac{I}{N}
\end{aligned} \tag{8}$$

3.3 Taking stock

By now, I have managed to recast the SIR model as a dynamic macroeconomic model where the individual's payoff depends on the constantly evolving individual state, whose change is driven by the interaction of individual dynamics in aggregation. In particular, it entails the value function $V(z, Z)$, the transition dynamics of the state variable z, namely π, and that of Z, namely Π.

A wide range of macroeconomic models can be essentially represented in the same manner. One of the best examples is the class of one-sided-matching models of the labor market, in which workers switch between various statuses such as unemployed and employed with exogenous separation and job-finding. The transition of the individual state depends on the evolution of the distribution of states among the population. For instance, the job-finding rate depends on how many people are currently unemployed.

Granted, many of these models are extended to incorporate these transitions are not entirely a consequence of exogenous transitions but partially depend on the choices of the agents, say job-accepting decisions or search efforts McCall (1970). Or it may depend on another type of agents' behaviors, i.e. firms, like in two-sided matching (Mortensen and Pissarides (1994)). But these extensions barely mask the core nature of the problem, as summarized above.

4 Make it even "more like" an economic model

Till this point, we have assumed that the utility consequence of agents V is a direct function of the state z and Z and no individual choices can be made to change the utility consequences either directly, or via altering the transitions of z and Z. To put it bluntly, each agent just takes the dynamics as given and bear the outcomes.

An economist would not be satisfied with stopping here, or even somewhat ironically, some dogmatic ones might think an economic model starts from here.

Either way, to make this model more of an economic model, we assume the agents at least could take certain actions given the states to maximize her expected utility. A minor modification of the value function above indicates such an extension.

$$V(z, Z) = u(a^*(z, Z)) + \beta E(V'(z', Z')) \tag{9}$$

The key difference between Equation 1 and Equation 9 is that the latter embeds an optimization problem of the agent, which involves choosing an action a^* given the states z and Z.[2] The true nature of a^* are contingent plans, or what economists would call "policies", instead of a fixed action.

And it goes without saying, the most common case in economics is to assume that the action a^* is the action that maximizes the expected utility.

$$a^*(z, Z) = \underset{a}{argmax} \quad u(a(z, Z)) + \beta E(V'(z', Z')) \tag{10}$$

Specifying the substance in this generic model structure and finding its solutions constitutes the main body of economic modeling. But the intrinsic nature of such problem is always the same.

[2]In dynamic programming terms, only the value function as written in Equation 9 is the value function, while we call the "value function" in Equation 1 Bellman equation as representing utility consequences of any state variable without the optimization involved.

5 Concluding remarks

Nevertheless, models sharing this structure span from those with analytical solutions to ones that can be only solved numerically.

Moving from a representative-agent model to heterogeneous agent models specifically complicates this problem by making the aggregate state Z high-dimensional, not to mention its transition dynamics, which obviously depend on the interactions of choices among agents and their induced transition of the aggregate state. In addition, modelers also need to carefully define the specific equilibrium under study. This is so because the perceived laws of the aggregate state that affect the actions of agents may generally not induce transitions that are perceived in the first place when they take actions. [3] The Rational Expectation assumption, which means the perceived law and induced law converge, is often invoked in these models. But the recent macroeconomic literature has gradually loosened such requirements and started exploring more interesting dynamics.

References

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[3] These points were famously made clear and loud by Krusell and Smith (1998).