

UNCONSTRAINED NONLINEAR OPTIMIZATION

Reference:

J.Nocedal and S.J. Wright, "*Numerical Optimization*," 2006. Chapter 3

UNCONSTRAINED NONLINEAR OPTIMIZATION METHODS

- For an arbitrary smooth function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ we want to solve the **unconstrained nonlinear programming** problem

$$\min_x f(x)$$

- There are fundamentally two classes of iterative methods:
 - line-search methods** choose a descent direction p_k , search a suitable scalar $\alpha_k > 0$ such that $f(x_k + \alpha_k p_k) < f(x_k)$, and set $x_{k+1} = x_k + \alpha_k p_k$
 - trust-region methods** compute a quadratic approximation $q(x)$ of f around x_k , solve

$$p_k = \arg \min_{p: \|p\|_2 \leq \Delta} q(x_k + p)$$

where the size Δ of the “trust region” of the model is shrunk until $f(x_k + p_k) < f(x_k)$, and set $x_{k+1} = x_k + p_k$

- The above methods converge to a local minimum (a global one if f convex)

LINE SEARCH METHODS: STEEPEST DESCENT

- **Steepest descent** is the most obvious method, as it picks up p_k orthogonal to the level sets of f

$$p_k = -\nabla f(x_k)$$

$$x_{k+1} = x_k - \alpha_k \nabla f(x_k)$$

- From Taylor's theorem

$$f(x_k + \alpha p_k) = f(x_k) - \alpha \|\nabla f(x_k)\|_2^2 + \alpha^2 p_k' \nabla^2 f(x_k + t\alpha p_k) p_k, t \in (0, 1)$$

- Note that the Hessian of f is not required to compute p_k
- The method can be very slow to converge

LINE SEARCH METHODS: NEWTON'S METHOD

- **Newton's method** chooses $p_k = -(\nabla^2 f(x_k))^{-1} \nabla f(x_k)$ (**Newton's direction**)

$$x_{k+1} = x_k - \alpha_k (\nabla^2 f(x_k))^{-1} \nabla f(x_k)$$

- Newton's direction provides the minimum of the quadratic Taylor's approximation q of f at x_k :

$$q(x_k + p) = f(x_k) + \nabla f(x_k)' p + \frac{1}{2} p' \nabla^2 f(x_k) p$$

- If $\nabla^2 f(x_k) \succ 0$ then for some $\sigma_k > 0$

$$\nabla f(x_k)' p_k = -\nabla f(x_k)' (\nabla^2 f(x_k))^{-1} \nabla f(x_k) \leq -\sigma_k \|p_k\|_2^2$$

so from Taylor's theorem we have $f(x_k + \alpha p_k) < f(x_k)$ for α small enough

LINE SEARCH METHODS: NEWTON'S METHOD

- The method converges very fast, especially close to x^* , where the function f and its quadratic approximation tend to coincide
- For $\alpha_k \equiv 1$ we have **pure Newton's method**. However line search over α is required to ensure convergence
- In case $\nabla^2 f(x_k)$ is not positive definite, a possibility is to use instead $\nabla^2 f(x_k) + \text{diag}(\delta_k)$.

For example δ_k can be computed during a Cholesky factorization to make intermediate diagonal entries $\geq \epsilon$ for some $\epsilon > 0$

LINE SEARCH METHODS: QUASI NEWTON METHODS

- Newton's method requires computing $\nabla^2 f(x_k)$, which could be expensive
- **Quasi-Newton methods** replace $\nabla^2 f(x_k)$ with a matrix B_k which is easier to compute, satisfying the **secant equation**

$$B_{k+1}s_k = y_k, \text{ where } s_k = x_{k+1} - x_k, y_k = \nabla f(x_{k+1}) - \nabla f(x_k)$$

and set $p_k = -B_k^{-1}\nabla f(x_k)$

- The **BFGS formula** (Broyden, Fletcher, Goldfarb, and Shanno) updates

$$B_{k+1} = B_k - \frac{B_k s_k s'_k B_k}{s'_k B_k s_k} + \frac{y_k y'_k}{y'_k s_k}$$

where $B_k \succ 0$ if $B_0 \succ 0$ and $s'_k y_k > 0$ for all k

LINE SEARCH METHODS: QUASI-NEWTON METHODS

- Since B_{k+1} differs from B_k by two one-rank updates, we can update a factorization of B_k recursively.

- In alternative, one can avoid B_k and directly update $H_k = B_k^{-1}$

$$H_{k+1} = H_k + \frac{s'_k y_k + y'_k H_k y_k}{(s'_k y_k)^2} s_k s'_k - \frac{H_k y_k s'_k + s_k H_k y_k}{s'_k y_k}$$

- For large-scale problems, **limited-memory BFGS** only stores a finite number m of past values of (s_k, y_k) (usually $m < 10$) and directly computes the descent direction $p_k = -H_k \nabla f(x_k)$ without storing H_k

LINE SEARCH METHODS: NONLINEAR CONJUGATE-GRADIENT

(Fletcher, Reeves, 1964)

- The **nonlinear conjugate gradient** (CG) method¹ updates p_k as follows:

$$\begin{aligned}\beta_k &= \frac{\|\nabla f(x_{k+1})\|_2^2}{\|\nabla f(x_k)\|_2^2} \\ p_{k+1} &= -\nabla f(x_{k+1}) + \beta_k p_k, \text{ with } p_0 = -\nabla f(x_0)\end{aligned}$$

- The method does not requires the storage of matrices
- The method is almost as simple as steepest descent but usually more efficient, although it does not converge as fast as (quasi-)Newton methods
- As for steepest descent, nonlinear CG may be sensitive to problem **scaling**

¹Vectors $s_1, \dots, s_n \neq 0$ of \mathbb{R}^n are conjugate to a matrix $G \succ 0$ if $s_i' G s_j = 0, \forall i \neq j$

- Given a descent direction p_k , ideally one should choose $x_{k+1} = x_k + \alpha_k p_k$ with

$$\alpha_k = \arg \min_{\alpha > 0} f(x_k + \alpha p_k)$$

- Such a scalar nonlinear optimization may be difficult to solve and require a lot of evaluations of f , so we look for simpler methods
- Simply imposing $f(x_k + \alpha_k p_k) < f(x_k)$ may not work, as the improvement may become smaller and smaller as k grows
- Sufficient decrease is provided by satisfying **Armijo condition**

$$f(x_k + \alpha p_k) \leq f(x_k) + c_1 \alpha \nabla f(x_k)' p_k \quad c_1 \in (0, 1)$$

where usually c_1 is small (e.g., $c_1 = 10^{-4}$)

BACKTRACKING LINE SEARCH

- The following is a simple practical algorithm for selecting a step size α_k satisfying Armijo formula:

Choose $\bar{\alpha} > 0, \rho \in (0, 1), c \in (0, 1)$. Set $\alpha = \bar{\alpha}$.

Repeat until $f(x_k + \alpha p_k) \leq f(x_k) + c\alpha \nabla f(x_k)' p_k$

$\alpha \leftarrow \rho\alpha$

end repeat

Set $\alpha_k = \alpha$.

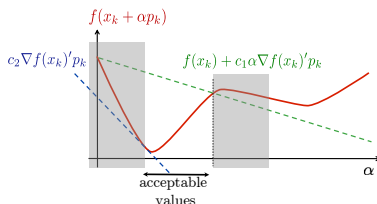
- Possible choices are $\bar{\alpha} = 1$ (e.g., in Newton's method) and $\rho = \frac{1}{2}$ (bisection)

- **Wolfe conditions** include Armijo condition + the **curvature condition**

$$\frac{df(x_k + \alpha p_k)}{d\alpha} = \nabla f(x_k + \alpha p_k)' p_k \geq c_2 \nabla f(x_k)' p_k \quad c_2 \in (c_1, 1)$$

(the condition is **strong** if $|\nabla f(x_k + \alpha p_k)' p_k| \leq c_2 |\nabla f(x_k)' p_k|$ is imposed)

- The curvature condition avoid values of α that are too small, when f is still decaying fast (=very negative derivative)
- Usually $c_1 = 0.9$ in (quasi-)Newton methods and $c_1 = 0.1$ in the nonlinear CG method
- It is possible to prove that if f is continuously differentiable and f bounded below along the descent direction $x_k + \alpha p_k$, $\alpha \geq 0$, the (strong) Wolfe conditions can be satisfied



THEOREM (ZOUTENDIJK)

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be bounded below and differentiable in an open set \mathcal{N} containing the level set $\mathcal{L} = \{x : f(x) \leq f(x_0)\}$, and let ∇f Lipschitz continuous on \mathcal{N} , that is

$$\|\nabla f(x_1) - \nabla f(x_2)\| \leq L\|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathcal{N}$$

for some $L > 0$. Any line search method with p_k a descent direction and α_k satisfying the Wolfe conditions is such that

$$\sum_{k=0}^{\infty} \cos^2(\theta_k) \|\nabla f(x_k)\|^2 < \infty, \quad \cos(\theta_k) = \frac{-\nabla f(x_k)' p_k}{\|\nabla f(x_k)\| \|p_k\|}$$

- If we choose p_k such that $\cos \theta_k \geq \delta > 0, \forall k \geq 0$, then $\lim_{k \rightarrow \infty} \|\nabla f(x_k)\| = 0$.

- The condition $\cos \theta_k \geq \delta > 0, \forall k \geq 0$, holds for the steepest descent method
- It also holds for (quasi-)Newton methods when $B_k \succ 0$ with uniformly bounded condition number
- The convergence result show that the algorithm converges to a stationary point $\nabla f(x) = 0$

CONVERGENCE RATES

- In analyzing the speed of convergence of iterative algorithms, we refer to **convergence rates**. Let $x_k : \mathbb{N} \rightarrow \mathbb{R}^n$ be a converging sequence, $\lim_{k \rightarrow \infty} x_k = x^*$. We define

$$\lim_{k \rightarrow \infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} = r, \quad r \in (0, 1) \quad \text{linear convergence}$$

$$\frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} = r_k, \quad \lim_{k \rightarrow \infty} r_k = 0 \quad \text{superlinear convergence}$$

$$\lim_{k \rightarrow \infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|^2} > 0 \quad \text{quadratic convergence}$$

- Convergence only relates to the **asymptotic behavior** of the algorithm. The transient is often more relevant, especially stopping tolerances are not small

LINE SEARCH - CONVERGENCE RATE

- When f is twice differentiable and $\nabla^2 f(x^*) \succ 0$ we can show that **steepest descent** has the **linear convergence rate**

$$f(x_{k+1}) - f(x^*) \leq r^2(f(x_k) - f(x^*)), \quad \frac{\lambda_{\max} - \lambda_{\min}}{\lambda_{\max} + \lambda_{\min}} < r < 1$$

where $\lambda_{\max}, \lambda_{\min}$ are the max/min eigenvalues of $\nabla^2 f(x^*)$

- When f is twice differentiable and $\nabla^2 f(x^*) \succ 0$ and x_0 is sufficiently close to x^* **Newton's method** has the **quadratic convergence rate**

$$\|x_{k+1} - x^*\| \leq \tilde{L}\|x_k - x^*\|^2, \quad \|\nabla f(x_{k+1})\| \leq 2L\|\nabla^2 f(x^*)^{-1}\|^2\|\nabla f(x_k)\|^2$$

while **quasi-Newton methods** have a **superlinear convergence rate**

LINE SEARCH METHODS: COORDINATE DESCENT

- **Coordinate descent** consists of successively optimizing **only one coordinate** x_{i_k} at each step $k, i_k \in \{1, \dots, n\}$
- The index i_k can be selected **cyclically** $i_{k+1} = [i_k \bmod n] + 1$ or **randomly**
- In case f is differentiable, the update is

$$x_{k+1} = x_k - \alpha_k \frac{\partial f(x_k)}{\partial x_{i_k}} e_{i_k}, \quad \alpha_k > 0, \quad e_i = i\text{th column of } I$$

- In case of perfect line search $x_{k+1} = \arg \min_{\alpha} f(x_k + \alpha e_{i_k})$
- The method can be applied even if f is nonsmooth and some x_i discrete
- Although $f(x^{k+1}) \leq f(x^k)$ the method may not converge to a local minimum
- The method stops if there is no improvement in $f(x^k)$ after one full cycle

NONLINEAR LEAST SQUARES AND GAUSS-NEWTON METHOD

- We want to solve the **nonlinear least-squares** problem

$$\min_x \frac{1}{2} \sum_{j=1}^m r_j^2(x) = \frac{1}{2} \|r(x)\|_2^2$$

where each **residual** $r_j : \mathbb{R}^n \rightarrow \mathbb{R}$ is smooth, $\forall j = 1, \dots, m$ (assume $m \geq n$)

- Let $J(x)$ be the **Jacobian** associated with $r(x)$

$$J(x) = \begin{bmatrix} \nabla r_1(x)' \\ \vdots \\ \nabla r_m(x)' \end{bmatrix}$$

- The gradient $\nabla f(x) = \sum_{j=1}^m r_j(x) \nabla r_j(x) = J(x)' r(x)$, the Hessian is

$$\nabla^2 f(x) = \sum_{j=1}^m \nabla r_j(x) \nabla r_j(x)' + r_j(x) \nabla^2 r_j(x) = J(x)' J(x) + \sum_{j=1}^m r_j(x) \nabla^2 r_j(x)$$

- Gauss-Newton method approximates $\nabla^2 f(x_k) \approx J(x_k)' J(x_k)$

NONLINEAR LEAST SQUARES AND GAUSS-NEWTON METHOD

- Gauss-Newton does not require computing the Hessian matrices $\nabla^2 r_j(x_k)$

$$x_{k+1} = x_k + \alpha_k p_k, \quad p_k = -(J(x_k)'J(x_k))^{-1} J(x_k)'r(x_k)$$

- In many problems $J(x_k)'J(x_k)$ dominates over the neglected term $\sum_{j=1}^m \nabla r_j(x_k) \nabla^2 r_j(x_k)$ close to x^* , so convergence speed can get very close to Newton method
- When $J(x_k)$ is full rank, p_k is a descent direction:

$$p_k' \nabla f(x_k) = p_k' J(x_k)' r(x_k) = -p_k' (J(x_k)' J(x_k)) p_k = -\|J(x_k) p_k\|_2^2$$

- Note that p_k can be obtained by solving the least-squares problem

$$p_k = \arg \min_p \frac{1}{2} \|J(x_k)p + r(x_k)\|_2^2$$

which is the linearized version of the problem at x_k , $r(x) \approx r(x_k) + J(x_k)p$

NONLINEAR LEAST SQUARES AND GAUSS-NEWTON METHOD

- Any technique can be used to solve each least-squares problem
- The Gauss-Newton (GN) method converges under mild assumptions

(Nocedal, Wright, 2006, Th. 10.1)

$$\lim_{k \rightarrow \infty} \nabla f(x_k) = \lim_{k \rightarrow \infty} J(x_k)' r(x_k) = 0$$

- The **Levenberg-Marquardt** (LM) method is a damped version of GN, based on selecting p_k by solving the regularized system

$$p_k = -(\rho_k I + J(x_k)' J(x_k))^{-1} J(x_k)' r(x_k)$$

The parameter ρ_k can be selected at each iteration by simple rules.

Note that $\text{LM} \approx \text{GN}$ for $\rho_k \ll 1$, $\text{LM} \approx \text{gradient descent}$ for $\rho_k \gg 1$.

The LM method can be reinterpreted also as a **trust-region** method.

GAUSS-NEWTON METHOD - EXAMPLE

- We have a data set of $N = 10000$ samples (u_k, y_k)

$$y_k = x_1^2 u_{1k} + x_1 x_2 u_{2k} - x_2 u_{3k}^2 + n_k$$

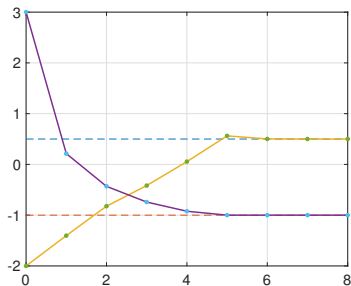
where $x_1 = 0.5, x_2 = -1$ are unknown and noise $n_k \sim N(0, \sigma^2), \sigma = 0.01$

- We want to estimate the parameter vector x by minimizing

$$\min_x \frac{1}{2} \sum_{i=1}^N \|y_k - x_1^2 u_{1k} + x_1 x_2 u_{2k} - x_2 u_{3k}^2\|_2^2$$

- Gauss-Newton method converges in 6 ms² after 8 iterations, with stopping tolerance $\|\nabla f(x_k)\| \leq 10^{-4}$

²Macbook 3 GHz Intel Core i7, MATLAB R2016b



NONLINEAR LEAST SQUARES - FITTING AN EPIDEMIC MODEL

- The spread of Coronavirus COVID-19 can be modeled by the **logistic model**³

$$n(t) = \frac{K}{1 + Ae^{-rt}}$$

where $n(t)$ = number of confirmed infected at time t and K = final epidemic size

- We want to fit (K, r, A) to data available for different countries^{4,5}

$$\min_{K,r,A} \frac{1}{2} \sum_{j=1}^m \left\| n(t_j) - \frac{K}{1 + Ae^{-rt_j}} \right\|_2^2$$

nonlinear least squares

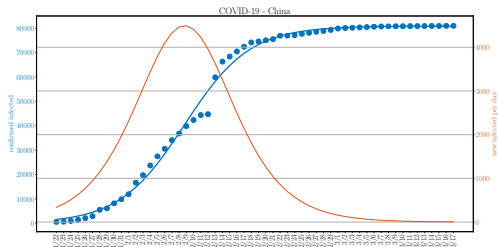
- Here $r_j = n(t_j) - \frac{K}{1+ Ae^{-rt_j}}$, $\nabla r_j = \frac{1}{(1+ Ae^{-rt_j})^2} \begin{bmatrix} 1+ Ae^{-rt_j} \\ -Ke^{-rt_j} \\ KAt_j e^{-rt_j} \end{bmatrix}$

³See also (Batista, 2020) <https://www.researchgate.net/publication/339240777>

⁴World data available at https://github.com/CSSEGISandData/COVID-19/raw/master/csse_covid_19_data/csse_covid_19_time_series/time_series_19-covid-Confirmed.csv

⁵Data for Italy available at <https://github.com/pcm-dpc/COVID-19>

NONLINEAR LEAST SQUARES - FITTING AN EPIDEMIC MODEL

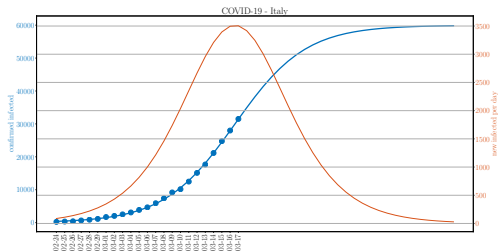


Results for China:

$$K = 80917.6142$$

$$r = 0.2221$$

$$A = 51.6394$$



Results for Italy:

$$K = 59939.3989$$

$$r = 0.2344$$

$$A = 157.5456$$

⁵Problem solved using derivative-free Particle Swarm Optimization (Eberhart, Kennedy, 1995) via the `pyswarm` interface <https://pythonhosted.org/pyswarm/>

SEQUENTIAL QUADRATIC PROGRAMMING

Reference:

J.Nocedal and S.J. Wright, "*Numerical Optimization*," 2006. Chapter 18

EQUALITY-CONSTRAINED NLP

- We consider the equality-constrained NLP problem

$$\begin{array}{ll}\min & f(x) \\ \text{s.t.} & h(x) = 0\end{array}$$

with $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$ smooth functions.

- The Lagrangian function and its derivatives are

$$\begin{aligned}\mathcal{L}(x, \nu) &= f(x) + \nu' h(x) \\ \nabla_x \mathcal{L}(x, \nu) &= \nabla f(x) + A'(x) \nu, \quad A'(x) = [\nabla h_1(x) \dots \nabla h_m(x)] \\ \nabla_{xx}^2 \mathcal{L}(x, \nu) &= \nabla^2 f(x) + \sum_{i=1}^m \nu_i \nabla^2 h_i(x)\end{aligned}$$

- Assume $A(x)$ full row rank, $d' \nabla_{xx}^2 \mathcal{L}(x, \nu) d > 0, \forall d \neq 0$ such that $A(x)d = 0$

QUADRATIC APPROXIMATION

- For all $\nu \in \mathbb{R}^m$, the original problem is equivalent to solving

$$\begin{array}{ll}\min & f(x) + \nu' h(x) = \mathcal{L}(x, \nu) \\ \text{s.t.} & h(x) = 0\end{array}$$

- Consider a pair (x_k, ν_k) and the quadratic approximation of the problem around x_k

$$\begin{array}{ll}\min & f(x_k) + \nu_k' h(x_k) + (\nabla f(x_k) + A'(x_k) \nu_k)' p + \frac{1}{2} p' \nabla_{xx}^2 \mathcal{L}(x_k, \nu_k) p \\ \text{s.t.} & h(x_k) + A(x_k) p = 0\end{array}$$

- By exploiting $\nu_k'(h(x_k) + A(x_k)p) = 0$, the QP is equivalent to

$$\begin{array}{ll}\min_p & \frac{1}{2} p' \nabla_{xx}^2 \mathcal{L}(x_k, \nu_k) p + \nabla f(x_k)' p \\ \text{s.t.} & h(x_k) + A(x_k) p = 0\end{array}$$

- The optimality conditions for the QP are

$$\begin{bmatrix} \nabla_{xx}^2 \mathcal{L}(x_k, \nu_k) & A'(x_k) \\ A(x_k) & 0 \end{bmatrix} \begin{bmatrix} p_k \\ \nu_{k+1} \end{bmatrix} = \begin{bmatrix} -\nabla f(x_k) \\ -h(x_k) \end{bmatrix}$$

- From the solution p_k we set $x_{k+1} = x_k + \alpha_k p_k$ (Newton's step)
- Also, we decided to update ν_{k+1} as the vector of Lagrange multipliers of the approximated QP
- **Sequential quadratic programming** (SQP) for equality-constrained NLP's iterates the above steps from an initial pair (x_0, ν_0) until convergence
- Note that in case $h(x) = Ax - b$ we have $\nabla_{xx}^2 \mathcal{L}(x, \nu) = \nabla^2 f(x)$

SQP FOR NLP WITH EQUALITY AND INEQUALITY CONSTRAINTS

- A similar reasoning applies to general NLP problems

$$\begin{array}{ll}\min & f(x) \\ \text{s.t.} & g_i(x) \leq 0, i \in I \\ & g_j(x) = 0, j \in E\end{array}$$

with $f : \mathbb{R}^n \rightarrow \mathbb{R}$, $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$ smooth functions, $\forall i = 1, \dots, m$.

- The quadratic approximation is

$$\begin{array}{ll}\min & \frac{1}{2}p' \nabla_{xx}^2 \mathcal{L}(x_k, \nu_k) p + \nabla f(x_k)' p \\ \text{s.t.} & g_i(x_k) + \nabla g_i(x_k)' p \leq 0, i \in I \\ & g_j(x_k) + \nabla g_j(x_k)' p = 0, j \in E\end{array}$$

- We solve the QP, get the primal-dual solution (p_k, ν_{k+1}) , and update $x_{k+1} = x_k + \alpha_k p_k$
- Several variants of the SQP method exist (including quasi-Newton methods)

INTERIOR-POINT METHODS

References:

S. Boyd, “Convex Optimization,” lecture notes, <http://ee364a.stanford.edu>

J. Nocedal and S.J. Wright, “*Numerical Optimization*,” 2006. Chapter 19

INTERIOR-POINT METHODS FOR CONVEX PROGRAMS

- Consider the **convex programming** problem

$$\begin{array}{ll}\min & f(x) \\ \text{s.t.} & g_i(x) \leq 0, \quad i = 1, \dots, m \\ & Ax = b\end{array}$$

- Assumptions:
 - f, g_i convex and twice continuously differentiable
 - $A \in \mathbb{R}^{p \times n}$ has rank $A = p$
 - an optimizer x^* exists and $f^* = f(x^*) \in \mathbb{R}$
 - the problem is strictly feasible

$$\exists x \text{ dom } f : g_i(x) < 0, \quad \forall i = 1, \dots, m, \quad Ax = b$$

which ensures strong duality, i.e., $f(x^*) = q(\lambda^*, \nu^*)$, q = dual function

LOGARITHMIC BARRIER

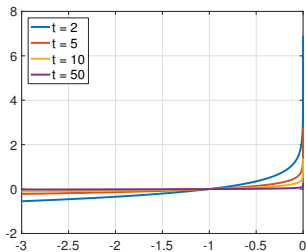
- Denote by $I : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ the **indicator function** of the negative reals ($I(\alpha) = 0$ if $\alpha \leq 0$, $I(\alpha) = +\infty$ if $\alpha > 0$). The problem can be rewritten as

$$\begin{aligned} \min \quad & f(x) + \sum_{i=1}^m I(g_i(x)) \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

- If we approximate $I(\alpha)$ with the smooth **logarithmic barrier** function $-\frac{1}{t} \log(-\alpha)$, $t > 0$, we get

$$\begin{aligned} \min \quad & f(x) - \frac{1}{t} \sum_{i=1}^m \log(-g_i(x)) \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

- The larger t the better the approximation



LOGARITHMIC BARRIER FUNCTION

- The **logarithmic barrier function**

$$\phi(x) = - \sum_{i=1}^m \log(-g_i(x))$$

has the following properties:

- $\text{dom } \phi = \{x : g_i(x) < 0, i = 1, \dots, m\}$
- ϕ is convex, since $-\log$ is monotonic and g_i is convex
- ϕ is twice continuously differentiable and

$$\begin{aligned}\nabla \phi(x) &= - \sum_{i=1}^m \frac{1}{g_i(x)} \nabla g_i(x) \\ \nabla^2 \phi(x) &= \sum_{i=1}^m \frac{1}{g_i(x)^2} \nabla g_i(x) \nabla g_i(x)' - \sum_{i=1}^m \frac{1}{g_i(x)} \nabla^2 g_i(x)\end{aligned}$$

CENTRAL PATH

- For $t \geq 0$ let $x^*(t)$ be the optimizer of the approximated problem

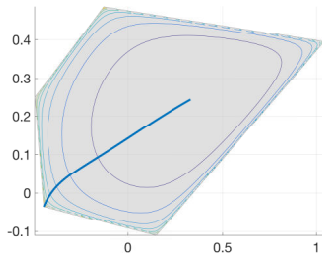
$$\begin{array}{ll}\min & tf(x) + \phi(x) \\ \text{s.t.} & Ax = b\end{array}$$

(assume for now that $x^*(t)$ is unique for all t)

- We call **central path** the curve $\{x^*(t)\}_{t>0}$
- Example: central path for the linear program

$$\begin{array}{ll}\min & c'x \\ \text{s.t.} & Gx \leq g, \quad g \in \mathbb{R}^5\end{array}$$

and level sets of $\phi(x)$



- The original problem satisfies the optimality conditions

$$\begin{aligned}\nabla f(x^*) + \sum_{i=1}^m \lambda_i^* \nabla g_i(x^*) + A' \nu^* &= 0 \\ \lambda^* \geq 0, Ax^* &= b, g_i(x^*) \leq 0, \lambda_i^* g_i(x^*) = 0\end{aligned}$$

- The approximated optimizer $x^*(t)$ satisfies the optimality conditions

$$\begin{aligned}\nabla f(x^*(t)) - \sum_{i=1}^m \frac{1}{tg_i(x^*(t))} \nabla g_i(x^*(t)) + \frac{1}{t} A' w^*(t) &= 0 \\ Ax^*(t) &= b\end{aligned}$$

where $w^*(t)$ is the corresponding vector of Lagrange multipliers

- If we set $\lambda_i^*(t) \triangleq -\frac{1}{tg_i(x^*(t))}$, $\nu^*(t) = \frac{1}{t}w^*(t)$, for all $t > 0$ we have

$$\nabla f(x^*(t)) + \sum_{i=1}^m \lambda_i^*(t) \nabla g_i(x^*(t)) + A' \nu^*(t) = 0, \quad Ax^*(t) = b$$

$$\lambda_i^*(t) \geq 0, \quad \lambda_i^*(t) g_i(x^*(t)) = -\frac{1}{t}$$

- These are the same KKT conditions of the original problem, except for the relaxation of the complementary slackness condition to $\lambda_i^*(t) g_i(x^*(t)) = -\frac{1}{t}$

- The dual function q of the original problem evaluated at $\lambda^*(t), \nu^*(t)$ is

$$\begin{aligned} q(\lambda^*(t), \nu^*(t)) &= \min_x \overbrace{\left\{ f(x) + \sum_{i=1}^m \lambda_i^*(t) g_i(x) + (Ax - b)' \nu^*(t) \right\}}^{\mathcal{L}(x, \lambda^*(t), \nu^*(t))} \\ &= f(x^*(t)) - \frac{m}{t} \end{aligned}$$

as $x^*(t)$ also satisfies the optimality condition $\nabla_x \mathcal{L}(x, \lambda^*(t), \nu^*(t)) = 0$

- Since $q(\lambda^*(t), \nu^*(t)) \leq f(x^*)$, and since $f(x^*) \leq f(x^*(t))$ as $x^*(t)$ is feasible, we get

$$f(x^*(t)) - \frac{m}{t} \leq f(x^*) \leq f(x^*(t))$$

which confirms the intuition $f(x^*(t)) \rightarrow f(x^*)$ as $t \rightarrow +\infty$

FINDING A FEASIBLE POINT (PHASE I)

- Consider the **feasibility problem**

find x such that $g_i(x) \leq 0, i = 1, \dots, m, \quad Ax = b$

- The **basic phase I method** consists of solving the following convex problem with $n + 1$ variables

$$\begin{aligned}(x_0^*, s_0^*) = \arg \min_{x, s} \quad & s \\ \text{s.t.} \quad & g_i(x) - s \leq 0, i = 1, \dots, m \\ & Ax = b\end{aligned}$$

from any initial guess x_0 such that $Ax_0 = b, s_0 > \max g_i(x_0)$

- If $s_0^* < 0$ then x_0^* is strictly feasible for the original problem

BARRIER METHOD

- **Barrier method:** Given an initial strictly feasible x , execute:

0. Let $t_0 > 0, t = t_0, \beta > 1$, tolerance $\epsilon > 0$

1. Compute $x \leftarrow \arg \min_{\text{s.t. } Ax = b} tf(x) + \phi(x)$ (centering step)

2. If $\frac{m}{t} \leq \epsilon$ stop

3. Otherwise increase $t \leftarrow \beta t$ and go to 1

- Newton's method solves the **centering step**, with the last x as initial guess
- Tradeoff: a large β makes fewer centering steps but more Newton iterations at each step. Typically $\beta = 10 \div 20$
- The algorithm terminates with $f(x) - f^*(x) \leq \frac{m}{t} \leq \epsilon$ in exactly $\left\lceil \frac{\log \frac{m}{\epsilon t_0}}{\log \beta} \right\rceil$ centering steps + computation of initial $x^*(t_0)$

PRIMAL-DUAL INTERIOR-POINT METHODS

- Consider the general NLP problem

$$\begin{array}{ll}\min & f(x) \\ \text{s.t.} & g(x) \leq 0 \\ & h(x) = 0\end{array}$$

with $f : \mathbb{R}^n \rightarrow \mathbb{R}, g : \mathbb{R}^n \rightarrow \mathbb{R}^m, h : \mathbb{R}^n \rightarrow \mathbb{R}^p$ smooth functions.

- The optimality condition for the NLP with slacks can be written as

$$\begin{array}{rclcl}\nabla f(x) + \nabla g(x)'z + \nabla h(x)'y & = & 0 & & \\ g(x) + s & = & 0 & S & = \text{diag}(s) \\ h(x) & = & 0 & Z & = \text{diag}(z) \\ SZe & = & 0 & e & = [1 \dots 1]' \\ s, z & \geq & 0 & & \end{array}$$

PRIMAL-DUAL INTERIOR-POINT METHODS

- Let us now relax the optimality conditions as

$$\begin{aligned}\nabla f(x) + \nabla g(x)'z + \nabla h(x)'y &= 0 \\ g(x) + s &= 0 \\ h(x) &= 0 \quad \mu \geq 0 \\ SZe &= \mu e \\ s, z &\geq 0\end{aligned}$$

- Let $x^*(\mu), s^*(\mu), y^*(\mu), z^*(\mu)$ be the solution of the relaxed KKT equations
- For $\mu > 0$ the curve $(x^*(\mu), s^*(\mu), y^*(\mu), z^*(\mu))$ is the **primal-dual central path**
- Note that $\mu = s_i^*(\mu)z_i^*(\mu) = -g_i(x^*(\mu))z_i^*(\mu) = \frac{1}{t}$
- For $\mu \rightarrow 0$, under suitable assumptions, the central path converges to the primal/dual optimizer (x^*, s^*, y^*, z^*)

PRIMAL-DUAL INTERIOR-POINT METHODS

- Primal-dual interior point methods apply a Newton step to solve the system of relaxed KKT equations with decreasing values of μ
- They are more efficient than barrier method when high accuracy is needed
- Often exhibit superlinear asymptotic convergence
- They can start at infeasible points

PRIMAL-DUAL INTERIOR-POINT METHOD FOR LP

- Let us consider the LP

$$\begin{array}{ll}\min_x & c'x \\ \text{s.t.} & Ax \leq b \\ & Ex = f\end{array}$$

- By introducing the slack vector $s = b - Ax$, the KKT conditions

$$\begin{array}{rcl}c + A'z + E'y & = & 0 \\ Ax + s & = & b \\ Ex & = & f \\ z_i s_i & = & 0, i = 1, \dots, m \\ z, s & \geq & 0\end{array}$$

can be rewritten as

$$F(x, z, y, s) = \begin{bmatrix} A'z + E'y + c \\ Ax + s - b \\ Ex - f \\ ZSe \end{bmatrix} = 0, \quad z, s \geq 0$$

where $Z = \text{diag}(z_1, \dots, z_m)$, $S = \text{diag}(s_1, \dots, s_m)$, $e = [1 \dots 1]'$

PRIMAL-DUAL INTERIOR-POINT METHOD FOR LP

- We want to solve the nonlinear system $F(x, z, y, s) = 0$ by Newton's method
- Starting from a candidate solution $z > 0, s > 0, x, y$, Newton's step $\Delta x, \Delta z, \Delta y, \Delta s$ is given by solving the linear system

$$0 = F(x, z, y, s) + \nabla F(x, z, y, s) \begin{bmatrix} \Delta x \\ \Delta z \\ \Delta y \\ \Delta s \end{bmatrix}$$

- Let $\begin{bmatrix} r^c \\ r^b \\ r^f \end{bmatrix} = \begin{bmatrix} A'z + E'y + c \\ Ax + s - b \\ Ex - f \end{bmatrix}$. The linear system to solve is

$$\begin{bmatrix} 0 & A' & E' & 0 \\ A & 0 & 0 & I \\ E & 0 & 0 & 0 \\ 0 & S & 0 & Z \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta z \\ \Delta y \\ \Delta s \end{bmatrix} = \begin{bmatrix} -r^c \\ -r^b \\ -r^f \\ -ZSe \end{bmatrix}$$

- To preserve $z, s \geq 0$ we set $\begin{bmatrix} x \\ z \\ y \\ s \end{bmatrix} \leftarrow \begin{bmatrix} x \\ z \\ y \\ s \end{bmatrix} + \alpha \begin{bmatrix} \Delta x \\ \Delta z \\ \Delta y \\ \Delta s \end{bmatrix}$ with α sufficiently small

PRIMAL-DUAL INTERIOR-POINT METHOD FOR LP

- To prevent excessively small α , given the current $x_k, s_k, \lambda_k, x_k, s_k > 0$, primal-dual interior-point method solve instead the relaxed system

$$\begin{bmatrix} 0 & A' & E' & 0 \\ A & 0 & 0 & I \\ E & 0 & 0 & 0 \\ 0 & S_k & 0 & Z_k \end{bmatrix} \begin{bmatrix} \Delta x_k \\ \Delta z_k \\ \Delta y_k \\ \Delta s_k \end{bmatrix} = \begin{bmatrix} -r_k^c \\ -r_k^b \\ -r_k^f \\ -Z_k S_k e + \sigma_k \mu_k e \end{bmatrix}$$

where $\mu_k = \frac{1}{m} z_k' s_k$ is the current **duality measure** and $\sigma_k \in [0, 1]$ is the **centering parameter**, that is the factor we want to reduce the current μ_k

- The performance of the method depends on how α_k and σ_k are chosen
- Mehrotra's predictor-corrector algorithm** is one of the most used IP methods for LP (Mehrotra, 1992)
- Homogeneous and self-dual formulations** are useful to easily recognize infeasibility and unboundedness (Yu, Todd, Mizuno, 1994) (Xu, Hung, Ye, 1996)

PRIMAL-DUAL INTERIOR-POINT METHOD FOR QP

- Consider the convex QP

$$\begin{array}{ll}\min_x & \frac{1}{2}x'Qx + c'x \\ \text{s.t.} & Ax \leq b \\ & Ex = f\end{array} \quad Q = Q' \succeq 0$$

- By introducing the slack vector $s = b - Ax$, the KKT conditions

$$\begin{array}{rcl}Qx + c + E'y + A'z & = & 0 \\ Ex & = & f \\ Ax + s & = & b \\ z_i s_i & = & 0, \quad i = 1, \dots, m \\ z, s & \geq & 0\end{array}$$

can be rewritten as

$$0 = F(x, z, y, s) = \begin{bmatrix} Qx + E'y + A'z + c \\ Ex - f \\ Ax + s - b \\ ZSe \end{bmatrix} \triangleq \begin{bmatrix} r_Q \\ r_E \\ r_A \\ r_S \end{bmatrix}, \quad z, s \geq 0$$

PRIMAL-DUAL INTERIOR-POINT METHOD FOR QP

(Nocedal, Wright, 2006) (Rao, Wright, Rawlings, 1998)

- Start from a candidate solution $z > 0, s > 0, x, y$
- As for LP, we want to solve $F(x, z, y, s) = 0$ by Newton's method
- We use a variant of Mehrotra's predictor-corrector algorithm (Mehrotra, 1992)
- First, we solve the linear system (**predictor step**)

$$\underbrace{\begin{bmatrix} Q & E' & A' & 0 \\ E & 0 & 0 & 0 \\ A & 0 & 0 & I \\ 0 & 0 & S & Z \end{bmatrix}}_{\nabla F} \begin{bmatrix} \Delta x_{\text{aff}} \\ \Delta y_{\text{aff}} \\ \Delta z_{\text{aff}} \\ \Delta s_{\text{aff}} \end{bmatrix} = \underbrace{\begin{bmatrix} -r_Q \\ -r_E \\ -r_A \\ -r_S \end{bmatrix}}_{-F}$$

PRIMAL-DUAL INTERIOR-POINT METHOD FOR QP

(Nocedal, Wright, 2006) (Rao, Wright, Rawlings, 1998)

- Next, we solve the linear system (**centering-corrector step**)

$$\begin{bmatrix} Q & E' & A' & 0 \\ E & 0 & 0 & 0 \\ A & 0 & 0 & I \\ 0 & 0 & S & Z \end{bmatrix} \begin{bmatrix} \Delta x_{cc} \\ \Delta y_{cc} \\ \Delta z_{cc} \\ \Delta s_{cc} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\Delta S_{\text{aff}} \Delta Z_{\text{aff}} e + \sigma \mu e \end{bmatrix}$$

where the **centering parameter** $\sigma \in [0, 1)$ is chosen as

$$\begin{aligned} \alpha_{\text{aff}} &= \arg \max_{\alpha} \{ \alpha \in [0, 1] : \begin{bmatrix} z + \alpha \Delta z_{\text{aff}} \\ s + \alpha \Delta s_{\text{aff}} \end{bmatrix} \geq 0 \} \\ \mu_{\text{aff}} &= (z + \alpha_{\text{aff}} \Delta z_{\text{aff}})' (s + \alpha_{\text{aff}} \Delta s_{\text{aff}}) / m \\ \mu &= z' s / m \quad \leftarrow \text{duality gap} \\ \sigma &= (\mu_{\text{aff}} / \mu)^3 \end{aligned}$$

- Note:** the same left-hand-side matrix is used to solve both linear systems.
So such a matrix can be factorized just once at each IP iteration

PRIMAL-DUAL INTERIOR-POINT METHOD FOR QP

(Nocedal, Wright, 2006) (Rao, Wright, Rawlings, 1998)

- Now set

$$\Delta x = \Delta x_{\text{aff}} + \Delta x_{\text{cc}}, \quad \Delta y = \Delta y_{\text{aff}} + \Delta y_{\text{cc}}$$

$$\Delta z = \Delta z_{\text{aff}} + \Delta z_{\text{cc}}, \quad \Delta s = \Delta s_{\text{aff}} + \Delta s_{\text{cc}}$$

and choose $\alpha_{\max} = \arg \max\{\alpha \in [0, 1] : \begin{bmatrix} z + \alpha \Delta z \\ s + \alpha \Delta s \end{bmatrix} \geq 0\}$, so that z, s remain nonnegative

- The actual step-length is chosen as $\alpha = \gamma \alpha_{\max}$, with the step-factor $\gamma \in (0, 1)$ close to 1, see (Mehrotra, 1992)
- For even better choices of the step-length α see (Curtis, Nocedal, 2007)
- For reducing the number of factorizations, execute multiple corrections steps (Gondzio, 1996)
- Given a starting point $\bar{x}, \bar{y}, \bar{z}, \bar{s}$, a good initial guess is to solve for $\Delta z_{\text{aff}}, \Delta s_{\text{aff}}$ and set $x_0 = \bar{x}, y_0 = \bar{y}, z_0 = \max\{1, |\bar{z} + \Delta z_{\text{aff}}|\}, s_0 = \max\{1, |\bar{s} + \Delta s_{\text{aff}}|\}$

PRIMAL-DUAL INTERIOR-POINT METHOD FOR QP

(Nocedal, Wright, 2006) (Rao, Wright, Rawlings, 1998) (Gondzio, Terlaki, 1994)

- Let $\Delta\tilde{z} = Z^{-1}\Delta z$. We can eliminate $\Delta s = Z^{-1}r_S - S\Delta\tilde{z}$ and get the system

$$\begin{bmatrix} Q & E' & A'Z \\ E & 0 & 0 \\ A & 0 & -S \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta\tilde{z} \end{bmatrix} = \begin{bmatrix} -r_Q \\ -r_E \\ Z^{-1}r_S - r_A \end{bmatrix}$$

- The above system can be made symmetric by multiplying the last rows by Z
- We can further easily eliminate $Z^{-1}\Delta z = S^{-1}(A\Delta x + r_A - Z^{-1}r_S)$ and get


$$\begin{bmatrix} Q + A'ZS^{-1}A & E' \\ E & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -r_Q + A'S^{-1}(r_S - Zr_A) \\ -r_E \end{bmatrix}$$

- Note that $Z^{-1}S$ is positive and diagonal.

EXAMPLE: NLP SOLUTION VIA IPOPT & CASADI

- **IPOPT** (Interior Point **OPT**imizer⁶) is a software package based on an IP method to solve the NLP (Wächter, Biegler, 2006)

$$\begin{aligned} \min_x \quad & f(x) \\ \text{s.t.} \quad & g_\ell \leq g(x) \leq g_u \\ & x_\ell \leq x \leq x_u \end{aligned}$$

- **CasADi**⁷ is a modeling language for NLP problems. It implements **automatic differentiation** for computing gradients (Andersson, Gillis, Horn, Rawlings, Diehl, 2019)
- CasADi + IPOPT greatly simplifies formulating and solving nonlinear optimization problems via interior-point methods in MATLAB,  python, or C++

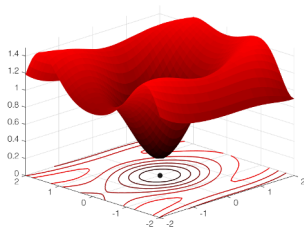
⁶<https://coin-or.github.io/Ipopt/>

⁷<https://web.casadi.org/>

EXAMPLE: NLP SOLUTION VIA IPOPT & CASADI

- Let us minimize the course-logo function

$$f(x, y) = -e^{-(x^2+y^2)} + 0.3 \sin\left(\frac{1}{10}x^3 + y^2\right) + 1.2$$



```
import casadi.*  
  
x= SX.sym('x')  
y= SX.sym('y')  
  
f=-exp(-(x^2+y^2))+.3*sin(x^3/10+y^2)+1.2;  
P=struct('f',f,'x',[x;y]);  
F=nlpso1('F','ipopt',P);  
r=F('x0',[-1;-1]);  
xopt=full(r.x);  
fopt=full(r.f);
```

```
from casadi import *  
  
x= SX.sym('x')  
y= SX.sym('y')  
  
f=-exp(-(x**2+y**2))+.3*sin(x**3/10+y**2)+1.2  
P=dict(x=vertcat(x,y), f=f)  
F=nlpso1('F','ipopt',P)  
r=F(x0=[-1,-1])  
xopt=r['x'].full()  
fopt=r['f'].full()
```

MATLAB

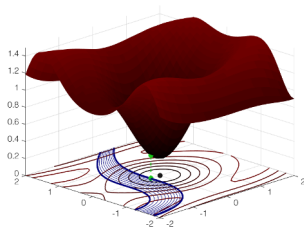


python

- Optimizer $x^* = 0, y^* = 0$, optimum $f^* = 0.2$

EXAMPLE: NLP SOLUTION VIA IPOPT & CASADI

Number of objective function evaluations	= 18
Number of objective gradient evaluations	= 11
Number of equality constraint evaluations	= 0
Number of inequality constraint evaluations	= 0
Number of equality constraint Jacobian evaluations	= 0
Number of inequality constraint Jacobian evaluations	= 0
Number of Lagrangian Hessian evaluations	= 10



- Let's add the constraint

$$1 \leq (x+2)^2 - \frac{1}{2}y^3 \leq 3$$

```
g=(x+2)^2-y^3/2;  
P=struct('f',f,'x',[x;y],'g',g);  
F=nlpsol('F','ipopt',P);  
r=F('x0',[-1;-1],'ubg',3,'lb主',1);  
xopt=full(r.x);  
fopt=full(r.f);  
lam_g_opt = full(r.lam_g);
```

```
g= (x+2)**2-y**3/2  
P=dict(x=vertcat(x,y), f=f, g=g)  
F=nlpsol('F','ipopt',P)  
r=F(x0=[-1,-1],ubg=3,lb主=1))  
xopt=r['x'].full()  
fopt=r['f'].full()  
lam_g_opt=r['lam_g'].full()
```

MATLAB



python

- New optimizer $x^* = -0.2679$, $y^* = 0$, optimum $f^* = 0.2687$

EXAMPLE: NLP SOLUTION VIA JAX/JAXOPT

- Use **JAX** for autodiff and **JAXopt** (<https://jaxopt.github.io>)

```
import jax
import jax.numpy as jnp
import jaxopt

def f(z):
    return -jnp.exp(-(z[0]**2+z[1]**2))+
           .3*jnp.sin(z[0]**3/10+z[1]**2)+1.2

z0=jnp.array([-1.,-1.])
solver=jaxopt.ScipyMinimize(fun=f,method="L-BFGS-B")
zopt,status=solver.run(z0)
fopt=status.fun_val
```


$$z^*(0,0), f^* = 0.2$$

$$\begin{aligned} \min_{z,t} \quad & f(z) + 1000(t - g(z))^2 \\ \text{s.t.} \quad & 1 \leq t \leq 3 \end{aligned}$$

```
def g(z):
    return (z[0]+2.)**2-z[1]**3/2.
def ft(zt):
    z=zt[0:2]; t=zt[2]
    return f(z)+1.e3*(t-g(z))**2
solver=jaxopt.ScipyBoundedMinimize(fun=ft,
                                   tol=1.e-10)
zt0=jnp.hstack((z0,g(z0)))
ztopt,status=solver.run(zt0,
                        bounds=([-jnp.inf,-jnp.inf,1.],
                               [jnp.inf,jnp.inf,3.]))
zopt=ztopt[0:2]
fopt=status.fun_val
```

$$z^*(-0.2679,0), f^* = 0.2687$$

EXAMPLE: NLP SOLUTION IN JULIA

- Use [Nonconvex.jl](https://julianonconvex.github.io/Nonconvex.jl) package and [IPOpt](#) in 
(<https://julianonconvex.github.io/Nonconvex.jl>)

```
using Nonconvex
Nonconvex.@load IPOpt
f(z) = - exp(-(z[1]^2+z[2]^2))+
        .3*sin(z[1]^3/10+z[2]^2)+1.2
z0 = [-1.,-1.]
model = Model(f)
u = [Inf,Inf]
ℓ = -u
addvar!(model,ℓ,u)
r = optimize(model, IPOptAlg(), z0)
zopt = r.minimizer
fopt = r.minimum
```

$$z^*(0,0), f^* = 0.2$$

$$\begin{aligned} \min_z \quad & f(z) \\ \text{s.t.} \quad & 1 \leq g(z) \leq 3 \end{aligned}$$

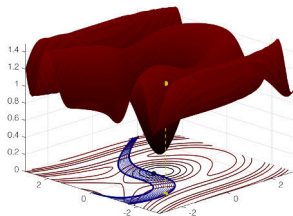
```
g(z) = (z[1] + 2.) ^ 2 - z[2] ^ 3 / 2.
add_ineq_constraint!(model, z-> g(z)-3)
add_ineq_constraint!(model, z-> -g(z)+1)
r = optimize(model, IPOptAlg(), z0)
zopt = r.minimizer
fopt = r.minimum
```

$$z^*(-0.2679,0), f^* = 0.2687$$

EXAMPLE: DEPENDENCE ON INITIAL GUESS

- **Caveat:** the NLP is non convex.
- If we start from $x_0 = -2, y_0 = -2$ we get the different local minimum

$$x^* = -1.5078, y^* = -1.7668, \text{optimum } f^* = 1.3019 !$$



- If function/constraints are not convex, one may need to test different initial conditions, or switch to **global optimization** methods