Portal

Note: there are a number of solutions for this problem, we present the one intended by the problem author.

Denote the portals as $p_1, p_2, \ldots, p_n$. The solution consists of a number of steps presented below.

Step 1: For any portals p_i and p_j, any x and $x + p_i - p_j$ must have the same colour

Your friend could walk from x to p_j. At that moment they could be at p_i or p_j. Then walk back. You friend thinks they are at x, but they could also be at $x + p_i - p_j$, thus these two cells must be of the same colour.

This step is necessary in all solutions known to the Scientific Committee.

Step 2: Only $n - 1$ vectors matter: $(p_1 - p_2), (p_1 - p_3), \ldots, (p_1 - p_n)$

This is because they form a connected component in the graph. For example, to see that x and $x + p_i - p_j$ must be the same colour, notice that x and $x + p_i - p_1$ must be the same colour, and so must $x + p_i - p_1$ and $x + p_i - p_1 + p_1 - p_j = x + p_i - p_j$.

This or an equivalent step is necessary in all solutions known to the Scientific Committee.

Step 3: If all the vectors are collinear, then infinitely many colours can be used

This means that our friend can only be teleported along that line, so we can use as many colours as we want along some other line.

This step is necessary in all solutions known to the Scientific Committee.

Step 4: If we have 2 non-collinear vectors, they generate a parallelogram tiling of the plane

Call those vectors a and b. The tiling statement means that we may use any colours we want in the parallelogram $[(0, 0), a, a + b, b]$ (where the edges from a to $a + b$ and from $a + b$ to b are not included), and the colouring of that parallelogram will need to be repeated throughout the plane.

This and the following steps are not present in some of the other solutions.

Step 5: If we have 2 non-collinear vectors, the answer is the area of the parallelogram they form

The number of integer coordinates inside that parallelogram is in fact the area of the parallelogram. We will not provide the proof here, but there are a few ways to see this. One is if you know Pick's theorem. Another way is to note that the number of integer coordinate points inside a rectangular region of a plane is the area of that region. And the number of parallelograms tiling that region will be approximately be the area of the region divided by the area of the parallelogram. So the number of integer points inside each parallelogram will (at least approximately) be the area of the parallelogram.

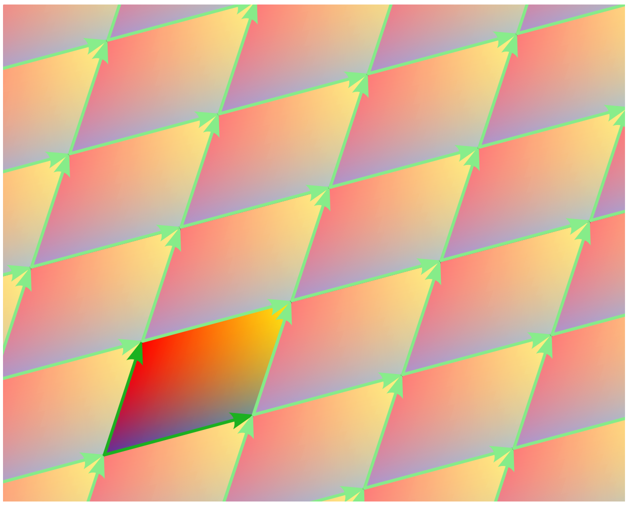

Figure 1: Parallelogram tiling

Step 6: Let the vectors be $v_1, v_2, \ldots, v_{n-1}$. Then any v_i can be replaced by $v_i + kv_j$ without loss of information

This statement is true when $i \neq j$ and k is any integer. To show this, we just need to show that replacing v_j with $v_i + kv_j$ still means that we can establish x and $x + v_i$ must be of the same colour for any x. That is because we know that x and $x + kv_j$ must have the same colour, also $x + lv_j$ and $x + (l-1)v_j$ must have the same colour for any integer l.

Step 7: For any vectors a, b, c, we can put c into a parallelogram generated by a and b. Using this we can make (a, b, c) smaller and smaller until two of them become collinear

For any vectors a, b, c we can find integers k and l such that $c - ka - lb$ is inside the parallelogram $[(0,0), a, a+b, b]$, excluding two edges as before. We can first find real numbers k' and l' such that $c = k'a + l'b$ (this is just a system of two linear equations with the unknowns being k' and l', so it's quite straightforward to solve). Then we can just take $k = \lfloor k' \rfloor$ and $l = \lfloor l' \rfloor$.

We can then replace c by $c - ka - lb$, shuffle the new (a, b, c) in any way, and repeat the above step. This can only stop if two of them become collinear (thus no longer forming a parallelogram to put the third vector into), as the area of $[(0,0), a, a+b, b]$ will decrease by an integer amount after each of these steps, and cannot drop below zero.

Step 8: Any collinear vectors a and b can be replaced by their greatest common divisor (gcd)

Their gcd makes sense as a concept since they are the same vector scaled by a different amount. In particular, there exists a vector v and integers a', b' such that $a = a'v$ and $b = b'v$. So $gcd(a, b) = gcd(a'v, b'v) = gcd(a', b')v$.

Step 9: Repeating the above two steps until only have 2 vectors left, and calculating the resulting parallelogram area completes the solution

Now we proceed using the previous two steps. Take three vectors. If no two of them are collinear, repeat step 7 until two of them become collinear. Then replace the two collinear vectors with their *gcd*. Repeat this only until two vectors are left. If they are collinear, the answer is -1 and if not, the answer is the area of the parralelogram they form, as step 5 explains.

Credits

- Task: Magnus Heghdal (Norway)
- Solutions and tests: Jonas Pukšta, Zigmas Bitinas (Lithuania)