EECE 5550 Mobile Robotics Lab #1

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Question 1: Object pose estimation

A 3D object O has feature points at the following locations, expressed in the object's body-centric coordinate frame:

$$_Op_1 = \begin{pmatrix} 2 \\ 3 \\ -3 \end{pmatrix}, \ _Op_2 = \begin{pmatrix} 0 \\ 0 \\ -3 \end{pmatrix}, \ _Op_3 = \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}, \ _Op_4 = \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}.$$

Using a stereo camera, a robot observes this object, and measures the locations of these feature points as:

$$_Sp_1 = \begin{pmatrix} -1.3840 \\ 4.5620 \\ -0.1280 \end{pmatrix}, \ _Sp_2 = \begin{pmatrix} -0.9608 \\ 1.3110 \\ -1.6280 \end{pmatrix}, \ _Sp_3 = \begin{pmatrix} 1.3250 \\ -2.3890 \\ 1.7020 \end{pmatrix}, \ _Sp_4 = \begin{pmatrix} -1.3140 \\ 0.2501 \\ -0.7620 \end{pmatrix}$$

in the stereocamera's body-centric frame S. What is the pose $T_{SO} \in \mathrm{SE}(3)$ of object O with respect to the camera frame S?

Question 2: Lie algebras and left-invariant vector fields

Let G be a Lie group with group operation $\star\colon G \times G \to G$. We saw in class that for each $g \in G$, the *left-translation map*:

$$\begin{aligned} L_g &\colon G \to G \\ L_g(x) &\triangleq g \star x \end{aligned} \tag{1}$$

is a diffeomorphism of G. We also saw that left-translation could be used to *identify*[1] the tangent space $T_e(G)$ of G at the identity element $e \in G$ with the Lie algebra $\mathrm{Lie}(G)$ (the set of *left-invariant vector fields* on G), as follows:

$$\begin{aligned} \varphi &\colon T_e(G) \to \mathrm{Lie}(G) \\ \varphi(\omega) &= V_\omega \end{aligned} \tag{2}$$

where V_ω is the left-invariant vector field on G determined by:

$$V_\omega(x) \triangleq d(L_x)_e(\omega). \tag{3}$$

In words: we associate to each element $\omega \in T_e(G)$ the left-invariant vector field $V_\omega \in \mathrm{Lie}(G)$ whose value $V_\omega(x)$ at $x \in G$ is the image of ω under the derivative of the left-translation map L_x that sends the identity $e \in G$ to x.

In this exercise, we will study the left-translation maps and left-invariant vector fields for our two favorite Lie group examples: $\mathbb{R}^n$ (with vector addition as the group operation) and $\mathrm{GL}(n)$.

[1] Because of the identification (2), many authors (including us) will often somewhat loosely refer to $T_e(G)$ itself as the "Lie algebra" of G.

(a) Given $v \in \mathbb{R}^n$, what is the corresponding left-translation map $L_v \colon \mathbb{R}^n \to \mathbb{R}^n$?

(b) What is the derivative dL_v of the map L_v you found in part (a)?

(c) Given a vector $\xi \in T_0(\mathbb{R}^n) \cong \mathbb{R}^n$ in $\mathbb{R}^n$'s Lie algebra, what is the left-invariant vector field V_ξ on $\mathbb{R}^n$ determined by ξ? Interpret this result geometrically.

(d) Given a matrix $A \in \mathrm{GL}(n)$, what is the corresponding left-translation map $L_A \colon \mathrm{GL}(n) \to \mathrm{GL}(n)$?

(e) What is the derivative dL_A of the map L_A you found in part (d)?

(f) The tangent space $T_I(\mathrm{GL}(n))$ of $\mathrm{GL}(n)$ at the identity $I \in \mathrm{GL}(n)$ is just $\mathbb{R}^{n \times n}$, the set of all $n \times n$ matrices.[2] Given a matrix $\Omega \in T_I(\mathrm{GL}(n))$, what is the left-invariant vector field V_Ω on $\mathrm{GL}(n)$ determined by Ω?

Question 3: Exponential map of the orthogonal group

We saw in class that the exponential map for the general linear group $\mathrm{GL}(n)$ is just the usual matrix exponential:

$$\begin{aligned} \exp \colon \mathbb{R}^{n \times n} &\to \mathrm{GL}(n) \\ \exp(X) &\triangleq \sum_{k=0}^{\infty} \frac{X^k}{k!}. \end{aligned} \tag{4}$$

However, we also mentioned that formula (4) can sometimes be significantly simplified when applied to a *subgroup* $G \subseteq \mathrm{GL}(n)$. In this exercise, we will explore what this simplification looks like for the orthogonal group $\mathrm{O}(2)$.

(a) We mentioned in class that the Lie algebra $\mathrm{Lie}(\mathrm{O}(n))$ of the orthogonal group $\mathrm{O}(n)$ is $\mathrm{Skew}(n)$, the set of n-dimensional skew-symmetric matrices:

$$\mathrm{Skew}(n) \triangleq \left\{ A \in \mathbb{R}^{n \times n} \mid A^{\mathsf{T}} = -A \right\}. \tag{5}$$

In particular, the Lie algebra of $\mathrm{O}(2)$ is:

$$\mathrm{Lie}(\mathrm{O}(2)) = \mathrm{Skew}(2) = \left\{ \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \in \mathbb{R}^{2 \times 2} \mid \omega \in \mathbb{R} \right\}. \tag{6}$$

Given an element:

$$\Omega \triangleq \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \tag{7}$$

of $\mathrm{Lie}(\mathrm{O}(2))$, derive an expression for its kth power Ω^k. (Hint: it may help to work out the first few powers of Ω. Can you spot a pattern?)

(b) Using the result of part (a), derive a simplified expression for $\exp(\Omega)$. (Hint: it may help to split the series in (4) into odd and even powers. Can you recognize these series?)

What is the geometric interpretation of $\exp(\Omega)$?

[2] Here's an easy way to see this: Recall that $\mathrm{GL}(n)$ is the group of invertible $n \times n$ matrices, and that a matrix M is invertible if and only if $\det(M) \neq 0$. This means that $\mathrm{GL}(n) = \det^{-1}(\mathbb{R} - \{0\})$, i.e., $\mathrm{GL}(n)$ is the *preimage* of the nonzero real numbers $\mathbb{R} - \{0\}$ under the determinant function. Since $\det \colon \mathbb{R}^{n \times n} \to \mathbb{R}$ is a continuous function and $\mathbb{R} - \{0\}$ is an open set, it follows that $\mathrm{GL}(n)$ is an *open subset* of $\mathbb{R}^{n \times n}$; this means that at each point $A \in \mathrm{GL}(n)$, we can take a small step in *any* direction while still staying within $\mathrm{GL}(n)$. In particular, we can locally move in *any* direction at the identity $I \in \mathrm{GL}(n)$ while staying within $\mathrm{GL}(n)$; this shows that $\mathrm{Lie}(\mathrm{GL}(n)) \cong T_I(\mathrm{GL}(n)) = \mathbb{R}^{n \times n}$.

Question 4: Motion on Lie groups

Let G be a Lie group with group operation $\star\colon G \times G \to G$ and Lie algebra $\mathrm{Lie}(G)$. We saw in class that each $\omega \in \mathrm{Lie}(G)$ generates a left-invariant vector field V_ω on G, and that the exponential map describes the *integral curves* (i.e. the *trajectories*) of this vector field. Specifically, the integral curve $\gamma\colon \mathbb{R} \to G$ of the left-invariant vector field V_ω that starts at the point $x \in G$ at time $t = 0$ is given by:

$$\gamma(t) \triangleq x \star \exp(t\omega). \tag{8}$$

Intuitively, equation (8) provides a prescription for "moving around" on the Lie group G along the "direction" determined by ω.

In this exercise, we will see how one can apply (8) to *interpolate* Lie group-valued data – this is an important operation for robot kinematics.

(a) If a point $x \in G$ lies in the image of G's exponential map,[3] we write "$\log(x)$" to denote one of x's preimages,[4] so that:

$$x = \exp(\log(x)). \tag{9}$$

If G's exponential map is *surjective*, then there is always *at least* one choice of $\log(x) \in \mathrm{Lie}(G)$ that will satisfy (9).

Now suppose that $x, y \in G$ and that G's exponential map is surjective. Using (8), derive a formula for a curve $\gamma\colon [0,1] \to G$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

(b) We mentioned in class that the exponential map for $\mathbb{R}^n$ is just the identity map:

$$\begin{aligned} \exp\colon \mathbb{R}^n &\to \mathbb{R}^n \\ \exp(\xi) &= \xi. \end{aligned} \tag{10}$$

Using (10), specialize your result from part (a) to derive a formula for a curve γ that joins x to y in $\mathbb{R}^n$. Interpret this result geometrically.

(c) We saw in Lecture 2 that the Lie group $\mathrm{SE}(3)$ of 3D robot poses can be modeled as the product manifold $M \triangleq \mathbb{R}^3 \times \mathrm{SO}(3)$,[5] equipped with the following group multiplication rule:

$$(t_1, R_1) \star (t_2, R_2) = (R_1 t_2 + t_1,\ R_1 R_2). \tag{11}$$

Given the two poses:

$$\begin{aligned} X_0 &= \left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.4330 & 0.1768 & 0.8839 \\ 0.2500 & 0.9186 & -0.3062 \\ -0.8660 & 0.3536 & 0.3536 \end{pmatrix} \right), \\ X_1 &= \left(\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 0.7500 & -0.0474 & 0.6597 \\ 0.4330 & 0.7891 & -0.4356 \\ -0.5000 & 0.6124 & 0.6124 \end{pmatrix} \right) \end{aligned} \tag{12}$$

apply the formula you derived in part (a) to calculate the "midpoint" $\gamma_{\mathrm{SE}(3)}(1/2)$ on the curve $\gamma_{\mathrm{SE}(3)}\colon [0,1] \to \mathrm{SE}(3)$ from X_0 to X_1. (Hint: you may find it helpful to use the *homogeneous* representation of $\mathrm{SE}(d)$ that we saw in Lecture 2.)

[3] Note that not *every* point $x \in G$ of a Lie group G will necessarily lie in the image of the exponential map – see for example Question 3.

[4] Note that a point $x \in G$ may have *more than one* preimage in $\mathrm{Lie}(G)$ – consider the example of $\mathrm{SO}(2) \cong S^1$, in which the exponential map "wraps" the Lie algebra $\mathbb{R}$ infinitely many times around the circle.

[5] That is, as the set of *pairs* (t, R) consisting of a 3-dimensional vector $t \in \mathbb{R}^3$ (giving the robot's *position*), and a 3×3 rotation matrix $R \in \mathrm{SO}(3)$ (giving the robot's *orientation*).

(d) Since $\mathbb{R}^3$ and SO(3) are themselves Lie groups (under vector addition and matrix multiplication, respectively), we can construct the *product* Lie group $P \triangleq \mathbb{R}^3 \times \mathrm{SO}(3)$: this is the group whose elements are pairs of the form $(t, R) \in \mathbb{R}^3 \times \mathrm{SO}(3)$, equipped with the multiplication law

$$(t_1, R_1) \star_P (t_2, R_2) = (t_1 + t_2, R_1 R_2). \tag{13}$$

That is, in the product group P, we simply apply the group operations from $\mathbb{R}^3$ and SO(3) *separately in each component*.

The Lie groups SE(3) and P thus have the same *manifold* structure (they are both built on the manifold $\mathbb{R}^3 \times \mathrm{SO}(3)$), but different *group* structures [compare the multiplication rules (11) and (13)].

Using the formula that you derived in part (a), compute the "midpoint" $\gamma_P(1/2)$ of the curve $\gamma_P \colon [0, 1] \to P$ from X_0 to X_1 in P.

(e) Plot the translational components of the curves $\gamma_{\mathrm{SE}(3)}$ and γ_P from parts (c) and (d) over two intervals: (i) $t \in [0, 1]$ and (ii) $t \in [0, 30]$. Describe these curves qualitatively.

Question 5: Bayesian inference with linear-Gaussian models

In this exercise we will study Bayesian estimation in linear-Gaussian models; as we will see later in the course, these play a fundamental role in robotic state estimation (most prominently in the celebrated Kalman filter).

We begin by recording a few useful facts. Recall that:

$$X \sim \mathcal{N}(\mu, \Sigma) \tag{14}$$

means that $X \in \mathbb{R}^n$ is a random variable that follows a Gaussian distribution with mean $\mu \in \mathbb{R}^n$ and covariance $\Sigma \in \mathbb{S}^n_{++}$. As we saw in class, X is described by the following probability density function:

$$\begin{gathered} p_X \colon \mathbb{R}^n \to \mathbb{R} \\ p_X(x) \triangleq \frac{1}{\sqrt{\det(2\pi\Sigma)}} \exp\left(-\frac{1}{2}(x-\mu)^\mathsf{T}\Sigma^{-1}(x-\mu)\right). \end{gathered} \tag{15}$$

If we expand the quadratic form and ignore the normalization constant in (15), we find:

$$p_X(x) \propto \exp\left(-\frac{1}{2}\left(x^\mathsf{T}\Sigma^{-1}x - 2\mu^\mathsf{T}\Sigma^{-1}x\right)\right). \tag{16}$$

It follows from (16) that *any* function of the form:

$$f(z) = \frac{1}{c} \exp\left(-\frac{1}{2}\left(z^\mathsf{T}\Lambda z - 2\eta^\mathsf{T} z\right)\right) \tag{17}$$

with $c > 0$ is an unnormalized density for a Gaussian random variable $Z \sim \mathcal{N}(\bar{\mu}, \bar{\Sigma})$ with parameters:

$$\bar{\Sigma} = \Lambda^{-1}, \qquad \bar{\mu} = \bar{\Sigma}\eta. \tag{18}$$

Equations (17) and (18) give an alternative way of parameterizing a Gaussian probability density, called the *information* or *canonical form*.

Now, suppose that Θ is a random variable with prior distribution:

$$\Theta \sim \mathcal{N}(\mu_0, \Sigma_0), \tag{19}$$

and that we collect a set of m noisy linear measurements $\tilde{Y}_1, \ldots, \tilde{Y}_m$ of Θ according to:

$$\tilde{Y}_i = A_i \Theta + b_i + \epsilon_i, \quad \epsilon_i \sim \mathcal{N}(\mu_i, \Sigma_i), \tag{20}$$

where A_i, b_i, μ_i, and Σ_i are known parameters for all $i = 1, \ldots, m$. In this exercise, you will determine the posterior distribution for Θ given the measurements $\tilde{Y}_1, \ldots, \tilde{Y}_m$.

(a) Use Bayes' Rule to express the posterior density $p(\Theta|\tilde{Y}_1, \ldots, \tilde{Y}_m)$ in terms of the prior $p(\Theta)$ and the measurement likelihoods $p(\tilde{Y}_i|\Theta)$ for each individual measurement. You may leave your result in an unnormalized form.

(b) Derive an expression for the likelihood function $p(\tilde{Y}_i|\Theta)$ of the ith measurement. (Hint: Notice that you can easily solve (20) for ϵ_i.)

(c) Using your results from parts (a) and (b), derive the parametric form of the posterior density $p(\Theta|\tilde{Y}_1, \ldots, \tilde{Y}_m)$. You should simplify your result by collecting linear and quadratic terms in Θ in the exponent. You may leave your result in an unnormalized form.

(Hint: Since your result need not be normalized, any term appearing in an exponent that does *not* involve Θ can be be discarded by absorbing it into the normalization constant. You can use this fact to dramatically simplify your work.)

(d) You should be able to recognize your expression for $p(\Theta|\tilde{Y}_1, \ldots, \tilde{Y}_m)$ in part (c) as an unnormalized Gaussian density in information form. This shows that the posterior distribution for Θ is Gaussian; that is, $\Theta|\tilde{Y}_1, \ldots, \tilde{Y}_m \sim \mathcal{N}(\bar{\mu}, \bar{\Sigma})$ for some mean $\bar{\mu}$ and covariance $\bar{\Sigma}$. What are the mean $\bar{\mu}$ and covariance $\bar{\Sigma}$ of this distribution?

Solutions

Question 1

Recall that the pose $T_{SO} \in \mathrm{SE}(3)$ of object O with respect to the camera frame S is defined to be the rigid transformation that maps coordinate expressions in frame O to coordinate expressions in frame S. Using the 4×4 homogeneous matrix representation for the rigid transformation T_{SO} and the corresponding (4-dimensional) homogeneous representations for each feature point, we thus have:

$$T_{SO} \begin{pmatrix} 2 & 0 & -1 & -1 \\ 3 & 0 & -2 & 0 \\ -3 & -3 & 2 & -2 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1.3840 & -0.9608 & 1.3250 & -1.3140 \\ 4.5620 & 1.3110 & -2.3890 & 0.2501 \\ -0.1280 & -1.6280 & 1.7020 & -0.7620 \\ 1 & 1 & 1 & 1 \end{pmatrix}. \tag{21}$$

Solving (21) for T_{SO}, we obtain:

$$\begin{aligned} T_{SO} &= \begin{pmatrix} -1.3840 & -0.9608 & 1.3250 & -1.3140 \\ 4.5620 & 1.3110 & -2.3890 & 0.2501 \\ -0.1280 & -1.6280 & 1.7020 & -0.7620 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 & -1 \\ 3 & 0 & -2 & 0 \\ -3 & -3 & 2 & -2 \\ 1 & 1 & 1 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 0.7068 & -0.6123 & 0.3536 & 0.1000 \\ 0.7072 & 0.6122 & -0.3537 & 0.2500 \\ 0.0000 & 0.5000 & 0.8660 & 0.9700 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{22}$$

Question 2

(a) Since the group operation in $\mathbb{R}^n$ is vector addition, we have

$$L_v(x) = x + v. \tag{23}$$

(b) Differentiating (23) shows that:

$$dL_v = I. \tag{24}$$

(c) Substituting the derivative map (24) into (3), we obtain:

$$V_\xi(x) = d(L_x)_e(\xi) = (I)(\xi) = \xi; \tag{25}$$

this is the vector field on $\mathbb{R}^n$ that takes the *constant* value ξ. This result shows that the left-invariant vector fields on $\mathbb{R}^n$ are precisely the *constant* vector fields.

(d) Since the group operation in $\mathrm{GL}(n)$ is matrix multiplication, we have

$$L_A(X) = AX. \tag{26}$$

(e) Since the left-translation map L_A in (26) is *already* a linear map, its derivative is simply:

$$dL_A = A. \tag{27}$$

(f) Substituting the derivative map (27) into (3), we obtain:

$$V_\Omega(X) = d(L_X)_I(\Omega) = X\Omega. \tag{28}$$

Question 3

(a) Direct computation shows that:

$$\Omega^2 = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} = \begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix} = -\omega^2 I. \tag{29}$$

Given any power Ω^k of Ω, we can apply (29) to reduce the exponent k on Ω to either 0 or 1 by pulling out a factor of $\Omega^{2p} = (\Omega^2)^p$ for $p \triangleq \lfloor k/2 \rfloor$:

$$\begin{aligned} \Omega^k &= (\Omega^2)^p \Omega^{k \bmod 2} \\ &= (-\omega^2 I)^p \Omega^{k \bmod 2} \\ &= (-1)^p \omega^{2p} \Omega^{k \bmod 2}. \end{aligned} \tag{30}$$

Writing:

$$\Omega = \omega \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\triangleq S} \tag{31}$$

we can substitute (31) into (30) to obtain:

$$\begin{aligned} \Omega^k &= (-1)^p \omega^{2p + (k \bmod 2)} S^{k \bmod 2} \\ &= (-1)^p \omega^k S^{k \bmod 2}. \end{aligned} \tag{32}$$

Finally, recalling the definition of k, we obtain:

$$\Omega^k = (-1)^{\lfloor k/2 \rfloor} \omega^k S^{k \bmod 2}, \tag{33}$$

which gives a simple formula for Ω^k in terms of the *scalar* ω and the 0th or 1st power of the *constant* matrix S defined in (31).

(b) We know from (4) that:

$$\exp(\Omega) = \sum_{k=0}^{\infty} \frac{\Omega^k}{k!}. \tag{34}$$

Substituting (33) into (34), we obtain:

$$\exp(\Omega) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor k/2 \rfloor} \omega^k S^{k \bmod 2}}{k!}. \tag{35}$$

Notice that we can split the series in (35) into even and odd powers k, where for k even $S^{k \bmod 2} = S^0 = I$, and for k odd $S^{k \bmod 2} = S^1 = S$. Thus, we can develop (35) as:

$$\begin{aligned} \exp(\Omega) &= \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor k/2 \rfloor} \omega^k S^{k \bmod 2}}{k!} \\ &= \sum_{p=0}^{\infty} \frac{(-1)^p \omega^{2p}}{(2p)!} I + \sum_{p=0}^{\infty} \frac{(-1)^p \omega^{2p+1}}{(2p+1)!} S. \end{aligned} \tag{36}$$

We may now recognize the two series appearing in (36) as the Taylor series for $\cos(\omega)$ and $\sin(\omega)$, respectively. Therefore, we can simplify (36) as:

$$\exp(\Omega) = \cos(\omega) I + \sin(\omega) S = \begin{pmatrix} \cos(\omega) & \sin(\omega) \\ -\sin(\omega) & \cos(\omega) \end{pmatrix}, \tag{37}$$

which we recognize as the 2D rotation matrix for the angle ω.

A few comments on this result: This is an interesting (and perhaps surprising) result because even though the orthogonal group $\mathrm{O}(2)$ has *both* orientation-**preserving** *and* orientation-**reversing** elements (the latter involving *reflections* in addition to rotations), equation (37) shows that the image of $\mathrm{O}(2)$'s exponential map is *always a rotation*. Put another way: the image of $\mathrm{O}(2)$'s exponential map $\exp(\Omega)$ always lies in the *subgroup* $\mathrm{SO}(2)$.

In fact, it turns out that $\exp(\Omega) \in \mathrm{SO}(n)$ for all $\Omega \in \mathrm{Lie}(\mathrm{O}(n))$ for *any* n (not just $n = 2$). This is a consequence of the fact that, geometrically, $\mathrm{O}(n)$ consists of two *disconnected components*: one consists of matrices having determinant $+1$ (this is $\mathrm{SO}(n)$, the subgroup of *rotations*), and the other consists of matrices having determinant -1. Since the exponential map is a continuous map that describes a "flow" originating at the identity I, and I lies in the $+1$ component of $\mathrm{O}(n)$ (the subgroup $\mathrm{SO}(n)$), it follows that $\exp(\Omega) \subseteq \mathrm{SO}(n)$.

In fact, this argument shows that given *any* Lie group G, the image of G's exponential map always lies in the connected component containing the identity element e.

Question 4

(a) We must determine a Lie algebra element ω in (8) so that $\gamma(1) = y$. Substituting the expression (8) for γ, we obtain:

$$y = \gamma(1) = x \exp(\omega). \tag{38}$$

Solving (38) for ω, we find:

$$\omega = \log(x^{-1}y). \tag{39}$$

Note that (39) is sensible since we have *assumed* that exp is surjective (and therefore $x^{-1}y$ is guaranteed to have *at least one* preimage under the exponential map).

Substituting (39) into (8), we thus find that:

$$\begin{aligned} \gamma \colon [0,1] &\to G \\ \gamma(t) &\triangleq x \exp\big(t \log(x^{-1}y)\big) \end{aligned} \tag{40}$$

is a curve on G satisfying $\gamma(0) = x$ and $\gamma(1) = y$.

(b) The inverse of x under vector addition $+$ is simply $-x$; therefore, using $\exp(v) = \log(v) = v$, the Euclidean specialization of (40) is:

$$\begin{aligned} \gamma(t) &= x \exp\big(t \log(x^{-1}y)\big) \\ &= x \exp(t \log(-x + y)) \\ &= x \exp(t(y - x)) \\ &= x + t(y - x). \end{aligned} \tag{41}$$

This is just the expression for the straight line segment joining x to y.

(c) Letting

$$
\begin{aligned}
T_0 &= \begin{pmatrix} 0.4330 & 0.1768 & 0.8839 & 1 \\ 0.2500 & 0.9186 & -0.3062 & 1 \\ -0.8660 & 0.3536 & 0.3536 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
T_1 &= \begin{pmatrix} 0.7500 & -0.0474 & 0.6597 & 2 \\ 0.4330 & 0.7891 & -0.4356 & 4 \\ -0.5000 & 0.6124 & 0.6124 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned} \tag{42}
$$

denote the (4×4)-dimensional homogeneous representations of X_0 and X_1, respectively, and defining:

$$
\Omega \triangleq \log(T_0^{-1} T_1) = \begin{pmatrix} 0.0000 & -0.3703 & -0.3703 & -0.4533 \\ 0.3702 & -0.0001 & -0.0000 & 4.1974 \\ 0.3702 & -0.0000 & -0.0001 & 1.2303 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{43}
$$

we have from part (a) that the curve $\gamma_{\mathrm{SE}(3)} \colon [0, 1] \to \mathrm{SE}(3)$ joining T_0 to T_1 in SE(3) is:

$$
\gamma_{\mathrm{SE}(3)}(t) = T_0 \exp(t\Omega). \tag{44}
$$

Evaluating (44) at $t = 1/2$, we find:

$$
\gamma_{\mathrm{SE}(3)}(1/2) = \begin{pmatrix} 0.6124 & 0.0795 & 0.7866 & 1.6710 \\ 0.3535 & 0.8624 & -0.3624 & 2.5988 \\ -0.7071 & 0.5000 & 0.5000 & 1.3443 \\ 0 & 0 & 0 & 1.0000 \end{pmatrix}, \tag{45}
$$

i.e., the midpoint $\gamma_{\mathrm{SE}(3)}(1/2)$ is the pair:

$$
\gamma_{\mathrm{SE}(3)}(1/2) = \left(\begin{pmatrix} 1.6710 \\ 2.5988 \\ 1.3443 \end{pmatrix}, \begin{pmatrix} 0.6124 & 0.0795 & 0.7866 \\ 0.3535 & 0.8624 & -0.3624 \\ -0.7071 & 0.5000 & 0.5000 \end{pmatrix} \right). \tag{46}
$$

(d) Letting

$$
t_0 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad t_1 = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}, \tag{47}
$$

and

$$
R_0 = \begin{pmatrix} 0.4330 & 0.1768 & 0.8839 \\ 0.2500 & 0.9186 & -0.3062 \\ -0.8660 & 0.3536 & 0.3536 \end{pmatrix}, \qquad R_1 = \begin{pmatrix} 0.7500 & -0.0474 & 0.6597 \\ 0.4330 & 0.7891 & -0.4356 \\ -0.5000 & 0.6124 & 0.6124 \end{pmatrix}, \tag{48}
$$

and defining:

$$
\begin{aligned}
\Phi &\triangleq \log(X_0^{-1} X_1) = \log(t_1 - t_0, R_0^{-1} R_1) \\
&= \left(\begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 0.0000 & -0.3703 & -0.3703 \\ 0.3702 & -0.0001 & -0.0000 \\ 0.3702 & -0.0000 & -0.0001 \end{pmatrix} \right),
\end{aligned} \tag{49}
$$

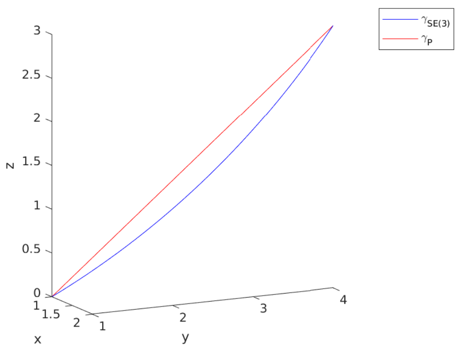
we have from part (a) that:

$$\gamma_P(t) = X_0 \exp(t\Phi). \tag{50}$$

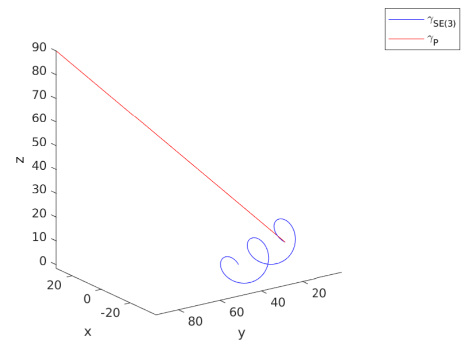
Evaluating (50) at $t = 1/2$, we obtain:

$$\gamma_P(1/2) = \left(\begin{pmatrix} 3/2 \\ 5/2 \\ 3/2 \end{pmatrix}, \quad \begin{pmatrix} 0.6124 & 0.0795 & 0.7866 \\ 0.3535 & 0.8624 & -0.3624 \\ -0.7071 & 0.5000 & 0.5000 \end{pmatrix} \right). \tag{51}$$

(e) Plotting the translational parts of the curves $\gamma_{\mathrm{SE}(3)}$ and γ_P from parts (c) and (d) in MATLAB over the intervals $[0, 1]$ and $[0, 30]$, we obtain the following plots:



(a) $t \in [0, 1]$



(b) $t \in [0, 30]$

The translational part of the curve γ_P is a straight line, while the translational part of $\gamma_{\mathrm{SE}(3)}$ looks like a "corkscrew" – indeed the curves on $\mathrm{SE}(d)$ corresponding to formula (40) are called "screw motions".

Problem 5

(a) From Bayes' Rule:

$$p(\Theta|\tilde{Y}_1, \ldots, \tilde{Y}_m) = \frac{p(\tilde{Y}_1, \ldots, \tilde{Y}_m|\Theta)p(\Theta)}{p(\tilde{Y}_1, \ldots, \tilde{Y}_m)} \propto p(\tilde{Y}_1, \ldots, \tilde{Y}_m|\Theta)p(\Theta). \tag{52}$$

Now observe that the measurement errors ϵ_i in (20) are assumed to be *independent* of one another. This implies that the measurements $\tilde{Y}_i$ are all *conditionally* independent *given* Θ, so the *joint* likelihood in (52) factors as the *product* of the likelihoods of each individual measurement $\tilde{Y}_i$:

$$p(\tilde{Y}_1, \ldots, \tilde{Y}_m|\Theta) = \prod_{i=1}^{m} p(\tilde{Y}_i|\Theta). \tag{53}$$

Substituting (53) into (52), we thus obtain:

$$p(\Theta|\tilde{Y}_1, \ldots, \tilde{Y}_m) \propto p(\Theta) \prod_{i=1}^{m} p(\tilde{Y}_i|\Theta). \tag{54}$$

(b) Solving (20) for the measurement error ϵ_i, we find:

$$\epsilon_i = \tilde{Y}_i - A_i\Theta - b_i. \tag{55}$$

Substituting (55) into the probability density function for $\epsilon_i \sim \mathcal{N}(\mu_i, \Sigma_i)$, we find that:

$$p(\tilde{Y}_i|\Theta) = \frac{1}{\sqrt{\det(2\pi\Sigma_i)}} \exp\left(-\frac{1}{2}(\tilde{Y}_i - A_i\Theta - b_i - \mu_i)^\mathsf{T}\Sigma_i^{-1}(\tilde{Y}_i - A_i\Theta - b_i - \mu_i)\right). \tag{56}$$

(c) Substituting the likelihoods (56) and the prior:

$$p(\Theta) = \frac{1}{\sqrt{\det(2\pi\Sigma_0)}} \exp\left(-\frac{1}{2}(\Theta - \mu_0)^\mathsf{T}\Sigma_0^{-1}(\Theta - \mu_0)\right) \tag{57}$$

for Θ into (54), we obtain:

$$\begin{aligned} p(\Theta|\tilde{Y}_1, \dots, \tilde{Y}_m) \propto \exp&\left(-\frac{1}{2}(\Theta - \mu_0)^\mathsf{T}\Sigma_0^{-1}(\Theta - \mu_0)\right) \\ &\times \prod_{i=1}^m \exp\left(-\frac{1}{2}(\tilde{Y}_i - A_i\Theta - b_i - \mu_i)^\mathsf{T}\Sigma_i^{-1}(\tilde{Y}_i - A_i\Theta - b_i - \mu_i)\right). \end{aligned} \tag{58}$$

Expanding the quadratic forms in the exponents and absorbing any terms not involving Θ into the normalization constant, we may write (58) equivalently as:

$$\begin{aligned} p(\Theta|\tilde{Y}_1, \dots, \tilde{Y}_m) \propto \exp&\left(-\frac{1}{2}\left(\Theta^\mathsf{T}\Sigma_0^{-1}\Theta - 2\mu_0^\mathsf{T}\Sigma_0^{-1}\Theta\right)\right) \\ &\times \prod_{i=1}^m \exp\left(-\frac{1}{2}\left(\Theta^\mathsf{T}A_i^\mathsf{T}\Sigma_i^{-1}A_i\Theta - 2(\tilde{Y}_i - b_i - \mu_i)^\mathsf{T}\Sigma_i^{-1}A_i\Theta\right)\right). \end{aligned} \tag{59}$$

Finally, we simplify (59) by summing the exponents and collecting like terms in Θ:

$$p(\Theta|\tilde{Y}_1, \dots, \tilde{Y}_m) \propto \exp\left(-\frac{1}{2}\left(\Theta^\mathsf{T}\left[\Sigma_0^{-1} + \sum_{i=1}^m A_i^\mathsf{T}\Sigma_i^{-1}A_i\right]\Theta - 2\left[\mu_0^\mathsf{T}\Sigma_0^{-1} + \sum_{i=1}^m (\tilde{Y}_i - b_i - \mu_i)^\mathsf{T}\Sigma_i^{-1}A_i\right]\Theta\right)\right). \tag{60}$$

(d) Comparing (60) with (17), we recognize $p(\Theta|\tilde{Y}_1, \dots, \tilde{Y}_m)$ as an unnormalized Gaussian density in canonical form, with parameters:

$$\Lambda = \Sigma_0^{-1} + \sum_{i=1}^m A_i^\mathsf{T}\Sigma_i^{-1}A_i, \qquad \eta = \Sigma_0^{-1}\mu_0 + \sum_{i=1}^m A_i^\mathsf{T}\Sigma_i^{-1}(\tilde{Y}_i - b_i - \mu_i). \tag{61}$$

It follows from (18) that the posterior distribution for Θ given $\tilde{Y}_1, \dots, \tilde{Y}_m$ is also Gaussian, with mean $\bar{\mu}$ and covariance $\bar{\Sigma}$ given by:

$$\bar{\Sigma} = \left(\Sigma_0^{-1} + \sum_{i=1}^m A_i^\mathsf{T}\Sigma_i^{-1}A_i\right)^{-1}, \qquad \bar{\mu} = \bar{\Sigma}\left(\Sigma_0^{-1}\mu_0 + \sum_{i=1}^m A_i^\mathsf{T}\Sigma_i^{-1}(\tilde{Y}_i - b_i - \mu_i)\right). \tag{62}$$

Remark 1. Note that the formulae for the parameters $\bar{\mu}$ and $\bar{\Sigma}$ of the posterior distribution in (62) both admit nice intuitive interpretations. The posterior covariance $\bar{\Sigma}$ is the inverse of the sum of the information matrices from the prior and the measurements; roughly speaking, this shows that *information is additive* for Gaussian distributions. Similarly, the posterior mean $\bar{\mu}$ takes the form of a *weighted average* of the prior mean μ_0 and the debiased measurements $\tilde{Y}_i$, where each of these vectors is weighted by their associated information matrix.