EECE 5550 Mobile Robotics Lab #1

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Question 1: Object pose estimation

A 3D calibration object O has feature points at the following locations, expressed in the object's body-centric coordinate frame:

$$_Op_1 = \begin{pmatrix} 2 \\ 3 \\ -3 \end{pmatrix}, \ _Op_2 = \begin{pmatrix} 0 \\ 0 \\ -3 \end{pmatrix}, \ _Op_3 = \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}, \ _Op_4 = \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}.$$

Using a stereo camera, a robot observes this object, and measures the locations of these feature points as:

$$_Sp_1 = \begin{pmatrix} -1.3840 \\ 4.5620 \\ -0.1280 \end{pmatrix}, \ _Sp_2 = \begin{pmatrix} -0.9608 \\ 1.3110 \\ -1.6280 \end{pmatrix}, \ _Sp_3 = \begin{pmatrix} 1.3250 \\ -2.3890 \\ 1.7020 \end{pmatrix}, \ _Sp_4 = \begin{pmatrix} -1.3140 \\ 0.2501 \\ -0.7620 \end{pmatrix}$$

in the stereocamera's body-centric frame S. What is the pose $T_{SO} \in \mathrm{SE}(3)$ of object O with respect to the camera frame S?

Question 2: Lie algebras and left-invariant vector fields

Let G be a Lie group. We saw in class that for each $g \in G$, the *left-translation map*:

$$\begin{aligned} L_g &\colon G \to G \\ L_g(x) &\triangleq gx \end{aligned} \tag{1}$$

determined by g is a diffeomorphism of G. We also saw that left-translation could be used to *identify* the Lie algebra $\mathrm{Lie}(G)$ of G with the set of *left-invariant vector fields* on G, as follows:

$$\begin{aligned} \varphi\colon\ \mathrm{Lie}(G) &\to \{\text{left-invariant vector fields on } G\} \\ \varphi(\omega) &= V_\omega \end{aligned} \tag{2}$$

where V_ω is the left-invariant vector field on G determined by:

$$V_\omega(x) \triangleq d(L_x)_e(\omega). \tag{3}$$

In words: we associate to each element $\omega \in \mathrm{Lie}(G)$ of the Lie algebra the left-invariant vector field V_ω whose value $V_\omega(x)$ at $x \in G$ is the image of ω under the derivative of the left-translation map L_x that sends the identity $e \in G$ to x.

In this exercise, we will study the left-translation maps and left-invariant vector fields for our two favorite Lie group examples: $\mathbb{R}^n$ (with vector addition as the group operation) and $\mathrm{GL}(n)$.

(a) Given $v \in \mathbb{R}^n$, what is the corresponding left-translation map $L_v : \mathbb{R}^n \to \mathbb{R}^n$?

(b) What is the derivative dL_v of the map L_v you found in part (a)?

(c) Given a vector $\xi \in \mathrm{Lie}(G) \cong \mathbb{R}^n$ in $\mathbb{R}^n$'s Lie algebra, what is the left-invariant vector field V_ξ on $\mathbb{R}^n$ determined by ξ? Interpret this result geometrically.

(d) Given a matrix $A \in \mathrm{GL}(n)$, what is the corresponding left-translation map $L_A : \mathrm{GL}(n) \to \mathrm{GL}(n)$?

(e) What is the derivative dL_A of the map L_A you found in part (d)?

(f) The Lie algebra $\mathrm{Lie}(\mathrm{GL}(n))$ of $\mathrm{GL}(n)$ is just $\mathbb{R}^{n \times n}$, the set of all $n \times n$ matrices.[1] Given a matrix $\Omega \in \mathrm{Lie}(\mathrm{GL}(n))$, what is the left-invariant vector field V_Ω on $\mathrm{GL}(n)$ determined by Ω?

Question 3: Exponential map of the orthogonal group

We saw in class that the exponential map for the general linear group $\mathrm{GL}(n)$ is just the usual matrix exponential:

$$\begin{aligned} \exp &: \mathbb{R}^{n \times n} \to \mathrm{GL}(n) \\ \exp(X) &\triangleq \sum_{k=0}^{\infty} \frac{X^k}{k!}. \end{aligned} \tag{4}$$

However, we also mentioned that formula (4) can sometimes be significantly simplified when applied to a *subgroup* $G \subseteq \mathrm{GL}(n)$. In this exercise, we will explore what this simplification looks like for the orthogonal group $\mathrm{O}(2)$.

(a) We showed in Lecture 2 that the Lie algebra $\mathrm{Lie}(\mathrm{O}(n))$ of the orthogonal group $\mathrm{O}(n)$ is $\mathrm{Skew}(n)$, the set of n-dimensional skew-symmetric matrices:

$$\mathrm{Skew}(n) \triangleq \left\{ A \in \mathbb{R}^{n \times n} \mid A^\mathsf{T} = -A \right\}. \tag{5}$$

In particular, the Lie algebra of $\mathrm{O}(2)$ is:

$$\mathrm{Lie}(\mathrm{O}(2)) = \mathrm{Skew}(2) = \left\{ \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \in \mathbb{R}^{2 \times 2} \mid \omega \in \mathbb{R} \right\}. \tag{6}$$

Given an element:

$$\Omega \triangleq \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \tag{7}$$

of $\mathrm{Lie}(\mathrm{O}(2))$, derive an expression for its kth power Ω^k. (Hint: it may help to work out the first few powers of Ω. Can you spot a pattern?)

(b) Using the result of part (a), derive a simplified expression for $\exp(\Omega)$. (Hint: it may help to split the series in (4) into odd and even powers. Can you recognize these series?)

What is the geometric interpretation of $\exp(\Omega)$?

[1] Here's an easy way to see this: Recall that $\mathrm{GL}(n)$ is the group of invertible $n \times n$ matrices, and that a matrix M is invertible if and only if $\det(M) \neq 0$. This means that $\mathrm{GL}(n) = \det^{-1}(\mathbb{R} - \{0\})$, i.e., $\mathrm{GL}(n)$ is the *preimage* of the nonzero real numbers $\mathbb{R} - \{0\}$ under the determinant function. Since $\det : \mathbb{R}^{n \times n} \to \mathbb{R}$ is a continuous function and $\mathbb{R} - \{0\}$ is an open set, it follows that $\mathrm{GL}(n)$ is an *open subset* of $\mathbb{R}^{n \times n}$; this means that at each point $A \in \mathrm{GL}(n)$, we can take a small step in *any* direction while still staying within $\mathrm{GL}(n)$. In particular, we can locally move in *any* direction at the identity $I \in \mathrm{GL}(n)$ while staying within $\mathrm{GL}(n)$; this shows that $\mathrm{Lie}(\mathrm{GL}(n)) \cong T_I(\mathrm{GL}(n)) = \mathbb{R}^{n \times n}$.

Question 4: Motion on Lie groups

Let G be a Lie group with Lie algebra $\mathrm{Lie}(G)$. We saw in class that each $\omega \in \mathrm{Lie}(G)$ generates a left-invariant vector field V_ω on G, and that the exponential map describes the *integral curves* (i.e. the *trajectories*) of this vector field. Specifically, the integral curve $\gamma \colon \mathbb{R} \to G$ of the left-invariant vector field V_ω that starts at the point $x \in G$ at time $t = 0$ is given by:

$$\gamma(t) \triangleq x \exp(t\omega). \tag{8}$$

Intuitively, equation (8) provides a prescription for "moving around" on the Lie group G along the "direction" determined by ω.

In this exercise, we will see how one can apply (8) to *interpolate* Lie group-valued data.

(a) If a point $x \in G$ lies in the image of G's exponential map,[2] we write "$\log(x)$" to denote one of x's preimages,[3] so that:

$$x = \exp(\log(x)). \tag{9}$$

If G's exponential map is *surjective*, then there is always *at least* one choice of $\log(x) \in \mathrm{Lie}(G)$ that will satisfy (9).

Now suppose that $x, y \in G$ and that G's exponential map is surjective. Using (8), derive a formula for a curve $\gamma \colon [0, 1] \to G$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

(b) We mentioned in class that the exponential map for $\mathbb{R}^n$ is just the identity map:

$$\begin{aligned} \exp \colon \mathbb{R}^n &\to \mathbb{R}^n \\ \exp(\xi) &= \xi. \end{aligned} \tag{10}$$

Using (10), specialize your result from part (a) to derive a formula for a curve γ that joins x to y in $\mathbb{R}^n$. Interpret this result geometrically.

(c) We saw in Lecture 1 that the Lie group $\mathrm{SE}(3)$ of 3D robot poses can be modeled as the product manifold $M \triangleq \mathbb{R}^3 \times \mathrm{SO}(3)$,[4] equipped with the following group multiplication rule:

$$(t_1, R_1) \cdot (t_2, R_2) = (R_1 t_2 + t_1,\ R_1 R_2). \tag{11}$$

Given the two poses:

$$\begin{aligned} X_0 &= \left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.4330 & 0.1768 & 0.8839 \\ 0.2500 & 0.9186 & -0.3062 \\ -0.8660 & 0.3536 & 0.3536 \end{pmatrix} \right), \\ X_1 &= \left(\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 0.7500 & -0.0474 & 0.6597 \\ 0.4330 & 0.7891 & -0.4356 \\ -0.5000 & 0.6124 & 0.6124 \end{pmatrix} \right) \end{aligned} \tag{12}$$

apply the formula you derived in part (a) to calculate the "midpoint" $\gamma_{\mathrm{SE}(3)}(1/2)$ on the curve $\gamma_{\mathrm{SE}(3)} \colon [0, 1] \to \mathrm{SE}(3)$ from X_0 to X_1.

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[2] Note that not *every* point $x \in G$ of a Lie group G will necessarily lie in the image of the exponential map – see for example Question 3.

[3] Note that a point $x \in G$ may have *more than one* preimage in $\mathrm{Lie}(G)$ – consider the case of $\mathrm{SO}(2)$, in which the exponential map "wraps" the Lie algebra $\mathbb{R}$ infinitely many times around the circle.

[4] That is, as the set of *pairs* (t, R) consisting of a 3-dimensional vector $t \in \mathbb{R}^3$ (giving the robot's *position*), and a 3×3 rotation matrix $R \in \mathrm{SO}(3)$ (giving the robot's *orientation*).

(d) Since $\mathbb{R}^3$ and SO(3) are themselves Lie groups (under vector addition and matrix multiplication, respectively), we can construct the *product* Lie group $P \triangleq \mathbb{R}^3 \times \mathrm{SO}(3)$: this is the group whose elements are pairs of the form $(t, R) \in \mathbb{R}^3 \times \mathrm{SO}(3)$, equipped with the multiplication law

$$(t_1, R_1) \cdot (t_2, R_2) = (t_1 + t_2, R_1 R_2). \tag{13}$$

That is, in the product group P, we simply apply the group operations from $\mathbb{R}^3$ and SO(3) *separately in each component.*

The Lie groups SE(3) and P thus have the same *manifold* structure (they are both built on the manifold $\mathbb{R}^3 \times \mathrm{SO}(3)$), but different *group* structures [compare the multiplication rules (11) and (13)].

Using the formula that you derived in part (a), compute the "midpoint" $\gamma_P(1/2)$ of the curve $\gamma_P : [0, 1] \to P$ from X_0 to X_1 in P.

(e) Plot the translational components of the curves $\gamma_{\mathrm{SE}(3)}$ and γ_P from parts (c) and (d) over two intervals: (i) $t \in [0, 1]$ and (ii) $t \in [0, 30]$. Describe these curves qualitatively.

Question 5: Bayesian state estimation

In this exercise, we will explore how one can estimate the position of a mobile robot from noisy sensor data, using tools from probability theory.

(a) Our robot begins with an initial estimate of its position $X_0 \in \mathbb{R}^2$, expressed in the form of a probability density function $p(X_0)$. It then moves by driving its motors according to some (known) motor commands u. The robot has access to a *motion model* that describes how its subsequent position $X_1 \in \mathbb{R}^2$ depends upon its initial position X_0 and the motor commands u, expressed in the form of a conditional probability density $p(X_1|X_0, u)$.

Using this information, derive a formula for $p(X_1|u)$, the conditional probability density for the robot's *next* position given the motor commands u. [Hint: You may want to use the fact that $p(X_0|u) = p(X_0)$; that is, the initial position of the robot is *independent* of the motor commands.]

(b) As we discussed in class, robot motion is an inherently noisy process (due to effects such as "slop" in the drivetrain, variations in the terrain over which the robot is driving that affect steering, etc.), and therefore driving a robot around tends to *increase* our uncertainty about its position. Consequently, we will often attempt to *reduce* this uncertainty by collecting additional information about the robot's position using a sensor (such as a GPS receiver).

Suppose that our robot has access to a sensor that generates a measurement Y, and that we have access to a *sensor model* that describes how measurement Y depends upon the robot's position X, again expressed in the form of a conditional probability density $p(Y|X)$.

Given $p(X_1|u)$ (our *prior* belief about the robot's position after applying the motor commands u) and the sensor model $p(Y_1|X_1)$, derive a formula for $p(X_1|Y_1, u)$, the *posterior* distribution for the robot's position X_1 given *both* a measurement Y_1 from the sensor *and* knowledge of the motor commands u.

[Hint: you will want to use the following "conditional" form of Bayes' Rule:

$$p(X|Y, Z) = \frac{p(Y|X, Z)p(X|Z)}{p(Y|Z)} \tag{14}$$

together with the fact that:

$$p(Y_1|X_1, u) = p(Y_1|X_1). \tag{15}$$

Formally, equation (15) states that the measurement Y_1 is *conditionally independent* of u *given* X_1 – this means that, if we knew the robot's *true* position X_1, then *also* knowing the motor commands u would not enable us to predict Y_1 any better than knowing X_1 alone. This simply reflects the fact that, while the measurement Y_1 does (indirectly) depend upon u, it does so only *through the effect of the motor commands* u *on the robot's position* X_1 – thus, if we are *given* X_1 itself, then u does not provide any *additional* useful information.]

Solutions

Question 1

Recall that the pose $T_{SO} \in \mathrm{SE}(3)$ of object O with respect to the camera frame S is defined to be the rigid transformation that maps coordinate expressions in frame O to coordinate expressions in frame S. Using the 4×4 homogeneous matrix representation for the rigid transformation T_{SO} and the corresponding (4-dimensional) homogeneous representations for each feature point, we thus have:

$$T_{SO} \begin{pmatrix} 2 & 0 & -1 & -1 \\ 3 & 0 & -2 & 0 \\ -3 & -3 & 2 & -2 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1.3840 & -0.9608 & 1.3250 & -1.3140 \\ 4.5620 & 1.3110 & -2.3890 & 0.2501 \\ -0.1280 & -1.6280 & 1.7020 & -0.7620 \\ 1 & 1 & 1 & 1 \end{pmatrix}. \tag{16}$$

Solving (16) for T_{SO}, we obtain:

$$\begin{aligned} T_{SO} &= \begin{pmatrix} -1.3840 & -0.9608 & 1.3250 & -1.3140 \\ 4.5620 & 1.3110 & -2.3890 & 0.2501 \\ -0.1280 & -1.6280 & 1.7020 & -0.7620 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 & -1 \\ 3 & 0 & -2 & 0 \\ -3 & -3 & 2 & -2 \\ 1 & 1 & 1 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 0.7068 & -0.6123 & 0.3536 & 0.1000 \\ 0.7072 & 0.6122 & -0.3537 & 0.2500 \\ 0.0000 & 0.5000 & 0.8660 & 0.9700 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{17}$$

Question 2

(a) Since the group operation in $\mathbb{R}^n$ is vector addition, we have

$$L_v(x) = x + v. \tag{18}$$

(b) Differentiating (18) shows that:

$$dL_v = I. \tag{19}$$

(c) Substituting the derivative map (19) into (3), we obtain:

$$V_\xi(x) = d(L_x)_e(\xi) = (I)(\xi) = \xi; \tag{20}$$

this is the vector field on $\mathbb{R}^n$ that takes the *constant* value ξ. This result shows that the left-invariant vector fields on $\mathbb{R}^n$ are precisely the *constant* vector fields.

(d) Since the group operation in $\mathrm{GL}(n)$ is matrix multiplication, we have

$$L_A(X) = AX. \tag{21}$$

(e) Since the left-translation map L_A in (21) is *already* a linear map, its derivative is simply:

$$dL_A = A. \tag{22}$$

(f) Substituting the derivative map (22) into (3), we obtain:

$$V_\Omega(X) = d(L_X)_I(\Omega) = X\Omega. \tag{23}$$

Question 3

(a) Direct computation shows that:

$$\Omega^2 = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} = \begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix} = -\omega^2 I. \tag{24}$$

Given any power Ω^k of Ω, we can apply (24) to reduce the exponent k on Ω to either 0 or 1 by pulling out a factor of $\Omega^{2p} = (\Omega^2)^p$ for $p \triangleq \lfloor k/2 \rfloor$:

$$\begin{aligned} \Omega^k &= (\Omega^2)^p \Omega^{k \bmod 2} \\ &= (-\omega^2 I)^p \Omega^{k \bmod 2} \\ &= (-1)^p \omega^{2p} \Omega^{k \bmod 2}. \end{aligned} \tag{25}$$

Writing:

$$\Omega = \omega \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\triangleq S} \tag{26}$$

we can substitute (26) into (25) to obtain:

$$\begin{aligned} \Omega^k &= (-1)^p \omega^{2p + (k \bmod 2)} S^{k \bmod 2} \\ &= (-1)^p \omega^k S^{k \bmod 2}. \end{aligned} \tag{27}$$

Finally, recalling the definition of k, we obtain:

$$\Omega^k = (-1)^{\lfloor k/2 \rfloor} \omega^k S^{k \bmod 2}, \tag{28}$$

which gives a simple formula for Ω^k in terms of the *scalar* ω and the 0th or 1st power of the *constant* matrix S defined in (26).

(b) We know from (4) that:

$$\exp(\Omega) = \sum_{k=0}^{\infty} \frac{\Omega^k}{k!}. \tag{29}$$

Substituting (28) into (29), we obtain:

$$\exp(\Omega) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor k/2 \rfloor} \omega^k S^{k \bmod 2}}{k!}. \tag{30}$$

Notice that we can split the series in (30) into even and odd powers k, where for k even $S^{k \bmod 2} = S^0 = I$, and for k odd $S^{k \bmod 2} = S^1 = S$. Thus, we can develop (30) as:

$$\begin{aligned} \exp(\Omega) &= \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor k/2 \rfloor} \omega^k S^{k \bmod 2}}{k!} \\ &= \sum_{p=0}^{\infty} \frac{(-1)^p \omega^{2p}}{(2p)!} I + \sum_{p=0}^{\infty} \frac{(-1)^p \omega^{2p+1}}{(2p+1)!} S. \end{aligned} \tag{31}$$

We may now recognize the two series appearing in (31) as the Taylor series for $\cos(\omega)$ and $\sin(\omega)$, respectively. Therefore, we can simplify (31) as:

$$\exp(\Omega) = \cos(\omega) I + \sin(\omega) S = \begin{pmatrix} \cos(\omega) & \sin(\omega) \\ -\sin(\omega) & \cos(\omega) \end{pmatrix}, \tag{32}$$

which we recognize as the 2D rotation matrix for the angle ω.

A few comments on this result: This is an interesting (and perhaps surprising) result because even though the orthogonal group $\mathrm{O}(2)$ has *both* orientation-**preserving** *and* orientation-**reversing** elements (the latter involving *reflections* in addition to rotations), equation (32) shows that the image of $\mathrm{O}(2)$'s exponential map is *always a rotation.* Put another way: the image of $\mathrm{O}(2)$'s exponential map $\exp(\Omega)$ always lies in the *subgroup* $\mathrm{SO}(2)$.

In fact, it turns out that $\exp(\Omega) \in \mathrm{SO}(n)$ for all $\Omega \in \mathrm{Lie}(\mathrm{O}(n))$ for *any* n (not just $n = 2$). This is a consequence of the fact that, geometrically, $\mathrm{O}(n)$ consists of two *disconnected components*: one consists of matrices having determinant $+1$ (this is $\mathrm{SO}(n)$, the subgroup of *rotations*), and the other consists of matrices having determinant -1. Since the exponential map is a continuous map that describes a "flow" originating at the identity I, and I lies in the $+1$ component of $\mathrm{O}(n)$ (the subgroup $\mathrm{SO}(n)$), it follows that $\exp(\Omega) \subseteq \mathrm{SO}(n)$.

In fact, this argument shows that given *any* Lie group G, the image of G's exponential map always lies in the connected component containing the identity element e.

Question 4

(a) We must determine a Lie algebra element ω in (8) so that $\gamma(1) = y$. Substituting the expression (8) for γ, we obtain:

$$y = \gamma(1) = x \exp(\omega). \tag{33}$$

Solving (33) for ω, we find:

$$\omega = \log(x^{-1} y). \tag{34}$$

Note that (34) is sensible since we have *assumed* that exp is surjective (and therefore $x^{-1}y$ is guaranteed to have *at least one* preimage under the exponential map).

Substituting (34) into (8), we thus find that:

$$\begin{gathered} \gamma \colon [0,1] \to G \\ \gamma(t) \triangleq x \exp\left(t \log(x^{-1} y)\right) \end{gathered} \tag{35}$$

is a curve on G satisfying $\gamma(0) = x$ and $\gamma(1) = y$.

(b) The inverse of x under vector addition $+$ is simply $-x$; therefore, using $\exp(v) = \log(v) = v$, the Euclidean specialization of (35) is:

$$\begin{aligned} \gamma(t) &= x \exp\left(t \log(x^{-1} y)\right) \\ &= x \exp(t \log(-x + y)) \\ &= x \exp(t(y - x)) \\ &= x + t(y - x). \end{aligned} \tag{36}$$

This is just the expression for the straight line segment joining x to y.

(c) Letting

$$T_0 = \begin{pmatrix} 0.4330 & 0.1768 & 0.8839 & 1 \\ 0.2500 & 0.9186 & -0.3062 & 1 \\ -0.8660 & 0.3536 & 0.3536 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$T_1 = \begin{pmatrix} 0.7500 & -0.0474 & 0.6597 & 2 \\ 0.4330 & 0.7891 & -0.4356 & 4 \\ -0.5000 & 0.6124 & 0.6124 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{37}$$

denote the (4×4)-dimensional homogeneous representations of X_0 and X_1, respectively, and defining:

$$\Omega \triangleq \log(T_0^{-1} T_1) = \begin{pmatrix} 0.0000 & -0.3703 & -0.3703 & -0.4533 \\ 0.3702 & -0.0001 & -0.0000 & 4.1974 \\ 0.3702 & -0.0000 & -0.0001 & 1.2303 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{38}$$

we have from part (a) that the curve $\gamma_{\mathrm{SE}(3)} \colon [0, 1] \to \mathrm{SE}(3)$ joining T_0 to T_1 in SE(3) is:

$$\gamma_{\mathrm{SE}(3)}(t) = T_0 \exp(t\Omega). \tag{39}$$

Evaluating (39) at $t = 1/2$, we find:

$$\gamma_{\mathrm{SE}(3)}(1/2) = \begin{pmatrix} 0.6124 & 0.0795 & 0.7866 & 1.6710 \\ 0.3535 & 0.8624 & -0.3624 & 2.5988 \\ -0.7071 & 0.5000 & 0.5000 & 1.3443 \\ 0 & 0 & 0 & 1.0000 \end{pmatrix}, \tag{40}$$

i.e., the midpoint $\gamma_{\mathrm{SE}(3)}(1/2)$ is the pair:

$$\gamma_{\mathrm{SE}(3)}(1/2) = \left(\begin{pmatrix} 1.6710 \\ 2.5988 \\ 1.3443 \end{pmatrix}, \begin{pmatrix} 0.6124 & 0.0795 & 0.7866 \\ 0.3535 & 0.8624 & -0.3624 \\ -0.7071 & 0.5000 & 0.5000 \end{pmatrix} \right). \tag{41}$$

(d) Letting

$$t_0 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad t_1 = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}, \tag{42}$$

and

$$R_0 = \begin{pmatrix} 0.4330 & 0.1768 & 0.8839 \\ 0.2500 & 0.9186 & -0.3062 \\ -0.8660 & 0.3536 & 0.3536 \end{pmatrix}, \qquad R_1 = \begin{pmatrix} 0.7500 & -0.0474 & 0.6597 \\ 0.4330 & 0.7891 & -0.4356 \\ -0.5000 & 0.6124 & 0.6124 \end{pmatrix}, \tag{43}$$

and defining:

$$\begin{aligned} \Phi &\triangleq \log(X_0^{-1} X_1) = \log(t_1 - t_0, R_0^{-1} R_1) \\ &= \left(\begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 0.0000 & -0.3703 & -0.3703 \\ 0.3702 & -0.0001 & -0.0000 \\ 0.3702 & -0.0000 & -0.0001 \end{pmatrix} \right), \end{aligned} \tag{44}$$

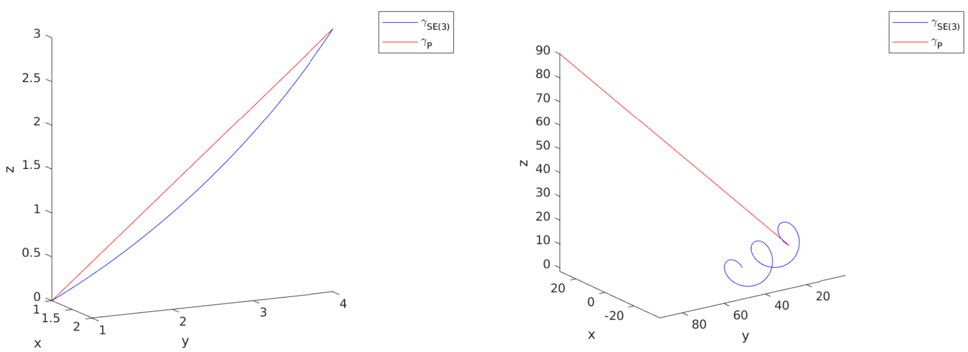
we have from part (a) that:

$$\gamma_P(t) = X_0 \exp(t\Phi). \tag{45}$$

Evaluating (45) at $t = 1/2$, we obtain:

$$\gamma_P(1/2) = \left(\begin{pmatrix} 3/2 \\ 5/2 \\ 3/2 \end{pmatrix}, \quad \begin{pmatrix} 0.6124 & 0.0795 & 0.7866 \\ 0.3535 & 0.8624 & -0.3624 \\ -0.7071 & 0.5000 & 0.5000 \end{pmatrix} \right). \tag{46}$$

(e) Plotting the translational parts of the curves $\gamma_{\mathrm{SE}(3)}$ and γ_P from parts (c) and (d) in MATLAB over the intervals $[0, 1]$ and $[0, 30]$, we obtain the following plots:



(a) $t \in [0, 1]$

(b) $t \in [0, 30]$

The translational part of the curve γ_P is a straight line, while the translational part of $\gamma_{\mathrm{SE}(3)}$ looks like a "corkscrew" – indeed the curves on SE(d) corresponding to formula (35) are formally called "screw motions".

Question 5

(a) From the definition of conditional probability, we have:

$$p(X_1, X_0|u) = p(X_1|X_0, u)p(X_0|u). \tag{47}$$

Now $p(X_0|u) = p(X_0)$, since the robot's initial position is independent of the motor commands. We can thus simplify (47) as:

$$p(X_1, X_0|u) = p(X_1|X_0, u)p(X_0), \tag{48}$$

which gives the *joint* distribution for X_0 *and* X_1 given u. Finally, since we are only interested in X_1, we can obtain $p(X_1|u)$ by marginalizing over the initial position X_0:

$$\begin{aligned} p(X_1|u) &= \int p(X_1, X_0|u)\, dX_0 \\ &= \int p(X_1|X_0, u)p(X_0)\, dX_0. \end{aligned} \tag{49}$$

Equation (49) gives a formula for computing $p(X_1|u)$, the density for the robot's next state given the motor commands u.

(b) Applying the conditional form of Bayes' Rule (14), we have:

$$p(X_1|Y_1, u) = \frac{p(Y_1|X_1, u)p(X_1|u)}{p(Y_1|u)} \tag{50}$$

Since $p(Y_1|X_1, u) = p(Y_1|X_1)$, we can simplify (50) as:

$$p(X_1|Y_1, u) = \frac{p(Y_1|X_1)p(X_1|u)}{p(Y_1|u)}. \tag{51}$$

Finally, we can obtain an expression for the evidence $p(Y_1|u)$ by integrating the numerator on the right-hand side of (51), as usual:

$$p(Y_1|u) = \int p(Y_1|X_1)p(X_1|u)\, dX_1. \tag{52}$$

Equations (51) and (52) provide formulae for computing $p(X_1|Y_1, u)$ in terms of the sensor model $p(Y_1|X_1)$ and $p(X_1|u)$, as desired.