Physica A 516 (2019) 37–49

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Fixed-time bipartite consensus of multi-agent systems with disturbances

Qun Deng, Jie Wu *, Tao Han, Qing-Sheng Yang, Xiu-Shan Cai

College of Mechatronics and Control Engineering, Hubei Normal University, Huangshi, 435002, PR China

HIGHLIGHTS

- The problem of fixed-time bipartite consensus with disturbances is investigated.
- The settling time is explicitly given whatever the initial conditions change.
- Fixed-time bipartite consensus is realized no matter under what topology.

ARTICLE INFO

Article history:
Received 28 June 2018
Received in revised form 14 August 2018
Available online 27 September 2018

Keywords:
Fixed-time bipartite consensus
Disturbances
Multi-agent systems

ABSTRACT

This article settles the fixed-time bipartite consensus for multi-agent systems that are subject to disturbances or not. The established protocols are aimed at controlling networked agents to reach agreement bipartitely, which reveals that a final state is the same in quantity but not in sign. By introducing algebraic graph theory, Lyapunov analysis and stability theory of fixed-time, the settling time associated with the design parameters and network connectivity rather than the initial states is explicitly given. The rationality of theoretical analysis is demonstrated via simulations.

1. Introduction

In the past few years, control systems including networked control system [1–4] and multi-agent system [5–8] have developed dramatically. Different from networked control systems, multi-agent systems not only provide reasonable explanations for complex phenomena in nature and society, but also can reveal natural and social laws more profoundly. Consensus, as one of typical collective behavior in multi-agent systems, means that all agents are ensured to reach a common quantity by local interactions. Moreover, it has received extensive concern owing to potential applications in many spheres (e.g. flocking control [9,10], formation control [11,12], containment control [13,14] etc.). In real system networks, however, it is noted that the agents not only collaborate but also compete. Compared with the consensus or agreement in cooperative interactions, a special type of consensus phenomenon is called bipartite consensus, where all the agents converge to final states with same magnitude but opposite sign. Wen et al. [15] addressed distributed bipartite tracking consensus problem in the presence of a dynamic leader, which is subject to bounded control inputs. Up to now, various control approaches have been applied to the distributed bipartite consensus protocols design such as adaptive control [16,17], feedback control [18,19] and forth on.

In the analysis of bipartite consensus, the convergence rate is considered as a key performance indicator. Unfortunately, the bipartite consensus in the most literatures are only guaranteed asymptotically when consider control accuracy [20] and robustness uncertainty [21]. In the view of time optimization, finite-time consensus has been widely studied, which

* Corresponding author.
E-mail address: Jiewu@hbnu.edu.cn (J. Wu).

https://doi.org/10.1016/j.physa.2018.09.066

can contribute desired behavior to some cases especially when consider the performance for networks of agents in a time interval. Meng et al. [22] took into account two types of protocols to address consensus problems within a finite time, where the interactions between agents are cooperative and antagonistic. Shang [23] presented sufficient criteria under fixed topology to ensure an agreement in finite time. Wang et al. [24] proposed two effective continuous finite-time consensus algorithms. Moreover, performance factors composed of convergence time, initial states and so on are all analyzed. However, it is investigated that settling time above is severely affected by the condition of initial states. For example, if initial states of agents increased numerically, the settling time would increase. Therefore, it will have difficulty in acquiring exact value of settling time for hardly getting information in advance.

Recently, on account of superiorities such as faster rate of convergence [25] and better disturbance rejection properties [26], more and more scholars are dedicated to the study of fixed-time consensus problem. As opposed to the finite-time consensus laws, the settling time is excellent in the independence on the initial states. That is to say, it is possible to predetermine accurate settling time by designating the appropriate parameters. Polyakov firstly explored fixed-time stability in 2012. Defoort et al. [27] dealt with fixed-time consensus of single-integrator agents with unknown dynamics. Zhang et al. [28] discussed both linear and nonlinear state measurements. Such laws are applicable to address fixed-time consensus problem. Meng et al. [29] focused on fixed-time consensus of networked agents, and revealed that interaction topologies are of importance in reaching consensus. However, as far as we know, it is of challenge to generalize the fixed-time consensus to fixed-time bipartite consensus since the agents will compete in networks with antagonistic interactions. In addition, it is unavoided affected by external disturbances in networked environments. Therefore, there are few studies on fixed-time bipartite consensus of networked agents with disturbances.

Inspired by the above discussions, we present two types of fixed-time bipartite consensus algorithms in networks with agents. The main contributions of the paper are threefold: (i) We make an endeavor to come up with fixed-time bipartite consensus laws, which guarantee all agents reach the agreement in the presence of disturbances or not. (ii) Under established algorithms, all the agents have the ability to converge to desired states with a bounded time whatever initial conditions change. (iii) Fixed-time bipartite consensus is achieved under structurally balanced and structurally unbalanced signed graph.

The remaining parts of this article are written as follows. In Section 2, basic notations and algebraic graph theory reviewed in brief. In Section 3, after introducing related lemmas, we drive the law to solve fixed-time bipartite consensus with disturbances. In Section 4, we construct a nonlinear protocol to handle the fixed-time bipartite consensus with no disturbances under different topology. The simulations results and conclusions are respectively developed in Sections 5 and 6.

2. Preliminaries

2.1. Notations

$\mathbb{R}^n$ stands for n-dimensional Euclidean space, $\mathcal{I}_n = \{1, 2, \ldots, n\}$ and $\mathbf{1} = [1, 1, \ldots, 1]^T$ are used to denote the set and unit vector, respectively. $\lambda(\cdot)$ represents the eigenvalue of the symmetric matrix. $\mathcal{D} = \left\{D = diag\left\{d_1, d_2, \ldots, d_n\right\}, d_i = \{\pm 1\}\right\}$ denotes a diagonal matrix with elements $d_1, d_2, \ldots d_n$. Let $sign\,()$ be sign function with the definition

$$sign\,(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}.$$

2.2. Graph theory

Without loss of generality, let networks consisting of n agents, which share identical state space $\mathbb{R}$. Weighted graph $\mathcal{G}$ is a triple $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where node set $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$, edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ and adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$. Furthermore, $\left(v_j, v_i\right) \in \mathcal{E}$ means $a_{ij} \neq 0$. $\mathcal{G}$ has no self-loops means $a_{ii} = 0$ and is undirected implies $a_{ij} = a_{ji}$. We call v_j a neighbor of v_i for $\left(v_j, v_i\right) \in \mathcal{E}$ and denote the set of neighbors as $N_i = \{j : (j, i) \in \mathcal{E}\}$. $\mathcal{G}$ has a path formed by a sequence of nodes $i_1, \ldots, i_m$ such that $(i_l, i_{l+1}) \in \mathcal{E}, \forall l = 1, \ldots, m-1$. Connected $\mathcal{G}$ renders any distinct nodes i and j communicated via paths. From [30], $\mathcal{G}$ is considered as structurally balanced when the nodes $\mathcal{V}_1, \mathcal{V}_2$ possess a bipartition, where $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$ and $V_1 \cap V_2 = \emptyset$, such that $a_{ij} \geq 0$ if $\forall v_i, v_j \in \mathcal{V}_k\,(k \in \{1, 2\})$, $a_{ij} \leq 0$ if $\forall v_i \in \mathcal{V}_k, v_j \in \mathcal{V}_l, k \neq l\,(k, l \in \{1, 2\})$. In addition, for structurally balanced graph $\mathcal{G}$, there exists diagonal matrix D satisfies all diagonal elements of DLD are nonnegative.

2.3. Some useful lemmas

To build fixed-time bipartite consensus algorithms, the following lemmas are required because they act a pivotal part in proof procedures.

Lemma 1 ([31]). *Let Laplacian matrix $L = \left[l_{ij}\right] \in R^{n\times n}$ related to $\mathcal{G}$, whose elements are*

$$l_{ij} = \begin{cases} \sum\limits_{k=1,k\neq j}^{n} |a_{ik}|\,, & j = i \\ -\,a_{ij}, & j \neq i \end{cases}.$$

For any $\omega = [\omega_1, \omega_2 \ldots, \omega_n]^T \in R^n$, $\omega^T L\omega = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\left(\omega_j - \omega_i\right)^2$. For undirected and connected graph, $\lambda_2(L) = \min_{\omega\neq 0,\,1^T\omega=0}\frac{\omega^T L\omega}{\omega^T\omega}$. Therefore, when $1^T\omega = 0$, one can obtain $\omega^T L\omega \geq \lambda_2(L)\,\omega^T\omega$.

Lemma 2 ([32]). *If $\xi_1, \xi_2, \ldots, \xi_n \geq 0$ and $0 \leq \delta \leq 1$, then*

$$\sum_{i=1}^{n} {\xi_i}^{\delta} \geq \left(\sum_{i=1}^{n} \xi_i\right)^{\delta}.$$

If $\xi_1, \xi_2, \ldots, \xi_n \geq 0$ and $\delta \geq 1$, then

$$\sum_{i=1}^{n} \xi_i^{\delta} \geq n^{1-\delta}\left(\sum_{i=1}^{n} \xi_i\right)^{\delta}.$$

Lemma 3. *For connected graph $\mathcal{G}$, if it is structurally balanced, a Laplacian candidate function is described by*

$$V(x) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|a_{ij}\right|\left(x_j - sign\left(a_{ij}\right)x_i\right)^2 = x^T Lx, \tag{1}$$

then there satisfies

$$x^T L^T Lx \geq \lambda_2(L)\,V(x).$$

Proof. Since $L = L^T$ is semipositive definite Laplacian, then a semipositive definite matrix $M = M^T$ is obtained, which satisfies $L = M^T M = M^2$. Due to $L1^n = 0$, we obtain $1_n^T M = 0$ which guarantees $1_n^T(Mx) = 0$. Thus, we use Lemma 1 to deduce

$$\begin{aligned} x^T L^2 x &= x^T M^4 x = (Mx)^T L\,(Mx) \geq \lambda_2(L)\,(Mx)^T\,(Mx) \\ &= \lambda_2(L)\,x^T Lx = \lambda_2(L)\,V(x). \end{aligned}$$

Since $\mathcal{A}$ is symmetrical, it can be expressed as

$$\frac{\partial V(x)}{\partial x_i} = -2\sum_{j=1}^{n} a_{ij}\left(x_j - sign\left(a_{ij}\right)x_i\right).$$

Further, the derivative of $V(x)$ is described by

$$\begin{aligned} \dot{V}(x) &= \textstyle\sum_{i=1}^{n}\frac{\partial V(x)}{\partial x}\dot{x}_i \\ &= -2\sum_{i=1}^{n}\left(\sum_{j\in N_i} a_{ij}\left(x_j - sign\left(a_{ij}\right)x_i\right)\right)(u_i + w_i). \end{aligned} \tag{2}$$

Lemma 4. *For all i and t, it follows that*

$$-w_i\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}\left(x_j - sign\left(a_{ij}\right)x_i\right)\right) \leq b_w\sum_{i=1}^{n}\left|\sum_{j=1}^{n} a_{ij}\left(x_j - sign\left(a_{ij}\right)x_i\right)\right|.$$

Proof. Due to direct result of the boundedness of the disturbance, i.e., $|w_i(t)| \leq b_w, \forall i, t$, we have

$$|w_i|\sum_{i=1}^{n}\left|\sum_{j=1}^{n} a_{ij}\left(x_j - sign\left(a_{ij}\right)x_i\right)\right| \leq b_w\sum_{i=1}^{n}\left|\sum_{j=1}^{n} a_{ij}\left(x_j - sign\left(a_{ij}\right)x_i\right)\right|.$$

Additionally,

$$-w_i\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}\left(x_j - sign\left(a_{ij}\right)x_i\right)\right) \leq |w_i|\sum_{i=1}^{n}\left|\sum_{j=1}^{n} a_{ij}\left(x_j - sign\left(a_{ij}\right)x_i\right)\right|.$$

As a result of inequality transitivity, we can deduce

$$-w_i \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} \left(x_j - sign\left(a_{ij}\right) x_i \right) \right) \leq b_w \sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij} \left(x_j - sign\left(a_{ij}\right) x_i \right) \right|.$$

Lemma 5 ([33]). *Given scalar system*

$$\dot{z} = -\beta z^{\frac{\varepsilon}{r}} - \gamma z^{\frac{p}{q}},$$

where $\beta > 0, \gamma > 0$, ε, r, p, and q are all positive integers, if $\varepsilon > r$ and $q > p$, then for any initial state $z(0)$, $\lim_{t\to T} z(t) = 0$ and $z(t) = 0 \ \ t \geq T$, where T is bounded by

$$T \leq \frac{1}{\beta} \frac{r}{\varepsilon - r} + \frac{1}{\gamma} \frac{q}{q-p}.$$

3. Fixed-time bipartite consensus with disturbances

In this section, let us begin with the following kinematics

$$\dot{x}_i(t) = u_i(t) + w_i(t), \tag{3}$$

where x_i and u_i are known as state and input, respectively. w_i denotes the external disturbances such that $|w_i(t)| \leq b_w$ and $b_w \geq 0$. The problem worked out in this section is that all agents is ensured to reach fixed-time bipartite consensus objective (4), i.e., $T \in [0, \infty)$ exists such that

$$\lim_{t\to T} x_i(t) = x^*(t),$$

and

$$x_i(t) = -x^*(t) \qquad t \geq T \ , \forall i \in \mathcal{I}_n, \tag{4}$$

where $x^*(t)$ is nontrivial trajectory.

Based on the properties of Lyapunov matrix and the graph theory, the neighbor-based algorithm is employed to handle the fixed-time bipartite consensus as

$$\begin{aligned} u_i = {} & \mu sign\left(\sum_{j\in N_i} a_{ij} \left(x_j - sign\left(a_{ij}\right) x_i \right) \right) \times \left| \sum_{j\in N_i} a_{ij} \left(x_j - sign\left(a_{ij}\right) x_i \right) \right|^{2-\frac{1}{m}} \\ & + \eta sign\left(\sum_{j\in N_i} a_{ij} \left(x_j - sign\left(a_{ij}\right) x_i \right) \right) \times \left| \sum_{j\in N_i} a_{ij} \left(x_j - sign\left(a_{ij}\right) x_i \right) \right|^{\frac{1}{m}} \\ & + \alpha sign\left(\sum_{j\in N_i} a_{ij} \left(x_j - sign\left(a_{ij}\right) x_i \right) \right), \end{aligned} \tag{5}$$

where $\mu > 0, \eta > 0, m > 1, \alpha \geq b_w$, m is positive odd integer.

In the following, according to the above discussion, the main result is stated detailedly.

Theorem 1. *Given the system (3), suppose $\mathcal{G}$ is structurally balanced. Then the protocol (5) renders objective (4) achieved under structurally balanced graph $\mathcal{G}$, where T_1 satisfies*

$$T_1 \leq \frac{1}{\lambda_2(L)} \left(\frac{n^{\frac{m-1}{2m}}}{\mu} + \frac{1}{\eta} \right) \frac{m}{m-1}. \tag{6}$$

Proof. Establishing the Lyapunov function as Eq. (1), and incorporate protocol (5) into (2). Its time derivative is expressed as

$$\begin{aligned} \dot{V}(x) = {} & -2\mu \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} \left(x_j - sign\left(a_{ij}\right) x_i \right) \right)^{3-\frac{1}{m}} - 2\eta \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} \left(x_j - sign\left(a_{ij}\right) x_i \right) \right)^{1+\frac{1}{m}} \\ & - 2\alpha \sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij} \left(x_j - sign\left(a_{ij}\right) x_i \right) \right| - 2w_i \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} \left(x_j - sign\left(a_{ij}\right) x_i \right) \right). \end{aligned}$$

Together with Lemma 4, we derive

$$\begin{aligned}\dot{V}(x) &\leq -2\mu\sum_{i=1}^{n}\left(\sum_{j=1}^{n}a_{ij}\left(x_j-sign\left(a_{ij}\right)x_i\right)\right)^{3-\frac{1}{m}}-2\eta\sum_{i=1}^{n}\left(\sum_{j=1}^{n}a_{ij}\left(x_j-sign\left(a_{ij}\right)x_i\right)\right)^{1+\frac{1}{m}}\\&\quad-2\left(\alpha-b_w\right)\sum_{i=1}^{n}\left|\sum_{j=1}^{n}a_{ij}\left(x_j-sign\left(a_{ij}\right)x_i\right)\right|.\end{aligned}$$

Since $\alpha \geq b_w$, we yield

$$\dot{V}(x) \leq -2\mu\sum_{i=1}^{n}\left(\left(\sum_{j=1}^{n}a_{ij}\left(x_j-sign\left(a_{ij}\right)x_i\right)\right)^2\right)^{\frac{3m-1}{2m}}-2\eta\sum_{i=1}^{n}\left(\left(\sum_{j=1}^{n}a_{ij}\left(x_j-sign\left(a_{ij}\right)x_i\right)\right)^2\right)^{\frac{m+1}{2m}}. \tag{7}$$

Owing to $m > 1$, thus

$$\frac{(3m-1)}{2m} > 1,$$

$$\frac{(m+1)}{2m} < 1. \tag{8}$$

With the application of Lemma 2, we obtain

$$\sum_{i=1}^{n}\left(\left(\sum_{j=1}^{n}a_{ij}\left(x_j-sign\left(a_{ij}\right)x_i\right)\right)^2\right)^{\frac{3m-1}{2m}} \geq n^{\frac{1-m}{2m}}\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}a_{ij}\left(x_j-sign\left(a_{ij}\right)x_i\right)\right)^2\right)^{\frac{3m-1}{2m}}. \tag{9}$$

Similarly,

$$\sum_{i=1}^{n}\left(\left(\sum_{j=1}^{n}a_{ij}\left(x_j-sign\left(a_{ij}\right)x_i\right)\right)^2\right)^{\frac{m+1}{2m}} \geq \left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}a_{ij}\left(x_j-sign\left(a_{ij}\right)x_i\right)\right)^2\right)^{\frac{m+1}{2m}}. \tag{10}$$

Substituting (9) and (10) into (7), we deduce

$$\dot{V}(x) \leq -2\mu n^{\frac{1-m}{2m}}\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}a_{ij}\left(x_j-sign\left(a_{ij}\right)x_i\right)\right)^2\right)^{\frac{3m-1}{2m}}-2\eta\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}a_{ij}\left(x_j-sign\left(a_{ij}\right)x_i\right)\right)^2\right)^{\frac{m+1}{2m}}. \tag{11}$$

Through simple calculations, we can see

$$\begin{aligned}a_{1j}\left(x_j-sign\left(a_{1j}\right)x_i\right) &= \left|a_{1j}\right| sign\left(a_{1j}\right)\left(x_j-sign\left(a_{1j}\right)x_i\right)\\&= \left|a_{1j}\right|\left(x_j sign\left(a_{1j}\right)-sign^2\left(a_{1j}\right)x_i\right)\\&= -\left|a_{1j}\right|\left(x_i-sign\left(a_{1j}\right)x_i\right).\end{aligned}$$

Next, we can deduce

$$\begin{bmatrix}\sum_{j=1}^{n}a_{1j}\left(x_j-sign\left(a_{1j}\right)x_i\right)\\ \vdots\\ \sum_{j=1}^{n}a_{nj}\left(x_j-sign\left(a_{nj}\right)x_i\right)\end{bmatrix} = -\begin{bmatrix}\sum_{j=1}^{n}\left|a_{1j}\right|\left(x_i-sign\left(a_{1j}\right)x_i\right)\\ \vdots\\ \sum_{j=1}^{n}\left|a_{nj}\right|\left(x_i-sign\left(a_{nj}\right)x_i\right)\end{bmatrix} = -Lx.$$

Additionally,

$$\begin{aligned}&\sum_{i=1}^{n}\left(\sum_{j=1}^{n}a_{ij}\left(x_j-sign\left(a_{ij}\right)x_i\right)\right)^2\\&=\left[\sum_{j=1}^{n}a_{1j}\left(x_j-sign\left(a_{1j}\right)x_i\right)\ldots\sum_{j=1}^{n}a_{nj}\left(x_j-sign\left(a_{nj}\right)x_i\right)\right]\begin{bmatrix}\sum_{j=1}^{n}a_{1j}\left(x_j-sign\left(a_{1j}\right)x_i\right)\\ \vdots\\ \sum_{j=1}^{n}a_{nj}\left(x_j-sign\left(a_{nj}\right)x_i\right)\end{bmatrix}.\end{aligned}$$

Hence,

$$\sum_{i=1}^{n}\left(\sum_{j=1}^{n}a_{ij}\left(x_j-sign\left(a_{ij}\right)x_i\right)\right)^2=(-Lx)^T(-Lx)=x^TL^TLx. \tag{12}$$

By incorporating (12) into (11), we can obtain

$$\dot{V}(x)\leq-2\mu n^{\frac{1-m}{2m}}\left(x^TL^TLx\right)^{\frac{3-\frac{1}{m}}{2}}-2\eta\left(x^TL^TLx\right)^{\frac{1+\frac{1}{m}}{2}}.$$

Due to $x^TL^TLx\geq\lambda_2(L)V(x)$ from Lemma 3, we get that

$$\dot{V}(x)\leq-2\mu n^{\frac{1-m}{2m}}(\lambda_2(L)V(x))^{\frac{3-\frac{1}{m}}{2}}-2\eta(\lambda_2(L)V(x))^{\frac{1+\frac{1}{m}}{2}} \tag{13}$$

Denoting $\Psi(x)=(\lambda_2(L)V(x))^{1/2}$, which leads (10) to

$$\begin{aligned}\dot{\Psi}(x)&\triangleq\frac{d\Psi(x)}{dt}\\&\leq\frac{\lambda_2(L)}{2\Psi(x)}\left(-2\mu n^{\frac{1-m}{2m}}(\Psi(x))^{\frac{3m-1}{m}}-2\eta(\Psi(x))^{\frac{m+1}{m}}\right)\\&=\lambda_2(L)\left(-\mu n^{\frac{1-m}{2m}}(\Psi(x))^{\frac{2m-1}{m}}-\eta(\Psi(x))^{\frac{1}{m}}\right).\end{aligned} \tag{14}$$

Based on (14), combine Lemma 5 with comparison principle of differential equations [34], one has

$$\lim_{t\to T}\Psi(x)=0\ \ and\ \ \Psi(x)=0,\ \ t\geq T_1, \tag{15}$$

where T_1 is a settling time satisfying (6). By $V(x)=[\Psi(x)]^2/\lambda_2$, we are in position to follow from (15) that $\lim_{t\to T}V(x)=0\ \ and\ \ V(x)=0,\ \forall t\geq T_1$. Therefore, the fixed-time bipartite consensus objective of (4) is derived. The proof is complete. ■

4. Fixed-time bipartite consensus with no disturbances

In this section, let us begin with the following kinematics

$$\dot{x}_i(t)=u_i(t), \tag{16}$$

where x_i and u_i are known as state and input, respectively. The problem worked out in this section is that all agents is ensured to reach fixed-time bipartite consensus objective (17), i.e., $T\in[0,\infty)$ exists such that

$$\lim_{t\to T}x_i(t)=\sigma_i c,$$

and

$$x_i(t)=\sigma_i c,\quad t\geq T, \tag{17}$$

where $\sigma_i\in\{1,-1\}$, $c=\frac{1}{n}\sum_{j=1}^{n}\sigma_j x_j(0)$, for structurally balanced $\mathcal{G}$, or

$$c=0,$$

for structurally unbalanced $\mathcal{G}$.

Based on the properties of Lyapunov matrix and the graph theory, the neighbor-based algorithm is employed to handle the fixed-time bipartite consensus problem as

$$\begin{aligned} u_i = \mu \sum_{j \in N_i} a_{ij} sign\left(x_j - sign\left(a_{ij}\right) x_i\right) \left|x_j - sign\left(a_{ij}\right) x_i\right|^{2-\frac{1}{m}} \\ + \eta \sum_{j \in N_i} a_{ij} sign\left(x_j - sign\left(a_{ij}\right) x_i\right) \left|x_j - sign\left(a_{ij}\right) x_i\right|^{\frac{1}{m}} \end{aligned}, \tag{18}$$

where μ, η *and* m have been stated in protocol (5).

Remark 1. Notice that the law (18) is particular case of law (5). Obviously, it reduces the result of Theorem 1 to networks in which the disturbances is zero.

Lemma 6 (*[32]*)**.** *Consider agents given by* (16) *under structurally balanced* $\mathcal{G}$, *if employ the protocol* (18), $\phi(t)$ *is time-invariant, where*

$$\phi(t) = \frac{1}{n} \sum_{i=1}^{n} \sigma_i x_i(t).$$

Moreover, assume that $\varepsilon_i(t) = \sigma_i x_i(t) - \phi$ *then* $\dot{\varepsilon}_i(t) = \sigma_i u_i(t)$ *and* $\sum_{i=1}^{n} \varepsilon_i(t) = 0$. *The proof is omitted.*

In the following, according to the above discussion, another result is stated detailedly.

Theorem 2. *Given the system* (16). *For structurally balanced* $\mathcal{G}$, *protocol* (18) *renders objective* (17) *achieved, where* T_2 *satisfies*

$$T_2 \le \frac{1}{\lambda_{\min}} \left(\frac{n^{\frac{m-1}{2m}}}{\mu} + \frac{1}{\eta} \right) \frac{m}{m-1}. \tag{19}$$

For structurally unbalanced $\mathcal{G}$, *protocol* (18) *renders each agent convergent to zero asymptotically in fixed time, where* T_3 *satisfies*

$$T_3 \le \left(\frac{n^{\frac{m-1}{2m}}}{\mu(\lambda_1 L(C))^{\frac{3m-1}{2m}}} + \frac{1}{\eta(\lambda_1 L(D))^{\frac{m+1}{m}}} \right) \frac{m}{m-1}.$$

Proof. For structurally balanced $\mathcal{G}$, we establish Lyapunov functional as $\Phi(\varepsilon) = \varepsilon^T \varepsilon$ through Lemma 6, where $\varepsilon = [\varepsilon_1, \varepsilon_2, \ldots \varepsilon_n]^T$. Its time derivative is expressed as

$$\dot{\Phi}(\varepsilon) = \frac{d\Phi(\varepsilon)}{dt} = 2 \sum_{i=1}^{n} \varepsilon_i \dot{\varepsilon}_i = 2 \sum_{i=1}^{n} \varepsilon_i (\sigma_i u_i). \tag{20}$$

Combined Eq. (18) with (20), we have

$$\begin{aligned} \sigma_i u_i &= \sigma_i \mu \sum_{j \in N_i} a_{ij} sign\left(x_j - sign\left(a_{ij}\right) x_i\right) \left|x_j - sign\left(a_{ij}\right) x_i\right|^{2-\frac{1}{m}} \\ &\quad + \sigma_i \eta \sum_{j \in N_i} a_{ij} sign\left(x_j - sign\left(a_{ij}\right) x_i\right) \left|x_j - sign\left(a_{ij}\right) x_i\right|^{\frac{1}{m}} \\ &= \mu \sum_{j \in N_i} \sigma_i a_{ij} sign\left(x_j - \sigma_i \sigma_j x_i\right) \left|x_j - \sigma_i \sigma_j x_i\right|^{2-\frac{1}{m}} \\ &\quad + \eta \sum_{j \in N_i} \sigma_i a_{ij} sign\left(x_j - \sigma_i \sigma_j x_i\right) \left|x_j - \sigma_i \sigma_j x_i\right|^{\frac{1}{m}} \\ &= \mu \sum_{j \in N_i} \sigma_i \sigma_j a_{ij} sign\left(\sigma_j x_j - \sigma_i x_i\right) \left|\sigma_j x_j - \sigma_i x_i\right|^{2-\frac{1}{m}} \\ &\quad + \eta \sum_{j \in N_i} \sigma_i \sigma_j a_{ij} sign\left(\sigma_j x_j - \sigma_i x_i\right) \left|\sigma_j x_j - \sigma_i x_i\right|^{\frac{1}{m}} \\ &= \mu \sum_{j \in N_i} \left|a_{ij}\right| sign\left(\varepsilon_j - \varepsilon_i\right) \left|\varepsilon_j - \varepsilon_i\right|^{2-\frac{1}{m}} \\ &\quad + \eta \sum_{j \in N_i} \left|a_{ij}\right| sign\left(\varepsilon_j - \varepsilon_i\right) \left|\varepsilon_j - \varepsilon_i\right|^{\frac{1}{m}}. \end{aligned}$$

After that, we obtain

$$\begin{aligned}
\dot{\Phi}(\varepsilon) &= 2\sum_{i=1}^{n}\varepsilon_i\mu\sum_{j\in N_i}\left|a_{ij}\right|sign\left(\varepsilon_j-\varepsilon_i\right)\left|\varepsilon_j-\varepsilon_i\right|^{2-\frac{1}{m}} \\
&\quad +2\sum_{i=1}^{n}\varepsilon_i\eta\sum_{j\in N_i}\left|a_{ij}\right|sign\left(\varepsilon_j-\varepsilon_i\right)\left|\varepsilon_j-\varepsilon_i\right|^{\frac{1}{m}} \\
&= \mu\sum_{i,j=1}^{n}\left|a_{ij}\right|\left(\varepsilon_i-\varepsilon_j\right)sign\left(\varepsilon_j-\varepsilon_i\right)\left|\varepsilon_j-\varepsilon_i\right|^{2-\frac{1}{m}} \\
&\quad +\eta\sum_{i,j=1}^{n}\left|a_{ij}\right|\left(\varepsilon_i-\varepsilon_j\right)sign\left(\varepsilon_j-\varepsilon_i\right)\left|\varepsilon_j-\varepsilon_i\right|^{\frac{1}{m}} \\
&= -\mu\sum_{i,j=1}^{n}\left|a_{ij}\right|\left|\varepsilon_j-\varepsilon_i\right|^{\frac{3m-1}{m}}-\eta\sum_{i,j=1}^{n}\left|a_{ij}\right|\left|\varepsilon_j-\varepsilon_i\right|^{\frac{m+1}{m}} \\
&= -\mu\textstyle\sum_{i,j=1}^{n}\left(\left|a_{ij}\right|^{\frac{2m}{3m-1}}\left(\varepsilon_j-\varepsilon_i\right)^2\right)^{\frac{3m-1}{2m}}-\eta\sum_{i,j=1}^{n}\left(\left|a_{ij}\right|^{\frac{2m}{m+1}}\left(\varepsilon_j-\varepsilon_i\right)^2\right)^{\frac{m+1}{2m}}.
\end{aligned} \tag{21}$$

Due to $m>1$, thus

$$\frac{(3m-1)}{2m}>1,$$

$$\frac{(m+1)}{2m}<1.$$

By employing Lemma 5 twice, we can further derive from (21) that

$$\dot{\Phi}(\varepsilon)\leq-\mu n^{\frac{1-m}{2m}}\left(\sum_{i,j=1}^{n}\left|a_{ij}\right|^{\frac{2m}{3m-1}}\left(\varepsilon_j-\varepsilon_i\right)^2\right)^{\frac{3m-1}{2m}}-\eta\left(\sum_{i,j=1}^{n}\left|a_{ij}\right|^{\frac{2m}{m+1}}\left(\varepsilon_j-\varepsilon_i\right)^2\right)^{\frac{m+1}{2m}}.$$

Suppose that $C=\left[c_{ij}\right]\in R^{n\times n}$, $D=\left[d_{ij}\right]\in R^{n\times n}$, where $c_{ij}=a_{ij}^{\frac{2m}{3m-1}}$, $d_{ij}=a^{\frac{m+1}{2m}}$. Since $1^T\varepsilon=\sum_{i=1}^{n}\varepsilon_i(t)=0$, we can employ Lemma 1 to deduce

$$\dot{\Phi}(\varepsilon)\leq-\mu n^{\frac{1-m}{2m}}\left(2\lambda_2(L(C))\Phi(\varepsilon)\right)^{\frac{3m-1}{2m}}-\eta\left(2\lambda_2(L(D))\Phi(\varepsilon)\right)^{\frac{m+1}{2m}}. \tag{22}$$

Letting $\lambda_{\min}=\min\left(\lambda_2(L(C))\lambda_2(L(D))\right)$, together with (22), implies that

$$\dot{\Phi}(\varepsilon)\leq-\mu n^{\frac{1-m}{2m}}\left(2\lambda_{\min}\Phi(\varepsilon)\right)^{\frac{3m-1}{2m}}-\eta\left(2\lambda_{\min}\Phi(\varepsilon)\right)^{\frac{m+1}{2m}}. \tag{23}$$

Let $\psi(x)=\left(2\lambda_{\min}\Phi(\varepsilon)\right)^{1/2}$, and then we can derive from (23) that

$$\dot{\psi}(x)\leq\lambda_{\min}\left(-\mu n^{\frac{1-m}{2m}}\left(\psi(x)\right)^{\frac{2m-1}{m}}-\eta\left(\psi(x)\right)^{\frac{1}{m}}\right). \tag{24}$$

Based on (24), combine Lemma 5 with the comparison principle of differential equations [32], we yield that

$$\lim_{t\to T}\psi(x)=0\ \ and\ \ \psi(x)=0,\quad\forall t\geq T_2, \tag{25}$$

where T_2 is a settling time satisfying (19). By $\psi(x)=\left(2\lambda_{\min}\Phi(\varepsilon)\right)^{1/2}$, we are in position to follow from (25) that $\lim_{t\to T}\Phi(\varepsilon)=0$ *and* $\Phi(\varepsilon)=0$, $\forall t\geq T_2$. Consequently, the fixed-time bipartite consensus objective (17) is obtained.

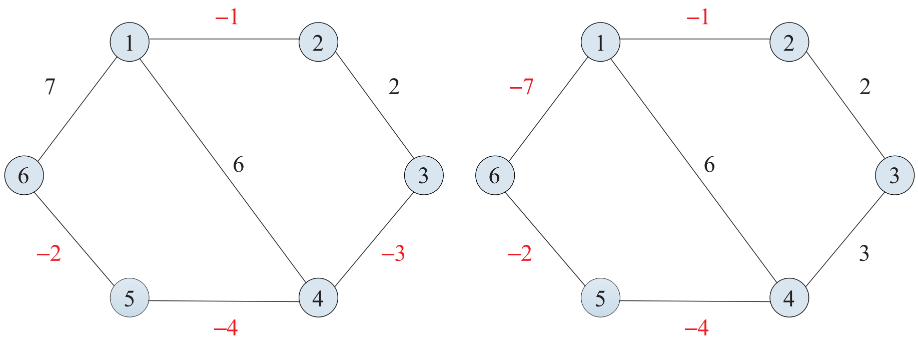
If $\mathcal{G}$ is structurally unbalanced, by similar calculation, one has that

$$\dot{\Phi}(\varepsilon)\leq-\mu n^{\frac{1-m}{2m}}\left(2\lambda_1(L(C))\Phi(\varepsilon)\right)^{\frac{3m-1}{2m}}-\eta\left(2\lambda_1(L(D))\Phi(\varepsilon)\right)^{\frac{m+1}{2m}}.$$

Note that (25) indicates (17) holds with $c=0$. The completes the proof. ■

5. Simulations

In this section, consider six agents in a competition network described in Fig. 1, where the agents are designated as 1, 2, …, 6. we implement fixed-time bipartite consensus controllers in two cases: disturbances and no disturbances. The case that constant disturbance and sinusoidal function disturbance are taken into consideration, respectively. The validity of related theory has been established by two illustrative examples.



(a) Structurally balanced graph. (b) Structurally unbalanced graph.

Fig. 1. The interaction topology of six agents.

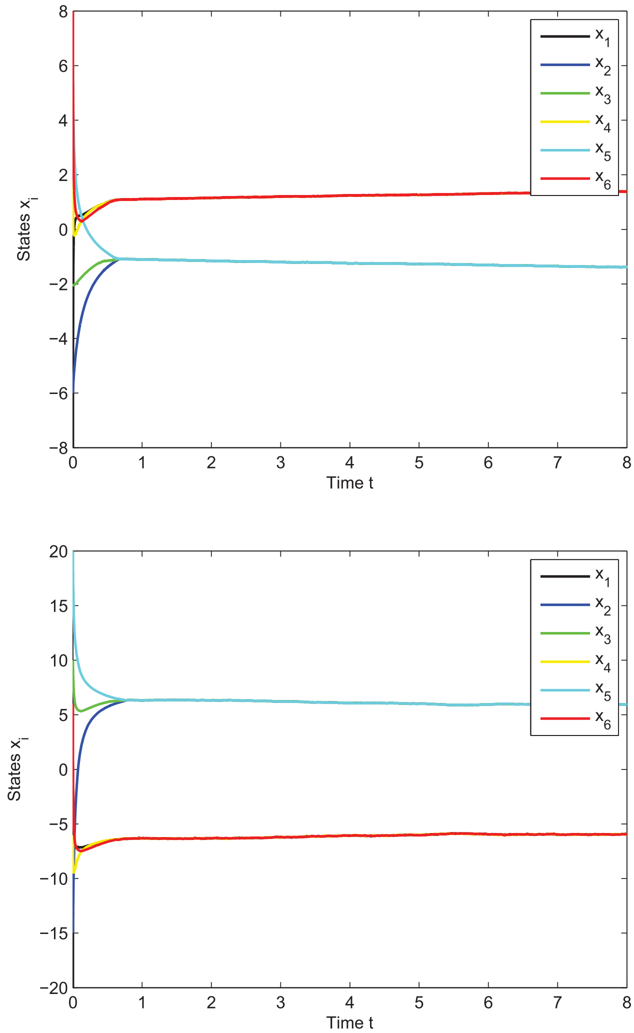


Fig. 2. Fixed time bipartite consensus with constant disturbance under Fig. 1(a).

Example 1 (*Disturbances Case*)**.** In this example, the protocol (5) is used to illustrate fixed-time bipartite consensus for dynamic systems of agents. From Theorem 1, protocol (5) can be applied to calculate the bound of setting time, which

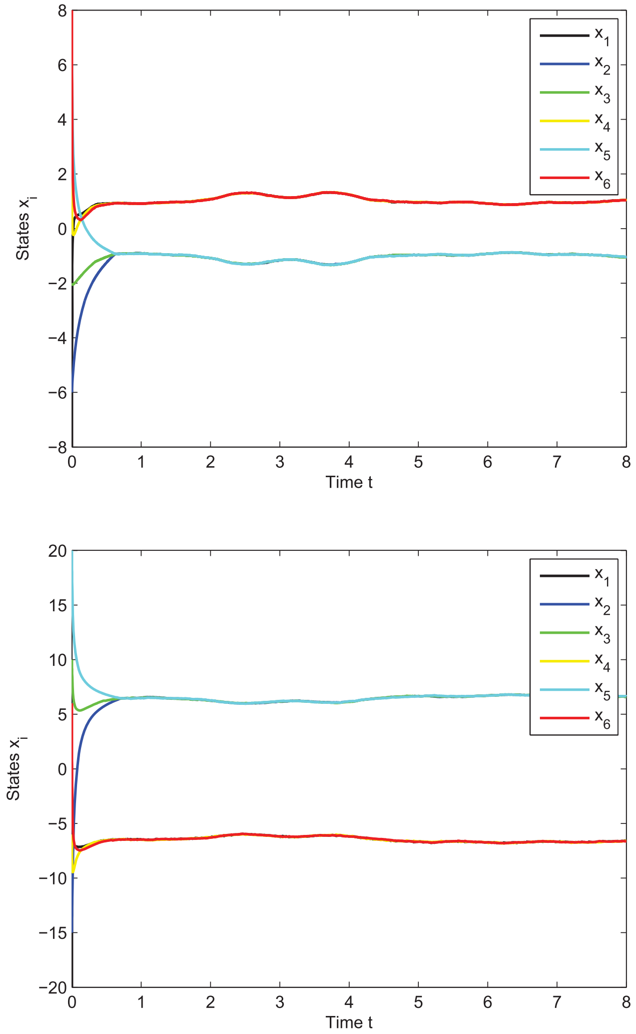


Fig. 3. Fixed-time bipartite consensus with sinusoidal function disturbance under Fig. 1(a).

yields with ease that settling time is irrelevant to the initial states of vehicles. To illustrate this observation, test results are presented in Figs. 2–3, where there are different initial states, as described below.

(1) $x(0) = [-8, -6, \ -2, \ 2, 6, 8]^T$.
(2) $x(0) = [-20, -15, 10, -6, 20, 6]^T$.

In what follows, we will consider the dynamic systems subjected to two types of disturbances in order to show the organized methods have better performance.

Case 1: $d_i(t) = 0.1, \ i \in \{1, 2, \ldots, 6\}$.

Case 2: $d_i(t) = \sin(it), i = \{1, 2, \ldots, 6\}$.

When it comes to the two cases, the designed parameters for the law (5) are chosen as $\mu = 0.5, \eta = 0.6, \alpha = 1.1, m = 3$, which satisfy the constraints that $\mu > 0, \eta > 0, m > 1, \alpha \geq b_w$, and m is positive odd integer. In spite of different disturbances in Cases 1–2, it can be clearly seen from Figs. 2–3 that the six agents equipped with designed law can reach fixed-time bipartite consensus, meanwhile, weaken the influences of different kinds of disturbances via making use of appropriate parameters.

Example 2 (*No Disturbances Case*)**.** In this example, we perform simulations to illustrate fixed-time bipartite consensus of protocol (18) with no disturbances in Figs. 4 and 5, where parameters and two initial states are the same in Example 1. It is shown that the algorithm (18) can guarantee agents achieve fixed-time bipartite consensus with different initial states

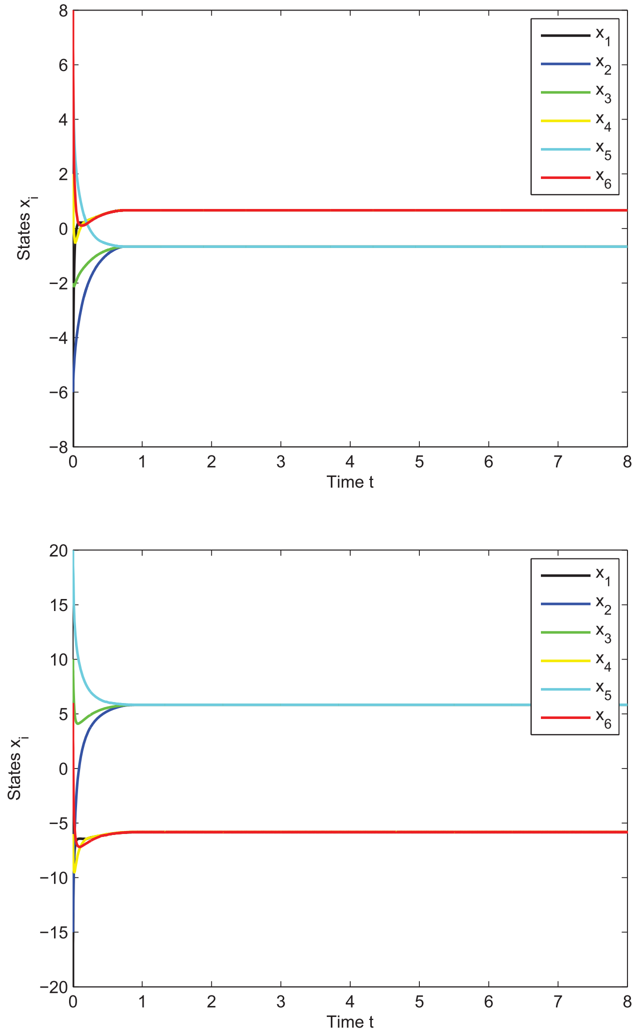


Fig. 4. Fixed-time bipartite consensus with no disturbances under Fig. 1(a).

under structurally balanced graph. On top of that, it renders each agent convergent to zero under structurally unbalanced graph, which correspond with Theorem 2.

6. Conclusion

In this study, by utilizing graph theory, Lyapunov analysis and stability theory of fixed-time, two different types of protocols are presented to handle fixed-time bipartite consensus for multi-agent systems with disturbances. Furthermore, the fixed-time bipartite consensus are guaranteed under structurally balanced and structurally unbalanced signed graph. We can find that only parameters and network connectivity have influence on the settling time. The validity of related theory has been verified via simulations tests. In addition, there are still exist many issues to consider, future works will be on the bipartite consensus in more complicated networks, such as networks with switching topology and quantization.

Acknowledgments

This work was partially supported by the National Natural Science Foundation of China under Grants 61602163, 61471163, 61673178 and 61773350, the Science Fund for Distinguished Young Scholars of Hubei Province, China under Grant 2017CFA034, the Youth Technology Innovation Team of Hubei Province under Grant T201710, the Hubei Provincial Education Department Science and Technology Research Program for Young Talents, China under Grant Q20182503.

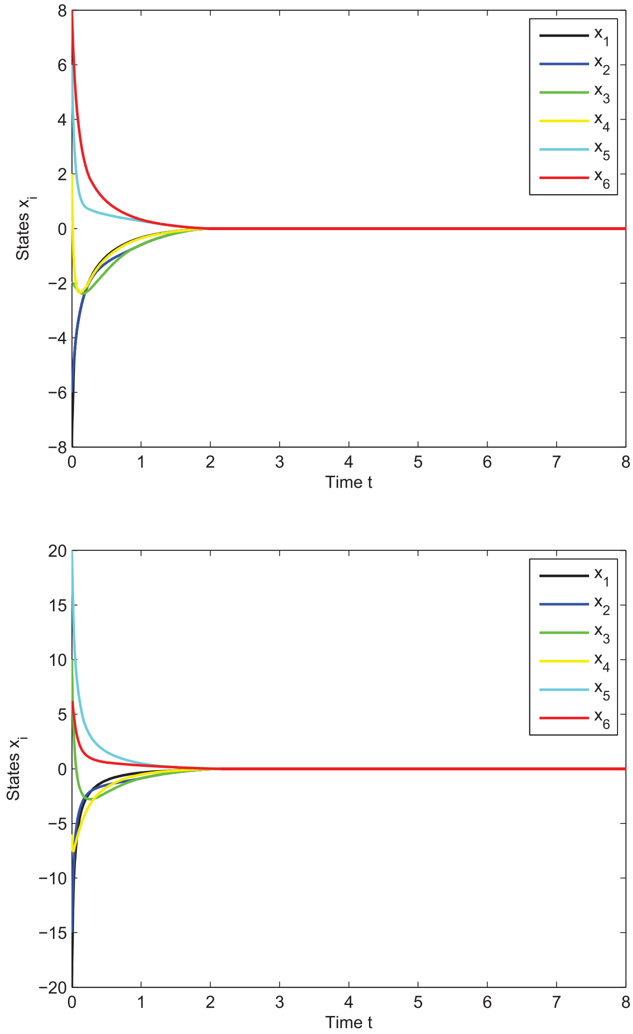


Fig. 5. Fixed-time bipartite consensus with no disturbances under Fig. 1(b).

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