GEOMETRY OF UNITARY SHIMURA VARIETIES AND ARITHMETIC LEVEL RAISING THEOREM

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Abstract. Let F be a real quadratic field in which a fixed prime p is inert, and E_0 be an imaginary quadratic field in which p splits; put $E = E_0F$. Let $\mathrm{Sh}_{1,n-1}$ be the special fiber over $\mathbb{F}_{p^2}$ of the Shimura variety for $G(U(1,n-1)\times U(n-1,1))$ with hyperspecial level structure at p for some integer $n \geq 2$. Let $\mathrm{Sh}_{1,n-1}(K_{\mathfrak{p}}^1)$ be the special fiber over $\mathbb{F}_{p^2}$ of a Shimura variety for $G(U(1,n-1)\times U(n-1,1))$ with parahoric level structure at p for some integer $n \geq 2$. We exhibit elements in the higher Chow group of the supersingular locus of $\mathrm{Sh}_{1,n-1}$ and study the stratification of $\mathrm{Sh}_{1,n-1}$. Moreover, we study the geometry of $\mathrm{Sh}_{1,n-1}(K_{\mathfrak{p}}^1)$ and prove a form of Ihara lemma. With Ihara lemma, we prove the the arithmetic level raising map is surjective for $n = 2$.

Contents

1. Introduction

The study of the geometry of Shimura varieties lies at the heart of the Langlands program. Arithmetic information of Shimura varieties builds a bridge relating the world of automorphic representations and the world of Galois representations.

One of the interesting topics in this area is to prove the surjectivity of the arithmetic level raising map for unitary Shimura varieties. Rong Zhou introduced a new method in [28] to prove the surjectivity by calculating the higher Chow group $\mathrm{Ch}^1(X^{ss}, 1, \mathbb{F}_l)$ with X to be the special fiber of a quaternionic Shimura variety. Ruiqi Bai and his collaborator Hao Fu calculated the higher Chow group $\mathrm{Ch}^1(X^{ss}, 1, \mathbb{F}_l)$ with X to be the special fiber of the unitary Shimura variety for $G(U(2r, 1))$ with hyperspecial level structure at an inert prime p. They both proved a form of Ihara lemma to show the surjectivity of the arithmetic level raising map after the calculation of the higher Chow group.

In our work, we calculate $\mathrm{Ch}^1(X^{ss}, 1, \mathbb{F}_l)$ with X to be the special fiber of the unitary Shimura varieties for $G(U(1, n-1) \times U(n-1, 1))$ with hyperspecial level structure at a split prime p. We adopt an approach which is largely inspired by Zhiyuan Ding's work on toyshtukas as in [6]. We use the correspondences constructed in the work of [11] to reduce the calculation to find the principal divisors of Deligne-Lusztig varieties.

In order to prove one form of the Ihara lemma, we study the Newton stratification and the Ekedahl-Oort stratification of the special fiber of the $G(U(1, n-1) \times U(n-1, 1))$-Shimura variety with hyperspecial level structure at a split prime p. Before us, many other cases of unitary Shimura varieties have been extensively studied. Viehmann and Wedhorn [27] developed general theory of the Newton and Ekedahl-Oort stratification for good reductions of Shimura varieties of PEL-type. Furthermore, Wooding studied the Newton and Ekedahl-Oort stratifications of $GU(m_1, m_2)$-Shimura variety with hyperspecial level at an unramified prime p for $0 \leq m_1 \leq m_2$. Bültel and Wedhorn [2] studied $GU(1, n-1)$-Shimura variety with hyperspecial level at an inert prime p and showed its Newton stratification, Ekedahl-Oort stratification and final stratification coincide in the nonsupersingular locus. However, there are few results for the the Newton and Ekedahl-Oort stratification of the special fiber of the $G(U(1,n-1)\times U(n-1,1))$-Shimura variety with hyperspecial level structure at a split prime p. We give an explicit description of the two stratifications and show their connections with each other. Moreover, we show the connections of the Ekedahl-Oort stratification with the correspondences Y_j's constructed in [11] for $1 \leq j \leq n$. We also introduce a unitary Shimura variety of parahoric level at a split prime p and study the geometry of it in order to prove the Ihara lemma. Via the Ihara lemma, we prove the surjectivity of the arithmetic level raising map for $n = 2$. The cases for $n \geq 3$ are left as a conjecture which may be proved in the future.

We explain the main results of this paper in more detail. Let F be a real quadratic field, E_0 be an imaginary quadratic field, and $E = E_0F$. Let p be a prime number inert in F, and split in E_0. Let $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ denote the two places of E above p so that $E_{\mathfrak{p}}$ and $E_{\bar{\mathfrak{p}}}$ are both isomorphic to $\mathbb{Q}_{p^2}$, the unique unramified quadratic extension of $\mathbb{Q}_p$. For an integer $n \geq 1$, let G be the similitude unitary group associated to a division algebra over E equipped with an involution of second kind. In the notation of Subsection 3.2, our G is denoted as $G_{1,n-1}$ (resp. $G_{0,n}$). This is an algebraic group over $\mathbb{Q}$ such that $G(\mathbb{Q}_p) \simeq \mathbb{Q}_p^\times \times \mathrm{GL}_n(E_{\mathfrak{p}})$ and $G(\mathbb{R})$ is the unitary similitude group with signature $(1, n-1)$ and $(n-1, 1)$ (resp. $(0, n)$ and $(n, 0)$) at the two archimedean places. (For a precise definition, see Subsection 3.2.)

Let $\mathbb{A}$ denote the ring of finite adeles of $\mathbb{Q}$, and $\mathbb{A}^\infty$ be its finite part. Fix a sufficiently small open compact subgroup $K \subseteq G(\mathbb{A}^\infty)$ with $K_p = \mathbb{Z}_p^\times \times \mathrm{GL}_n(\mathbb{Z}_{p^2}) \subseteq G(\mathbb{Q}_p)$, where $\mathbb{Z}_{p^2}$ is the ring of integers of $\mathbb{Q}_{p^2}$. Let $\mathcal{S}h(G)_K$ be the Shimura variety associated to G of level K.

According to Kottwitz [12], when K^p is neat, $\mathcal{S}h(G)_K$ admits a proper and smooth integral model over $\mathbb{Z}_{p^2}$ which parametrizes certain polarized abelian schemes with K-level structure (See Subsection 3.3). Let $\mathrm{Sh}_{1,n-1}$ (resp. $\mathrm{Sh}_{0,n}$) denote the special fiber of $\mathcal{S}h(G_{1.n-1})_K$ (resp. $\mathcal{S}h(G_{0.n})_K$) over $\mathbb{F}_{p^2}$ and let $\overline{\mathrm{Sh}}_{1,n-1}$ (resp. $\overline{\mathrm{Sh}}_{0,n}$) be its geometric special fiber. This is a proper smooth variety over $\mathbb{F}_{p^2}$ of dimension $2(n-1)$. Let $\mathrm{Sh}^{\mathrm{ss}}_{1,n-1}$ denote the supersingular locus of $\mathrm{Sh}_{1,n-1}$, i.e. the reduced closed subvariety of $\mathrm{Sh}_{1,n-1}$ that parametrizes supersingular abelian varieties. As illustrated in [11], $\mathrm{Sh}^{\mathrm{ss}}_{1,n-1}$ is equidimensional of dimension $n-1$.

Let $\mathrm{Ch}^1(\mathrm{Sh}^{\mathrm{ss}}_{1,n-1}, 1, \mathbb{F}_l)$ be the higher Chow group of the Shimura variety $\mathrm{Sh}_{1,n-1}$. Let $\mathrm{T} = \mathrm{Sh_{0,n}}(\mathrm{K}^1_{\mathfrak{p}})$ be the unitary Shimura variety group of level $K^1_{\mathfrak{p}}$ with signature $(0, n)$ and $(n, 0)$ as in Definition 3.5.1. As in [11], there is a correspondence between T and $\mathrm{Sh}_{0,n}$ which can be expressed by the diagram as below:

$$\begin{tikzcd} & \mathrm{T} \arrow[dl, "\overleftarrow{p}_*"'] \arrow[dr, "\overrightarrow{p}_*"] & \\ \mathrm{Sh}_{0,n} & & \mathrm{Sh}_{0,n}, \end{tikzcd}$$

Then one of our main theorem with respect to the higher Chow group can be stated as follows:

Theorem 1.1. *With notations as above, we have*

$$\mathrm{Ch}^1(\mathrm{Sh}^{\mathrm{ss}}_{1,n-1}, 1, \mathbb{F}_l) = \mathrm{Ker}(\mathrm{H}^0_{\acute{e}t}(\mathrm{T}, \mathbb{F}_l) \xrightarrow{\psi} \mathrm{H}^0_{\acute{e}t}(\mathrm{Sh}_{0,n}, \mathbb{F}_l)^{\oplus n}),$$

where $\psi = (\overleftarrow{p}_*, \overrightarrow{p}_*, \overrightarrow{p}_* A, \overrightarrow{p}_*(A \circ A), \cdots, \overrightarrow{p}_*(\underbrace{A \circ \cdots \circ A}_{n-2}))$ *with* $(\overleftarrow{p}_*, \overrightarrow{p}_*)$ *given by the correpsondence above and* A *is a correpondence defined between different* T*'s. The composition* $\circ$ *of* A *is defined in Section* 5.

In particular, for $n = 2$*, we have*

$$\mathrm{Ch}^1(\mathrm{Sh}^{\mathrm{ss}}_{1,1}, 1, \mathbb{F}_l) = \mathrm{Ker}(\mathrm{H}^0_{\acute{e}t}(\mathrm{T}, \mathbb{F}_l) \xrightarrow{(\overleftarrow{p}_*, \overrightarrow{p}_*)} \mathrm{H}^0_{\acute{e}t}(\mathrm{Sh}_{0,2}, \mathbb{F}_l)^{\oplus 2})$$

, where $(\overleftarrow{p}_*, \overrightarrow{p}_*)$ *is the map given by the correspondence above.*

The theorem gives a relation of unitary Shimura varieties of different signatures. With this theorem, we prove a form of Ihara lemma, which is a key ingredient in the proof of the arithmetic level raising theorem inspired by [28]. We first need the following notations.

Definition 1.2. We say that a (complex) representation Π of $\mathrm{GL}_n(\mathbb{A}_E)$ is *RASDC* (that is, regular algebraic conjugate self-dual cuspidal) if

(1) Π is an irreducible cuspidal automorphic representation;
(2) $\Pi \circ \mathrm{c} \cong \Pi^\vee$;
(3) for every archimedean place w of F, Π_w is regular algebraic.

Definition 1.3. For any signature $a_\bullet$ defined in Section 3.2. An irreducible representation π of $G_{a_\bullet}(\mathbb{A}^\infty)$ is *relevant* if it satistfies:

(1) There exists an admissible irreducible representation π_∞ of $G_{a_\bullet}(\mathbb{R})$ such that $\pi \otimes \pi_\infty$ is a cuspidal automorphic representation of $G_{a_\bullet}(\mathbb{A})$.
(2) π_∞ is *cohomological in degree* $d(a_\bullet)$ as defined in Section 3.6.
(3) $\pi \otimes \pi_\infty$ admits a cuspidal base change to a representation Π of $\mathrm{GL}_n(\mathbb{A}_E) \times \mathbb{A}^\times_{E_0}$, which can be assumed to be *RASDC* and $\Pi_{\mathfrak{p}} = \pi_p$ by [8].

After base change to Π, we can talk about the Satake parameters associated to $\Pi_{\mathfrak{p}}$ and we denote them by $\{\alpha_1, \cdots, \alpha_n\}$. We can also associate to Π a Hecke maximal ideal $\mathfrak{m}$ which is the kernel of a homomorphism ϕ^Π_R as defined in Section 3.7

Hypothesis 1.4. (1) The prime $l \nmid p(p^{2n-2} - 1)$;
(2) The Hecke maximal ideal $\mathfrak{m}$ is non-Eisenstein such that for every $i \neq d(a_\bullet)$, $H^i_{\acute{e}t}(\overline{\mathrm{Sh}}_{a_\bullet}, \mathbb{F}_l)_{\mathfrak{m}} = 0$ with $\overline{\mathrm{Sh}}_{a_\bullet}$ to be the geometric specical fiber of the unitary Shimura variety of signature $a_\bullet$ to be defined in Subsection 3.3 and $d(a_\bullet)$ to be its dimension and $H^{d(a_\bullet)}_{\acute{e}t}(\mathrm{Sh}_{a_\bullet}, \mathbb{F}_l)_{\mathfrak{m}}$ is torsion-free.
(3) The Satake parameters $\alpha_{\pi_{\mathfrak{p}},1}, \cdots, \alpha_{\pi_{\mathfrak{p}},n}$ mod $\mathfrak{m}$ at $\mathfrak{p}$ are distinct and for any $1 \leq i \neq j \leq n$, $\alpha_{\mathfrak{p},i}/\alpha_{\mathfrak{p},j}$ is not a root of unity.
(4) The multiplicities for π for $a_\bullet = (1, n-1)$ and $a_\bullet = (0, n)$, denoted by $m_{1,n-1}(\pi), m_{0,n}(\pi)$, are equal.
(5) $\phi^\pi_R(S_{\mathfrak{p}}) \equiv 1 \mod \mathfrak{m}$ with ϕ^π_R to be defined in Subsection 3.7.

Theorem 1.5. *Under the the Hypothesis 1.4, we have:*

(1) *(Definite Ihara) The map*

$$\mathrm{H}^0_{ét}(\mathrm{T}, \mathbb{F}_l)_{\mathfrak{m}} \xrightarrow{\psi} \mathrm{H}^0_{ét}(\mathrm{Sh}_{0,n}, \mathbb{F}_l)^{\oplus n}_{\mathfrak{m}}$$

is surjective, where $\psi = (\overleftarrow{p}_*, \overrightarrow{p}_*, \overrightarrow{p}_*A, \overrightarrow{p}_*(A\circ A), \cdots, \overrightarrow{p}_*\underbrace{(A\circ\cdots\circ A)}_{n-2})$ *with* $(\overleftarrow{p}_*, \overrightarrow{p}_*)$ *given by the correpsondence above and A is a correpondence defined between different* T*'s. The composition* $\circ$ *of A is defined in Section 6.*

(2) *(Indefinite Ihara) The map*

$$\mathrm{H}^{2(n-1)}_{ét}(\overline{Sh}_{1,n-1}(K^1_{\mathfrak{p}}), \mathbb{F}_l(n))_{\mathfrak{m}} \xrightarrow{\psi} \mathrm{H}^{2(\mathrm{n}-1)}_{\mathrm{ét}}(\overline{Sh}_{1,n-1}, \mathbb{F}_l(n))^{\oplus n}_{\mathfrak{m}}$$

is surjective, where $\overline{Sh}_{1,n-1}, \overline{Sh}_{1,n-1}(K^1_{\mathfrak{p}})$ *are the generic fibers of* $\mathcal{S}h_{1,n-1}, \mathcal{S}h_{1,n-1}(K^1_{\mathfrak{p}})$, $\psi = (\overleftarrow{p}_*, \overrightarrow{p}_*, \overrightarrow{p}_*A, \overrightarrow{p}_*(A\circ A), \cdots, \overrightarrow{p}_*\underbrace{(A\circ\cdots\circ A)}_{n-2})$ *with all the maps induced from those in* (1) *and we use the same notation for simplicity.*

The proof of the indefinite Ihara lemma from the definite Ihara lemma follows from the result of [11] on Tate conjecture.

In order to prove the Definite Ihara lemma, we study the Newton and Ekedahl-Oort stratification of $\mathrm{Sh}_{1,n-1}$. We prove that there are n^2 Ekedahl-Oort strata and $\frac{n(n-1)}{2}$ of them are contained in the nonsupersingular locus which are in bijection with the Newton strata contained in the non-supersingular locus. Moreover, we give the connection of the Ekedahl-Oort strata contained in the supersingular locus with the correspondences Y_j's constructed in [11]. Based on [18, Theorem 4.7], we give an explicit construction for the Dieudonneé module corresponds to each Ekedahl-Oort stratum. We also prove that the μ-ordinary locus of $\mathrm{Sh}_{1,n-1}$ is affine by considering the Hasse invariants. As a direct corollary of our result, we show the Newton stratification, Ekedahl-Oort stratification coincide in the nonsupersingular locus.

Additionally, we study the geometry of $\mathrm{Sh}_{1,n-1}(K^1_{\mathfrak{p}})$, which is the special fiber of a unitary Shimura variety with parahoric level structure at p. More explicitly, we construct n correspondences C_j with $1 \le j \le n$ between $\mathrm{Sh}_{1,n-1}(K^1_{\mathfrak{p}})$ and $\mathrm{Sh}_{0,n}$ and show the image of them in $\mathrm{Sh}_{1,n-1}(K^1_{\mathfrak{p}})$ is just the supersingular locus $\mathrm{Sh}_{1,n-1}(K^1_{\mathfrak{p}})$. Moreover, we shown under the natrual map from $\mathrm{Sh}_{1,n-1}(K^1_{\mathfrak{p}})$ to $\mathrm{Sh}_{1,n-1}$, the C_j's are mapped onto Y_j's with $1 \le j \le n$. Thus the C_j's play a similar role in $\mathrm{Sh}_{1,n-1}(K^1_{\mathfrak{p}})$ as Y_j's in $\mathrm{Sh}_{1,n-1}$. We get the definite Ihara lemma by considering the cohomology groups of C_j's and the Tate conjecture proved in [11, Theorem 4.18].

With the Ihara lemma, we try to prove the surjectivity of the Arithmetic level raising map.

As in [11], $\mathrm{Sh}_{1,n-1}$ is of even dimension $d = 2(n-1)$, and the supersingular locus $\mathrm{Sh}^{\mathrm{ss}}_{1,\mathrm{n}-1}$ is of codimension $n-1$. Fix $l \neq p$. We get the cycle class map:

$$\beta : \mathrm{Ch}^1(\mathrm{Sh}^{\mathrm{ss}}_{1,n-1}, 1, \mathbb{F}_l) \to \mathrm{Ch}^n(\mathrm{Sh}_{1,n-1}, 1, \mathbb{F}_l) \to \mathrm{H}^{2n-1}_{ét}(\mathrm{Sh}_{1,n-1}, \mathbb{F}_l(n)).$$

By combining the Hochschild–Serre spectral sequence and localizing at a maximal ideal $\mathfrak{m}$ of the Hecke algebra in Hypothesis 1.4, we obtain the following diagram:

$$\begin{tikzcd}
 & & 0 = \mathrm{H}^0(\mathbb{F}_{p^2}, \mathrm{H}^{2n-1}_{\acute{e}t}(\overline{\mathrm{Sh}}_{1,n-1}, \mathbb{F}_l(n))_{\mathfrak{m}}) \\
\mathrm{Ch}^n(\mathrm{Sh}_{1,n-1}, 1, \mathbb{F}_l)_{\mathfrak{m}} \arrow[r, equal] & \mathrm{H}^{2n-1}_{\mathcal{M}}(\mathrm{Sh}_{1,n-1}, \mathbb{F}_l(n))_{\mathfrak{m}} \arrow[r] \arrow[dr, "Abel-Jacobi\ map"'] & \mathrm{H}^{2n-1}_{\acute{e}t}(\mathrm{Sh}_{1,n-1}, \mathbb{F}_l(n))_{\mathfrak{m}} \arrow[u, two heads] \\
\mathrm{Ch}^1(\mathrm{Sh}^{\mathrm{ss}}_{1,n-1}, 1, \mathbb{F}_l)_{\mathfrak{m}} \arrow[u] \arrow[rr, "level-raising\ map"] & & \mathrm{H}^1(\mathbb{F}_{p^2}, \mathrm{H}^{2n-2}_{\acute{e}t}(\overline{\mathrm{Sh}}_{1,n-1}, \mathbb{F}_l(n))_{\mathfrak{m}}) \arrow[u, hook]
\end{tikzcd}$$

Under the hypothesis 1.4, it is reasonable to assume $\mathrm{H}^i_{\acute{e}t}(\overline{\mathrm{Sh}}_{1,n-1}, \mathbb{F}_l(r))_{\mathfrak{m}} = 0$ whenever $i \neq d$. So the injection in the diagram is an isomorphism, and we get the level-raising map. For the level raising map we have the following conjecture which is well known as arithmetic level raising theorem:

Conjecture 1.6. *Under the Hypothesis 1.4, the level raising map*

$$\mathrm{Ch}^1(\mathrm{Sh}^{\mathrm{ss}}_{1,n-1}, 1, \mathbb{F}_l)_{\mathfrak{m}} \to \mathrm{H}^1(\mathbb{F}_{p^2}, \mathrm{H}^{2n-2}_{\acute{e}t}(\overline{\mathrm{Sh}}_{1,n-1}, \mathbb{F}_l(n))_{\mathfrak{m}})$$

is surjective.

In particular, we have proved the arithmetic level raising theorem for $n = 2$.

Theorem 1.7. *Under the Hypothesis 1.4, the level raising map*

$$\mathrm{Ch}^1(\mathrm{Sh}^{\mathrm{ss}}_{1,1}, 1, \mathbb{F}_l)_{\mathfrak{m}} \to \mathrm{H}^1(\mathbb{F}_{p^2}, \mathrm{H}^2_{\acute{e}t}(\overline{\mathrm{Sh}}_{1,1}, \mathbb{F}_l(2))_{\mathfrak{m}})$$

is surjective.

We also determine when the surjectivity of the level raising map is not trivial, that is, when $\mathrm{Ch}^1(\mathrm{Sh}^{\mathrm{ss}}_{1,n-1}, 1, \mathbb{F}_l)_{\mathfrak{m}}$ for $n = 2, 3$.

Theorem 1.8. *When $n = 2, 3$, the higher Chow group $\mathrm{Ch}^1(\mathrm{Sh}^{\mathrm{ss}}_{1,n-1}, 1, \mathbb{F}_l)_{\mathfrak{m}}$ is nonzero if and only if there exist two Satake parameters α_i, α_j such that $\alpha_i = p^2\alpha_j$. If $n = 2$, we can further show $\Pi_{\mathfrak{p}}$ is isomorphic to a twisted Steinberg representation and if $n = 3$ and futher assume there exists only one pair such (α_i, α_j) we can show $\Pi_{\mathfrak{p}}$ is isomorphic to the isobaric sum of a 2 dimensional twisted Steinberg representation and a 1 dimensional reprentation, denoted by $St_2(\gamma) \boxplus \beta$.*

For $n > 3$, we can also show if $\mathrm{Ch}^1(\mathrm{Sh}^{\mathrm{ss}}_{1,n-1}, 1, \mathbb{F}_l)_{\mathfrak{m}}$ is nonzero, there exist two Satake parameters α_i, α_j such that $\alpha_i = p^2\alpha_j$. For $n = 2$, we can also assume $\phi^{\Pi}_R(T^1_{\mathfrak{p}}) \equiv p^2 + 1 \mod \mathfrak{m}$.

We briefly describe the structure of the paper. In Section 3, we consider a more general setup of unitary Shimura varieties of PEL type and describe Hecke actions and correspondences on $\mathrm{Sh}_{0,n}$ and $\mathrm{Sh}_{1,n-1}$. We also recall the Tate conjectures proved in [11]. In Section 4, we introduce basic knowledge of Deligne-Lusztig varieties. In Section 5, We recall basic notions of Dieudonné modules and Grothendieck-Messing deformation theory. In Section 6, we recall basic properties of higher Chow groups and calculate $\mathrm{Ch}^1(\mathrm{Sh}^{\mathrm{ss}}_{1,n-1}, 1, \mathbb{F}_l)$. Section 7 is devoted to the proof of the Ihara lemma for $n = 2$. In section 7, we study the Newton and Ekedahl-Oort stratification of $\mathrm{Sh}_{1,n-1}$. In Section 9, we study the geometry of $\mathrm{Sh}_{K^1_{\mathfrak{p}}}$ and in Section 10 we give the proof of the Ihara lemma for $n \geq 3$. In Section 11, we prove the arithmetic level raising map is surjective for $n = 2$. Section 12 is devoted for the proof of theorem 1.8.

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3. Unitary Shimura varieties

We will discuss unitary Shimura varieties of PEL type following [11] and assuming $f = 2$. More general setting can be found in [12].

3.1. **Notation.** We fix a prime number p throughout this paper. We fix an isomorphism $\iota_p : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$. Let $\mathbb{Q}_p^{\mathrm{ur}}$ be the maximal unramified extension of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$.

Let F be a totally real field of degree 2 in which p is inert. We label all real embeddings of F, or equivalently (via ι_p), all p-adic embeddings of F (into $\mathbb{Q}_p^{\mathrm{ur}}$) by τ_1, τ_2 so that post-composition by the Frobenius map takes τ_1 to τ_2 (resp. τ_2 to τ_1). Let E_0 be an imaginary quadratic extension of $\mathbb{Q}$, in which p splits. Put $E = E_0F$. Denote by v and $\bar{v}$ the two p-adic places of E_0. Then p splits into two primes $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ in E, where $\mathfrak{p}$ (resp. $\bar{\mathfrak{p}}$) is the p-adic place above v (resp. $\bar{v}$). Let q_i denote the embedding $E \to E_{\mathfrak{p}} \cong F_p \xrightarrow{\tau_i} \overline{\mathbb{Q}}_p$ and $\bar{q}_i$ the analogous embedding which factors through $E_{\bar{\mathfrak{p}}}$ instead. Composing with ι_p^{-1}, we regard q_i and $\bar{q}_i$ as complex embeddings of E, and we put $\Sigma_{\infty,E} = \{q_1, q_2, \bar{q}_1, \bar{q}_2\}$.

3.2. **Shimura data.** Let D be a division algebra of dimension n^2 over its center E, equipped with a positive involution $*$ which restricts to the complex conjugation c on E. In particular, $D^{\mathrm{opp}} \cong D \otimes_{E,c} E$. We assume that D splits at $\mathfrak{p}$ and $\bar{\mathfrak{p}}$, and we fix an isomorphism

$$D \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \mathrm{M}_n(E_{\mathfrak{p}}) \times \mathrm{M}_n(E_{\bar{\mathfrak{p}}}) \cong \mathrm{M}_n(\mathbb{Q}_{p^2}) \times \mathrm{M}_n(\mathbb{Q}_{p^2}),$$

where $*$ switches the two direct factors. We use $\mathfrak{e}$ to denote the element of $D \otimes_{\mathbb{Q}} \mathbb{Q}_p$ corresponding to the $(1,1)$-elementary matrix by which we mean an $n \times n$-matrix whose $(1,1)$-entry is 1 and whose other entries are zero in the first factor. Let $a_\bullet = (a_1, a_2)$ be a tuple of 2 numbers with $a_i \in \{0, \dots, n\}$ for $1 \le i \le 2$. Assume that there exists an element $\beta_{a_\bullet} \in (D^\times)^{*=-1}$ such that the following condition is satisfied:[1]

Let $G_{a_\bullet}$ be the algebraic group over $\mathbb{Q}$ such that $G_{a_\bullet}(R)$ for a $\mathbb{Q}$-algebra R consists of elements $g \in (D^{\mathrm{opp}} \otimes_{\mathbb{Q}} R)^\times$ with $g\beta_{a_\bullet}g^* = c(g)\beta_{a_\bullet}$ for some $c(g) \in R^\times$. If $G^1_{a_\bullet}$ denotes the kernel of the similitude character $c : G_{a_\bullet} \to \mathbb{G}_{m,\mathbb{Q}}$, then there exists an isomorphism

$$G^1_{a_\bullet}(\mathbb{R}) \simeq U(a_1, n-a_1) \times U(a_2, n-a_2),$$

where the i-th factor corresponds to the real embedding $\tau_i : F \hookrightarrow \mathbb{R}$.

Note that the assumption on D at p implies that

$$G_{a_\bullet}(\mathbb{Q}_p) \simeq \mathbb{Q}_p^\times \times \mathrm{GL}_n(E_{\mathfrak{p}}) \cong \mathbb{Q}_p^\times \times \mathrm{GL}_n(\mathbb{Q}_{p^2}).$$

We put $V_{a_\bullet} = D$ and view it as a left D-module. Let $\langle -,- \rangle_{a_\bullet} : V_{a_\bullet} \times V_{a_\bullet} \to \mathbb{Q}$ be the perfect alternating pairing given by

$$\langle x, y \rangle_{a_\bullet} = \mathrm{Tr}_{D/\mathbb{Q}}(x\beta_{a_\bullet}y^*), \quad \text{for } x, y \in V_{a_\bullet}.$$

Then $G_{a_\bullet}$ is identified with the similitude group associated to $(V_{a_\bullet}, \langle -,- \rangle_{a_\bullet})$, i.e. for all $\mathbb{Q}$-algebra R, we have

$$G_{a_\bullet}(R) = \big\{ g \in \mathrm{End}_{D\otimes_{\mathbb{Q}}R}(V_{a_\bullet} \otimes_{\mathbb{Q}} R) \;\big|\; \langle gx, gy \rangle_{a_\bullet} = c(g)\langle x, y \rangle_{a_\bullet} \text{ for some } c(g) \in R^\times \big\}.$$

Consider the homomorphism of $\mathbb{R}$-algebraic groups $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m) \to G_{a_\bullet,\mathbb{R}}$ given by

$$h(z) = \mathrm{diag}(\underbrace{z, \dots, z}_{a_1}, \underbrace{\bar{z}, \dots, \bar{z}}_{n-a_1}) \times \mathrm{diag}(\underbrace{z, \dots, z}_{a_2}, \underbrace{\bar{z}, \dots, \bar{z}}_{n-a_2}), \quad \text{for } z = x + \sqrt{-1}y.$$

[1] As explained in the proof of [8, Lemma I.7.1], when n is odd, such $\beta_{a_\bullet}$ always exists, and when n is even, existence of $\beta_{a_\bullet}$ depends on the parity of $a_1 + a_2$.

Let $\mu_h : \mathbb{G}_{m,\mathbb{C}} \to G_{a_\bullet,\mathbb{C}}$ be the composite of $h_\mathbb{C}$ with the map $\mathbb{G}_{m,\mathbb{C}} \to \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)_\mathbb{C} \cong \mathbb{C}^\times \times \mathbb{C}^\times$ given by $z \mapsto (z, 1)$. Here, the first copy of $\mathbb{C}^\times$ in $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)_\mathbb{C}$ is the one indexed by the identity element in $\mathrm{Aut}_\mathbb{R}(\mathbb{C})$, and the other copy of $\mathbb{C}^\times$ is indexed by the complex conjugation.

Let E_h be the reflex field of μ_h, i.e. the minimal subfield of $\mathbb{C}$ where the conjugacy class of μ_h is defined. It has the following explicit description. The group $\mathrm{Aut}_\mathbb{Q}(\mathbb{C})$ acts naturally on $\Sigma_{\infty,E}$, and hence on the functions on $\Sigma_{\infty,E}$. Then E_h is the subfield of $\mathbb{C}$ fixed by the stabilizer of the $\mathbb{Z}$-valued function a on $\Sigma_{\infty,E}$ defined by $a(q_i) = a_i$ and $a(\bar{q}_i) = n - a_i$. The isomorphism $\iota_p : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$ defines a p-adic place $\wp$ of E_h. By our hypothesis on E, the local field $E_{h,\wp}$ is an unramified extension of $\mathbb{Q}_p$ contained in $\mathbb{Q}_{p^2}$, the unique unramified extension over $\mathbb{Q}_p$ of degree 2.

3.3. **Unitary Shimura varieties of PEL-type.** Let $\mathcal{O}_D$ be a $*$-stable order of D and $\Lambda_{a_\bullet}$ an $\mathcal{O}_D$-lattice of $V_{a_\bullet}$ such that $\langle \Lambda_{a_\bullet}, \Lambda_{a_\bullet} \rangle_{a_\bullet} \subseteq \mathbb{Z}$ and $\Lambda_{a_\bullet} \otimes_\mathbb{Z} \mathbb{Z}_p$ is self-dual under the alternating pairing induced by $\langle -, - \rangle_{a_\bullet}$. We put $K_p = \mathbb{Z}_p^\times \times \mathrm{GL}_n(\mathcal{O}_{E_\mathfrak{p}}) \subseteq G_{a_\bullet}(\mathbb{Q}_p)$, and fix an open compact subgroup $K^p \subseteq G_{a_\bullet}(\mathbb{A}^{\infty,p})$ such that $K = K^p K_p$ is *neat*, i.e. $G_{a_\bullet}(\mathbb{Q}) \cap gKg^{-1}$ is torsion free for any $g \in G_{a_\bullet}(\mathbb{A}^\infty)$.

Following [12], we have a unitary Shimura variety $\mathcal{S}h_{a_\bullet}$ defined over $\mathbb{Z}_{p^2}$;[2] it represents the functor that takes a locally noetherian $\mathbb{Z}_{p^2}$-scheme S to the set of isomorphism classes of tuples (A, λ, η), where

(1) A is an $2n^2$-dimensional abelian variety over S equipped with an action of $\mathcal{O}_D$ such that the induced action on $\mathrm{Lie}(A/S)$ satisfies the *Kottwitz determinant condition*, that is, if we view the *reduced* relative de Rham homology $H_1^{\mathrm{dR}}(A/S)^\circ := \mathfrak{e} H_1^{\mathrm{dR}}(A/S)$ and its quotient $\mathrm{Lie}^\circ_{A/S} := \mathfrak{e} \cdot \mathrm{Lie}_{A/S}$ as a module over $F_p \otimes_{\mathbb{Z}_p} \mathcal{O}_S \cong \bigoplus_{i=1}^2 \mathcal{O}_S$, they, respectively, decompose into the direct sums of locally free $\mathcal{O}_S$-modules $H_1^{\mathrm{dR}}(A/S)^\circ_i$ of rank n and, their quotients, locally free $\mathcal{O}_S$-modules $\mathrm{Lie}^\circ_{A/S,i}$ of rank $n - a_i$;
(2) $\lambda : A \to A^\vee$ is a prime-to-p $\mathcal{O}_D$-equivariant polarization such that the Rosati involution induces the involution $*$ on $\mathcal{O}_D$;
(3) η is a collection of, for each connected component S_j of S with a geometric point $\bar{s}_j$, a $\pi_1(S_j, \bar{s}_j)$-invariant K^p-orbit of isomorphisms $\eta_j : \Lambda_{a_\bullet} \otimes_\mathbb{Z} \widehat{\mathbb{Z}}^{(p)} \simeq T^{(p)}(A_{\bar{s}_j})$ such that the following diagram commutes for an isomorphism $\nu(\eta_j) \in \mathrm{Hom}(\widehat{\mathbb{Z}}^{(p)}, \widehat{\mathbb{Z}}^{(p)}(1))$:

$$\begin{array}{ccc} \Lambda_{a_\bullet} \otimes_\mathbb{Z} \widehat{\mathbb{Z}}^{(p)} \times \Lambda_{a_\bullet} \otimes_\mathbb{Z} \widehat{\mathbb{Z}}^{(p)} & \xrightarrow{\langle -,- \rangle} & \widehat{\mathbb{Z}}^{(p)} \\ \Big\downarrow {\scriptstyle \eta_j \times \eta_j} & & \Big\downarrow {\scriptstyle \nu(\eta_j)} \\ T^{(p)} A_{\bar{s}_j} \times T^{(p)} A_{\bar{s}_j} & \xrightarrow{\text{Weil pairing}} & \widehat{\mathbb{Z}}^{(p)}(1), \end{array}$$

where $\widehat{\mathbb{Z}}^{(p)} = \prod_{\ell \neq p} \mathbb{Z}_\ell$ and $T^{(p)}(A_{\bar{s}_j})$ denotes the product of the ℓ-adic Tate modules of $A_{\bar{s}_j}$ for all $\ell \neq p$.

The Shimura variety $\mathcal{S}h_{a_\bullet}$ is smooth and projective over $\mathbb{Z}_{p^f}$ of relative dimension $d(a_\bullet) := \sum_{i=1}^2 a_i(n - a_i)$. Note that if $a_i \in \{0, n\}$ for all i, then $\mathcal{S}h_{a_\bullet}$ is of relative dimension zero; we call it a *discrete Shimura variety*.

We denote by $\mathcal{S}h_{a_\bullet}(\mathbb{C})$ the complex points of $\mathcal{S}h_{a_\bullet}$ via the embedding $\mathbb{Z}_{p^2} \hookrightarrow \overline{\mathbb{Q}}_p \xrightarrow{\iota_p^{-1}} \mathbb{C}$. Let $K_\infty \subseteq G_{a_\bullet}(\mathbb{R})$ be the stabilizer of h (3.2) under the conjugation action, and let X_∞ denote the $G_{a_\bullet}(\mathbb{R})$-conjugacy class of h. Then K_∞ is a maximal compact-modulo-center subgroup of $G_{a_\bullet}(\mathbb{R})$. According to [12, page 400], the complex manifold $\mathcal{S}h_{a_\bullet}(\mathbb{C})$ is the disjoint union of $\#\ker^1(\mathbb{Q}, G_{a_\bullet})$ copies of

$$G_{a_\bullet}(\mathbb{Q}) \backslash \big(G_{a_\bullet}(\mathbb{A}^\infty) \times X_\infty\big) / K \cong G_{a_\bullet}(\mathbb{Q}) \backslash G_{a_\bullet}(\mathbb{A}) / K \times K_\infty.$$

[2] Although one can descend $\mathcal{S}h_{a_\bullet}$ to the subring $\mathcal{O}_{E_{h,\wp}}$ of $\mathbb{Z}_{p^2}$, we ignore this minor improvement here.

Here, if n is even, then $\ker^1(\mathbb{Q}, G_{a_\bullet}) = (0)$, while if n is odd then

$$\ker^1(\mathbb{Q}, G_{a_\bullet}) = \operatorname{Ker}\Big(F^\times/\mathbb{Q}^\times N_{E/F}(E^\times) \to \mathbb{A}_F^\times/\mathbb{A}^\times N_{E/F}(\mathbb{A}_E^\times)\Big).$$

In either case, $\ker^1(\mathbb{Q}, G_{a_\bullet})$ depends only on the CM extension E/F and the parity of n but not on the tuple $a_\bullet$.

Let $\mathrm{Sh}_{a_\bullet} := \mathcal{S}h_{a_\bullet} \otimes_{\mathbb{Z}_{p^2}} \mathbb{F}_{p^2}$ denote the special fiber of $\mathcal{S}h_{a_\bullet}$, and let $\overline{\mathrm{Sh}}_{a_\bullet} := \mathrm{Sh}_{a_\bullet} \otimes_{\mathbb{F}_{p^2}} \overline{\mathbb{F}}_p$ denote the geometric special fiber. We let $Sh_{a_\bullet} := \mathcal{S}h_{a_\bullet} \otimes_{\mathbb{Z}_{p^2}} \mathbb{Q}_{p^2}$ denote the generic fiber of $\mathcal{S}h_{a_\bullet}$ and let $\overline{Sh}_{a_\bullet}$ denote the geometric generic fiber.

This paper mainly focuses on the cases $a_\bullet = (0, n)$ and $(1, n-1)$. From now on, we fix an isomorphism $G_{1,n-1}(\mathbb{A}^\infty) \cong G_{0,n}(\mathbb{A}^\infty)$ by [11, Lemma 2.9] and denote them by $G(\mathbb{A}^\infty)$.

3.4. **Cycles on** $\mathrm{Sh}_{1,n-1}$ **and** $\mathrm{Sh}_{0,n}$**.** For a smooth variety X over $\mathbb{F}_{p^2}$, we denote by T_X the tangent bundle of X, and for a locally free $\mathcal{O}_X$-module M, we put $M^* = \mathcal{H}om_{\mathcal{O}_X}(M, \mathcal{O}_X)$.

Definition 3.4.1. We first construct cycles on $\mathrm{Sh}_{1,n-1}$. For each integer j with $1 \le j \le n$, we first define the variety Y_j we briefly mentioned in the introduction. Let Y_j be the moduli space over $\mathbb{F}_{p^2}$ that associates to each locally noetherian $\mathbb{F}_{p^2}$-scheme S, the set of isomorphism classes of tuples $(A, \lambda, \eta, B, \lambda', \eta', \phi)$, where

- (A, λ, η) is an S-point of $\mathrm{Sh}_{1,n-1}$,
- (B, λ', η') is an S-point of $\mathrm{Sh}_{0,n}$, and
- $\phi : B \to A$ is an $\mathcal{O}_D$-equivariant isogeny whose kernel is contained in $B[p]$,

such that

- $p\lambda' = \phi^\vee \circ \lambda \circ \phi$,
- $\phi \circ \eta' = \eta$, and
- the cokernels of the maps
$$\phi_{*,1} : H_1^{\mathrm{dR}}(B/S)_1^\circ \to H_1^{\mathrm{dR}}(A/S)_1^\circ \quad \text{and} \quad \phi_{*,2} : H_1^{\mathrm{dR}}(B/S)_2^\circ \to H_1^{\mathrm{dR}}(A/S)_2^\circ$$
are locally free $\mathcal{O}_S$-modules of rank $j-1$ and j, respectively.

There is a unique isogeny $\psi : A \to B$ such that $\psi \circ \phi = p \cdot \mathrm{id}_B$ and $\phi \circ \psi = p \cdot \mathrm{id}_A$. We have

$$\operatorname{Ker}(\phi_{*,i}) = \operatorname{Im}(\psi_{*,i}) \quad \text{and} \quad \operatorname{Ker}(\phi_{*,i}) = \operatorname{Im}(\psi_{*,i}),$$

where $\psi_{*,i}$ for $i = 1, 2$ is the induced homomorphism on the reduced de Rham homology in the evident sense. Th1is moduli space Y_j is represented by a scheme of finite type over $\mathbb{F}_{p^2}$. We have a natural diagram of morphisms:

$$\begin{tikzcd} & \mathrm{Y}_j \arrow[dl, "\overleftarrow{p}_j"'] \arrow[dr, "\overrightarrow{p}_j"] & \\ \mathrm{Sh}_{1,n-1} & & \mathrm{Sh}_{0,n}, \end{tikzcd}$$

where pr_j and pr'_j send a tuple $(A, \lambda, \eta, B, \lambda', \eta', \phi)$ to (A, λ, η) and to (B, λ', η'), respectively. Letting K^p vary, we see easily that both pr_j and pr'_j are equivariant under prime-to-p Hecke actions given by the double cosets $K^p \backslash G(\mathbb{A}^{\infty,p})/K^p$. Y_j gives a correspondence between $\mathrm{Sh}_{1,n-1}$ and $\mathrm{Sh}_{0,n}$.

The moduli problem for Y_j is slightly complicated. In [11], there is a more explicit moduli space Y'_j as below which is isomorphic to Y_j for $1 \le j \le n$.

Consider the functor $\underline{\mathrm{Y}}'_j$ which associates to each locally noetherian $\mathbb{F}_{p^2}$-scheme S, the set of isomorphism classes of tuples $(B, \lambda', \eta', H_1, H_2)$, where

- (B, λ', η') is an S-valued point of $\mathrm{Sh}_{0,n}$;

- $H_1 \subset H_1^{\mathrm{dR}}(B/S)_1^\circ$ and $H_2 \subset H_1^{\mathrm{dR}}(B/S)_2^\circ$ are $\mathcal{O}_S$-subbundles of rank j and $j-1$ respectively such that
$$V^{-1}(H_2^{(p)}) \subseteq H_1, \quad H_2 \subseteq F(H_1^{(p)}).$$
Here, $F : H_1^{\mathrm{dR}}(B/S)_1^{\circ,(p)} \xrightarrow{\sim} H_1^{\mathrm{dR}}(B/S)_2^\circ$ and $V : H_1^{\mathrm{dR}}(B/S)_1^\circ \xrightarrow{\sim} H_1^{\mathrm{dR}}(B/S)_2^{\circ,(p)}$ are respectively the Frobenius and Verschiebung homomorphisms, which are actually isomorphisms because of the signature condition on $\mathrm{Sh}_{0,n}$.

It follows from the moduli problem that the quotients $H_1/V^{-1}(H_2^{(p)})$ and $F(H_1^{(p)})/H_2$ are both locally free $\mathcal{O}_S$-modules of rank one.

There is a natural projection $\pi_j' \colon \underline{\mathrm{Y}}_j' \to \mathrm{Sh}_{0,n}$ given by $(B, \lambda', \eta', H_1, H_2) \mapsto (B, \lambda', \eta')$. In [11, Proposition 4.4], it is shown that $\underline{\mathrm{Y}}_j'$ is representable by a scheme Y_j' smooth and projective over $\mathrm{Sh}_{0,n}$ of dimension $n-1$.

There is a natural morphism $\alpha : Y_j \to Y_j'$ for $1 \leq j \leq n$ defined as follows. For a locally noetherian $\mathbb{F}_{p^2}$-scheme S and an S-point $(A, \lambda, \eta, B, \lambda', \eta', \phi)$ of Y_j, we define
$$H_1 := \phi_{*,1}^{-1}(\omega^\circ_{A^\vee/S,1}) \subseteq H_1^{\mathrm{dR}}(B/S)_1^\circ, \quad \text{and} \quad H_2 := \psi_{*,2}(\omega^\circ_{A^\vee/S,2}) \subseteq H_1^{\mathrm{dR}}(B/S)_2^\circ.$$
In particular, H_1 and H_2 are $\mathcal{O}_S$-subbundles of rank j and $j-1$, respectively. Also, there is a canonical isomorphism $\omega^\circ_{A^\vee/S,2}/\mathrm{Im}(\phi_{*,2}) \xrightarrow{\sim} H_2$. As the proof of [11, Lemma 4.6], it is easy to see that $F(H_1^{(p)}) \subseteq \mathrm{Ker}(\phi_{*,2}) = \mathrm{Im}(\psi_{*,2})$, but comparing the rank forces this to be an equality. It follows that $H_2 \subseteq F(H_1^{(p)})$. Similarly, $V^{-1}(H_2^{(p)})$ is identified with $\mathrm{Im}(\psi_{*,1}) = \mathrm{Ker}(\phi_{*,1})$, hence $V^{-1}(H_2^{(p)}) \subseteq H_1$. From these, we deduce two canonical isomorphisms:
$$H_1/V^{-1}(H_2^{(p)}) \xrightarrow{\sim} \omega^\circ_{A^\vee/S,1}, \quad \text{and} \quad F(H_1^{(p)})/H_2 \xrightarrow{\sim} H_1^{\mathrm{dR}}(A/S)_2^\circ/\omega^\circ_{A^\vee/S,2} \cong \mathrm{Lie}^\circ_{A/S,2}.$$
Therefore, we have a well-defined map $\alpha \colon \mathrm{Y}_j \to \mathrm{Y}_j'$ given by
$$\alpha \colon (A, \lambda, \eta, B, \lambda', \eta', \phi) \longmapsto (B, \lambda', \eta', H_1, H_2).$$
Moreover, it is clear from the definition that $\pi_j' \circ \alpha = \overrightarrow{p}_j$.

In [11, Proposition 4.8], it is shown that α is an isomorphism. Moreover, we have the following proposition:

Proposition 3.4.2. The cycles Y_j for $1 \leq j \leq n$ satisfies:

1. For each fixed closed point $z \in \mathrm{Sh}_{0,n}$ and $1 \leq j \leq n$, $\mathrm{Y}_{j,z}$ is isomorphic to $Z_j^{\langle n \rangle}$ which is the closed subscheme of $\mathrm{Gr}(n,i) \times \mathrm{Gr}(n,i-1)$ defined in Definition 4.1.
2. For each $1 \leq j \leq n$, the map $\overleftarrow{p}_j|_{\mathrm{Y}_{j,z}} : \mathrm{Y}_{j,z} \to \mathrm{Sh}_{1,n-1}$ is a closed immersion.
3. The union of the images of $\overleftarrow{p}_j$ for all $1 \leq j \leq n$ is the supersingular locus $\mathrm{Sh}^{\mathrm{ss}}_{1,n-1}$.

Proof. We refer to [11, Proposition 4.14] for the proof of (2) and (3).

Now we give the proof of (1): Since α induces an isomorphism between Y_j and Y_j' and $\pi_j' \circ \alpha = \overrightarrow{p}_j$, it suffices to show $\mathrm{Y}_{j,z}' = \pi_j'^{-1}(z)$ is isomorphic to $Z_j^{\langle n \rangle}$.

In fact, for any $\mathbb{F}_{p^2}$ scheme S and any S-point of Y_j':$(B, \lambda', \eta', H_1, H_2)$, we can choose an appropriate basis of $H_1^{\mathrm{dR}}(B/S)_1^\circ$ and $H_1^{\mathrm{dR}}(B/S)_2^\circ$ such that $F, V : H_1^{\mathrm{dR}}(B/S)_1^\circ \to H_1^{\mathrm{dR}}(B/S)_2^\circ$ are both given by the identity matrix. Now, $0 \subseteq H_2 \subseteq F(H_1^{(p)}) \subseteq H_1^{\mathrm{dR}}(B/S)_2^\circ$ and $0 \subseteq V^{-1}(H_2^{(p)}) \subseteq H_1 \subseteq H_1^{\mathrm{dR}}(B/S)_1^\circ$ gives an S-point of $Z_j^{\langle n \rangle}$ with an isomorphism between $H_1^{\mathrm{dR}}(B/S)_1^\circ$ and $H_1^{\mathrm{dR}}(B/S)_2^\circ$ $\square$

3.5. **Hecke action on** $\mathrm{Sh}_{0,n}$ **at** p**.** We follow the construction in [11, Section 6] here.

We assume that the tame level structure K^p is taken sufficiently small so that [11, lemma 4.13] holds with $N = 2$, i.e., given any $\overline{\mathbb{F}}_p$-point of $\mathrm{Sh}_{0,n}$ and an $\mathcal{O}_D$-quasi-isogeny $f : B \to B$ such that if $p^2 f \in \mathrm{End}_{\mathcal{O}_D}(B)$, $f^\vee \circ \lambda \circ f = \lambda$ and $f \circ \eta = \eta$, then $f = \mathrm{id}$.

Recall that we have an isomorphism

$$G(\mathbb{Q}_p) \simeq \mathbb{Q}_p^\times \times \mathrm{GL}_n(E_\mathfrak{p}) \cong \mathbb{Q}_p^\times \times \mathrm{GL}_n(\mathbb{Q}_{p^2}).$$

Put $K_\mathfrak{p} = \mathrm{GL}_n(\mathcal{O}_{E_\mathfrak{p}})$ and $K_p = \mathbb{Z}_p^\times \times K_\mathfrak{p}$. The Hecke algebra $\mathbb{Z}[K_\mathfrak{p}\backslash \mathrm{GL}_n(E_\mathfrak{p})/K_\mathfrak{p}]$ can be viewed as a subalgebra of $\mathbb{Z}[K_p\backslash G(\mathbb{Q}_p)/K_p]$ (with trivial factor at the $\mathbb{Q}_p^\times$-component).

Definition 3.5.1. For $\gamma \in \mathrm{GL}_n(E_\mathfrak{p})$, the double coset $T_\mathfrak{p}(\gamma) := K_\mathfrak{p}\gamma K_\mathfrak{p}$ defines a Hecke correspondence on $\mathrm{Sh}_{0,n}$. By [11, Remark 4.12], $\mathrm{Sh}_{0,n}(\overline{\mathbb{F}}_p)$ is a union of $\#\ker^1(\mathbb{Q}, G_{0,n})$-isogeny classes of abelian varieties. For any $z \in \mathrm{Sh}_{0,n}(\overline{\mathbb{F}}_p)$. Let

$$\Theta_z : \mathrm{Isog}(z) \xrightarrow{\sim} G_{0,n}(\mathbb{Q})\big\backslash\big(G(\mathbb{A}^{\infty,p}) \times G(\mathbb{Q}_p)\big)/K^p \times K_p.^3$$

be the bijection constructed as in [11, Corollary 4.11]. Write $K_\mathfrak{p}\gamma K_\mathfrak{p} = \coprod_{i\in I} \gamma_i K_\mathfrak{p}$. Then $T_\mathfrak{p}(\gamma)$ can be expressed as a set of pairing (z, z'), such that:

(1) $z \in \mathrm{Sh}_{0,n}(\overline{\mathbb{F}}_p)$,
(2) $z' \in \mathrm{Sh}_{0,n}(\overline{\mathbb{F}}_p)$ such that $\Theta_z(z') = (1, \gamma_i)$ for some $i \in I$.

To express it more geometrically, we have the following description: Write $z = (A, \lambda, \eta)$, and let $\mathbb{L}_z$ denote the $\mathbb{Z}_{p^2}$-free module $\tilde{\mathcal{D}}(A)_1^{\circ, F^2=p}$.[4] Then a point $z' = (B, \lambda', \eta') \in \mathrm{Sh}_{0,n}(\overline{\mathbb{F}}_p)$ belongs to $T_\mathfrak{p}(\gamma)(z)$ if and only if there exists an $\mathcal{O}_D$-equivariant p-quasi-isogeny $\phi : B' \to B$ (i.e. $p^m\phi$ is an isogeny of p-power order for some integer m) such that

(1) $\phi^\vee \circ \lambda \circ \phi = \lambda'$,
(2) $\phi \circ \eta' = \eta$,
(3) $\phi_*(\mathbb{L}_{z'})$ is a lattice of $\mathbb{L}_z[1/p] = \mathbb{L}_z \otimes_{\mathbb{Z}_{p^2}} \mathbb{Q}_{p^2}$ with the property: there exists a $\mathbb{Z}_{p^2}$-basis $(e_1, \dots, e_n)$ for $\mathbb{L}_z$ such that $(e_1, \dots, e_n)\gamma$ is a $\mathbb{Z}_{p^2}$-basis for $\phi_*(\mathbb{L}_{z'})$.

When $\gamma = \mathrm{diag}(p^{a_1}, \dots, p^{a_n})$ with $a_i \in \{-1, 0, 1\}$, For given z and z', such a ϕ is necessarily unique if it exists. Therefore, $T_\mathfrak{p}(\gamma)(z)$ is in natural bijection with the set of $\mathbb{Z}_{p^2}$-lattices $\mathbb{L}' \subseteq \mathbb{L}_z[1/p]$ satisfying property (3) above.

For each integer i with $0 \le i \le n$, we put

$$T_\mathfrak{p}^{(i)} = T_\mathfrak{p}\big(\mathrm{diag}(\underbrace{p, \dots, p}_{i}, \underbrace{1, \dots, 1}_{n-i})\big).$$

By the discussion above, one has a natural bijection

$$T_\mathfrak{p}^{(i)}(z) \xrightarrow{\sim} \big\{\mathbb{L}_{z'} \subseteq \mathbb{L}_z[1/p] \;\big|\; p\mathbb{L}_z \subseteq \mathbb{L}_{z'} \subseteq \mathbb{L}_z,\ \dim_{\mathbb{F}_{p^2}}(\mathbb{L}_z/\mathbb{L}_{z'}) = i\big\}$$

for $z \in \mathrm{Sh}_{0,n}(\overline{\mathbb{F}}_p)$. Note that $\mathrm{T}_\mathfrak{p}^{(0)} = \mathrm{id}$, and we put $S_\mathfrak{p} := T_\mathfrak{p}^{(n)}$ and $T := T_\mathfrak{p}^{(1)}$. It is easy to check $T \cong \mathrm{Sh}_{0,n}(K_\mathfrak{p}^1)$, where $K_\mathfrak{p}^1 := K_\mathfrak{p} \bigcap \mathrm{diag}(p, \underbrace{1, \dots, 1}_{n-1})^{-1} K_\mathfrak{p} \mathrm{diag}(p, \underbrace{1, \dots, 1}_{n-1})$. Then the Satake isomorphism implies $\mathbb{Z}[K_\mathfrak{p}\backslash \mathrm{GL}_n(E_\mathfrak{p})/K_\mathfrak{p}] \cong \mathbb{Z}[T_\mathfrak{p}^{(1)}, \dots, T_\mathfrak{p}^{(n-1)}, S_\mathfrak{p}, S_\mathfrak{p}^{-1}]$. More generally, for $0 \le a \le b \le n$, we put

$$R_\mathfrak{p}^{(a,b)} = T_\mathfrak{p}\big(\mathrm{diag}(\underbrace{p^2, \dots, p^2}_{a}, \underbrace{p, \dots, p}_{b-a}, \underbrace{1, \dots, 1}_{n-b})\big).$$

---

[3] For defintion of $\mathrm{Isog}(z)$, we refer to [11, subsection 4.11]

[4] [11, Remark 3.7]

Note that $R_{\mathfrak{p}}^{(0,i)} = T_{\mathfrak{p}}^{(i)}$, and $R_{\mathfrak{p}}^{(a,b)} S_{\mathfrak{p}}^{-1}$ is Hecke operator $T_{\mathfrak{p}}\big(\mathrm{diag}(\underbrace{p,\dots,p}_{a},\underbrace{1,\dots,1}_{b-a},\underbrace{p^{-1},\dots,p^{-1}}_{n-b})\big)$.

By abuse of notations, we may simply write $z' = R_{\mathfrak{p}}^{(a,b)} S_{\mathfrak{p}}^{-1} z$ to mean $z' \in R_{\mathfrak{p}}^{(a,b)} S_{\mathfrak{p}}^{-1}(z)$ for any a, b.

Proposition 3.5.2. Let i, j be integers with $1 \le i \le j \le n$ and $z, z' \in \mathrm{Sh}_{0,n}(\overline{\mathbb{F}}_p)$. The subvarieties $\mathrm{Y}_{i,z}$ and $\mathrm{Y}_{j,z'}$ of $\mathrm{Sh}_{1,n-1}$ have non-empty intersection of dimension $n-2$ if and only if $j = i+1$ and $(z, z') \in T_{\mathfrak{p}}^{(1)}$. Furthermore, if we identify $\mathrm{Y}_{i,z}$ with $Z_n^{\langle i \rangle}$, then the intersection $\mathrm{Y}_{i,z} \times_{\overline{\mathrm{Sh}}_{1,n-1}} \mathrm{Y}_{i+1,z'}$ (resp. $\mathrm{Y}_{i,z} \times_{\overline{\mathrm{Sh}}_{1,n-1}} \mathrm{Y}_{i-1,z'}$) belongs to the special divisor class $\mathtt{SD}_+$ (resp. $\mathtt{SD}_-$) on $Z_i^{\langle n \rangle}$ ($\mathtt{SD}_+$ and $\mathtt{SD}_-$ is defined in Definition 4.3.2).

Proof. As in [11, Proposition 6.4], where we take $j \ge i$ and δ satisfies $0 \le \delta \le \min\{n-j, i-1\}$, we have $\mathrm{Y}_{i,z} \times_{\overline{\mathrm{Sh}}_{1,n-1}} Y_{j,z'}$ is isomorphic to the variety $Z_{i-\delta}^{\langle n+i-j-2\delta \rangle}$. Now we require the dimension of $Z_{i-\delta}^{\langle n+i-j-2\delta \rangle}$ to be $n+i-j-2\delta-1 = n-2$, i.e., $i-j-2\delta = -2$. We get $0 \le 2\delta = i-j+1 \le 2$ and $j = i+1$ since $j \ge i$.

If we identify $\mathrm{Y}_{i,z}$ with $Z_n^{\langle i \rangle}$ as in Proposition 3.4.2, we want to show $\mathrm{Y}_{i,z} \times_{\overline{\mathrm{Sh}}_{1,n-1}} \mathrm{Y}_{i+1,z'}$ is ismorphic to $\mathtt{SD}_+$.

Let $(\mathcal{B}_z, \lambda_z, \eta_z)$ and $(\mathcal{B}_{z'}, \lambda_{z'}, \eta_{z'})$ be the universal polarized abelian varieties on $\overline{\mathrm{Sh}}_{0,n}$ at z and z', respectively. Then $\mathrm{Y}_{i,z} \times_{\overline{\mathrm{Sh}}_{1,n-1}} \mathrm{Y}_{j,z'}$ is the moduli space of tuples $(A, \lambda, \eta, \phi, \phi')$ where $\phi : \mathcal{B}_z \to A$ and $\phi' : \mathcal{B}_{z'} \to A$ are isogenies such that $(A, \lambda, \eta, \mathcal{B}_z, \lambda_z, \eta_z, \phi)$ and $(A, \lambda, \eta, \mathcal{B}_{z'}, \eta_{z'}, \phi')$ are points of $\mathrm{Y}_{i,z}$ and $\mathrm{Y}_{j,z'}$ respectively. We take

$$M_k = \big(\tilde{\mathcal{D}}(\mathcal{B}_z)_k^{\circ} \cap \tilde{\mathcal{D}}(\mathcal{B}_{z'})_k^{\circ}\big) / p\big(\tilde{\mathcal{D}}(\mathcal{B}_z)_k^{\circ} + \tilde{\mathcal{D}}(\mathcal{B}_{z'})_k^{\circ}\big)$$

for $k = 1, 2$. Then one has $\dim_{\overline{\mathbb{F}}_p}(M_k) = n-1$, since we require $\mathrm{Y}_{i,z} \times_{\overline{\mathrm{Sh}}_{1,n-1}} \mathrm{Y}_{i+1,z'}$ is isomorphic to $Z_i^{\langle n-1 \rangle}$. The Frobenius and Verschiebung on $\tilde{\mathcal{D}}(\mathcal{B}_z)$ induce two bijective Frobenius semi-linear maps $F : M_1 \to M_2$ and $V^{-1} : M_2 \to M_1$. We denote their linearizations by the same notation if no confusions arise. Let $Z(M_\bullet)$ be the moduli space which attaches to each locally noetherian $\overline{\mathbb{F}}_p$-scheme S the set of isomorphism classes of pairs (L_1, L_2), where $L_1 \subseteq M_1 \otimes_{\overline{\mathbb{F}}_p} \mathcal{O}_S$ and $L_2 \subseteq M_2 \otimes_{\overline{\mathbb{F}}_p} \mathcal{O}_S$ are subbundles of rank i and $i-$ respectively such that

$$L_2 \subseteq F(L_1^{(p)}), \quad V^{-1}(L_2^{(p)}) \subseteq L_1.$$

Note that there exists a basis $(\varepsilon_{k,1}, \dots, \varepsilon_{k,n-1})$ of M_k for $k = 1, 2$ under which the matrices of F and V^{-1} are both identity. Indeed, by solving a system of equations of Artin–Schreier type, one can take a basis $(\varepsilon_{1,\ell})_{1 \le \ell \le n-1}$ for M_1 such that

$$V^{-1}(F(\varepsilon_{1,\ell})) = \varepsilon_{1,\ell} \quad \text{for all } 1 \le \ell \le n-1.$$

We put $\varepsilon_{2,\ell} = F(\varepsilon_{1,\ell})$. Using these bases to identify both M_1 and M_2 with $\overline{\mathbb{F}}_p^{n-2}$, it is clear that $Z(M_\bullet)$ is isomorphic to the variety $\overline{Z}_i^{\langle n-1 \rangle}$.

Now since we identify $\mathrm{Y}_{i,z}$ with $Z_n^{\langle i \rangle}$, we get the closed immersion of $\mathrm{Y}_{i,z} \times_{\overline{\mathrm{Sh}}_{1,n-1}} \mathrm{Y}_{i+1,z'}$ into $\mathrm{Y}_{i,z}$ can be identified with a closed immersion of $\overline{Z}_i^{\langle n-1 \rangle}$ into $\overline{Z}_i^{\langle n \rangle}$ which is induced by $\tilde{\mathcal{D}}(\mathcal{B}_z)_k^{\circ} \cap \tilde{\mathcal{D}}(\mathcal{B}_{z'})_k^{\circ} \subseteq \tilde{\mathcal{D}}(\mathcal{B}_z)_k^{\circ}$ for $k = 1, 2$.

The proof of $\mathrm{Y}_{i,z} \times_{\overline{\mathrm{Sh}}_{1,n-1}} \mathrm{Y}_{i-1,z'}$ is ismorphic to $\mathtt{SD}_-$ is similar with a closed immersion $\overline{Z}_{i-1}^{\langle n-1 \rangle}$ into $\overline{Z}_i^{\langle n \rangle}$ induced by $p\tilde{\mathcal{D}}(\mathcal{B}_z)_k^{\circ} \subseteq p\big(\tilde{\mathcal{D}}(\mathcal{B}_z)_k^{\circ} + \tilde{\mathcal{D}}(\mathcal{B}_{z'})_k^{\circ}\big)$ for $k = 1, 2$ and we omit here. □

3.6. **ℓ-adic cohomology.** We introduce the ℓ-adic cohomology of unitary Shimura varieties. In this subsection we do not assume K_p is hyperspecial and use additional notation (K) to express Shimura varieties with respect to K.

We fix a prime number $\ell \neq p$, and an isomorphism $\iota_\ell : \mathbb{C} \simeq \overline{\mathbb{Q}}_l$. Let ξ be an algebraic representation of $G_{a_\bullet}$ over $\overline{\mathbb{Q}}_l$, and $\xi_{\mathbb{C}}$ be the base change via ι_ℓ^{-1}. The theory of automorphic sheaves or just reading off from the rational ℓ-adic Tate modules of the universal abelian variety allows us to attach to ξ a lisse $\overline{\mathbb{Q}}_l$-sheaf $\mathcal{L}_\xi$ over $\mathcal{S}h_{a_\bullet}$. For example, if ξ is the representation of $G_{a_\bullet}$ on the vector space $V_{a_\bullet}$ (Subsection 3.2), the corresponding ℓ-adic local system is given by the rational ℓ-adic Tate module (tensored with $\overline{\mathbb{Q}}_l$) of the universal abelian scheme over $\mathcal{S}h_{a_\bullet}(K)$.

We assume that ξ is irreducible. Let $\mathscr{H}_K = \mathscr{H}(K, \overline{\mathbb{Q}}_l)$ be the Hecke algebra of compactly supported K-bi-invariant $\overline{\mathbb{Q}}_l$-valued functions on $G_{a_\bullet}(\mathbb{A}^\infty)$. It is known that the étale cohomology group $H^{d(a_\bullet)}_{\text{et}}(\overline{\text{Sh}}_{a_\bullet}(K), \mathcal{L}_\xi)$ is equipped with a natural action of $\mathscr{H}_K \times \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})$. Since $\text{Sh}_{a_\bullet}(K)$ is proper and smooth, there is no continuous spectrum and we have a canonical decomposition of $\mathscr{H}_K \times \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})$-modules (see e.g. [8, Proposition III.2.1])

$$H^{d(a_\bullet)}_{\text{et}}(\overline{\text{Sh}}(K)_{a_\bullet}, \mathcal{L}_\xi) = \bigoplus_{\pi \in \text{Irr}(G_{a_\bullet}(\mathbb{A}^\infty))} \iota_\ell(\pi^K) \otimes R_{a_\bullet,\ell}(\pi), \tag{3.6.1}$$

where $\text{Irr}(G_{a_\bullet}(\mathbb{A}^\infty))$ denotes the set of irreducible admissible representations of $G_{a_\bullet}(\mathbb{A}^\infty)$ with coefficients in $\mathbb{C}$, π^K is the K-invariant subspace of $\pi \in \text{Irr}(G_{a_\bullet}(\mathbb{A}^\infty))$, and $R_{a_\bullet,\ell}(\pi)$ is a certain ℓ-adic representation of $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})$ which we specify below.

We write $H^{d(a_\bullet)}_{\text{et}}(\overline{\text{Sh}}_{a_\bullet}(K), \mathcal{L}_\xi)_\pi$ for the π-*isotypic component* of the cohomology group, that is the direct summand of (3.6.1) labeled by π. We make the following assumption on π.

Hypothesis 3.6.1. (1) We have $\pi^K \neq 0$.

(2) There exists an admissible irreducible representation π_∞ of $G_{a_\bullet}(\mathbb{R})$ such that $\pi \otimes \pi_\infty$ is a cuspidal automorphic representation of $G_{a_\bullet}(\mathbb{A})$,

(2a) π_∞ is *cohomological in degree* $d(a_\bullet)$ *for* ξ in the sense that

$$H^{d(a_\bullet)}\big(\text{Lie}(G_{a_\bullet}(\mathbb{R})), K_\infty, \pi_\infty \otimes \xi_{\mathbb{C}}\big) \neq 0,^{5} \tag{3.6.2}$$

where K_∞ is a maximal compact subgroup of $G_{a_\bullet}(\mathbb{R})$,

(2b) and $\pi \otimes \pi_\infty$ admits a base change to a *cuspidal* automorphic representation of $\text{GL}_n(\mathbb{A}_E) \times \mathbb{A}^\times_{E_0}$, which can be assumed to be *RASDC* and $\Pi_{\mathfrak{p}} = \pi_p$ by [8].

Hypothesis 3.6.1 (2)(a) ensures that $R_{a_\bullet,\ell}(\pi)$ is non-trivial. Moreover, by [3, Theorem 1.2], this hypothesis implies that the base change of $\pi \otimes \pi_\infty$ to $\text{GL}_{n,E}$ is tempered at all finite places, and hence π_p is tempered. By [15], we have Π is tempered at all finite places and for any rational primme *ell* and every isomorphism $\iota_\ell : \mathbb{C} \to \overline{\mathbb{Q}}_\ell$, there is a semisimple continuous homomorphism $\rho_{\Pi,\iota_\ell} : \text{Gal}_E \to \text{GL}_n(\overline{\mathbb{Q}_\ell})$ unique uo to conjugation such that for every nonarchimedean palce w of E, the Frobenious semisimplification of the associated Weil-Deligne representation of $\rho_{\Pi,\iota_\ell}|_{\text{Gal}_{E_w}}$ corresponds to the irreducible admissible representation $\iota_\ell \Pi_w |\det|_w^{\frac{1-n}{2}}$ of $GL_n(E_w)$ under the local Langlands correpsondence. Moreover, $\rho^{c}_{\Pi,\iota_\ell}$ and $\rho^{\vee}_{\Pi,\iota_\ell}(1-n)$ are conjugate. Since $\Pi_{\mathfrak{p}}$ is an irreducible representation of $\text{GL}_n(\mathbb{Q}_{p^2})$, we can talk about its Satake parameters and denote them by $\{\alpha_1, \cdots, \alpha_n\}$.

We recall now an explicit description of the Galois module $R_{a_\bullet,\ell}(\pi)$. Let $\overline{\mathbb{Q}}_\ell(\frac{1}{2})$ denote the unramified representation of $W_{\mathbb{Q}_{p^2}}$ which sends Frob_{p^2} to $(\sqrt{p})^{-2}$, $r_{a_\bullet} = \bigotimes_{i=1}^2 \wedge^{a_i}\text{Std}$, and $\chi_{\pi_{p,0}}$ denotes the character of $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})$ sending Frob_{p^2} to $\iota_\ell(\pi_{p,0}(p^2))$. Then we have $R_{a_\bullet,\ell}(\pi) = \#\text{Ker}^1(\mathbb{Q}, G_{a_\bullet}) m_{a_\bullet(\pi)}[(r_{a_\bullet} \circ \rho_{\Pi_{\mathfrak{p}}}) \otimes \chi^{-1}_{\pi_{p,0}} \otimes \overline{\mathbb{Q}}_\ell(\sum_i \frac{a_i(a_i-1)}{2})]$.

3.7. **Tate conjecture for** $\text{Sh}_{1,n-1}$ **and** $\text{Sh}_{0,n}$. In [11], they show the Tate conjecture is true for $\text{Sh}_{1,n-1}$ and $\text{Sh}_{0,n}$.

[5] This automatically implies that π_∞ has the same central and infinitesimal characters as the *contragradient* of $\xi_{\mathbb{C}}$.

3.7.1. *Gysin/trace maps.* Before stating the Tate conjecture, we recall the general definition of Gysin maps. Let $f : Y \to X$ be a proper morphism of smooth varieties over an algebraically closed field k. Let d_X and d_Y be the dimensions of X and Y respectively. Recall that the derived direct image Rf_* on the derived category of constructible ℓ-adic étale sheaves has a left adjoint $f^!$. Since both X and Y are smooth, the ℓ-adic dualizing complex of X (resp. Y) is $\overline{\mathbb{Q}}_\ell(d_X)[2d_X]$ (resp. $\overline{\mathbb{Q}}_l(d_Y)[2d_Y]$). Therefore, one has

$$f^!\big(\overline{\mathbb{Q}}_l(d_X)[2d_X]\big) = \overline{\mathbb{Q}}_l(d_Y)[2d_Y].$$

The adjunction map $Rf_*f^!\overline{\mathbb{Q}}_l \to \overline{\mathbb{Q}}_l$ induces a canonical morphism

$$\mathrm{Tr}_f\colon Rf_*\overline{\mathbb{Q}}_l \to \overline{\mathbb{Q}}_l(d_X - d_Y)[2(d_X - d_Y)].$$

More generally, if $\mathcal{L}$ is a lisse $\overline{\mathbb{Q}}_l$-sheaf on X, it induces a Gysin/trace map

$$Rf_*(f^*\mathcal{L}) \cong \mathcal{L} \otimes Rf_*(\overline{\mathbb{Q}}_l) \xrightarrow{1\otimes \mathrm{Tr}_f} \mathcal{L}(d_X - d_Y)[2(d_X - d_Y)],$$

where the first isomorphism is the projection formula. When f is flat with equidimensional fibers of dimension $d_Y - d_X$, this is the canonical trace map. When f is a closed immersion of codimension $r = d_X - d_Y$, it is the usual Gysin map. For any integer q, the Gysin/trace map induces a morphism on cohomology groups:

$$f_!\colon H^q_{\mathrm{et}}(Y, f^*\mathcal{L}) \longrightarrow H^{q+2(d_X-d_Y)}_{\mathrm{et}}\big(X, \mathcal{L}(d_X - d_Y)\big).$$

Now we introduce the Tate conjecture for $\mathrm{Sh}_{1,n-1}$ and $\mathrm{Sh}_{0,n}$.

Let $\ell \neq p$ be a prime number. For $1 \le j \le n$, there is a natural morphism

$$\mathcal{JL}_j\colon H^0_{\mathrm{et}}\big(\overline{\mathrm{Sh}}_{0,n}, \overline{\mathbb{Q}}_l\big) \xrightarrow{\vec{p}_j^{\,*}} H^0_{\mathrm{et}}\big(\overline{Y}_j, \overline{\mathbb{Q}}_l\big) \xrightarrow{\vec{p}_{j,!}} H^{2(n-1)}_{\mathrm{et}}\big(\overline{\mathrm{Sh}}_{1,n-1}, \overline{\mathbb{Q}}_l(n-1)\big),$$

where $\mathrm{pr}_{j,!}$ is the Gysin map defined above, whose restriction to each $H^0_{\mathrm{et}}(\mathrm{Y}_{j,z}, \overline{\mathbb{Q}}_l)$ for $z \in \mathrm{Sh}_{0,n}(\overline{\mathbb{F}}_p)$ is the Gysin map associated to the closed immersion $\mathrm{Y}_{j,z} \hookrightarrow \overline{\mathrm{Sh}}_{1,n-1}$. It is clear that the image of $\mathcal{JL}_j$ is the subspace generated by the cycle classes of $[\mathrm{Y}_{j,z}] \in A^{n-1}(\overline{\mathrm{Sh}}_{1,n-1})$ with $z \in \mathrm{Sh}_{0,n}(\overline{\mathbb{F}}_p)$. According to [10], $\mathcal{JL}_j$ should be considered as a certain geometric realization of the Jacquet–Langlands transfer from $G_{0,n}$ to $G_{1,n-1}$. Putting all $\mathcal{JL}_j$'s together, we get a morphism

$$\mathcal{JL} = \sum_j \mathcal{JL}_j\colon \bigoplus_{j=1}^n H^0_{\mathrm{et}}\big(\overline{\mathrm{Sh}}_{0,n}, \overline{\mathbb{Q}}_l\big) \longrightarrow H^{2(n-1)}_{\mathrm{et}}\big(\overline{\mathrm{Sh}}_{1,n-1}, \overline{\mathbb{Q}}_l(n-1)\big). \tag{3.7.1}$$

Recall that we have fixed an isomorphism $G_{1,n-1}(\mathbb{A}^\infty) \simeq G_{0,n}(\mathbb{A}^\infty)$, which we write uniformly as $G(\mathbb{A}^\infty)$. Denote by $\mathscr{H}(K^p, \overline{\mathbb{Q}}_l) = \overline{\mathbb{Q}}_l[K^p\backslash G(\mathbb{A}^{\infty,p})/K^p]$ the prime-to-p Hecke algebra. Then the homomorphism is a homomorphism of $\mathscr{H}(K^p, \overline{\mathbb{Q}}_l)$-modules. For an irreducible admissible representation π of $G(\mathbb{A}_\infty)$, we write $\pi = \pi^p \otimes \pi_p$, where π^p (resp. π_p) is the prime-to-p part (resp. the p-component) of π.

Lemma 3.7.2. *Let π_1 and π_2 be two admissible irreducible representations of $G(\mathbb{A}^\infty)$, and (r_i, s_i) for $i = 1, 2$ be two pairs of integers with $0 \le r_i, s_i \le n$ and $r_1 + s_1 \equiv r_2 + s_2 \mod 2$. Assume that π_1 satisfies Hypothesis 3.6.1 with $\mathrm{a}_\bullet = (r_1, s_1)$, and there exists an admissible irreducible representation $\pi_{2,\infty}$ of $G_{(r_2,s_2)}(\mathbb{R})$ such that $\pi_2 \otimes \pi_{2,\infty}$ is a cuspidal automorphic representation of $G_{(r_2,s_2)}(\mathbb{A})$. If π_1^p and π_2^p are isomorphic as representations of $G(\mathbb{A}^{p,\infty})$, then $\pi_{1,p} \simeq \pi_{2,p}$, and $\pi_2 \otimes \pi_{2,\infty}$ admits a base change to a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E) \times \mathbb{A}^\times_{E_0}$; in particular, π_2 satisfies Hypothesis 3.6.1 for $\mathrm{a}_\bullet = (r_2, s_2)$.*

Proof. We refer to [11, Lemma 4.17]. □

Let $\mathscr{A}_K$ be the set of isomorphism classes of irreducible admissible representations π of $G(\mathbb{A}^\infty)$ satisfying Hypothesis 3.6.1 with $a_\bullet = (0,n)$. In particular, each $\pi \in \mathscr{A}_K$ is the finite part of an automorphic cuspidal representation of $G_{0,n}(\mathbb{A})$.

We fix such a $\pi \in \mathscr{A}_K$. Let

$$\mathcal{JL}_\pi\colon \bigoplus_{i=1}^n H^0_{\mathrm{et}}\big(\overline{\mathrm{Sh}}_{0,n}, \overline{\mathbb{Q}}_l\big)_{\pi^p} \longrightarrow H^{2(n-1)}_{\mathrm{et}}\big(\overline{\mathrm{Sh}}_{1,n-1}, \overline{\mathbb{Q}}_l(n-1)\big)_{\pi^p}$$

denote the morphism on the $(\pi^p)^{K^p}$-isotypic components induced by $\mathcal{JL}$, where for an $\mathscr{H}(K^p, \overline{\mathbb{Q}}_\ell)$-module M we put

$$M_{\pi^p} := \mathrm{Hom}_{\mathscr{H}(K^p,\overline{\mathbb{Q}}_\ell)}((\pi^p)^{K^p}, M) \otimes (\pi^p)^{K^p}.$$

Then Lemma3.7.2 implies that π is completely determined by its prime-to-p part. Hence, taking the π^p-isotypic components is the same as taking the π-isotypic components. We can thus write M_π instead of M_{π^p} for a $\mathscr{H}(K, \overline{\mathbb{Q}}_\ell)$-module M.

The image of $\mathcal{JL}_\pi$ is included in $H^{2(n-1)}_{\mathrm{et}}\big(\overline{\mathrm{Sh}}_{1,n-1}, \overline{\mathbb{Q}}_l(n-1)\big)^{\mathrm{fin}}_\pi$, which is the maximal subspace of $H^{2(n-1)}_{\mathrm{et}}\big(\overline{\mathrm{Sh}}_{1,n-1}, \overline{\mathbb{Q}}_l(n-1)\big)_\pi$ where the action of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})$ factors through a finite quotient.

In [11], it is shown that this inclusion is actually an equality under certain genericity conditions on π_p. To make this precise, write $\pi_p = \pi_{p,0} \otimes \pi_{\mathfrak{p}}$ as a representation of $G(\mathbb{Q}_p) \simeq \mathbb{Q}_p^\times \times \mathrm{GL}_n(E_{\mathfrak{p}})$. Let

$$\rho_{\pi_{\mathfrak{p}}} : W_{\mathbb{Q}_{p^2}} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$$

be the unramified representation of the Weil group of $\mathbb{Q}_{p^2}$ defined above. It induces a continuous ℓ-adic representation of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})$, which we denote by the same notation. Then $\rho_{\pi_{\mathfrak{p}}}(\mathrm{Frob}_{p^2})$ is semisimple. Using this, we get an explicit description of $H^{2(n-1)}_{\mathrm{et}}\big(\overline{\mathrm{Sh}}_{1,n-1}, \overline{\mathbb{Q}}_l(n-1)\big)_\pi$ and $H^0_{\mathrm{et}}(\overline{\mathrm{Sh}}_{0,n}, \overline{\mathbb{Q}}_l)_\pi$ in terms of $\rho_{\pi_{\mathfrak{p}}}$.

We can now state the theorem proved in [11].

Theorem 3.7.3. *Fix a π in $\mathscr{A}_K$. Let $\alpha_{\pi_{\mathfrak{p}},1}, \ldots, \alpha_{\pi_{\mathfrak{p}},n}$ denote the eigenvalues of $\rho_{\pi_{\mathfrak{p}}}(\mathrm{Frob}_{p^2})$.*

1. *If $\alpha_{\pi_{\mathfrak{p}},1}, \ldots, \alpha_{\pi_{\mathfrak{p}},n}$ are distinct, then the map $\mathcal{JL}_\pi$ is injective,*
2. *Let $m_{1,n-1}(\pi)$ (resp. $m_{0,n}(\pi)$) denote the multiplicity for π for $a_\bullet = (1, n-1)$ (resp. for $a_\bullet = (0,n)$). Assume that $m_{1,n-1}(\pi) = m_{0,n}(\pi)$ and that $\alpha_{\pi_{\mathfrak{p}},i}/\alpha_{\pi_{\mathfrak{p}},j}$ is not a root of unity for all $1 \le i,j \le n$. Then the map*

$$\mathcal{JL}_\pi : \bigoplus_{j=1}^n H^0_{\mathrm{et}}\big(\overline{\mathrm{Sh}}_{0,n}, \overline{\mathbb{Q}}_l\big)_\pi \longrightarrow H^{2(n-1)}_{\mathrm{et}}\big(\overline{\mathrm{Sh}}_{1,n-1}, \overline{\mathbb{Q}}_l(n-1)\big)^{\mathrm{fin}}_\pi$$

is an isomorphism. In other words, $H^{2(n-1)}_{\mathrm{et}}\big(\overline{\mathrm{Sh}}_{1,n-1}, \overline{\mathbb{Q}}_l(n-1)\big)^{\mathrm{fin}}_\pi$ is generated by the cycle classes of the irreducible components of Y_j for $1 \le j \le n$.

As the end of the section, we give the definition of a maximal ideal $\mathfrak{m}$ of the Hecke algebra which is non-Eisenstein. Let Π be the base change of the representation π to $\mathrm{GL}_n(\mathbb{A}_E) \times \mathbb{A}^\times_{E_0}$.

Let R be a finite set of places of F away from which Π is unramified and K is hyperspecial, Let $\mathbb{T}_R$ denote the Hecke algebra away from R; i.e., the polynomial ring over $\mathbb{Z}$ generated by $T_{\mathfrak{q}}, S_{\mathfrak{q}}$ where $\mathfrak{q}$ is a prime away from R. The representation Π determines a homomorphism

$$\phi^\pi_R : \mathbb{T}_R \to \mathcal{O}_E$$

via the Hecke eigenvalue of Π. Suppose λ is a prime of E lying over l, then we define $\mathfrak{m} \subseteq \mathbb{T}_{R\cup\{\mathfrak{p}\}}$ to be the preimage of the ideal $(\lambda) \subseteq \mathcal{O}_E$ under the map of ϕ^Π_R. It is easy to see that it is a maximal ideal of $\mathbb{T}_{R\cup\{\mathfrak{p}\}}$. For any $\mathbb{T}_{R\cup\{\mathfrak{p}\}}$-module M, we will write $M_{\mathfrak{m}}$ for the localization of M at the ideal $\mathfrak{m}$.

It follows from [25, Theorem 4.3.1] and [1, Theorem 2.3.3] that there exists a continuous semisimple n-dimension Galois representation

$$\overline{\rho}_{\mathfrak{m}} : Gal(\overline{E}/E) \to \mathrm{GL}_n(\overline{\mathbb{F}}_l),$$

which coincides with $\rho_{\Pi_{\mathfrak{p}}}$ defined before. The maximal ideal $\mathfrak{m}$ is called non-Eisenstein if $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible. By [3], the cohomology group $H^i_{\acute{e}t}(\mathrm{Sh}_{a_\bullet}, \mathbb{Z}_l)_{\mathfrak{m}}$ is zero if $i \neq d(a_\bullet)$ and is non-torsion otherwise. We use subscript $\mathfrak{m}$ to express both localizing at $\mathfrak{m}$ and π.

4. Deligne-Lusztig Variety

General Deligne-Lusztig varieties are defined as follows:

Definition 4.1. Let G be a reductive group over $\overline{\mathbb{F}}_{p^2}$ with a $\mathbb{F}_q$-structure. Let F be the corresponding Frobenius morphism. Fix a F-stable Borel subgroup B and a torus $T \subset B$. Let W and Δ be the Weyl group and set of simple roots. For any $w \in W$, the Deligne-Lusztig variety of type w is defined by

$$X(w) := \{gB \in G/B | g^{-1}F(g) \in BwB\}.$$

In general, let P_I be the standard parabolic subgroup defined by a subset I of Δ. Define

$$X_I(w) := \{gP_I \in G/P_I | g^{-1}F(g) \in P_I w P_{F(I)}\}.$$

In our case, by the method of Dieudonné modules, we can characterize fibers of $\overrightarrow{p}_j : Y_i \to \mathrm{Sh}_{0,n}$ as the following space.

Definition 4.2. We define $Z_i^{\langle n\rangle}$(resp. $\tilde{Z}_i^{\langle n\rangle}$, $\widehat{Z}_i^{\langle n\rangle}$) as a closed subscheme of $\mathbf{Gr}(n,i) \times \mathbf{Gr}(n,i-1)$ over $\mathbb{F}_{p^2}$ whose S-valued points are the isomorphism classes of pairs (H_1, H_2) where H_1 and H_2 are respectively subbundles of $\mathcal{O}_S^{\oplus n}$ of rank i and $i-1$ satifying $H_2 \subseteq H_1^{(p)}$ and $H_2^{(p)} \subseteq H_1$(resp. $H_2 \subseteq H_1$ and $H_2 \subseteq H_1^{(p^2)}$, $H_2 \subseteq H_1$ and $H_2^{(p^2)} \subseteq H_1$). There are morphisms called 'relative Frobenius':

$$\begin{tikzcd} \widehat{Z}_i^{\langle n\rangle} \arrow[r, shift left, "H_2^{(p)}"] & Z_i^{\langle n\rangle} \arrow[l, shift left, "H_1^{(p)}"] \arrow[r, shift left, "H_2^{(p)}"] & \tilde{Z}_i^{\langle n\rangle} \arrow[l, shift left, "H_1^{(p)}"] \end{tikzcd}$$

$$(H_1, H_2) \longrightarrow (H_1, H_2^{(p)}) \longrightarrow (H_1, H_2^{(p^2)})$$

$$(H_1^{(p^2)}, H_2) \longleftarrow (H_1^{(p)}, H_2) \longleftarrow (H_1, H_2)$$

Since these morphisms are purely inseparable, we can study the divisors of $Z_n^{\langle i\rangle}$ by studying divisors of $\tilde{Z}_i^{\langle n\rangle}$ and $\widehat{Z}_i^{\langle n\rangle}$.

Remark 4.3. It should be noted that the Frobenius map here is defined over $\mathbb{F}_p$, which implies the total vector space should have a $\mathbb{F}_p$ -structure. It does hold in our cases since we take $H_1^{dR}(A/S)_1^\circ$ and $H_1^{dR}(A/S)_2^\circ$ to be whole space and we can take a non-canonical basis of $H_1^{dR}(A/S)_i^\circ$ for $i = 1, 2$. such that FV^{-1} acts as identity on $H_1^{dR}(A/S)_1^\circ$ and $H_1^{dR}(A/S)_2^\circ$.

The spaces $Z_i^{\langle n\rangle}$, $\tilde{Z}_i^{\langle n\rangle}$ and $\widehat{Z}_i^{\langle n\rangle}$ can be realized as the disjoint union of Deligne-Lusztig varieties. Furthermore, it can be shown the three schemes are irreducible and smooth of dimension $n-1$ over $\mathbb{F}_{p^2}$. Here we study $\tilde{Z}_i^{\langle n\rangle}$ as an example:

Example 4.3.1. *Let $V = \overline{\mathbb{F}}_{p^2} e_1 \oplus \cdots \oplus \overline{\mathbb{F}}_{p^2} e_n$ and let V_i be the subspace generated by $e_1, ..., e_i$. Let B be the Borel subgroup that stabilizes this flag and $\Delta = \{s_1, ..., s_{n-1}\}$. The stabilizer of (V_{i-1}, V_i) under the action of GL_n is the parabolic subgroup P_I defined by $I = \Delta \setminus \{s_{i-1}, s_i\}$, i.e. matrices of form* $\begin{bmatrix} i-1 & & \\ & 1 & \\ & & n-i \end{bmatrix}$. *Let F be the Frobenius morphism associated to the $\mathbb{F}_{p^2}$-structure of V. We can show that for $g \in \mathrm{GL_n}$, $g(V_i, V_{i+1}) \in \tilde{Z}_i^{\langle n \rangle}$ if and only if $g^{-1}F(g) \in P_I w P_I$ for some $w \in \{\mathrm{Id}, s_{i-1}, s_i, s_i s_{i-1}\}$. So $\tilde{Z}_i^{\langle n \rangle}$ is the disjoint union of four Deligne-Lusztig varieties:*

$$\tilde{Z}_i^{\langle n \rangle} = \left[\begin{array}{ccc} & X_I(s_{i-1}) & \\ X_I(Id) & & X_I(s_i s_{i-1}) \\ & X_I(s_i) & \end{array} \right] = \overline{X_I(s_i s_{i-1})}.$$

Define two special classes of divisors on $Z_i^{\langle n \rangle}$ as follows.

Definition 4.3.2. (1) When $i < n$, for every subbundles H in $\mathcal{O}_{\mathbb{F}_{p^2}}^{\oplus n}$ of rank $n-1$, denote by $[H]$ the locus on $Z_i^{\langle n \rangle}$ where $H_1 \subseteq H \oplus_{\mathcal{O}_{\mathbb{F}_{p^2}}} \mathcal{O}_S$ for any $\mathbb{F}_{p^2}$scheme S. Then $[H] \subset Z_i^{\langle n \rangle}$ is a closed subvariety of codimension 1, and we have $[H] \cong Z_i^{\langle n-1 \rangle}$.Let $\mathtt{SD}_+ = \{[H] :$ H is a subbundle of rank $n-1$ in $\mathcal{O}_{\mathbb{F}_{p^2}^{\oplus n}}\}$.

(2) When $i > 1$, for every line bundle L in $\mathcal{O}_{\mathbb{F}_{p^2}}^{\oplus n}$, denote by $[L]$ the locus on $Z_i^{\langle n \rangle}$ where $L \oplus_{\mathcal{O}_{\mathbb{F}_{p^2}}} \mathcal{O}_S \subset H_2$. Then $[L] \subset Z_i^{\langle n \rangle}$ is a closed subvariety of codimension 1, and we have $[L] \cong Z_{i-1}^{\langle n-1 \rangle}$. Let $\mathtt{SD}_- = \{[L] : L$ is a line bundle of $\mathcal{O}_{\mathbb{F}_{p^2}}^{\oplus n}\}$.

For the cohomology groups of $Z_i^{\langle n \rangle}$, we have the following proposition:

Proposition 4.4. *The cohomology groups $H^j_{\acute{e}t}(Z_i^{\langle n \rangle}) = 0$ is zero if j is odd.*

Proof. Since relative Frobenius induce isomorphism on ℓ-adic cohomology groups, $H^j_{\acute{e}t}(Z_i^{\langle n \rangle}) = H^j_{\acute{e}t}(\widehat{Z}_i^{\langle n \rangle}) = H^j_{\acute{e}t}(\tilde{Z}_i^{\langle n \rangle})$. Now we prove the proposition by induction on i.

For $i = 1$, we have $Z_1^{\langle n \rangle} = \mathbb{P}^{n-1}$ and the proposition is satisfied automatically.

Suppose we have proved the result for i, we try to prove the proposition for $i+1$.

We consider the moduli space W over $\mathbb{F}_{p^2}$ whose S-points are tuples (L_1, L_2, L_3), where L_1, L_2 and L_3 are repsectively subbundles of $\mathcal{O}_S^{\oplus n}$ of rank $i, i-1, i-2$ satisfying $L_3 \subseteq L_2, L_2^{(p^2)} \subseteq L_1$. It is easy to use deformation theory to check that W is a smooth variety of dimension $n-1$. There are two morphisms

$$\begin{tikzcd} & W \arrow[dl, "\psi_{12}"'] \arrow[dr, "\psi_{23}"] & \\ \widehat{Z}_i^{\langle n \rangle} & & \tilde{Z}_{i-1}^{\langle n \rangle} \end{tikzcd} \qquad \begin{tikzcd} & (L_1, L_2 = L_2', L_3') \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ (L_1, L_2) & & (L_2', L_3') \end{tikzcd}.$$

Let E denote the subspace of W whose Let E denote the subspace of W whose closed points $x \in W(\overline{\mathbb{F}}_p)$ are those such that $L_{2,x} = L_{2,x}^{(p^2)}$, i.e., $L_{2,x}$ is an $\mathbb{F}_{p^2}$-rational subspace of $\mathbb{F}_{p^2}^{\oplus n}$ of dimension

$i-1$. It is clear that E is a disjoint union of $\#\mathbb{P}^{i-1}(\mathbb{F}_{p^2})$ copies of $\mathbb{P}^{n-i}\times\mathbb{P}^{k-2}$. It gives rise to a smooth divisor on W.

For a point $x\in(W\backslash E)(\overline{\mathbb{F}}_p)$, we ahve $L_{2,x}\neq L_{2,x}^{p^2}$ and hence it uniquely determines both $L_{1,x}$ and $L'_{3,x}$; so ψ_{12} and ψ_{23} are isomoprhisms restricted to $W\ E$. On the other hand, when restricted to E, ψ_{12} contracts each copy of $\mathbb{P}^{n-i}\times\mathbb{P}^{k-2}$ of E into the first factor; whereas ψ_{23} contracts each copy of $\mathbb{P}^{n-i}\times\mathbb{P}^{k-2}$ of E into the second factor. It is clear from this and the deformation theory that ψ_{12} is the blowing-up of $\widehat{Z}^{\langle n\rangle}$ along $\psi_{12}(E)$ and ψ_{23} is the blowing-up of $\tilde{Z}_{i-1}^{\langle n\rangle}$ along $\psi_{23}(E)$. Thus by blowing up sequence and induction, we conclude our proof easily. $\square$

5. Dieudonné modules and Grothendieck-Messing deformation theory

In this section we focus on the connection between Dieudonné modules and abelian varieties. We refer to [5] for general construction of Dieudonné modules with respect to p-divisible groups, which we omit here.[6]

5.1. **Dieudonné modules.** As in Subsection 3.2, we have the following isomorphism

$$\mathcal{O}_D\otimes_{\mathbb{Z}}\mathbb{Z}_{p^2}\cong\bigoplus_{i=1}^{2}\big(\mathcal{O}_D\otimes_{\mathcal{O}_E,q_i}\mathbb{Z}_{p^2}\oplus\mathcal{O}_D\otimes_{\mathcal{O}_E,\bar{q}_i}\mathbb{Z}_{p^2}\big)\simeq\bigoplus_{i=1}^{2}\big(\mathrm{M}_n(\mathbb{Z}_{p^2})\oplus\mathrm{M}_n(\mathbb{Z}_{p^2})\big).$$

Let S be a locally noetherian $\mathbb{Z}_{p^2}$-scheme. An $\mathcal{O}_D\otimes_{\mathbb{Z}}\mathcal{O}_S$-module M admits a canonical decomposition

$$M=\bigoplus_{i=1}^{2}\big(M_{q_i}\oplus M_{\bar{q}_i}\big),$$

where M_{q_i} (resp. $M_{\bar{q}_i}$) is the direct summand of M on which $\mathcal{O}_E$ acts via q_i (resp. via $\bar{q}_i$). Then each M_{q_i} has a natural action by $\mathrm{M}_n(\mathcal{O}_S)$. Let $\mathfrak{e}$ denote the element of $\mathrm{M}_n(\mathcal{O}_S)$ whose $(1,1)$-entry is 1 and whose other entries are 0. We put $M_i^\circ:=\mathfrak{e}M_{q_i}$, which is called the *reduced part* of M_{q_i}.[7].

Let A be an $2n^2$-dimensional abelian variety over an $\mathbb{F}_{p^2}$-scheme S, equipped with an $\mathcal{O}_D$-action. The de Rham homology $H_1^{\mathrm{dR}}(A/S)$ has a Hodge filtration[8]

$$0\to\omega_{A^\vee/S}\to H_1^{\mathrm{dR}}(A/S)\to\mathrm{Lie}_{A/S}\to0,$$

compatible with the natural action of $\mathcal{O}_D\otimes_{\mathbb{Z}}\mathcal{O}_S$ on $H_1^{\mathrm{dR}}(A/S)$. When $A\to S$ satisfies the moduli problem in Subsection 3.3, $H_1^{\mathrm{dR}}(A/S)_i^\circ$ is locally free of rank n and $\omega^\circ_{A^\vee/S,i}$ is subbundle of rank a_i.

When $S=\mathrm{Spec}(k)$ with k a perfect field containing $\mathbb{F}_{p^2}$, let $W(k)$ denote the ring of Witt vectors in k. Let $\tilde{\mathcal{D}}(A)$ denote the (covariant) Dieudonné module associated to the p-divisible group of A. This is a free $W(k)$-module of rank $4n^2$ equipped with a Frob-linear action of F and a Frob^{-1}-linear action of V such that $FV=VF=p$. The $\mathcal{O}_D$-action on A induces a natural action of $\mathcal{O}_D$ on $\tilde{\mathcal{D}}(A)$ that commutes with F and V. For each $i\in\mathbb{Z}/2\mathbb{Z}$, the Verschiebung and the Frobenius morphism on A induce natural maps

$$V:\tilde{\mathcal{D}}(A)_i^\circ\longrightarrow\tilde{\mathcal{D}}(A)_{i-1}^\circ,\quad F:\tilde{\mathcal{D}}(A)_i^\circ\longrightarrow\tilde{\mathcal{D}}(A)_{i+1}^\circ.$$

Moreover, there is a canonical isomorphism $\tilde{\mathcal{D}}(A)_i/p\tilde{\mathcal{D}}(A)_i\cong H_1^{\mathrm{dR}}(A/k)$ compatible with all structures on both sides. The action of F and V on $\mathcal{D}(A)_i$ induces the Frobenius and Vershibung morphism on the de Rham homology:

$$F:H_1^{\mathrm{dR}}(A/S)_{i-1}^{\circ,(p)}\longrightarrow H_1^{\mathrm{dR}}(A/S)_i^\circ.$$

$$V:H_1^{\mathrm{dR}}(A/S)_i^\circ\longrightarrow H_1^{\mathrm{dR}}(A/S)_{i-1}^{\circ,(p)},$$

[6]The definition of the Dieudonne module we use in this paper is in fact the Serre dual of that in [5]

[7]The idea of taking the reduced part comes from Morita Equivalence

[8]The exact sequence splits as direct sum in fact.

where by "(p)" we mean the pullback via the absolute Frobenius $\sigma$of S and there is an isomorphism $H_1^{\mathrm{dR}}(A/S)_i^{\circ,(p)} \cong H_1^{\mathrm{dR}}(A/S)_i^\circ \otimes_{S,\sigma} S$.

It can be shown $\mathrm{Ker}(F) = \mathrm{Im}(V) = \omega^{\circ,(p)}_{A^\vee/S,i-1}$ and $\mathrm{Ker}(V) = \mathrm{Im}(F)$. This implies isomorphisms:

$$V\tilde{\mathcal{D}}(A)_{i-1}/p\tilde{\mathcal{D}}(A)_i \cong \omega^\circ_{A^\vee/S,i}$$

$$\tilde{\mathcal{D}}(A)_i/V\tilde{\mathcal{D}}(A)_{i-1} \cong \mathrm{Lie}^\circ_{A/S,i}$$

For any $2n^2$-dimensional abelian variety A' over k equipped with an $\mathcal{O}_D$-action, an $\mathcal{O}_D$-equivariant isogeny $A' \to A$ induces a morphism $\tilde{\mathcal{D}}(A')_i^\circ \to \tilde{\mathcal{D}}(A)_i^\circ$ compatible with the actions of F and V. Conversely, [11] provides a proposition to obtain a new abelian variety corresponds to submodules of $\tilde{\mathcal{D}}(A)$. Here we give a similar proposition.

For any $2n^2$-dimensional abelian variety A' over k equipped with an $\mathcal{O}_D$-action, an $\mathcal{O}_D$-equivariant isogeny $A' \to A$ induces a morphism $\tilde{\mathcal{D}}(A')_i^\circ \to \tilde{\mathcal{D}}(A)_i^\circ$ compatible with the actions of F and V. Conversely, we have the following.

Proposition 5.1.1. Let A be an abelian variety of dimension $2n^2$ over prefect field k which contains $\mathbb{F}_{p^2}$, equipped with an $\mathcal{O}_D$-action and an $\mathcal{O}_D$-compatible prime-to-p polarization λ. Suppose given an integer $m \geq 1$ and a $W(k)$-submodule $\tilde{\mathcal{E}}_i \subseteq \tilde{\mathcal{D}}(A)_i^\circ$ for each $i \in \mathbb{Z}/2\mathbb{Z}$ such that

$$p^m\tilde{\mathcal{D}}(A)_i^\circ \subseteq \tilde{\mathcal{E}}_i, \quad F(\tilde{\mathcal{E}}_i) \subseteq \tilde{\mathcal{E}}_{i+1}, \quad \text{and} \quad V(\tilde{\mathcal{E}}_i) \subseteq \tilde{\mathcal{E}}_{i-1}. \tag{5.1.1}$$

Then there exists a unique abelian variety A' over k (depending on m) equipped with an $\mathcal{O}_D$-action, a prime-to-p polarization λ', and an $\mathcal{O}_D$-equivariant p-isogeny $\phi : A' \to A$ such that the natural inclusion $\tilde{\mathcal{E}}_i \subseteq \tilde{\mathcal{D}}(A)_i^\circ$ is naturally identified with the map $\phi_{*,i}\colon \tilde{\mathcal{D}}(A')_i^\circ \to \tilde{\mathcal{D}}(A)_i^\circ$ induced by ϕ and such that $\phi^\vee \circ \lambda \circ \phi = p^m\lambda'$. Moreover, we have

(1) If $\dim \omega^\circ_{A^\vee/k,i} = a_i$ and $\mathrm{length}_{W(k)}\big(\tilde{\mathcal{D}}(A)_i^\circ/\tilde{\mathcal{E}}_i\big) = \ell_i$ for $i \in \mathbb{Z}/2\mathbb{Z}$, then

$$\dim \omega^\circ_{A'^\vee/k,i} = a_i + \ell_i - \ell_{i+1}. \tag{5.1.2}$$

(2) If A is equipped with a prime-to-p level structure η (in the sense of Subsection 3.3(3)), then there exists a unique prime-to-p level structure η' on A' such that $\eta = \phi \circ \eta'$.

Proof. The proof can be found in [11, Proposition 3.2]. $\square$

Let $\mathfrak{D}(A)^\circ$ be the Dieudonné module corresponds to $A[p]$. It should be noted that Proposition 5.1.1 also holds for a submodule of $\mathfrak{E} \subseteq \mathfrak{D}(A)^\circ$. By Proposition 5.1.1, we have a corollary, which is also useful:

Corollary 5.1.2. *Let A be an abelian variety of dimension $2n^2$ over prefect field k which contains $\mathbb{F}_{p^2}$, equipped with an $\mathcal{O}_D$-action and an $\mathcal{O}_D$-compatible prime-to-p polarization λ. Suppose given a $W(k)$-submodule $\tilde{\mathcal{D}}(A)_i^\circ \subseteq \tilde{\mathcal{E}}_i \subseteq p^{-1}\tilde{\mathcal{D}}(A)_i^\circ$ (resp. $p\tilde{\mathcal{D}}(A)_i^\circ \subseteq \tilde{\mathcal{E}}_i \subseteq p^{-1}\tilde{\mathcal{D}}(A)_i^\circ$) for each $i \in \mathbb{Z}/2\mathbb{Z}$ such that*

$$F(\tilde{\mathcal{E}}_i) \subseteq \tilde{\mathcal{E}}_{i+1}, \quad \text{and} \quad V(\tilde{\mathcal{E}}_i) \subseteq \tilde{\mathcal{E}}_{i-1}.$$

Then there exists a unique abelian variety A' over k equipped with an $\mathcal{O}_D$-action, a prime-to-p polarization λ', and an $\mathcal{O}_D$-equivariant p-isogeny $\phi : A \to A'$ (resp. an $\mathcal{O}_D$-equivariant p-quasi-isogeny $\phi : A \to A'$) such that $\phi^\vee \circ \lambda' \circ \phi = p\lambda$ (resp. $\phi^\vee \circ \lambda' \circ \phi = \lambda$). Moreover, we have

(1) *If $\dim \omega^\circ_{A^\vee/k,i} = a_i$ and $\mathrm{length}_{W(k)}\big(\tilde{\mathcal{D}}(A)_i^\circ/p\tilde{\mathcal{E}}_i\big) = \ell_i$ for $i \in \mathbb{Z}/2\mathbb{Z}$, then*

$$\dim \omega^\circ_{A'^\vee/k,i} = a_i + \ell_i - \ell_{i+1}. \tag{5.1.3}$$

(2) *If A is equipped with a prime-to-p level structure η (in the sense of Subsection 3.3(3)), then there exists a unique prime-to-p level structure η' on A' such that $\eta' = \phi \circ \eta$.*

Proof. The two modules $p\tilde{\mathcal{E}}_1, p\tilde{\mathcal{E}}_2$ satisfies (5.1.1). Applying Proposition 5.1.1 with $m = 1$, we get there is an abelian variety A' over k equipped with an $\mathcal{O}_D$-action, a prime-to-p polarization λ', a unique prime-to-p level structure η', and an $\mathcal{O}_D$-equivariant p-isogeny $\psi : A' \to A$ such that the natural inclusion $p\tilde{\mathcal{E}}_i \subseteq \tilde{\mathcal{D}}(A)^\circ_i$ is naturally identified with the map $\psi_{*,i}\colon \tilde{\mathcal{D}}(A')^\circ_i \to \tilde{\mathcal{D}}(A)^\circ_i$ induced by ψ and such that $\psi^\vee \circ \lambda \circ \psi = p\lambda', p\eta = \psi \circ \eta'$(The last equation holds since we can simply multiply p on the level structure η' we get by Proposition 5.1.1.). There is an isogeny $\phi : A \to A'$ such that $\phi \circ \psi = p\mathrm{id}_{A'}$ and $\psi \circ \phi = p \circ \mathrm{id}_A$. Therefore $\phi^\vee \circ \lambda' \circ \phi = p \circ \lambda$. Moreover, $\eta' = \phi \circ \eta$.

Moreover, if we apply Proposition 5.1.1 with $m = 2$, we get there is an abelian variety A' over k equipped with an $\mathcal{O}_D$-action, a prime-to-p polarization λ', a unique prime-to-p level structure η', and an $\mathcal{O}_D$-equivariant p-isogeny $\psi' : A' \to A$ such that the natural inclusion $p\tilde{\mathcal{E}}_i \subseteq \tilde{\mathcal{D}}(A)^\circ_i$ is naturally identified with the map $\psi'_{*,i}\colon \tilde{\mathcal{D}}(A')^\circ_i \to \tilde{\mathcal{D}}(A)^\circ_i$ induced by ψ' and such that $\psi'^\vee \circ \lambda \circ \psi' = p^2\lambda', p\eta = \psi' \circ \eta'$(The last equation holds since we can simply multiply p on the level structure η' we get by Proposition 5.1.1.). There is an isogeny $\phi' : A \to A'$ such that $\phi' \circ \psi' = p^2\mathrm{id}_{A'}$ and $\psi' \circ \phi' = p^2 \circ \mathrm{id}_A$. Therefore $\phi'^\vee \circ \lambda' \circ \phi' = p^2 \circ \lambda$. Moreover, $p\eta' = \phi' \circ \eta$.

Take a p-quasi-isogeny $\phi : A \to A'$ such that $p \circ \phi = \phi'$. Then $\phi^\vee \circ \lambda' \circ \phi = \lambda$. Moreover, $\eta' = \phi \circ \eta$. Hence we finish the proof. ☐

5.2. **Grothendieck-Messing deformation theory.** Grothendieck-Messing deformation theory is important to compare the tangent spaces of moduli spaces. We state the theory following [11]. We shall frequently use Grothendieck–Messing deformation theory to compare the tangent spaces of moduli spaces. We make this explicit in our setup.

Let $\hat{R}$ be a noetherian $\mathbb{F}_{p^2}$-algebra and $\hat{I} \subset \hat{R}$ an ideal such that $\hat{I}^2 = 0$. Put $R = \hat{R}/\hat{I}$. Let $\mathscr{C}_{\hat{R}}$ denote the category of tuples $(\hat{A}, \hat{\lambda}, \hat{\eta})$, where $\hat{A}$ is an $2n^2$-dimensional abelian variety over $\hat{R}$ equipped with an $\mathcal{O}_D$-action, $\hat{\lambda}$ is a polarization on $\hat{A}$ such that the Rosati involution induces the $*$-involution on $\mathcal{O}_D$, and $\hat{\eta}$ is a level structure as in Subsection 3.3(3). We define $\mathscr{C}_R$ in the same way. For an object (A, λ, η) in the category $\mathscr{C}_R$, let $H_1^{\mathrm{cris}}(A/\hat{R})$ be the evaluation of the first relative crystalline homology (i.e. dual crystal of the first crystalline cohomology) of A/R at the divided power thickening $\hat{R} \to R$, and $H_1^{\mathrm{cris}}(A/\hat{R})^\circ_i := \mathfrak{e}H_1^{\mathrm{cris}}(A/\hat{R})_{q_i}$ be the i-th reduced part. We denote by $\mathscr{D}\mathrm{ef}(R, \hat{R})$ the category of tuples $(A, \lambda, \eta, (\hat{\omega}^\circ_i)_{i=1,2})$, where (A, λ, η) is an object in $\mathscr{C}_R$, and $\hat{\omega}^\circ_i \subseteq H_1^{\mathrm{cris}}(A/\hat{R})^\circ_i$ for each $i \in \mathbb{Z}/2\mathbb{Z}$ is a subbundle that lifts $\omega^\circ_{A^\vee/R,i} \subseteq H_1^{\mathrm{dR}}(A/R)^\circ_i$. The following is a combination of Serre–Tate and Grothendieck–Messing deformation theory.

Theorem 5.2.1 (Serre–Tate, Grothendieck–Messing)**.** *Functor* $(\hat{A}, \hat{\lambda}, \hat{\eta}) \mapsto (\hat{A} \otimes_{\hat{R}} R, \lambda, \eta, \omega^\circ_{\hat{A}^\vee/\hat{R},i})$, *where* λ *and* η *are the natural induced polarization and level structure on* $\hat{A} \otimes_{\hat{R}} R$*, is an equivalence of categories between* $\mathscr{C}_{\hat{R}}$ *and* $\mathscr{D}\mathrm{ef}(R, \hat{R})$.

Proof. The proof can be found in [11, Theorem 3.4]. ☐

Corollary 5.2.2. *If* $\mathcal{A}_{a_\bullet}$ *denotes the universal abelian variety over* $\mathrm{Sh}_{a_\bullet}$*, then the tangent space* $T_{\mathrm{Sh}_{a_\bullet}}$ *of* $\mathrm{Sh}_{a_\bullet}$ *is*

$$\bigoplus_{i=1}^{f} \mathrm{Lie}^\circ_{\mathcal{A}^\vee_{a_\bullet}/\mathrm{Sh}_{a_\bullet},i} \otimes \mathrm{Lie}^\circ_{\mathcal{A}_{a_\bullet}/\mathrm{Sh}_{a_\bullet},i}.$$

Proof. The proof can be found in [11, Corollary 3.5]. ☐

Remark 5.2.3. Even though we omit the proof here, it should be highlighted that the proof provides us an explit example about how to caluate the tangent sheaf from the deformation theory.

6. Description for the Higher Chow group

In this section, we give a proof of Theorem 1.1. First we give the definition of higher Chow groups.

Definition 6.1. Let X be a quasi-projective variety over a field k and Δ^n be the standard $n-$th simplex Spec $k[x_0, \cdots, x_n]/(\sum_{i=0}^n x_i - 1)$. For all $0 \leqslant i \leqslant n$, there is an embedding $\partial_{n,i} : \Delta_{n-1,i} \xrightarrow{\cong} \{x_i = 0\} \subset \Delta^n$. For $n, r \in \mathbb{N}$, we define $Z^r(X, n)$ to be the abelian group generated by $\{Z \subset X \times \Delta^n$ integral closed subvarieties $\mid \forall$ face $F \subset \Delta^n$, we have $\mathrm{codim}_{X\times F}(Z \cap (X \times F)) \geqslant r\}$

Since $\partial_{X,n,i} = id_X \times \partial_{n,i} : X \times \Delta^{n-1} \to X \times \Delta^n$ is an effective Cartier divisor, there is a Gysin homomorphism: $\partial^*_{X,n,i} : Z^r(X, n) \to Z^r(X, n-1)$ maps a subvariety V to the inersection $(X \times \{x_i = 0\}) \cap V$. Define the boundary operator $d_n = \sum_{i=0}^n (-1)^i \partial^*_{X,n,i}$, then get a chain complex:

$$\cdots \to Z^r(X, n) \to Z^r(X, n-1) \to \cdots \to Z^r(X, 0) \to 0$$

The *higher Chow group* $\mathrm{Ch}^r(X, n)$ is defined to be the n^{th} homology of the above complex. Moreover, for any ring R, we can get the higher Chow group with coefficients in R by tensoring the above chain complex with R. We denote it by $\mathrm{Ch}^r(X, n, R)$.

Here we list some basic property of higher Chow group.

Proposition 6.2. *Suppose X is a quasi-projective variety over a field k and R is a ring, then we have*

1. $\mathrm{Ch}^i(X, 0, R) = \mathrm{Ch}^i(X, R)$, *where* $\mathrm{Ch}^i(\mathrm{X}, R)$ *is the Chow group with coefficients in* R *as usual.*
2. *For X smooth, we have* $\mathrm{Ch}^1(X, 1) = \mathrm{H}^0_{Zar}(X, \mathcal{O}_X^\times)$.
3. *The motivic cohomology* $\mathrm{H}^i_{\mathcal{M}}(X, R(j)) = \mathrm{Ch}^j(X, 2j - i, R)$.
4. *If $Y \subseteq X$ is a closed subscheme smooth of codimension c, then the pushforward of cycles along $Z \times \Delta^n \to X \times \Delta^n$ induces the Gysin map* $\mathrm{Ch}^i(Y, j, R) \to \mathrm{Ch}^{i+c}(X, j, R)$.

In our cases, We mainly concern about $\mathrm{Ch}^1(\mathrm{Sh}^{\mathrm{ss}}_{1,n-1}, 1, \mathbb{F}_l)$. By Proposition 3.4.2, we have $\mathrm{Sh}^{\mathrm{ss}}_{1,n-1}$ is equi-dimensional and its irreducible components can be expressed as the images of closed immersions $\overrightarrow{p}_j|_{\mathrm{Y}_{j,z}} : \mathrm{Y}_{j,z} \to \mathrm{Sh}_{1,n-1}$ $(1 \leqslant j \leqslant n, z \in \mathrm{Sh}_{0,n}(\overline{\mathbb{F}}_p))$, denote as $C_{j,z}$. Taking the Hecke correspondence $\mathrm{T} = \mathrm{T}^{(1)}_{\mathfrak{p}}$, we can define

$$\mathcal{D} := \{D \in \mathrm{Div}(\mathrm{Sh}^{\mathrm{ss}}_{1,n-1}) \mid D \subseteq C_{j,z} \cap C_{i,z'} \text{ for some } (j,z) \neq (i,z')\},$$

$$D_{i,i+1} := (Y_i \times_{\mathrm{Sh}_{1,n-1}} Y_{i+1}) \times_{\mathrm{Sh}_{0,n}\times \mathrm{Sh}_{0,n}} T = \coprod_{(z,z')\in T} (\mathrm{Y}_{i,z} \times_{\mathrm{Sh}_{1,n-1}} Y_{i+1,z'}).$$

Then $\mathrm{H}^0(D_{i,i+1}, \mathbb{F}_l) \cong \mathrm{H}^0(T, \mathbb{F}_l)$ and from Proposition 3.5.2 $\coprod_{D\in\mathcal{D}} D = \coprod_{i=1}^{n-1} D_{i,i+1}$. By Nart [20], we get the following expression of $\mathrm{Ch}^1(\mathrm{Sh}^{\mathrm{ss}}_{1,n-1}, 1, \mathbb{F}_l)$:

Proposition 6.3. *Let $Y_i^\circ = Y_i \backslash (\cup_{j\neq i} Y_i)$ $(1 \leqslant i \leqslant n)$, then*

$$\mathrm{Ch}^1(\mathrm{Sh}^{\mathrm{ss}}_{1,n-1}, 1, \mathbb{F}_l) = \mathrm{Ker}\left(\bigoplus_i R(\mathrm{Y}_i^\circ)^* \xrightarrow{div} \bigoplus_{i=1}^{n-1} \mathrm{H}^0(T, \mathbb{F}_l)\right),$$

where the div : $\bigoplus_i R(\mathrm{Y}_i^\circ)^* \xrightarrow{div} \bigoplus_{i=1}^{n-1} \mathrm{H}^0(T, \mathbb{F}_l)$, *is induced by the div map on each Y_i° as usual.*

Proof. By [20, Corollary 1.2], the higher Chow group

$$\mathrm{Ch}^1(\mathrm{Sh}^{ss}_{1,n-1}, 1, \mathbb{F}_\ell) = \mathrm{Ker}(R(\mathrm{Sh}^{ss}_{1,n-1})^* \xrightarrow{div} Z^1(\mathrm{Sh}^{ss}_{1,n-1})).$$

Here $R(\mathrm{Sh}^{ss}_{1,n-1})$ stands for the ring of rational functions of $\mathrm{Sh}^{ss}_{1,n-1}$. $Z^1(\mathrm{Sh}^{ss}_{1,n-1})$ is the group of 1-codimensional cycles. Since $\{\mathrm{Y}_{i,z} \mid 1 \leq i \leq n, z \in \mathrm{Sh}_{0,n}(\overline{\mathbb{F}}_p)\}$ are irreducible components of $\mathrm{Sh}^{ss}_{1,n-1}$, we get further

$$\mathrm{Ch}^1(\mathrm{Sh}^{ss}_{1,n-1}, 1, \mathbb{F}_\ell) = \mathrm{Ker}(\bigoplus_{i=1}^n R(\mathrm{Y}_i)^* \xrightarrow{div} Z^1(\mathrm{Sh}^{ss}_{1,n-1})).$$

Since Y_i° is an open subset of Y_i, the ring of rational functions $R(\mathrm{Y}_i^\circ) = R(\mathrm{Y}_i)$. For any $1 \leq i \leq n$ and any $f \in R(\mathrm{Y}_i^\circ)^* \bigcap \mathrm{Ch}^1(\mathrm{Sh}_{1,n-1}^{ss}, 1, \mathbb{F}_\ell)$, the principal divisor $div(f)$ can be expressed sums of divisors contained in $\mathrm{Y}_{i,z} \bigcap \mathrm{Y}_{j,z'}$ for some $1 \leq i, j \leq n$ and $z, z' \in \mathrm{Sh}_{1,n-1}$ such that $(i,z) \neq (j,z')$. Since each summation in $div(f)$ has codimension 1, we get $(z,z') \in T$ with $j = i+1$ or $(z',z) \in T$ with $j = i-1$ from Proposition 3.5.2. Furthermore, we have

$$\begin{aligned}\mathrm{Ch}^1(\mathrm{Sh}_{1,n-1}^{\mathrm{ss}}, 1, \mathbb{F}_l) &= \mathrm{Ker}\left(\bigoplus_i R(\mathrm{Y}_i^\circ)^* \xrightarrow{div} \bigoplus_{D \in \mathcal{D}} \mathbb{F}_l\right) \\ &= \mathrm{Ker}\left(\bigoplus_i R(\mathrm{Y}_i^\circ)^* \xrightarrow{div} \bigoplus_{i=1}^{n-1} \mathrm{H}^0(D_{i,i+1}, \mathbb{F}_l)\right) \\ &= \mathrm{Ker}\left(\bigoplus_i R(\mathrm{Y}_i^\circ)^* \xrightarrow{div} \bigoplus_{i=1}^{n-1} \mathrm{H}^0(T, \mathbb{F}_l)\right).\end{aligned}$$

□

Hence we need to consider the principal divisors on $\mathrm{Sh}_{1,n-1}$. More explicitly, if we identify $\mathrm{Y}_{i,z}$ with $Z_i^{\langle n\rangle}$, we need to consider when a divisor in $\mathbb{Q}[\mathtt{SD}_-] \oplus \mathbb{Q}[\mathtt{SD}_+]$ can be expressed as a principal divisor of Y_i°.

In fact we have the following proposition:

Proposition 6.4. *For all $1 \leqslant i \leqslant n-1$, in $Z_n^{\langle i\rangle}$, the divisor*

$$\sum_{[L] \in \mathtt{SD}_-} a_L[L] + \sum_{[H] \in \mathtt{SD}_+} b_H[H] \in \mathbb{Q}[\mathtt{SD}_-] \oplus \mathbb{Q}[\mathtt{SD}_+]$$

is principal if and only if:

$$\sum_{[L] \in \mathtt{SD}_-} a_L = 0,$$

and

$$b_H = p^{i+1-n} \cdot \sum_{\substack{L \subseteq H \\ [L] \in \mathtt{SD}_-}} a_L \ , \ \forall [H] \in \mathtt{SD}_+;$$

or equivalently:

$$\sum_{[H] \in \mathtt{SD}_+} b_H = 0,$$

and

$$a_L = p^{1-i} \cdot \sum_{\substack{L \subseteq H \\ [H] \in \mathtt{SD}_+}} b_H \ , \ \forall [L] \in \mathtt{SD}_-$$

Remark 6.5. The two conditions in Proposition 6.4 are equivalent by finite Randon transforms.

We want to prove Proposition 6.4 by induction on n.

Firstly, we observe that for any $1 \leq i \leq n$, both $\widehat{Z}_i^{\langle n\rangle}$ and $\tilde{Z}_i^{\langle n\rangle}$ have an open dense subscheme isomorphic to each other. More explicitly, we have the following defintion:

Definition 6.6. Let ${}^\circ\tilde{Z}_i^{\langle n\rangle}(resp.{}^\circ\widehat{Z}_i^{\langle n\rangle})$ be the nontrivial locus of $\tilde{Z}_n^{\langle i\rangle}, (resp.\widehat{Z}_i^{\langle n\rangle})$, i.e. ${}^\circ\tilde{Z}_n{}^{\langle i\rangle} := \{(H_1, H_2) \in \tilde{Z}_n{}^{\langle i\rangle} \mid H_1 \neq H_1^{(p^2)}\}$ and ${}^\circ\widehat{Z}_i^{\langle n\rangle} := \{(H_1, H_2) \in \widehat{Z}_i^{\langle n\rangle} \mid H_2 \neq H_2^{(p^2)}\}$.

To simplify our computation, we introduce a new scheme to contact $\widehat{Z}_i^{\langle n\rangle}$, $\tilde{Z}_i^{\langle n\rangle}$ and $\widehat{Z}_i^{\langle n\rangle}$ and $Z_n^{\langle i\rangle}$.

Definition 6.7. For each n and $1 \leq i \leq n$, we define $Y_i^{\langle n \rangle}$ to be the closed subscheme of $\mathbf{Gr}(n,k)$ whose S-valued points are the ismorphism classes of H, where H is subbundles of $\mathcal{O}_S^{\oplus n}$ of rank i with locally free quotient satisfying $H^{(p^2)} \hookrightarrow \mathcal{O}_S^{\oplus n} \twoheadrightarrow \mathcal{O}_S^{\oplus n}/H$ has rank at most 1. Similarly, we define ${}^{\circ}Y_i^{\langle n \rangle} := \{H \in Y_i^{\langle n \rangle} \mid H \neq H^{(p^2)}\}$.

It is easy to observe that we have natural morphisms $\tilde{Z}_i^{\langle n \rangle} \to Y_i^{\langle n \rangle} : (H_1, H_2) \mapsto H_1$ and $\widehat{Z}_i^{\langle n \rangle} \to Y_n^{\langle i-1 \rangle} : (H_1, H_2) \mapsto H_2$. These morphisms induce isomorphisms as below:

Lemma 6.8. *We have ismorphisms which are induced by morphisms given above:* ${}^{\circ}\tilde{Z}_i^{\langle n \rangle} \cong {}^{\circ}Y_i^{\langle n \rangle}$ *and* ${}^{\circ}\widehat{Z}_n^{\langle i+1 \rangle} \cong {}^{\circ}Y_i^{\langle n \rangle}$.

Proof. Check by defintion directly. □

We denote $\widehat{\phi} : \widehat{Z}_i^{\langle n \rangle} \xrightarrow{H_2^{(p)}} Z_n^{\langle i \rangle}$ and $\widehat{\psi} : Z_n^{\langle i \rangle} \xrightarrow{H_1^{(p)}} \widehat{Z}_i^{\langle n \rangle}$ to be the relative Frobenius map defined in Definition 4.2. The $\tilde{\phi}$ and $\tilde{\psi}$ are similar. The four morphisms connect $Z_n^{\langle i \rangle}$, $\tilde{Z}_n^{\langle i \rangle}$ and $\widehat{Z}_n^{\langle i \rangle}$ together.

To differ $[H] \in \mathtt{SD}_+$ and $[L] \in \mathtt{SD}_-$ in Z_n^i of different i, we denote them by $[H]_i^n$ and $[L]_i^n$. However, for simplicity, we use $[H]$ and $[L]$ to denote all the $[H] \in \mathtt{SD}_+$ and $[L] \in \mathtt{SD}_-$ in $Z_i^{\langle n \rangle}$, $\tilde{Z}_n^{\langle i \rangle}$ and $\widehat{Z}_n^{\langle i \rangle}$.

Lemma 6.9. *For* $1 \leqslant i \leqslant n-1$ *and any* $[H] \in \mathtt{SD}_+$, *we have* $\tilde{\phi}^*[H]_i^n = [H]_i^n$ *and* $\widehat{\phi}^*[H]_i^n = [H]_i^n$. *For* $1 \leqslant i \leqslant n-1$, *and any* $[L] \in \mathtt{SD}_-$, *we have* $\tilde{\psi}^*[L]_i^n = [L]_i^n$ *and* $\widehat{\psi}^*[L]_i^n = [L]_i^n$.

Proof. We only prove $\widehat{\phi}^*[H]_i^n = [H]_i^n$ as an example. Others are similar. We only need to show $\widehat{\phi}^{-1}[H]_i^n \subseteq [H]_i^n$ since $[H]_i^n$ is irreducible and $\widehat{\phi}^*[H]_i^n$ has the same dimension with $[H]_i^n$.

In fact, for any S-point (H_1, H_2) of $\widehat{\phi}^*[H]_i^n$, we get (H_1, H_2) as a S-point of $\widehat{Z}_n^{\langle i \rangle}$ with $H_2^{(p)} \subseteq H_1 \subseteq H$ as an element in $Z_i^{\langle n \rangle}$. Therefore, as a S-point in $\widehat{Z}_n^{\langle i \rangle}$, (H_1, H_2) is a S-point in $[H]_i^n$. □

Lemma 6.10. *For* $1 \leqslant i \leqslant n-1$, *and any* $[L] \in \mathtt{SD}_-$, *we have* $\tilde{\phi}^*[L]_i^n = p[L]_i^n$ *and* $\widehat{\phi}^*[L]_i^n = p[L]_i^n$ *For* $1 \leqslant i \leqslant n-1$, *and any* $[H] \in \mathtt{SD}_+$, *we have* $\tilde{\psi}^*[H]_i^n = p[H]_i^n$ *and* $\widehat{\psi}^*[H]_i^n = p[H]_i^n$.

Proof. Since $\tilde{\phi} \circ \tilde{\psi}$, $\hat{\phi} \circ \hat{\psi}$ is the Frobenius morphism, the statements follow from Lemma 6.9. □

Definition 6.11. Let Y be a smooth scheme over $\mathbb{F}_{p^2}$ and $U \subset Y$ be a dense open subscheme. Then we have a short complex

$$0 \to C^0(Y, U) \to C^1(Y, U) \to 0,$$

where $C^0(Y, U) := \mathrm{H}^0(U, \mathcal{O}_U^{\times})$, $C^1(Y, U) := \{\text{divisors on } Y \text{ with zero restriction to } U\}$. Then the complex $C^{\bullet}(Y, U)$ is a contravariant functor in (Y, U).

Remark 6.12. Let $Z, Z' \subset Y$ be closed subsets of codimension at least 2 such that $U - Z' \subset Y - Z$. Then the map $C^{\bullet}(Y, U) \to C^{\bullet}(Y - Z, U - Z')$ is isomorphism.

Corollary 6.13. *With notations as above, we have the following isomorphisms of the complexes defined above:*

(1) *For* $2 \leqslant i \leqslant n-1$, *we ahve*

$$C^{\bullet}(\tilde{Z}_i^{\langle n \rangle}, {}^{\circ\circ}Y_i^{\langle n \rangle}) \xrightarrow{\cong} C^{\bullet}({}^{\circ}Y_i^{\langle n \rangle}, {}^{\circ\circ}Y_i^{\langle n \rangle}).$$

(2) *For* $1 \leqslant i \leqslant n-2$, *we have*

$$C^{\bullet}(\widehat{Z}_n^{\langle i+1 \rangle}, {}^{\circ\circ}Y_i^{\langle n \rangle}) \xrightarrow{\cong} C^{\bullet}({}^{\circ}Y_i^{\langle n \rangle}, {}^{\circ\circ}Y_i^{\langle n \rangle}).$$

where ${}^{\circ\circ}Y_i^{\langle n \rangle} = {}^{\circ}Y_i^{\langle n \rangle} \backslash (\bigcup_{[H] \in \mathtt{SD}_+} Y_i^{\langle H \rangle}) \cup (\bigcup_{[L] \in \mathtt{SD}_-} Y_{i-1}^{\langle L \rangle}) \subset {}^{\circ}Y_i^{\langle n \rangle}$. *Here* $Y_{i-1}^{\langle L \rangle} := \{H \in Y_i^{\langle n \rangle} \mid L \subseteq H\}$ *and* $Y_i^{\langle H \rangle} := \{L \in Y_i^{\langle n \rangle} \mid L \subseteq H\}$.

Now we can give the proof of Proposition 6.4

Proof of Proposition 5.4. Via Lemma 6.9 and Lemma 6.10, we can prove the proposition by induction on i. Thus it suffices to prove the case $i = n-1$.

We know that the natural map from $\tilde{Z}_i^{\langle n\rangle}$ to $Y_i^{\langle n\rangle}$ is the blow up of $Y_i^{\langle n\rangle}$ at the $\mathbb{F}_{p^2}$ -points. We denote it as π here. For any $H \in Y_n^{\langle n-1\rangle}$, the exceptional divisor with center H is $[H]_n^{n-1}$with $[H] \in \mathrm{SD}_+$. Thus for $[L] \in \mathrm{SD}_-$, we have an identity of divisor of Z_n^{n-1}:

$$(\pi)^* Y_{n-2}^{\langle n-2\rangle} = [L]_n^{n-1} + \sum_{L\subseteq H} [H]_n^{n-1}.$$

The equations in the proposition can be obtained directly now. $\square$

For $1 < i < n$, we have the map $div : R(\mathrm{Y}_i^\circ)^* \xrightarrow{div} \mathrm{H}^0(T, \mathbb{F}_l)^{\oplus 2}$ as in 6.3. We can express any $(x, y) \in \mathrm{H}^0(T, \mathbb{F}_l)^{\oplus 2}$ as:

$$x = \sum_{(z',z)\in \mathrm{T}} a_{z',z} \cdot [\mathrm{Y}_{i-1,z'} \bigcap \mathrm{Y}_{i,z}],\ y = \sum_{(z,z'')\in \mathrm{T}} b_{z,z'} \cdot [\mathrm{Y}_{i,z} \bigcap \mathrm{Y}_{i+1,z''}]$$

. With Proposition 6.4, we get the following corollary about the coeffcients $a_{z',z}$ and $b_{z,z''}$:

Corollary 6.14. *For $1 < i < n$, an element $(x, y) \in \mathrm{H}^0(T, \mathbb{F}_l)^{\oplus 2}$ is in the image of the map $div : \mathcal{O}(\mathrm{Y}_i^\circ)^\times \otimes \mathbb{F}_l \to \mathrm{H}^0(T, \mathbb{F}_l)^{\oplus 2}$, i.e the divisors corresponds to x and y are all principal divisors defined by rational functions on Y_i° if and only if*

$$\sum_{(z,z')\in \mathrm{T}} a_{z',z} = 0,$$

and

$$b_{z,z''} = p^{i+1-n} \cdot \sum_{\substack{(z',z,z'')\in \mathrm{A} \\ (z',z)\in \mathrm{T}}} a_{z',z}\ ,\ \forall (z, z'') \in \mathrm{T},$$

where A *is a correspondence defined on* $\mathrm{Sh}_{1,n-1}(K_{\mathfrak{p}}^1)$ *as below.*

Remark 6.15. We can also prove Corollary 6.14 with a similar method as in Proposition 4.4.

Similar to the construction of the Hecke correspondence T, we construct A as follows:
Write $z = (B, \lambda, \eta)$, $z' = (B', \lambda', \eta')$ and $z'' = (B'', \lambda'', \eta'')$, we say $(z', z, z'') \in A$ if and only if

(1) The pair $(z', z) \in T$ and $(z, z'') \in T$. We use $\phi : B' \to B$ and $\phi' : B \to B''$ to denote the quasi-isogeny in the definition of T and $\mathbb{L}_z, \mathbb{L}_{z'}$ and $\mathbb{L}_{z''}$ to denote the $\mathbb{Z}_{p^2}$ lattice in the definition of T.
(2) We require $p\mathbb{L}_{z''} \subseteq \phi' \circ \phi(\mathbb{L}_{z'})$.

To express the action of A more explicitly, we have the following diagram:

$$\begin{tikzcd}
 & & \mathrm{A} \arrow[dl, "\overleftarrow{p}_{A,1}"'] \arrow[dr, "\overrightarrow{p}_{A,1}"] & & \mathrm{A} \arrow[dl, "\overleftarrow{p}_{A,2}"'] \arrow[dr, "\overrightarrow{p}_{A,2}"] & & \\
 & \mathrm{T} \arrow[dl, "\overleftarrow{p}"'] \arrow[dr, "\overrightarrow{p}"] & & \mathrm{T} \arrow[dl, "\overleftarrow{p}"'] \arrow[dr, "\overrightarrow{p}"] & & \mathrm{T} \arrow[dl, "\overleftarrow{p}"'] \arrow[dr, "\overrightarrow{p}"] & \\
\mathrm{Sh_{0,n}} & & \mathrm{Sh_{0,n}} & & \mathrm{Sh_{0,n}} & & \mathrm{Sh_{0,n}}
\end{tikzcd}$$

where for simplicity we only draw 'two' A in the diagram.[9]

The first conditions in Corollary 6.14 can be expressed as the commutativity of one part of the diagram, i,e $\overleftarrow{p}_* \overrightarrow{p}_{A,1,*} \overleftarrow{p}^*_{A,1} = \overrightarrow{p}_*$ and $\overleftarrow{p}_* \overrightarrow{p}_{A,2,*} \overleftarrow{p}^*_{A,2} = \overrightarrow{p}_*$. The operator A on T corresponds to

[9]It is exactly the case $n = 4$.

$\overrightarrow{p}_{A,1,*}\overleftarrow{p}^*_{A,1}$ and $\overrightarrow{p}_{A,2,*}\overleftarrow{p}^*_{A,2}$ in the diagram. We define the operator '$\circ$' as the composition of two adjacent A' s, which corresponds to the composition of $\overrightarrow{p}_{A,1,*}\overleftarrow{p}^*_{A,1}$ and $\overrightarrow{p}_{A,2,*}\overleftarrow{p}^*_{A,2}$ in the diagram.

Theorem 6.16. *With notations as above, we have*

$$\mathrm{Ch}^1(\mathrm{Sh}^{\mathrm{ss}}_{1,n-1},1,\mathbb{F}_l)=\mathrm{Ker}(\mathrm{H}^0(\mathrm{T},\mathbb{F}_l)\xrightarrow{\alpha}\mathrm{H}^0(\mathrm{Sh}_{0,n},\mathbb{F}_l)^{\oplus n}),$$

where $\alpha=(\overleftarrow{p}_*,\overrightarrow{p}_*,\overrightarrow{p}_*A,\overrightarrow{p}_*(A\circ A),\cdots,\underbrace{\overrightarrow{p}_*(A\circ\cdots\circ A)}_{n-2})$ *with* $(\overleftarrow{p}_*,\overrightarrow{p}_*)$ *given by the correpsondence.*

In particular, for $n=2$*, we have*

$$\mathrm{Ch}^1(\mathrm{Sh}^{\mathrm{ss}}_{1,n-1},1,\mathbb{F}_l)=\mathrm{Ker}(\mathrm{H}^0(\mathrm{T},\mathbb{F}_l)\xrightarrow{(\overleftarrow{p}_*,\overrightarrow{p}_*)}\mathrm{H}^0(\mathrm{Sh}_{0,n},\mathbb{F}_l)^{\oplus 2}),$$

where $(\overleftarrow{p}_*,\overrightarrow{p}_*)$ *is the map given by the correspondence above.*

Remark 6.17. We cannot write $A\circ A$ simply as A^2 since they cannot be viewed as the same 'A'. More explicily, we can check that $A=\mathrm{Sh}_{0,n}(K^{1,1}_{\mathfrak{p}})$, where the level group can be expressed as

$$K^{1,1}_{\mathfrak{p}}=\underbrace{\begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1\\ p & & & \end{pmatrix}K\begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1\\ p^{-1} & & & \end{pmatrix}}_{K^-}\bigcap K\bigcap\underbrace{\begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1\\ p^{-1} & & & \end{pmatrix}K\begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1\\ p & & & \end{pmatrix}}_{K^+}.$$

Then we can express the operator A as:

$$\begin{tikzcd}
 & & \mathrm{Sh}_{0,\mathrm{n}}(K^{1,1}_{\mathfrak{p}}) \arrow[dl, "\overleftarrow{p}_{A,1}"'] \arrow[dr, "\overrightarrow{p}_{A,1}"] & & \\
 & \mathrm{Sh}_{0,n}(K^1_{\mathfrak{p}}) \arrow[dl, "\overleftarrow{p}"'] \arrow[dr, "\overrightarrow{p}"] & & \mathrm{Sh}_{0,n}(K^1_{\mathfrak{p}}) \arrow[dl, "\overleftarrow{p}"'] \arrow[dr, "\overrightarrow{p}"] & \\
\mathrm{Sh}_{0,\mathrm{n}}(K^-) & & \mathrm{Sh}_{0,\mathrm{n}}(K) & & \mathrm{Sh}_{0,\mathrm{n}}(K^+)
\end{tikzcd}$$

and 'adjacent' A's are combined together with the conjugate action of $\begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1\\ p & & & \end{pmatrix}$. Hence we can not composite A together simply by multiplication.

Finaly, we get a proof of Theorem 6.16

Proof of Theorem 5.16. First, we have $\mathcal{O}(\mathrm{Y}_1^\circ)^\times\otimes\mathbb{F}_l\cong\mathrm{Ker}(\mathrm{H}^0(\mathrm{T},\mathbb{F}_l)\xrightarrow{\overleftarrow{p}_*}\mathrm{H}^0(\mathrm{Sh}_{0,n},\mathbb{F}_l))$.

In fact, we have for every $f\in\mathcal{O}(\mathrm{Y}_1)^\times$, the principal divisor $div f=\sum_{(z',z)\in\mathrm{T}}a_{z',z}[\mathrm{Y}_{1,z'}\bigcap\mathrm{Y}_{2,z}]\in\mathrm{H}^0(D_{1,2},\mathbb{F}_l)=\mathrm{H}^0(\mathrm{T},\mathbb{F}_l)$ satisfies $\sum_{(z',z)\in\mathrm{T}}a_{z',z}=0$ for any fixed $z\in\mathrm{Sh}_{0,n}$. The converse is also true. Therefore, we have $\mathcal{O}(\mathrm{Y}_1^\circ)^\times\otimes\mathbb{F}_l\cong\mathrm{Ker}(\mathrm{H}^0(\mathrm{T},\mathbb{F}_l)\xrightarrow{\overleftarrow{p}_*}\mathrm{H}^0(\mathrm{Sh}_{0,n},\mathbb{F}_l))$.

Now by induction on $1<i<n$, we have that for any function $f\in\mathcal{O}(\mathrm{Y}_i)$, the map $div:R(\mathrm{Y}_i^\circ)^*\to\mathrm{H}^0(T,\mathbb{F}_l)^{\oplus 2}$ maps f to (x,y) with

$$x=\sum_{(z',z)\in T}a_{z,z'}\cdot[\mathrm{Y}_{i-1,z'}\bigcap\mathrm{Y}_{i,z}],\ y=\sum_{(z,z'')\in T}b_{z,z''}\cdot[\mathrm{Y}_{i+1,z''}\bigcap\mathrm{Y}_{i,z}],$$

if $div f = \sum_{(z',z)\in T} a_{z,z'} \cdot [\mathrm{Y}_{i-1,z'} \bigcap \mathrm{Y}_{i,z}] + \sum_{(z,z'')\in T} b_{z,z''} \cdot [\mathrm{Y}_{i+1,z''} \bigcap \mathrm{Y}_{i,z}]$. Then by 8.8, we have $y = Ax$ and $\overrightarrow{p}_* x = 0$.

Thus by induction we get

$$\mathrm{Ch}^1(\mathrm{Sh}^{\mathrm{ss}}_{1,n-1}, 1, \mathbb{F}_l) = \mathrm{Ker}(\mathrm{H}^0(\mathrm{T}, \mathbb{F}_l) \xrightarrow{\alpha} \mathrm{H}^0(\mathrm{Sh}_{0,n}, \mathbb{F}_l)^{\oplus n}),$$

where $\alpha = (\overleftarrow{p}_*, \overrightarrow{p}_*, \overrightarrow{p}_* A, \overrightarrow{p}_*(A \circ A), \cdots, \overrightarrow{p}_*(\underbrace{A \circ \cdots \circ A}_{n-2}))$ with $(\overleftarrow{p}_*, \overrightarrow{p}_*)$ given by the correpsondence.

$\square$

Remark 6.18. As a remark, we point out that the above expression can also be written as

$$\mathrm{Ch}^1(\mathrm{Sh}^{\mathrm{ss}}_{1,n-1}, 1, \mathbb{F}_l) \cong \mathrm{Ker}(\mathrm{H}^0(\mathrm{Sh}_{0,n}, \Omega_l) \xrightarrow{\overrightarrow{p}_* \circ \overleftarrow{p}^*, \overrightarrow{p}_* \circ A \circ \overleftarrow{p}^*, \ldots, \overrightarrow{p}_* \circ (\overbrace{A \circ \cdots \circ A}^{n-2}) \circ \overleftarrow{p}^*} \mathrm{H}^0(\mathrm{Sh}_{0,n}, \mathbb{F}_l)),$$

Indeed, if $\ell \nmid p^{2n-2} - 1$ we have the decomposition of the pushforward sheaf $\overleftarrow{p}_* \mathbb{F}_l = \mathbb{F}_l \oplus \Omega_l$ which is induced from that of the parabolic induction $\mathrm{Ind}_{K_p^1}^{K_p} \mathbb{1} = \mathbb{1} \oplus \rho_{(n-1,1)}$ where $\rho_{(n-1,1)}$ is the unipotent representation of $\mathrm{GL}_n(\mathbb{F}_{p^2})$ labelled by the partition $(n-1, 1)$ of n.

7. Ihara Lemma for $n = 2$

The proof of Ihara lemma for $n = 2$ and $n \geq 3$ are different. In section 6, we assume $n = 2$.

Let $\mathrm{T} = \mathrm{T}_{\mathfrak{p}}^{(1)}$ as in Definition 3.5.1. It can be checked directly that $T = \mathrm{Sh}_{0,2}(K_{\mathfrak{p}}^1)$. To prove Ihara lemma, we introduce $Sh_{1,1}(K_{\mathfrak{p}}^1)$, which is defined as the following moduli space:

Definition 7.1. Let $Sh_{1,1}(K_{\mathfrak{p}}^1)$ be the moduli space over $\mathbb{F}_{p^2}$ that associates to each locally noetherian $\mathbb{F}_{p^2}$-scheme S, the set of isomorphism classes of tuples $(A, \lambda, \eta, A', \lambda', \eta', \phi)$, where

- (A, λ, η) is an S-point of $\mathrm{Sh}_{1,1}$,
- (A', λ', η') is an S-point of $\mathrm{Sh}_{1,1}$, and
- $\phi : A \to A'$ is an $\mathcal{O}_D$-equivariant p-quasi-isogeny(i.e., $p^m\phi$ is an isogeny of p-power order for some integer m),

such that

- $\lambda' = \phi^\vee \circ \lambda \circ \phi$,
- $\phi \circ \eta' = \eta$, and
- the cokernels of the maps

$$\phi_{*,1} : H_1^{\mathrm{dR}}(A/S)_1^\circ \to H_1^{\mathrm{dR}}(A'/S)_1^\circ \quad \text{and} \quad \phi_{*,2} : H_1^{\mathrm{dR}}(A/S)_2^\circ \to H_1^{\mathrm{dR}}(A'/S)_2^\circ$$

  are both locally free $\mathcal{O}_S$-modules of rank 1.

$\mathrm{Sh}_{1,1}(K_{\mathfrak{p}}^1)$ is union of four closed schemes $\mathrm{Sh}_{1,1}(K_{\mathfrak{p}}^1) = \mathrm{Y}_{00} \bigcup \mathrm{Y}_{01} \bigcup \mathrm{Y}_{10} \bigcup \mathrm{Y}_{11}$, where the the four closed subschemes are defined as moduli subspace such that for each locally noetherian $\mathbb{F}_{p^2}$-scheme S an S-point $(A, \lambda, \eta, A', \lambda', \eta', \phi)$ in Y_{ij} for $0 \leq i, j \leq 1$ is an S-point of $\mathrm{Sh}_{1,1}(K_{\mathfrak{p}}^1)$ satisfying $(1.i)(2.j)$ below:

- (1.0) $\omega^\circ_{A^\vee,1} = \mathrm{Ker}(\phi_{*,1})$, (1.1) $\omega^\circ_{A'^\vee,1} = \mathrm{Im}(\phi_{*,1})$.
- (2.0) $\mathrm{Ker}(\phi_{*,2}) = \omega^\circ_{A^\vee,2}$, (2.1) $\mathrm{Im}(\phi_{*,2}) = \omega^\circ_{A'^\vee,2}$.

It can be checked directly that if $(A, \lambda, \eta, A', \lambda', \eta', \phi)$ is an S-point of Y_{00} or Y_{11}, then A and A' are all supersingular.

Our main result in this section is:

Theorem 7.2. *Under the Hypothesis 1.4 for $n = 2$, we have*

(1) *(Definite Ihara) The map*

$$\mathrm{H}^0(\mathrm{T}, \mathbb{F}_l)_{\mathfrak{m}} \xrightarrow{(\overleftarrow{p}_*, \overrightarrow{p}_*)} \mathrm{H}^0(\mathrm{Sh}_{0,2}, \mathbb{F}_l)_{\mathfrak{m}}^{\oplus 2}$$

is surjective.

(2) *(Indefinite Ihara) The map*

$$\mathrm{H}^2(\overline{Sh}_{1,1}(K_{\mathfrak{p}}^1), \mathbb{F}_l(2))_{\mathfrak{m}} \xrightarrow{(\overleftarrow{p}_*, \overrightarrow{p}_*)} (\overline{Sh}_{1,1}, \mathbb{F}_l(2))_{\mathfrak{m}}^{\oplus 2}$$

is surjective, with the map induced by projection of $\overline{Sh}_{1,1}(K_{\mathfrak{p}}^1)$ to $\overline{Sh}_{1,1}$.

To prove Theorem 7.2, we need to analyze the structure of Y_{ij} more carefully.

Using deformation theory in Subsection 5.2, we get:

Proposition 7.3. *For $1 \leq i, j \leq 2$,Y_{ij} are all smooth of dimension 2, which comes from calculation of the tangent sheaf $T_{\mathrm{Y}_{ij}} = \mathfrak{F}_i \oplus \mathfrak{G}_j$ where $\mathfrak{F}_i$ and $\mathfrak{G}_j$ are:*

(1) $\mathfrak{F}_0 = \mathcal{H}om\big(\omega^\circ_{\mathcal{A}'^\vee/\mathrm{Sh}_{1,1},1}, \mathrm{Lie}^\circ_{\mathcal{A}'/\mathrm{Sh}_{1,1},1}\big)$, $\mathfrak{F}_1 = \mathcal{H}om\Big(\omega^\circ_{\mathcal{A}^\vee/\mathrm{Sh}_{1,1},1}, \dfrac{\phi^{-1}_{*,1}(\omega^\circ_{\mathcal{A}^\vee/\mathrm{Sh}_{1,1},1})}{\omega^\circ_{\mathcal{A}^\vee/\mathrm{Sh}_{1,1},1}}\Big)$,

(2) $\mathfrak{G}_0 = \mathcal{H}om\big(\omega^\circ_{\mathcal{A}^\vee/\mathrm{Sh}_{1,1},2}, \mathrm{Lie}^\circ_{\mathcal{A}/\mathrm{Sh}_{1,1},2}\big)$, $\mathfrak{G}_1 = \mathcal{H}om\Big(\dfrac{\omega^\circ_{\mathcal{A}'^\vee/\mathrm{Sh}_{1,1},2}}{\phi_{*,2}(\omega^\circ_{\mathcal{A}^\vee/\mathrm{Sh}_{1,1},2})}, \mathrm{Lie}^\circ_{\mathcal{A}'/\mathrm{Sh}_{1,1},2}\Big)$.

Here we suppose $(\mathcal{A}, \lambda, \eta, \mathcal{A}'.\lambda', \eta', \phi)$ is the universal object over $\mathrm{Sh}_{1,1}$. The tangent sheaves $T_{\mathrm{Y}_{ij}}$ are all locally free of rank 2.

Proof. Let S be an affine noetherian $\mathbb{F}_{p^2}$-scheme and let $y = (A, \lambda, \eta, A', \lambda', \eta', \phi)$ be an S-point of Y_{00}. Put $\hat{S} = S \times_{\mathrm{Spec}(\mathbb{F}_{p^2})} \mathrm{Spec}(\mathbb{F}_{p^2}[t]/t^2)$. Then we have a natural bijection

$$\mathscr{D}\mathrm{ef}(y, \hat{S}) \cong \Gamma(S, y^* T_{\mathrm{Y}_{00}}),$$

where $\mathscr{D}\mathrm{ef}(y, \hat{S})$ is the set of deformations of y to $\hat{S}$.

By deformation theory in Subsection 5.2, it suffices to calculate $\mathscr{D}\mathrm{ef}(y, \hat{S})$.

We calculate Y_{00} as an example and the other three are similar.

Giving a point of $\mathscr{D}\mathrm{ef}(y, \hat{S})$ is equivalent to giving $\mathcal{O}_{\hat{S}}$-subbundles $\hat{\omega}^\circ_{A^\vee,i} \subseteq H_1^{\mathrm{cris}}(A/\hat{S})^\circ_i, \hat{\omega}^\circ_{A'^\vee,i} \subseteq H_1^{\mathrm{cris}}(A'/\hat{S})^\circ_i$ over $\hat{S}$ for $i = 1, 2$ such that

- $\hat{\omega}^\circ_{A^\vee/\hat{S},i}$ lifts $\omega^\circ_{A^\vee/S,i}$ and $\hat{\omega}^\circ_{A'^\vee/\hat{S},i}$ lifts $\omega^\circ_{A'^\vee/S,i}$;
- $\hat{\omega}^\circ_{A^\vee/\hat{S},1} = \mathrm{Ker}\phi_{*,1} \otimes \mathbb{F}_{p^2}[t]/t^2$;
- $\hat{\omega}^\circ_{A'^\vee/\hat{S},2} = \mathrm{Im}\phi_{*,2} \otimes \mathbb{F}_{p^2}[t]/t^2$.

Hence, one sees easily that

$$Def(y, \hat{S}) \cong \mathrm{Hom}_{\mathcal{O}_S}\left(\omega^\circ_{A'^\vee/S,1}, \mathrm{Lie}^\circ_{A'/S,1}\right) \oplus \mathrm{Hom}_{\mathcal{O}_S}\left(\omega^\circ_{A^\vee/S,2}, \mathrm{Lie}^\circ_{A/\hat{S}_{1,1},2}\right).$$

Now applying the argument above to the affine open subsets of Y_{00}, then we get

$T_{\mathrm{Y}_{00}} = \mathcal{H}om\big(\omega^\circ_{\mathcal{A}'^\vee/\mathrm{Sh}_{1,1},1}, \mathrm{Lie}^\circ_{\mathcal{A}'/\mathrm{Sh}_{1,1},1}\big) \oplus \mathcal{H}om\big(\omega^\circ_{\mathcal{A}^\vee/\mathrm{Sh}_{1,1},2}, \mathrm{Lie}^\circ_{\mathcal{A}/\mathrm{Sh}_{1,1},2}\big)$

□

Before we introduce the geometry of $\mathrm{Sh}_{1,1}(K_{\mathfrak{p}}^1)$, we need to define an action on $\mathrm{Sh}_{1,1}$ which is exactly the 'essential Frobenius' as in [28].

Definition 7.4. We define F to be the morphism on $Sh_{1,1}$ which maps its any S-point (A, λ, η) to $(A^{(p)}, \lambda', \eta')$ satisfying F acts on A as the Frobenius morphism, $F \circ \eta' = \eta$ and $F^\vee \circ \lambda' \circ F = \lambda$. Such λ' and η' exist by [19, Theorem 2, Section 23] and F on A corresponds to the Frobenius map which is purely inseparable with trivial kernel.

Proposition 7.5. *The four closed subscheme $\mathrm{Y}_{00}, \mathrm{Y}_{11}, \mathrm{Y}_{01}, \mathrm{Y}_{10}$ have the following properties:*

- Y_{00} *is a* $\mathbb{P}^1 \times \mathbb{P}^1$ *bundle over* $\mathrm{Sh}_{0,2}$. *More explicitly, if we define* C_2 *as a closed subscheme of* $\mathrm{Sh}_{1,1}(K^1_{\mathfrak{p}})$ *satisfying for any* $\mathbb{F}_{p^2}$ *scheme* S*, any* S*-point* $(A, \lambda, \eta, A', \lambda', \eta', \phi)$*, there exists* S*-points of* $\mathrm{Sh}_{0,2}$ (B_1, λ_1, η_1) *and* (B_2, λ_2, η_2) *such that there exists isogenies* $B_1 \to A \in Y_2$*,* $B_2 \to A' \in Y_1$ *and* $B_1 \in S_{\mathfrak{p}}(B_2)$*, we have the following diagram:*

$$\begin{tikzcd}
\mathrm{Y}_{00} \arrow[r, "\cong"] \arrow[d] & C_2 \arrow[r, "\mathbb{P}^1\times\mathbb{P}^1 bundle"] \arrow[d] & \mathrm{Sh}_{0,2} \arrow[d, "(1{,}S_{\mathfrak{p}})"] \\
\mathrm{Sh}_{1,1}\times\mathrm{Sh}_{1,1} & \mathrm{Y}_2\times\mathrm{Y}_1 \arrow[l, "\overleftarrow{p}_2\times\overleftarrow{p}_1"'] \arrow[r, "\overrightarrow{p}_2\times\overrightarrow{p}_1"] & \mathrm{Sh}_{0,2}\times\mathrm{Sh}_{0,2}
\end{tikzcd}$$

- Y_{11} *is a* $\mathbb{P}^1 \times \mathbb{P}^1$ *bundle over* $\mathrm{Sh}_{0,2}$. *More explicitly, if we define* C_1 *as a closed subscheme of* $\mathrm{Sh}_{1,1}(K^1_{\mathfrak{p}})$ *satisfying for any* $\mathbb{F}_{p^2}$ *scheme* S*, any* S*-point* $(A, \lambda, \eta, A', \lambda', \eta', \phi)$*, there exists a* S*-point of* $\mathrm{Sh}_{0,2}$ (B_1, λ_1, η_1) *such that there exists isogenies* $B_1 \to A \in Y_1$ *and* $B_1 \to A' \in Y_2$*, we have the following diagram:*

$$\begin{tikzcd}
\mathrm{Y}_{11} \arrow[r, "\cong"] \arrow[d] & C_1 \arrow[r, "\mathbb{P}^1\times\mathbb{P}^1 bundle"] \arrow[d] & \mathrm{Sh}_{0,2} \arrow[d, "(1{,}1)"] \\
\mathrm{Sh}_{1,1}\times\mathrm{Sh}_{1,1} & \mathrm{Y}_1\times\mathrm{Y}_2 \arrow[l, "\overleftarrow{p}_1\times\overleftarrow{p}_2"'] \arrow[r, "\overrightarrow{p}_1\times\overrightarrow{p}_2"] & \mathrm{Sh}_{0,2}\times\mathrm{Sh}_{0,2}
\end{tikzcd}$$

- Y_{01} *and* Y_{10} *are all isomorphic to* $\mathrm{Sh}_{1,1}$ *and they induce a morphism from* $\mathrm{Sh}_{1,1}^{\oplus 2} \to \mathrm{Sh}_{1,1}^{\oplus 2}$ *characterized by* $\begin{pmatrix} 1 & F \\ S_{\mathfrak{p}}^{-1}F & 1 \end{pmatrix}$*, where* $S_{\mathfrak{p}}$ *is the standard Hecke action at* p.

Proof. Firstly, we show Y_{00} and Y_{11} are all $\mathbb{P}^1 \times \mathbb{P}^1$ bundles over $\mathrm{Sh}_{0,2}$. For simplicity, we only prove it for Y_{00}. There is a natrual map from Y_{00} to $\mathrm{Sh}_{0,2}$ such that for any $\mathbb{F}_{p^2}$-scheme S, an S-point $(A, \lambda, \eta, A', \lambda', \eta', \phi)$ of Y_{00} is sent to B which is given by Proposition 5.1.1 with $\tilde{\mathcal{D}}(B)^\circ_1 = \tilde{\mathcal{D}}(A')^\circ_1$ and $\tilde{\mathcal{D}}(B)^\circ_2 = V\tilde{\mathcal{D}}(A')^\circ_2$. With a simple argument of deformation theory, we can see such a map gives Y_{00} the structure of $\mathbb{P}^1 \times \mathbb{P}^1$ bundles over $\mathrm{Sh}_{0,2}$.

Secondly, we show $\mathrm{Y}_{00} \cong C_2$. The proof of $\mathrm{Y}_{11} \cong C_1$ is also similar. In fact, given any S-point $(A, \lambda, \eta, A', \lambda', \eta', \phi)$, we can construct B, B' as follows. By Proposition 5.1.1, we get two S-points of $\mathrm{Sh}_{0,2}$ B, B' from two pairs of dieudonneé modules $(\tilde{\mathcal{D}}(A')^\circ_1, V\tilde{\mathcal{D}}(A')^\circ_1)$ and $(V\tilde{\mathcal{D}}(A)^\circ_2, p\tilde{\mathcal{D}}(A)^\circ_2)$ respectively. It is easy to check this gives us the desired isomophism. Thus we get the diagram.

Thirdly, we show Y_{01} and Y_{10} are all isomorphic to $\mathrm{Sh}_{1,1}$. Let α_1 and β_2 be morphisms from Y_{01} and Y_{10} to $\mathrm{Sh}_{1,1}$ such that for any $\mathbb{F}_{p^2}$-scheme S, an S-point $(A, \lambda, \eta, A', \lambda', \eta', \phi)$ of Y_{01} or Y_{10} is sent to A' and let β_1 and α_2 be morphisms from Y_{01} and Y_{10} to $\mathrm{Sh}_{1,1}$ such that for any $\mathbb{F}_{p^2}$-scheme S, an S-point $(A, \lambda, \eta, A', \lambda', \eta', \phi)$ of Y_{01} or Y_{10} is sent to A. In fact we have the following diagram:

$$\begin{tikzcd}
\mathrm{Y}_{01} \arrow[rr, "\alpha_1{,}\cong"] \arrow[dr, "\beta_1"'] & & \mathrm{Sh}_{1,1} & \\
& \mathrm{Sh}_{1,1} & & \mathrm{Y}_{10} \arrow[ll, "\alpha_2{,}\cong"] \arrow[ul, "\beta_2"']
\end{tikzcd}$$

, where α_1 and α_2 are all isomorphism and β_1 and β_2 are all purely inseparable morphisms which are bijective on points.

What remains to show is that $\begin{pmatrix} 1 & \beta_2 \circ \alpha_2^{-1} \\ \beta_1 \circ \alpha_1^{-1} & 1 \end{pmatrix}$ on $\mathrm{Sh}_{1,1}^{\oplus 2}$ induces $\begin{pmatrix} 1 & F \\ S_{\mathfrak{p}}^{-1}F & 1 \end{pmatrix}$.

In fact, for any $\mathbb{F}_{p^2}$-scheme S, we have the following claim:

(1) Any S point $(A, \lambda, \eta, A', \lambda', \eta', \phi)$ in Y_{01} is isomorphic to $(A, \lambda, \eta, A', \lambda', \eta', F^{-1}S_{\mathfrak{p}})$[10]
(2) Any S point $(A, \lambda, \eta, A', \lambda', \eta', \phi)$ in Y_{10} is isomorphic to $(A, \lambda, \eta, A^{(p)}, \lambda', \eta', F)$

As to (1), we find that $F \circ \phi$ maps the dieudonné of A $(\tilde{\mathcal{D}}(A)_1^\circ, \tilde{\mathcal{D}}(A)_2^\circ)$ to $(p\tilde{\mathcal{D}}(A'^{(p)})_1^\circ, p\tilde{\mathcal{D}}(A'^{(p)})_2^\circ)$, since $\phi_{1,*}\tilde{\mathcal{D}}(A)_1^\circ = F\tilde{\mathcal{D}}(A')_2^\circ$ and $\phi_{2,*}\tilde{\mathcal{D}}(A)_2^\circ = F\tilde{\mathcal{D}}(A')_1^\circ$. Moreover, we can see $F \circ \phi$ gives an isogeny from A to $A'^{(p)}$ such that $A \in S_{\mathfrak{p}}(A')$.

(2) can be proved similarly by consider the Frobenius action on A and the uniqueness of Proposition 5.1.1.

With the claim, we finish the proof of the proposition. $\square$

With Proposition 7.5, we can describe the intersections of the four closed subschemes as below:

Proposition 7.6. *For the intersections of the four closed subschemes, we have:*
(1) $\mathrm{Y}_{00} \bigcap \mathrm{Y}_{01} \cong \mathrm{Y}_1$; $\mathrm{Y}_{00} \bigcap \mathrm{Y}_{10} \cong \mathrm{Y}_2$;
(2) $\mathrm{Y}_{11} \bigcap \mathrm{Y}_{01} \cong \mathrm{Y}_2$; $\mathrm{Y}_{11} \bigcap \mathrm{Y}_{10} \cong \mathrm{Y}_1$;
(3) $\mathrm{Y}_{00} \bigcap \mathrm{Y}_{11} \cong \mathrm{T}$.

More explicitly, we can express the intersections as the following diagram:

$$\begin{tikzcd}
\mathrm{Y}_{00} = \mathbb{P}^1 \times \mathbb{P}^1/\mathrm{S} \arrow[r, dash, "\mathrm{Y}_2 = \mathbb{P}^1/\mathrm{Sh}_{0,2}"] \arrow[d, dash, "\mathrm{Y}_1 = \mathbb{P}^1/\mathrm{Sh}_{0,2}"'] \arrow[dr, phantom, "\mathrm{T}"] & \mathrm{Y}_{10} = \mathrm{Sh}_{1,1} \arrow[d, dash, "\mathrm{Y}_1 = \mathbb{P}^1/\mathrm{Sh}_{0,2}"] \\
\mathrm{Y}_{01} = \mathrm{Sh}_{1,1} \arrow[r, dash, "\mathrm{Y}_2 = \mathbb{P}^1/\mathrm{Sh}_{0,2}"'] & \mathrm{Y}_{11} = \mathbb{P}^1 \times \mathbb{P}^1/\mathrm{S}
\end{tikzcd}$$

Proof. The proof of (1) and (2) are similar. For simplicity, we only give the proof of $\mathrm{Y}_{00} \bigcap \mathrm{Y}_{01} \cong \mathrm{Y}_{01}$ and (3).

First, we show $\mathrm{Y}_{00} \bigcap \mathrm{Y}_{01} \cong \mathrm{Y}_{01}$. We define a morphism $\alpha : \mathrm{Y}_{00} \bigcap \mathrm{Y}_{01} \to \mathrm{Y}_1$ as following: Let k be a perfect field containinng $\mathbb{F}_{p^2}$, suppose $y = (A, \lambda, \eta, A', \lambda', \eta', \phi)$ is a k-point of $\mathrm{Y}_{00} \bigcap \mathrm{Y}_{01}$. We let $\tilde{\mathcal{E}}_1 = \tilde{\mathcal{D}}(A')_1^\circ$ and $\tilde{\mathcal{E}}_2 = V\tilde{\mathcal{D}}(A')_1^\circ \subseteq \tilde{\mathcal{D}}(A')_2^\circ$. Then it can be checked that $F(\tilde{\mathcal{E}}_i) \subseteq \tilde{\mathcal{E}}_{3-i}$ and $V(\tilde{\mathcal{E}}_i) \subseteq \tilde{\mathcal{E}}_{3-i}$ for $i = 1, 2$. Applying PropositionProposition 5.1.1 with $\tilde{\mathcal{E}}_1, \tilde{\mathcal{E}}_2$, we get a triple (B, λ'', η'') and an $\mathcal{O}_D$-equivariant isogeny $\phi' : B \to A'$, where B is an abelian variety over k with an action of $\mathcal{O}_D$, λ'' is a prime-to-p polarization on B, and η'' is a prime-to-p level structure on B respectively, such that $\phi'^\vee \circ \lambda' \circ \phi' = p\lambda''$. Moreover we have $\phi' \circ \eta'' = \eta'$. Moreover, the dimension formula (5.1.3) implies that $\omega^\circ_{B^\vee/k,1}$ has dimension 0, and $\omega^\circ_{B^\vee/k,2}$ has dimension n. Therefore, (B, λ'', η'') is a point of $\mathrm{Sh}_{0,n}$. and $(A', \lambda', \eta', B, \lambda'', \eta'', \phi') \in \mathrm{Y}_1$. In this way, we define $\alpha(y) = (A', \lambda', \eta', B, \lambda'', \eta'', \phi')$. Now we construct β and check it is the converse of α. Suppose $y' = (A', \lambda', \eta', B, \lambda'', \eta''.\phi')$ is a k-point of Y_1. Let $\tilde{\mathcal{E}}_1 = F\tilde{\mathcal{D}}(A')_2^\circ$ and $\tilde{\mathcal{E}}_2 = F\tilde{\mathcal{D}}(A')_1^\circ \subseteq \tilde{\mathcal{D}}(A')_2^\circ$. Then it can be checked that $F(\tilde{\mathcal{E}}_i) \subseteq \tilde{\mathcal{E}}_{3-i}$ and $V(\tilde{\mathcal{E}}_i) \subseteq \tilde{\mathcal{E}}_{3-i}$ for $i = 1, 2$. Applying PropositionProposition 5.1.1 with $\tilde{\mathcal{E}}_1, \tilde{\mathcal{E}}_2$, we get a triple (A, λ, η) and an $\mathcal{O}_D$-equivariant isogeny $\phi : A \to A'$, where A is an abelian variety over k with an action of $\mathcal{O}_D$, λ is a prime-to-p polarization on B, and η is a prime-to-p level structure on A respectively, such that $\phi^\vee \circ \lambda' \circ \phi = p\lambda$. Moreover we have $\phi' \circ \eta = \eta'$. Moreover, the dimension formula (5.1.3) implies that $\omega^\circ_{A^\vee/k,1}$ has dimension 1, and $\omega^\circ_{A^\vee/k,2}$ has dimension 1. Therefore, (A, λ, η) is a point of $\mathrm{Sh}_{1,1}$. and $(A, \lambda, \eta, A', \lambda', \eta', \phi) \in \mathrm{Y}_{00} \bigcap \mathrm{Y}_{01}$. In this way, we define $\beta(y') = (A, \lambda, \eta, A', \lambda', \eta', \phi')$. It can be checked directly α and β are inverse of each other on points.

To show α gives isomorphism between $\mathrm{Y}_{00} \bigcap \mathrm{Y}_{01}$ and Y_1, it suffices to check α induces an isomorphism on tangent spaces as the two schemes are smooth. Let $y' = (A', \lambda', \eta', B, \lambda'', \eta''.\phi') \in \mathrm{Y}_1(k)$ be

[10]It may not be valid to write F^{-1} as an isogeny. It just means after the Hecke action $S_{\mathfrak{p}}$ acts on A, it is isomorphic to the image of A' under F.

a closed point. Consider the infinitesimal deformation over $k[\epsilon]=k[t]/t^2$. Note that (B,λ'',η'') has a unique deformation to $k[\epsilon]$, namely the trivial deformation. By the Serre-Tate and Grothendieck-Messing deformation theory in Subsection 5.2, giving a deformation $(\hat{A}',\hat{\lambda}',\eta')$ of (A',λ',η') to $k[\epsilon]$ is equivalent to giving free $k[\epsilon]$-submodules $\hat{\omega}^\circ_{A'^\vee,i}\subseteq H_1^{cris}(A'/k[\epsilon])^\circ_i$ for $i=1,2$ which lift $\omega^\circ_{A'^\vee/k,i}$. The isogeny ϕ' and the polarization λ' deform to an isogeny $\hat{\phi}:\hat{B}\to\hat{A}'$ and a polarization $\hat{\lambda}:\hat{A}'^\vee\to\hat{A}'$, necessarily unique if they exist. In the other way, we see for any point $y=(A,\lambda,\eta,A',\lambda',\eta',\phi)\in \mathrm{Y}_{00}\bigcap\mathrm{Y}_{01}(k)$, the deformation is completely determined by the deformation of (A',λ',η'). Hence α induces a bijection on tangent spaces. Hence we finish the proof of $\mathrm{Y}_{00}\bigcap\mathrm{Y}_{01}\cong\mathrm{Y}_1$.

Now we give the proof of (3). Similar as above, we construct a morpshim $\alpha':\mathrm{Y}_{00}\bigcap\mathrm{Y}_{11}$ as follows: Let k be a perfect field containing $\mathbb{F}_{p^2}$, suppose $y=(A,\lambda,\eta,A',\lambda',\eta',\phi)\in\mathrm{Y}_{00}\bigcap\mathrm{Y}_{11}(k)$. We let $\tilde{\mathcal{E}}_1=V\tilde{\mathcal{D}}(A)^\circ_2,\tilde{\mathcal{E}}_1'=\tilde{\mathcal{D}}(A')^\circ_1$ and $\tilde{\mathcal{E}}_2=p\tilde{\mathcal{D}}(A)^\circ_2,\tilde{\mathcal{E}}_2'=V\tilde{\mathcal{D}}(A')^\circ_1$. Applying Proposition 5.1.1 with $\tilde{\mathcal{E}}_1,\tilde{\mathcal{E}}_2$ and $\tilde{\mathcal{E}}_1',\tilde{\mathcal{E}}_2'$ repectively, we get two triples $(B_1,\lambda_1,\eta_1),(B_2,\lambda_2,\eta_2)$ and two $\mathcal{O}_D$-equivariant isogenies $\phi_1:B_1\to A,\phi_2:B_2\to A'$ where B_1,B_2 are two abelian varieties over k with an action of $\mathcal{O}_D$, λ_1,λ_2 are prime-to-p polarizations on B_1,B_2, and η_1,η_2 are two prime-to-p level structures on B_1,B_2 respectively, such that $\phi_1^\vee\circ\lambda\circ\phi_1=p\lambda_1$ and $\phi_2^\vee\circ\lambda'\circ\phi_2=p\lambda_2$. Moreover we have $\phi_1\circ\eta_1=\eta$ and $\phi_2\circ\eta_2=\eta'$. Moreover, the dimension formula (5.1.3) implies that $\omega^\circ_{B_1^\vee/k,1}$ and $\omega^\circ_{B_2^\vee/k,1}$ have dimension 0, and $\omega^\circ_{B_1^\vee/k,2}$ and $\omega^\circ_{B_2^\vee/k,2}$ have dimension n. Therefore, (B_1,λ_1,η_1) and (B_2,λ_2,η_2) is two points of $\mathrm{Sh}_{0,n}$. Then $(A,\lambda,\eta,B_1,\lambda_1,\eta_1,\phi_1)\in\mathrm{Y}_2$ and $(A',\lambda',\eta',B_2,\lambda_2,\eta_2,\phi_2)\in\mathrm{Y}_2$. In this way, we define $\alpha'(y)=(B_1,\lambda_1,\eta_1,B_2,\lambda_2,\eta_2,\phi_2^{-1}\circ\phi\circ\phi_1)\in\mathrm{T}$. Using a similar argument as above, we can show α' is an isomorphism. This concludes the proof. $\square$

It can be checked directly that the intersection $\mathrm{Y}_{00}\bigcap\mathrm{Y}_{01}$, $\mathrm{Y}_{00}\bigcap\mathrm{Y}_{10}$, $\mathrm{Y}_{11}\bigcap\mathrm{Y}_{01}$, $\mathrm{Y}_{11}\bigcap\mathrm{Y}_{10}$ are all of dimension 1, but $\mathrm{Y}_{00}\bigcap\mathrm{Y}_{11}=\mathrm{Y}_{10}\bigcap\mathrm{Y}_{01}$ are of dimension 0. To prove the Ihara lemma, we hope to get a strictly semi-stable scheme which is defined as below and use weight spectral sequence [23].

Following [9], we get:

Proposition 7.7. *Let x be a closed geometric point of $Sh_{1,1}(K^1_{\mathfrak{p}})$, then the completion of the strictly henselization of the local ring of $Sh_{1,1}(K^1_{\mathfrak{p}})$ is*

$$W(\overline{\mathbb{F}}_p)[\![X_1,Y_1,X_2,Y_2]\!]/(X_1Y_1-p,X_2Y_2-p).$$

Without loss of generality, we only need to consider the case $\operatorname{Spec}\mathbb{Z}_p[\![X_1,Y_1,X_2,Y_2]\!]/(X_1Y_1-p,X_2Y_2-p)$. Blowing up at (X_1,X_2), we get $\operatorname{Proj}\mathbb{Z}_p[\![X_1,Y_1,X_2,Y_2]\!][T_1,T_2]/(X_1Y_1-p,X_2Y_2-p,X_1T_1-X_2T_2,Y_1T_2-Y_2T_1)$. It is easy to see it is strictly semistable as in the following diagram:

$$\begin{tikzcd} \mathbb{A}^2 \arrow[r, dash, "\mathbb{P}^1"] \arrow[d, dash, "\mathbb{P}^1"'] & \mathbb{A}^2 \arrow[d, dash, "\mathbb{P}^1"] \\ \mathbb{A}^2 \arrow[r, dash, "\mathbb{P}^1"'] & \mathbb{A}^2 \end{tikzcd} \Longrightarrow \begin{tikzcd} Bl_{pt}\mathbb{A}^2 \arrow[r, dash, "\mathbb{P}^1"] \arrow[d, dash, "\mathbb{P}^1"'] \arrow[dr, dash, "\mathbb{P}^1" description] & \mathbb{A}^2 \arrow[d, dash, "\mathbb{P}^1"] \\ \mathbb{A}^2 \arrow[r, dash, "\mathbb{P}^1"'] & Bl_{pt}\mathbb{A}^2 \end{tikzcd}$$

.

The four vertices of each square denotes the closed subschemes determined by the ideal sheaves $(X_1,X_2),(X_1,Y_2),(Y_1,X_2),(Y_1,Y_2)$. And each edge means the intersection of two vertices adjacent to it.

As illustrated in this toy model, we can get a strictly semi-stable model by blowing up the integral model $Sh_{1,1}(K^1_{\mathfrak{p}})$ at the closed subscheme corresponds to Y_{00} in the special fiber. We denote it (resp. its special fiber) by $\widetilde{\mathrm{Sh}}_{1,1}(K^1_{\mathfrak{p}})$(resp. $\widetilde{\mathrm{Sh}}_{1,1}(K^1_{\mathfrak{p}})$). We have $\widetilde{\mathrm{Sh}}_{1,1}(K^1_{\mathfrak{p}})=\widetilde{\mathrm{Y}}_{00}\bigcup\mathrm{Y}_{01}\bigcup\mathrm{Y}_{10}\bigcup\widetilde{\mathrm{Y}}_{11}$. The special fiber after blowing up can be decribed by the following diagram:

$$\begin{tikzcd}
\widetilde{\mathrm{Y}}_{00}=\widetilde{\mathbb{P}^1\times\mathbb{P}^1}/\mathrm{Sh}_{0,2} \arrow[rr, dash, "\mathrm{Y}_2=\mathbb{P}^1/\mathrm{Sh}_{0,2}"] \arrow[d, dash, "\mathrm{Y}_1=\mathbb{P}^1/\mathrm{Sh}_{0,2}"'] \arrow[drr, dash, "\mathrm{T}\times\mathbb{P}^1" description] & & \mathrm{Y}_{10}=\mathrm{Sh}_{1,1} \arrow[d, dash, "\mathrm{Y}_1=\mathbb{P}^1/\mathrm{Sh}_{0,2}"] \\
\mathrm{Y}_{01}=\mathrm{Sh}_{1,1} \arrow[rr, dash, "\mathrm{Y}_2=\mathbb{P}^1/\mathrm{Sh}_{0,2}"'] & & \widetilde{\mathrm{Y}}_{11}=\widetilde{\mathbb{P}^1\times\mathbb{P}^1}/\mathrm{Sh}_{0,2}
\end{tikzcd}$$

Applying Proposition B.4 to $\mathcal{S}h_{1,n-1}(K^1_{\mathfrak{p}})$, we can get a weight spectral sequence.
we can write E_1-page of the weight spectral sequence as the following diagram:

$$\begin{array}{ccccccccc}
\mathrm{H}^0_{\acute{e}t}(Y^{(2)}_{\overline{\mathbb{F}}_p})(-2) & \longrightarrow & \mathrm{H}^2_{\acute{e}t}(Y^{(1)}_{\overline{\mathbb{F}}_p})(-1) & \longrightarrow & \mathrm{H}^4_{\acute{e}t}(Y^{(0)}_{\overline{\mathbb{F}}_p}) & & 0 & & 0 \\
0 & & \mathrm{H}^1_{\acute{e}t}(Y^{(1)}_{\overline{\mathbb{F}}_p})(-1) & \longrightarrow & \mathrm{H}^3_{\acute{e}t}(Y^{(0)}_{\overline{\mathbb{F}}_p}) & & 0 & & 0 \\
0 & & \mathrm{H}^0_{\acute{e}t}(Y^{(1)}_{\overline{\mathbb{F}}_p})(-1) & \longrightarrow & \mathrm{H}^2_{\acute{e}t}(Y^{(0)}_{\overline{\mathbb{F}}_p})\oplus\mathrm{H}^0_{\acute{e}t}(Y^{(2)}_{\overline{\mathbb{F}}_p})(-1) & \longrightarrow & \mathrm{H}^2_{\acute{e}t}(Y^{(1)}_{\overline{\mathbb{F}}_p}) & & 0 \\
0 & & 0 & & \mathrm{H}^1_{\acute{e}t}(Y^{(0)}_{\overline{\mathbb{F}}_p}) & \longrightarrow & \mathrm{H}^1_{\acute{e}t}(Y^{(1)}_{\overline{\mathbb{F}}_p}) & & 0 \\
0 & & 0 & & \mathrm{H}^0_{\acute{e}t}(Y^{(0)}_{\overline{\mathbb{F}}_p}) & \longrightarrow & \mathrm{H}^0_{\acute{e}t}(Y^{(1)}_{\overline{\mathbb{F}}_p}) & \longrightarrow & \mathrm{H}^0_{\acute{e}t}(Y^{(2)}_{\overline{\mathbb{F}}_p})
\end{array}$$

with the middle term of the last row at index $(0,0)$ and $E_1^{p,q}=0$ for $q>2$ or $q<-2$.

It is easy to see that at the index $(1,0)$, the spectral sequence degenerates at E_2 page. Hence the short sequence at the bottom of the E_1-page is exact after localizing at a non-Eisenstein and 'generic 'm as in 1.5. Now we give a proof of theorem 6.2(1) by analyzing such a short exact sequence:

Proof of theorem 6.2(1). After localization, the short exact sequence above is equivalent to

$$H^0(\overline{\mathrm{Sh}}_{0,2})^{\oplus 2}_m \xrightarrow{\alpha} H^0(\overline{\mathrm{Sh}}_{0,2})^{\oplus 4}_m \oplus H^0(\overline{\mathrm{T}})_m \xrightarrow{\beta} H^0(\overline{\mathrm{T}})^{\oplus 2}_m$$

with $\alpha^t=\begin{pmatrix} -S_{\mathfrak{p}} & -1 & 0 & 0 & -\overleftarrow{p}^* \\ 0 & 0 & 1 & 1 & \overrightarrow{p}^* \end{pmatrix}$ and $\beta=\begin{pmatrix} \overleftarrow{p}^* & 0 & \overrightarrow{p}^* & 0 & -\mathrm{T}S_{\mathfrak{p}} \\ 0 & \overrightarrow{p} & 0 & \overleftarrow{p}^* & -1 \end{pmatrix}$, where we use α^t to express the transverse of α. We use $\overline{T}$ to express the geometric special fiber of T. Since $\mathrm{Im}\alpha=\mathrm{Ker}\beta$, we get for any $(x,y,z,w,r)\in H^0(\overline{\mathrm{Sh}}_{0,2})^{\oplus 4}_m\oplus H^0(\overline{\mathrm{T}})_m$ satisfying $\overleftarrow{p}^* x+\overrightarrow{p}^* z-\mathrm{T}S_{\mathfrak{p}}r=\overleftarrow{p}^* y+\overrightarrow{p}^* w-r=0$, i.e $(x,y,z,w,r)\in\mathrm{Ker}\beta$, we get that there exists $(s,t)\in H^0(\overline{\mathrm{Sh}}_{0,2})^{\oplus 2}_m$ such that $(x,y,z,w,r)=(-S_{\mathfrak{p}}s,-s,t,t,,-\overleftarrow{p}^* s+\overrightarrow{p}^* t)$. Therefore, $x=S_{\mathfrak{p}}y$ and $z=w$. Since $S_{\mathfrak{p}}$ is an isormorphism as a morphism, we get if $-\overleftarrow{p}^* s+\overrightarrow{p}^* t=0$, then $s=t=0$. Hence the map $H^0(\overline{\mathrm{Sh}}_{0,2})^{\oplus 2}_m \xrightarrow{(\overleftarrow{p}^*,\overrightarrow{p}^*)} H^0(\overline{\mathrm{T}})_m$ is injective. By Poincaré duality, we get the definite Ihara lemma. □

The proof of Indefinite Ihara lemma from the definite Ihara lemma makes no difference from the case $n\geq 3$, so we omit the proof of Indefinite Ihara lemma here.

8. Stratification of $\mathrm{Sh}_{1,n-1}$

In this section, we analyze the Ekedahl-Oort stratification and the Newton stratification of $\mathrm{Sh}_{1,n-1}$. For general theory of these two stratifications of unitary Shimura varieties, we refer to [27] as a reference.

Firstly, we analyze the Ekedahl-Oort stratification of $\mathrm{Sh}_{1,n-1}$.

We take $G = \mathrm{Res}_{\mathbb{Q}_{p^2}/\mathbb{Q}} \mathrm{GL}_n \times \mathbb{G}_m$. Then G has Weyl group $W = S_n \times S_n$ and the p-Frobenius morphism of G induces a map $\Psi : W \to W$ by switching the two components of the Weyl group.

Let $\underline{A} = (A, \lambda, \eta) \in \mathrm{Sh}_{1,n-1}(\overline{\mathbb{F}}_p)$ and let $\mathfrak{D}^\circ = \mathfrak{D}_1(A)^\circ \bigoplus \mathfrak{D}_2(A)^\circ$ be the summation of the Dieudonné module of $A[p]$ of rank $2n$. Then there is a canonical action of G on $\mathfrak{D}^\circ$. By applying F, V^{-1} to $(0) \subseteq \mathfrak{D}^\circ$ until it stabilizes, we obtain an F, V^{-1}-stable flag of $\mathfrak{D}^\circ$,

$$\mathcal{C}_\bullet : \mathcal{C}_0 = (0) \subseteq \cdots \subseteq \mathcal{C}_n = \mathfrak{D}^\circ[V] = F(\mathfrak{D}^\circ) \subseteq \cdots \subseteq \mathcal{C}_{2n} = \mathfrak{D}^\circ,$$

where $\dim \mathcal{C}_i = i$, called the canonical flag of $\underline{A}$. More details on the canonical filtration can be found in [18, Section 2.5,4.4,6.3].

Let any extension of $\mathcal{C}_\bullet$ to a complete $\mathcal{O}_K$-invariant symplectic flag of $\mathfrak{D}^\circ$ be called a conjugate flag of $\underline{A}$. Let $\mathcal{C}_\bullet$ denote a conjugate filtration of $\underline{A}$ and let $Q = \mathit{Stab}(\mathcal{C}_\bullet)$. It is easy to see that Q is a Borel group as $\mathcal{C}_\bullet$ is a complete filtration.

Let J be the type of $P = \mathit{Stab}(\mathfrak{D}^\circ[F] = V\mathfrak{D}^\circ \subseteq \mathfrak{D}^\circ)$ which is independent on the choice of $\underline{A}$ but only dependent on the moduli problem of $\mathrm{Sh}_{1,n-1}$. Now for each $\underline{A}$, we get an element $\omega(\underline{(}A))$ in ${}^JW = W_J\ W$ which represents the relative position of P and Q defined in Appendix A. Since $\mathfrak{D}(A)_1^\circ / V\tilde{\mathcal{D}}(A)_1^\circ$ has rank $n-1$ over $\overline{\mathbb{F}}_p$ and $\mathfrak{D}(A)_2^\circ / V\mathfrak{D}(A)_2^\circ$ has rank 1 over $\overline{\mathbb{F}}_p$, we get $W_J = S_n \backslash \{s_1\} \times S_n \backslash \{s_{n-1}\}$ where $s_1 = (1,2)$ and $s_{n-1} = (n-1, n)$. Any $(\omega_1, \omega_2) \in {}^JW$ satisfies $w_1^{-1}(2) < w_1^{-1}(3) \cdots < w_1^{-1}(n)$ and $w_2^{-1}(1) < w_2^{-1}(2) < \cdots < w_2^{-1}(n-1)$.

There is a partial order on ${}^JW$, denoted by $\leqslant_\Psi$: For any $(\omega_1, \omega_2), (\omega_1', \omega_2') \in {}^JW$, we say $(\omega_1', \omega_2') \leqslant_\Psi (\omega_1, \omega_2)$ if and only if there exists $y \in W_J$ such that

$$y(\omega_1', \omega_2')x\Psi(y^{-1})x^{-1} \leqslant (\omega_1, \omega_2),$$

where $\leqslant$ is the Bruhat order and $x = \omega_0 \omega_{0,\Psi(J)}$ with ω_0 and $\omega_{0,\Psi(J)}$ to be the element of maximal length in W and W_J.

Definition 8.1. In $\mathrm{Sh}_{1,n-1}$, the Ekedahl-Oort stratum associated to $\omega \in^J W$ is a locally closed reduced subscheme V^ω with geometric points given by

$$V^\omega := \{\underline{A} \in \mathrm{Sh}_{1,n-1} | \omega(\underline{A}) = \omega\}.$$

By [27, Theorem 2,3], we get the following theorem:

Theorem 8.2. (1) *For all $\omega \in^J W$, the Ekedahl-Oort stratum V^ω is non-empty and equidimensional of dimension $\ell(\omega)$, which is the length of $\omega \in W$ and is equal to $\ell(\omega_1) + \ell(\omega_2)$ if $\omega = (\omega_1, \omega_2)$.*

(2) *The Ekedahl-Oort strata are non-singular and quasi-affine.*

(3) *The closure of an Ekedahl-Oort stratum is a union of Ekedahl-Oort strata with respect to the partial order $\leqslant_\Psi$ on ${}^JW$. That is,*

$$\overline{V}^\omega = \coprod_{\omega' \leqslant_\Psi \omega} V^{\omega'}.$$

By convention, we will call the minimal Ekedahl-Oort stratum the core locus and the maximal Ekedahl-Oort stratum the μ-ordinary locus.

Based on Theorem 8.2, we have the following proposition:

Proposition 8.3. (1) *There are n^2 Ekedahl-Oort strata in $\mathrm{Sh}_{1,n-1}$ which is characterized by $\omega_1^{-1}(1)$ and $\omega_2^{-1}(n)$ for $(\omega_1, \omega_2) \in {}^JW$. From now on, we use the notation $(\omega_1, \omega_2) = (a, b)$ to mean $\omega_1^{-1}(1) = a$ and $\omega_2^{-1}(n) = b$.*

(2) *The Ekedahl-Oort stratum corresponds to* $(\omega_1, \omega_2) \in {}^JW$ *has dimension* $d = \omega_1^{-1}(1) - \omega_2^{-1}(n) + n - 1$.

(3) *Suppose* $(\omega_1, \omega_2) = (a, b)$ *and* $(\omega_1', \omega_2') = (a', b')$*, then* $V^{\omega'} \subseteq \overline{V}^{\omega}$ *if and only if it satisfies one of the following conditions:*
   - $a \geq a'$ *and* $b \leq b'$;
   - $a' - b' \leq 0 \leq a - b$.

Proof. Since any $(\omega_1, \omega_2) \in {}^JW$ satisfies $w_1^{-1}(2) < w_1^{-1}(3) \cdots < w_1^{-1}(n)$ and $w_2^{-1}(1) < w_2^{-1}(2) < \cdots < w_2^{-1}(n-1)$, we get (1) easily.

For (2), we notice that $\ell(\omega_1) = \sum_{i=1}^{n-1} \omega_1^{-1}(i) - i$ and $\ell(\omega_2) = \omega_2^{-1}(1) - 1$. Thus we can get (2) directly.

For (3), even though we can compute it with Theorem 8.2 directly, we will prove it with Newton stratification and some more detailed analysis of Ekedahl-Oort strata in the supersingular locus of $\mathrm{Sh}_{1,n-1}$. Hence we delay the proof later. □

By [18, Theorem 4.7], we get the following explicit description of the action of F, V on $\mathfrak{D}^\circ$:

Proposition 8.4. *For every* $(\omega_1, \omega_2) \in {}^JW$ *and each* $\underline{A} \in V^{(\omega_1, \omega_2)}$*, we get there is a model for* $\mathfrak{D}^\circ$ *such that each* $\mathfrak{D}_i^\circ$ *has a basis* $e_{i,1} \dots e_{i,n}$ *for* $i = 1, 2$ *and* F, V *act on* $\mathfrak{D}^\circ$ *as follows:*

$$F(e_{i,j}) = \begin{cases} 0 & \omega_i(j) \leq f(i) \\ e_{i+1,a} & \omega_i(j) = f(i) + a \end{cases}$$

and

$$V(e_{i+1,j}) = \begin{cases} 0 & j \leq n - f(i) \\ e_{i+1,a} & j = n - f(i) + \omega_i(a) \end{cases}$$

where $i \in \mathbb{Z}/2\mathbb{Z}$ *and* $f(1) = 1, f(2) = n - 1$.

Proof. It can be checked directly by taking $\mathcal{F} = \{1, 2\}$ and $f(1) = 1, f(2) = n - 1$ in [18, Theorem 4.7]. □

With Proposition 8.4, we get the following proposition:

Proposition 8.5. *For* $(\omega_1, \omega_2) = (a, b) \in {}^JW$*, we have* $V^{(\omega_1, \omega_2)} \in \overleftarrow{p}_{n+1-i}(\mathrm{Y}_{n+1-i})$ *if and only if* $a \leq i \leq b$*. In particular, we have* $V^{(\omega_1, \omega_2)} \in \mathrm{Sh}_{1,n-1}^{ss}$ *if and only if* $a \leq b$.

Proof. The "if" part can be obtained by direct calculation with Proposition 8.4.

Let k be a finite extension of $\mathbb{F}_{p^2}$. By Proposition 8.4, we get for any $\underline{A} = (A, \lambda, \eta) \in V^{(\omega_1, \omega_2)}(k)$, there is a basis of $\mathfrak{D}(A)_1^\circ \oplus \mathfrak{D}(A)_2^\circ$, denoted by $\{e_{i,j} \mid i = 1, 2; 1 \leq j \leq n\}$ such that F, V act on $\mathfrak{D}(A)_1^\circ \oplus \mathfrak{D}(A)_2^\circ$ by

$$F(e_{1,i}) = \begin{cases} e_{2,i} & \text{if } 1 \leq i \leq a - 1; \\ 0 & \text{if } i = a; \\ e_{2,i-1} & \text{if } i \geq a + 1. \end{cases} \quad F(e_{2,i}) = \begin{cases} 0 & \text{if } 1 \leq i \leq b - 1; \\ e_{1,1} & \text{if } i = b; \\ 0 & \text{if } i \geq b + 1. \end{cases}$$

$$V(e_{1,i}) = \begin{cases} 0 & \text{if } i = 1; \\ e_{2,i-1} & \text{if } 2 \leq i \leq b; \\ e_{2,i} & \text{if } b + 1 \leq i \leq n; \end{cases} \quad V(e_{2,i}) = \begin{cases} 0 & \text{if } 1 \leq i \leq a - 1; \\ 0 & \text{if } a \leq i \leq n - 1; \\ e_{1,a} & \text{if } i = n. \end{cases}$$

Therefore, for any $a \leq i \leq b$, we let $\mathfrak{E}_1 = k\{e_{1,1}, e_{1,2}, \dots, e_{1,i}\}$ and $\mathfrak{E}_2 = k\{e_{2,1}, \dots, e_{2,i-1}\}$. Then it can be checked that $F(\mathfrak{E}_i) \subseteq \mathfrak{E}_{3-i}$ and $V(\mathfrak{E}_i) \subseteq \mathfrak{E}_{3-i}$ for $i = 1, 2$. Applying Proposition 5.1.1 and the remark below with $\mathfrak{E}_1, \mathfrak{E}_2$, we get a triple (B, λ', η') and an $\mathcal{O}_D$-equivariant isogeny $\phi : B \to A$, where B is an abelian variety over k with an action of $\mathcal{O}_D$, λ' is a prime-to-p polarization on B, and η' is a prime-to-p level structure on B respectively, such that $\phi^\vee \circ \lambda \circ \phi = p\lambda'$. Moreover we have $\phi \circ \eta' = \eta$. Moreover, the dimension formula (5.1.2) implies that $\omega_{B^\vee/k,1}^\circ$ has dimension 0, and $\omega_{B^\vee/k,2}^\circ$

has dimension n. Therefore, (B, λ', η') is a point of $\mathrm{Sh}_{0,n}$. and $(A, \lambda, \eta, B, \lambda', \eta', \phi) \in \mathrm{Y}_{n+1-i}$.. This finish the proof of the 'if' part.

Conversely, by [11, Proposition 6.4], we get the intersection of Y_i and Y_j has dimension at most of $n+i-j$ for $1 \leq i \leq j \leq n$. This restricts the 'only if' part must hold. $\square$

Now we begin to analyze the Newton stratification of $\mathrm{Sh}_{1,n-1}$.

Via [21, Theorem 3.8], the Newton stratification of $\mathrm{Sh}_{1,n-1}$ coincides with the Newton polygon stratification $\mathrm{Sh}_{1,n-1}$. So we study the Newton polygon stratification below and call it the Newton stratification of $\mathrm{Sh}_{1,n-1}$.

Definition 8.6. Let k be a perfect field of characteristic p and $W(k)$ be the witt vector ring corresponding to k. We say a pair (P, π) is a $Q(k) = Frac(W(k))$-isocrystal if P is a finite-dimensional $Q(k)$-vector space together with a σ-linear automorphism F. In particular, for any abelian variety A over k, the p-divisible group $A[p^\infty]$ gives an isocrystal $(\tilde{\mathcal{D}}^\circ \otimes_{W(k)} Q(k) = (\tilde{\mathcal{D}}(A)_1^\circ \oplus \tilde{\mathcal{D}}(A)_2^\circ) \otimes_{W(k)} Q(k), F \otimes 1)$.

Following [5], for each field k of finite extension over $\mathbb{F}_p$ and abelian variety A over k, the isocrystal $(\tilde{\mathcal{D}}^\circ \otimes_{W(k)} Q(k) = (\tilde{\mathcal{D}}(A)_1^\circ \oplus \tilde{\mathcal{D}}(A)_2^\circ) \otimes_{W(k)} Q(k), F \otimes 1)$ has a unique decomposition such that $\tilde{\mathcal{D}}^\circ \otimes_{W(k)} Q(k) = \bigoplus_{i=1}^{r} N(\lambda_i)$, where $0 \leq \lambda_1 < \cdots < \lambda_r$ are the slopes corresponds to the isocrystal and $N(\lambda_i)$ is the simplest isocrystal of slope λ_i. The multiplicity of each λ_i is equal to the dimension of $N(\lambda_i)$ over $Q(k)$. Considering the multiplicity, we use $\lambda_\bullet$ to denote a sequence of slopes $0 \leq \lambda_1 \leq \cdots \leq \lambda_{2n}$ and $\lambda_\bullet(A)$ to denote the sequence of slopes of the isocrystal corresponds to A. Each slope $\lambda_\bullet$ defines a Newton polygon by connecting the points in x-y-plane with coordinates (i, λ_i) by line segments. Now we give the definition of Newton stratification:

Definition 8.7. For every sequence of slopes $\lambda_\bullet$, the Newton stratum associated to it is a locally closed reduced subscheme of $\mathrm{Sh}_{1,n-1}$, denoted by $N^{\lambda_\bullet}$ such that

$$N^{\lambda_\bullet} = \{\underline{A} \in \mathrm{Sh}_{1,n-1} \mid \lambda_\bullet(\underline{A}) = \lambda_\bullet\}.$$

For two different Netwon strata determined by $\lambda_\bullet$ and $\lambda'_\bullet$, we have $N^{\lambda'_\bullet} \subseteq \overline{N}^{\lambda_\bullet}$ if and only if the Newton polygon defined by $\lambda_\bullet$ is below the one of $\lambda'_\bullet$.

Proposition 8.8. *There are* $\frac{n(n-1)}{2}+1$ *Newton strata in* $\mathrm{Sh}_{1,n-1}$*. More explicitly, we have* $\mathrm{Sh}^{ss}_{1,n-1}$ *corresponds to the Newton stratum with all the slopes to be* $\frac{1}{2}$*. The others can be expressed as* $N^{a,b}$*, the sequence of slopes of which is* $((\frac{a-1}{2a})^{2a}, (\frac{1}{2})^{2b-2a}, (\frac{n-b}{2n-2b})^{2n-2b})$ *with* $1 \leq a, b \leq n-1$*. The dimension of* $N_{a,b}$ *is* $b-a+n$*. Moreover, we have the Ekedahl-Oort strata contained in the nonsupersingular locus are in bijection with the Newton strata contained in the nonsupersingular locus, given by* $N^{a,b} \cong V^{(\omega_1,\omega_2)}$ *with* $(\omega_1, \omega_2) = (b, a)$*.*

Proof. Calculate by definition as in [27], we have each slope must have even multiplicities and the μ-ordinary locus corresponds to the sequence of slopes $(0^2, (\frac{1}{2})^{2n-4}, 1^2)$. This forces admissible sequences of slopes corresponds to nonsupersingular locus can only be $((\frac{a-1}{2a})^{2a}, (\frac{1}{2})^{2b-2a}, (\frac{n-b+1}{2n-2b})^{2n-2b})$ with $1 \leq a \leq b \leq n-1$. Thus we get there can only be $\frac{n(n-1)}{2}+1$ Newton strata in $\mathrm{Sh}_{1,n-1}$. It is well known that $\mathrm{Sh}^{ss}_{1,n-1}$ corresponds to the Newton stratum with all the slopes to be $\frac{1}{2}$.

Assume $1 \leq a \leq b \leq n-1$. We want to show there is a bijection between the Ekedahl-Oort strata contained in the nonsupersingular locus and the Newton strata contained in the nonsupersingular locus, given by $N^{a,b} \cong V^{(\omega_1,\omega_2)}$ with $(\omega_1, \omega_2) = (b, a)$. Suppose $\{e_{i,j} \mid i = 1, 2; 1 \leq j \leq n\}$ is a basis as in Proposition 8.5. Take $\tilde{\mathcal{D}}'^\circ = W(\overline{\mathbb{F}}_p)\{e_{i,j} \mid i = 1, 2; 1 \leq j \leq n\}$ such that

$$F(e_{1,i}) = \begin{cases} e_{2,i} & \text{if } 1 \leq i \leq b-1; \\ pe_{2,n} & \text{if } i = b; \\ e_{2,i-1} & \text{if } i \geq b+1. \end{cases} \qquad F(e_{2,i}) = \begin{cases} pe_{1,i+1} & \text{if } 1 \leq i \leq a-1; \\ e_{1,1} & \text{if } i = a; \\ pe_{1,i} & \text{if } i \geq a+1. \end{cases}$$

$$V(e_{1,i}) = \begin{cases} pe_{2,a} & \text{if } i=1; \\ e_{2,i-1} & \text{if } 2 \le i \le a; \\ e_{2,i} & \text{if } a+1 \le i \le n; \end{cases} \quad V(e_{2,i}) = \begin{cases} pe_{1,i} & \text{if } 1 \le i \le b-1; \\ pe_{1,i+1} & \text{if } b \le i \le n-1; \\ e_{1,b} & \text{if } i=n. \end{cases}$$

Here we use the same notation to express the basis for simplicity. Then we have $F^{2a}e_{i,j} = p^{a-1}e_{i,j}$ for $1 \le i \le 2$ and $1 \le j \le n$ and $V^{2n-2b}e_{i,j} = p^{n-b}e_{i,j}$ for $1 \le i \le 2$ and $1 \le j \le n$. It can be checked by Remark 8.9 that the sequence of slopes of $\tilde{\mathcal{D}}'^{\circ}$ is $((\frac{a-1}{2a})^{2a}, (\frac{1}{2})^{2b-2a}, (\frac{n-b}{2n-2b})^{2n-2b})$. Generally, since $\mathfrak{D}^{\circ} = \tilde{\mathcal{D}}^{\circ}/p\tilde{\mathcal{D}}^{\circ}$, we can take $\{\tilde{e}_{i,j} \mid i=1,2; 1 \le j \le n\}$ such that $\tilde{e}_{i,j} = e_{i,j}$ in in $\mathfrak{D}^{\circ}$. For $\underline{A} \in V^{\omega_1,\omega_2}$ with $(\omega_1,\omega_2) = (a,b)$, we have F, V acts on $\tilde{\mathcal{D}}^{\circ}$ by

$$F(\tilde{e}_{1,i}) = \begin{cases} \tilde{e}_{2,i} & \text{if } 1 \le i \le b-1; \\ p\tilde{e}_{2,n} & \text{if } i=b; \\ \tilde{e}_{2,i-1} & \text{if } i \ge b+1. \end{cases} \quad F(\tilde{e}_{2,i}) = \begin{cases} p\tilde{e}_{1,i+1} & \text{if } 1 \le i \le a-1; \\ \tilde{e}_{1,1} & \text{if } i=a; \\ p\tilde{e}_{1,i} & \text{if } i \ge a+1. \end{cases}$$

$$V(e_{1,i}) = \begin{cases} p\tilde{e}_{2,a} & \text{if } i=1; \\ \tilde{e}_{2,i-1} & \text{if } 2 \le i \le a; \\ \tilde{e}_{2,i} & \text{if } a+1 \le i \le n; \end{cases} \quad V(e_{2,i}) = \begin{cases} p\tilde{e}_{1,i} & \text{if } 1 \le i \le b-1; \\ p\tilde{e}_{1,i+1} & \text{if } b \le i \le n-1; \\ \tilde{e}_{1,b} & \text{if } i=n. \end{cases}$$

Here we omit summations $p\alpha$ for some $\alpha \in \tilde{\mathcal{D}}^{\circ}$ which can not delete the summation above in each formula. For $m = 2at+s$ and $1 \le s \le 2a$, $\min\{k \mid F^m(\tilde{\mathcal{D}}_1^{\circ}(A) \oplus \tilde{\mathcal{D}}_2^{\circ}(A)) \subseteq p^k(\tilde{\mathcal{D}}_1^{\circ}(A) \oplus \tilde{\mathcal{D}}_2^{\circ}(A))\} = (a-1)(t+1)$. For $m = (2n-2b)t+s$ and $1 \le s \le 2n-2b$, $\min\{k \mid V^m(\tilde{\mathcal{D}}_1^{\circ}(A) \oplus \tilde{\mathcal{D}}_2^{\circ}(A)) \subseteq p^k(\tilde{\mathcal{D}}_1^{\circ}(A) \oplus \tilde{\mathcal{D}}_2^{\circ}(A))\} = (n-b)(t+1)$. Therefore, we get by Remark 8.9, the minmal slope and maximal slope are $\frac{a-1}{2a}$ and $\frac{n-b}{2n-2b}$ respectively, Hence we finish the proof. □

Now we give the proof of (3) in Proposition 8.3:

Proof of (3) *in Proposition 8.3.* With Proposition 8.8,we get (3) is true when $a'-b' \le 0 \le a-b$ or $a \ge a', b \le b'$ and $a-b, a'-b' \ge 0$, since for two different sequences of slopes $\lambda_{\bullet}, \lambda'_{\bullet}$, $N^{\lambda'_{\bullet}} \subseteq N^{\lambda_{\bullet}}$ if and only if the Newton polygon of $\lambda'_{\bullet}$ is above that of $\lambda_{\bullet}$. It remains to show the proposition is true for $a \ge a', b \le b'$ and $a-b, a'-b' \le 0$. It comes from Proposition 8.5 directly. □

Remark 8.9. For any S-point of $\mathrm{Sh}_{1,n-1}$, the minimal slope $\lambda_{\min}$ of the isocrystal $(\tilde{\mathcal{D}}_1^{\circ}(A) \oplus \tilde{\mathcal{D}}_2^{\circ}(A), F)$ can be computed by $\lambda_{\min} = \lim_{m\to\infty} \frac{1}{m}\min\{k \mid F^m(\tilde{\mathcal{D}}_1^{\circ}(A) \oplus \tilde{\mathcal{D}}_2^{\circ}(A)) \subseteq p^k(\tilde{\mathcal{D}}_1^{\circ}(A) \oplus \tilde{\mathcal{D}}_2^{\circ}(A))\}$.

Let S be the Serre dual of the p-divisible group $A[p^{\infty}]$. Then by [5], the maximal slope of the dieudonné module associated to S with respect to F is $1-\lambda_{\max}$. Here $\lambda_{\max}$ is the maximal slope $\lambda_{\min}$ of the isocrystal $(\tilde{\mathcal{D}}_1^{\circ}(A) \oplus \tilde{\mathcal{D}}_2^{\circ}(A), F)$. Since the action of F on the dieudonné module associated to S is induced by the action of V on $\tilde{\mathcal{D}}(A)^{\circ}$, we get $\lambda_{\max} = 1 - \lim_{m\to\infty} \frac{1}{m}\min\{k \mid V^m(\tilde{\mathcal{D}}_1^{\circ}(A) \oplus \tilde{\mathcal{D}}_2^{\circ}(A)) \subseteq p^k(\tilde{\mathcal{D}}_1^{\circ}(A) \oplus \tilde{\mathcal{D}}_2^{\circ}(A))\}$.

As the end of this section, we show the μ-ordinary locus of $\mathrm{Sh}_{1,n-1}$ is affine:

Proposition 8.10. *The μ-ordinary locus of* $\mathrm{Sh}_{1,n-1}$ *is affine.*

By Proposition 8.4, it can be checked directly that any S-point (A,λ,η) of $\mathrm{Sh}_{1,n-1}$ is in the Newton strata of minimal slope not less than $\frac{1}{4}$ if and only if $F^3\tilde{\mathcal{D}}(A)_2^{\circ} \subseteq V\tilde{\mathcal{D}}_1^{\circ}$. Moreover, the S-point (A,λ,η) is in the Newton strata of maximal slope not larger than $\frac{3}{4}$ if and only if $V^2D_2^{\circ} \subseteq F\tilde{\mathcal{D}}_1^{\circ}$(In fact, we can get the result from Proposition 8.8 since any such point must be in $\mathrm{Y}_{10}\bigcap\mathrm{Y}_{11}$ or $\mathrm{Y}_{01}\bigcap\mathrm{Y}_{11}$.).

Take $(\mathcal{A},\lambda,\eta)$ to be the universal object of $\mathrm{Sh}_{1,n-1}$. It is in the Newton strata of minimal slope not less than $\frac{1}{4}$ is equivalent to $F^2\tilde{\mathcal{D}}(\mathcal{A})_2^{\circ} \subseteq V\tilde{\mathcal{D}}(\mathcal{A})_1^{\circ}$. This is equivalent to the map $F^2 : \mathrm{Lie}^{\circ,(p^2)}_{\mathcal{A}/\mathrm{Sh}_{1,n-1},2} \to \mathrm{Lie}^{\circ}_{\mathcal{A}/\mathrm{Sh}_{1,n-1},2}$ is trivial. In other words, F^2 is a trivial section in $\Gamma(\mathrm{Sh}_{1,n-1}, \mathrm{Lie}^{\circ,(1-p^2)}_{\mathcal{A}/\mathrm{Sh}_{1,n-1},2})$ if and only if it is in the Newton strata of minimal slope not less than $\frac{1}{4}$.

In the other way, the universal object $(\mathcal{A},\lambda,\eta)$ is in the Newton strata of maximal slope not larger than $\frac{3}{4}$ is equivalent to $V^3\tilde{\mathcal{D}}(\mathcal{A})_2^{\circ} \subseteq p\tilde{\mathcal{D}}(\mathcal{A})_1^{\circ}$. This is equivalent to $V^2 : \omega^{\circ}_{\mathcal{A}^{\vee}/\mathrm{Sh}_{1,n-1},1} \to \omega^{\circ,(p^2)}_{\mathcal{A}^{\vee}/\mathrm{Sh}_{1,n-1},1}$

is trivial. In other words, V^2 is a trivial section in $\Gamma(\mathrm{Sh}_{1,n-1}, \omega^{\circ,(p^2-1)}_{\mathcal{A}^\vee/\mathrm{Sh}_{1,n-1},1})$ if and only if it is in the Newton strata of maximal slope not larger than $\frac{3}{4}$.

Thus the universal object is in the μ-ordinary locus is the nonvanishing locus of $F^2 \otimes V^2 \in \Gamma(\mathrm{Sh}_{1,n-1}, \mathrm{Lie}^{\circ,(1-p^2)}_{\mathcal{A}/\mathrm{Sh}_{1,n-1},2} \otimes \omega^{\circ,(p^2-1)}_{\mathcal{A}^\vee/\mathrm{Sh}_{1,n-1},1})$. Since $\wedge^n \mathcal{H}^{dR}_1(\mathcal{A}/\mathrm{Sh}_{1,n-1})^\circ_2$ is trivial, we have the line bundle $\mathrm{Lie}^{\circ,-1}_{\mathcal{A}/\mathrm{Sh}_{1,n-1},2} \otimes \omega^\circ_{\mathcal{A}^\vee/\mathrm{Sh}_{1,n-1},1}$ is ample if and only if $\wedge^n \omega^\circ_{\mathcal{A}/\mathrm{Sh}_{1,n-1}}$ is ample, which is a result of [14, Theorem 7.2.4.1]. Since the the nonvanishing locus of a section of an ample line bundle on a projective scheme is affine, we have the μ-ordinary locus of $\mathrm{Sh}_{1,n-1}$ is affine.

Remark 8.11. Usually, we call F^2 and V^2 Hasse invariants of $\mathrm{Sh}_{1,n-1}$.

9. Geometry of $\mathrm{Sh}_{1,n-1}(K^1_\mathfrak{p})$

In this section, we analyze the geometry of $\mathrm{Sh}_{1,n-1}(K^1_\mathfrak{p})$ for $n \geq 3$.

Let $\mathrm{T} = T^{(1)}_\mathfrak{p}$ as in Definition 3.5.1. It can be checked directly that $T = \mathrm{Sh}_{0,n}(K^1_\mathfrak{p})$. To prove Ihara lemma, we introduce $Sh_{1,n-1}(K^1_\mathfrak{p})$, which is defined as the following moduli space:

Definition 9.1. Let $Sh_{1,n-1}(K^1_\mathfrak{p})$ be the moduli space over $\mathbb{F}_{p^2}$ that associates to each locally noetherian $\mathbb{F}_{p^2}$-scheme S, the set of isomorphism classes of tuples $(A, \lambda, \eta, A', \lambda', \eta', \phi)$, where

- (A, λ, η) is an S-point of $\mathrm{Sh}_{1,n-1}$,
- (A', λ', η') is an S-point of $\mathrm{Sh}_{1,n-1}$, and
- $\phi : A \to A'$ is an $\mathcal{O}_D$-equivariant p-quasi-isogeny (i.e., $p^m\phi$-quasi-isogeny is an isogeny of p-power order fr some integer m),

such that

- $\lambda' = \phi^\vee \circ \lambda \circ \phi$,
- $\phi \circ \eta' = \eta$, and
- the cokernels of the maps
$$\phi_{*,1} : H^{\mathrm{dR}}_1(A/S)^\circ_1 \to H^{\mathrm{dR}}_1(A'/S)^\circ_1 \quad \text{and} \quad \phi_{*,2} : H^{\mathrm{dR}}_1(A/S)^\circ_2 \to H^{\mathrm{dR}}_1(A'/S)^\circ_2$$
are both locally free $\mathcal{O}_S$-modules of rank 1.

$\mathrm{Sh}_{1,n-1}(K^1_\mathfrak{p})$ is union of four closed schemes $\mathrm{Sh}_{1,n-1}(K^1_\mathfrak{p}) = \mathrm{Y}_{00} \bigcup \mathrm{Y}_{01} \bigcup \mathrm{Y}_{10} \bigcup \mathrm{Y}_{11}$, where the four closed subschemes are defined as a moduli subspace such that for each locally noetherian $\mathbb{F}_{p^2}$-scheme S an S-point $(A, \lambda, \eta, A', \lambda', \eta', \phi)$ in Y_{ij} for $0 \leq i, j \leq 1$ is an S-point of $\mathrm{Sh}_{1,1}(K^1_\mathfrak{p})$ satisfying $(1.i)(2.j)$ below:

- (1.0) $\omega^\circ_{A^\vee,1} = \mathrm{Ker}(\phi_{*,1})$, (1.1) $\omega^\circ_{A'^\vee,1} = \mathrm{Im}(\phi_{*,1})$.
- (2.0) $\mathrm{Ker}(\phi_{*,2}) = \omega^\circ_{A^\vee,2}$, (2.1) $\mathrm{Im}(\phi_{*,2}) = \omega^\circ_{A'^\vee,2}$.

It can be checked directly that if $(A, \lambda, \eta, A', \lambda', \eta', \phi)$ is an S-point of Y_{00}, then A and A' are all supersingular. However, A and A' can be nonsupersingular if $(A, \lambda, \eta, A', \lambda', \eta', \phi)$ is an S-point of Y_{11}.[11] Using deformation theory in Subsection5.2, we get:

Proposition 9.2. *For $1 \leq i, j \leq 2$,Y_{ij} are all smooth of dimension $2n-2$, which comes from calculation of the tangent sheaf $T_{\mathrm{Y}_{ij}} = \mathfrak{F}_i \oplus \mathfrak{G}_j$ where $\mathfrak{F}_i$ and $\mathfrak{G}_j$ satisfy:*

(1) $\mathfrak{F}_0 = \mathcal{H}om\big(\omega^\circ_{\mathcal{A}'^\vee/\mathrm{Sh}_{1,n-1},1}, \mathrm{Lie}^\circ_{\mathcal{A}'/\mathrm{Sh}_{1,n-1},1}\big)$,

(2) $0 \to \mathcal{H}om\big(\omega^\circ_{\mathcal{A}^\vee/\mathrm{Sh}_{1,n-1},1}, \dfrac{\phi^{-1}_{*,1}(\omega^\circ_{\mathcal{A}^\vee/\mathrm{Sh}_{1,n-1},1})}{\omega^\circ_{\mathcal{A}^\vee/\mathrm{Sh}_{1,n-1},1}}\big) \to \mathfrak{F}_1 \to \mathcal{H}om\big(\omega^\circ_{\mathcal{A}'^\vee/\mathrm{Sh}_{1,n-1},1}, \dfrac{\mathrm{Im}\phi_{*,1}}{\omega^\circ_{\mathcal{A}'^\vee/\mathrm{Sh}_{1,n-1},1}}\big) \to 0$ *is exact,*

(3) $\mathfrak{G}_0 = \mathcal{H}om\big(\omega^\circ_{\mathcal{A}^\vee/\mathrm{Sh}_{1,n-1},2}, \mathrm{Lie}^\circ_{\mathcal{A}/\mathrm{Sh}_{1,n-1},2}\big)$,

[11]It is the main difference between the case $n \geq 3$ and the case $n = 2$.

(4) $0 \to \mathcal{H}om\big(\frac{\omega^{\circ}_{\mathcal{A}'^{\vee}/\mathrm{Sh}_{1,n-1},2}}{\phi_{*,2}(\omega^{\circ}_{\mathcal{A}^{\vee}/\mathrm{Sh}_{1,n-1},2})}, \mathrm{Lie}^{\circ}_{\mathcal{A}'/\mathrm{Sh}_{1,n-1},2}\big) \to \mathfrak{G}_1 \to \mathcal{H}om\big(\frac{\omega^{\circ}_{\mathcal{A}^{\vee}/\mathrm{Sh}_{1,n-1},2}}{\mathrm{Im}(\psi_{*,2})}, \mathrm{Lie}^{\circ}_{\mathcal{A}/\mathrm{Sh}_{1,n-1},2}\big) \to$ 0 *is exact.*

The tangent sheaves $T_{\mathrm{Y}_{ij}}$ are all locally free of rank $2n-2$.

Proof. The proof is the same as the case $n = 2$, so we omit here. We should remark that the reason the sheaves $\mathfrak{F}_2$ and $\mathfrak{G}_2$ can not be written as direct sums is that we can not lift the two sheaves simultaneously. □

Remark 9.3. By [11, Remark 3.7], points in Y_{00} are supersingular.

We introduce some new correspondences now.

Definition 9.4. For $1 \le i \le n-1$, let C_i be the moduli space over $\mathbb{F}_{p^2}$ that associates to each locally noetherian $\mathbb{F}_{p^2}$-scheme S, the set of isomorphism classes of $(A, \lambda, \eta, A', \lambda', \eta', \phi, B, \lambda'', \eta'', \psi, \psi')$, where

- $(A, \lambda, \eta, A', \lambda', \eta', \phi) \in \mathrm{Sh}_{1,n-1}(K^1_{\mathfrak{p}})$,
- $(B, \lambda'', \eta'') \in \mathrm{Sh}_{0,n}$,
- ψ is an isogeny from B to A such that $(A, \lambda, \eta, B, \lambda'', \eta'', \psi) \in Y_i$ and ψ' is an isogeny from B to A such that $(A', \lambda', \eta', B, \lambda'', \eta'', \psi') \in Y_{i+1}$.
- $\phi \circ \psi = \psi'$

We also let C_n be the moduli space over $\mathbb{F}_{p^2}$ that associates to each locally noetherian $\mathbb{F}_{p^2}$-scheme S, the set of isomorphism classes of tuples $(A, \lambda, \eta, A', \lambda', \eta', \phi, B_1, \lambda''_1, \eta''_1, B_2, \lambda''_2, \eta''_2, \psi, \psi')$, where

- $(A, \lambda, \eta, A', \lambda', \eta', \phi) \in \mathrm{Sh}_{1,n-1}(K^1_{\mathfrak{p}})$,
- $(B_1, \lambda''_1, \eta''_1), (B_2, \lambda''_2, \eta''_2) \in \mathrm{Sh}_{0,n}$,
- ψ is an isogeny from B_1 to A such that $(A, \lambda, \eta, B_1, \lambda''_1, \eta''_1, \psi) \in Y_n$ and ψ' is an isogeny from B_2 to A' such that $(A', \lambda', \eta', B_2, \lambda''_2, \eta''_2, \psi') \in Y_1$.
- $B_1 \in S_{\mathfrak{p}}(B_2)$.

It can be shown for $1 \le i \le n$, each C_i is representable by a smooth and projective scheme over $\mathrm{Sh}_{0,n}$. There is a natural projection from C_i to $\mathrm{Sh}_{0,n}$ mapping $(A, \lambda, \eta, A', \lambda', \eta', \phi, B, \lambda'', \eta'', \psi, \psi')$ to (B, λ'', η'')which we denote by pr'_i. And there is a natrual projection from C'_i to $\mathrm{Sh}_{1,n-1}(K^1_{\mathfrak{p}})$ mapping $(A, \lambda, \eta, A', \lambda', \eta', \phi, B, \lambda'', \eta'', \psi, \psi')$ to $(A, \lambda, \eta, A', \lambda', \eta', \phi)$, denoted by pr_i.

The moduli problem for C_i for $1 \le i \le n$ is slightly complicated. We will introduce a more explict moduli space and then show they are isomorphic.

For $1 \le i \le n-1$, consider the functor $\underline{C'_i}$ which associates to each locally noetherian $\mathbb{F}_{p^2}$-scheme S, the set of isomorphism classes of tuples $(B, \lambda'', \eta'', H_1, H_2, H'_1, H'_2)$, where

- (B, λ'', η'') is an S-valued point of $\mathrm{Sh}_{0,n}$;
- $H_1, H'_1 \subset H^{\mathrm{dR}}_1(B/S)^{\circ}_1$ are $\mathcal{O}_S$-subbundles of rank i and $i+1$ respectively and $H_2, H'_2 \subset H^{\mathrm{dR}}_1(B/S)^{\circ}_2$ are $\mathcal{O}_S$-subbundles of rank $i-1$ and i respectively. They satisfy:
  (1) $V^{-1}(H_2^{(p)}) \subseteq V^{-1}(H_2^{'(p)}) \bigcap H_1, V^{-1}(H_2^{'(p)}) \bigcup H_1 \subseteq H'_1$,
  (2) $H_2 \subseteq H'_2 \bigcap F(H_1^{(p)}), H_2^{(p)} \bigcup F(H_1^{(p)}) \subseteq F(H_1^{'(p)})$.
  Here, $F : H^{\mathrm{dR}}_1(B/S)^{\circ,(p)}_1 \xrightarrow{\sim} H^{\mathrm{dR}}_1(B/S)^{\circ}_2$ and $V : H^{\mathrm{dR}}_1(B/S)^{\circ}_1 \xrightarrow{\sim} H^{\mathrm{dR}}_1(B/S)^{\circ,(p)}_2$ are respectively the Frobenius and Verschiebung homomorphisms, which are actually isomorphisms because of the signature condition on $\mathrm{Sh}_{0,n}$.

There is a natural projection $\pi'_i \colon \underline{C'_i} \to \mathrm{Sh}_{0,n}$ given by $(B, \lambda', \eta', H_1, H_2, H'_1, H'_2) \mapsto (B, \lambda', \eta')$.

Proposition 9.5. *For $1 \le i \le n-1$, the functor $\underline{C'_i}$ is representable by a scheme C'_i smooth and projective over $\mathrm{Sh}_{0,n}$ of dimension n. Moreover, if $(\mathcal{B}, \lambda'', \eta'', \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}'_1, \mathcal{H}'_2)$ denotes the universal object over C'_i, then the tangent bundle of C'_i is*

$$T_{Y'_j, y_0} \cong \mathfrak{F} \oplus \mathfrak{G},$$

where $\mathfrak{F}, \mathfrak{G}$ satisfies:

- $0 \to \mathcal{H}om_{\mathrm{Sh}_{0,n}}\big(\mathcal{H}_1'/V^{-1}(\mathcal{H}_2^{'(p)}), \mathcal{H}_1^{\mathrm{dR}}(B_0/\mathrm{Sh}_{0,n})_1^{\circ}/\mathcal{H}_1'\big)^* \to \mathfrak{F}$
  $\to \mathcal{H}om_{\mathrm{Sh}_{0,n}}\big(\mathcal{H}_1/V^{-1}(\mathcal{H}_2^{(p)}), \mathcal{H}_1'/\mathcal{H}_1\big) \to 0$ *is exact.*
- $0 \to \mathcal{H}om_{\mathrm{Sh}_{0,n}}\big(\mathcal{H}_2'/\mathcal{H}_2, F(\mathcal{H}_1^{',(p)})/\mathcal{H}_2'\big) \to \mathfrak{G} \to \mathcal{H}om_{\mathrm{Sh}_{0,n}}\big(\mathcal{H}_2, F(\mathcal{H}_1^{(p)})/\mathcal{H}_2\big) \to 0$ *is exact.*

Proof. For each integer m with $0 \le m \le n$ and $i = 1, 2$, let $\mathbf{Gr}(H_1^{\mathrm{dR}}(\mathcal{B}/\mathrm{Sh}_{0,n})_i^{\circ}, m)$ be the Grassmannian scheme over $\mathrm{Sh}_{0,n}$ that parametrizes subbundles of the universal de Rham homology $H_1^{\mathrm{dR}}(\mathcal{B}/\mathrm{Sh}_{0,n})_i^{\circ}$ of rank m. Then $\underline{C}_i'$ is a closed subfunctor of product of Grassmannian schemes. The representability of $\underline{C}_i'$ follows. Moreover, C_j' is projective.

We show now that the structural map $\pi_i' : C_i' \to \mathrm{Sh}_{0,n}$ is smooth of relative dimension n. Let $S_0 \hookrightarrow S$ be an immersion of locally noetherian $\mathbb{F}_{p^2}$-schemes with ideal sheaf I satisfying $I^2 = 0$. Suppose we are given a commutative diagram

$$\begin{tikzcd} S_0 \arrow[r, "g_0"] \arrow[d] & C_i' \arrow[d, "\pi_i'"] \\ S \arrow[r, "h"] \arrow[ur, dashed, "g"] & \mathrm{Sh}_{0,n} \end{tikzcd}$$

with solid arrows. We have to show that, locally for the Zariski topology on S_0, there is a morphism $g : S \to C_i'$ making the diagram commute. Let B be the abelian scheme over S given by h, and B_0 be the base change to S_0. The morphism g_0 gives rises to subbundles $\overline{H}_1, \overline{H}_1' \subset H_1^{\mathrm{dR}}(B_0/S_0)_1^{\circ}$ and $\overline{H}_2, \overline{H}_2' \subset H_1^{\mathrm{dR}}(B_0/S_0)_2^{\circ}$ with

(1) $V^{-1}(\overline{H}_2^{(p)}) \subseteq V^{-1}(\overline{H}_2^{'(p)}) \bigcap \overline{H}_1, V^{-1}(\overline{H}_2^{'(p)}) \bigcup \overline{H}_1 \subseteq \overline{H}_1'$,

(2) $\overline{H}_2 \subseteq \overline{H}_2' \bigcap F(\overline{H}_1^{(p)}), \overline{H}_2^{(p)} \bigcup F(\overline{H}_1^{(p)}) \subseteq F(\overline{H}_1^{'(p)})$.

Finding g is equivalent to finding a subbundle $H_i, H_i' \subset H_1^{\mathrm{dR}}(B/S)_i^{\circ}$ which lifts each $\overline{H}_i$ for $i = 1, 2$ and satisfies the same conditions above; this is certainly possible when passing to small enough affine open subsets of S_0. Thus $\pi_i' : C_i' \to \mathrm{Sh}_{0,n}$ is formally smooth, and hence smooth. We note that $F_S^* : \mathcal{O}_S \to \mathcal{O}_S$ factors through $\mathcal{O}_{S_0}$. Hence $V^{-1}(H_2^{(p)})$, $V^{-1}(H_2^{'(p)})$, $F(H_1^{(p)})$ and $F(H_1^{'(p)})$ actually depend only on $\overline{H}_1, \overline{H}_1', \overline{H}_2$ and $\overline{H}_2$, but not on the lifts H_1, H_1' and H_2, H_2'. Therefore, the possible lifts H_2 form a torsor under the group

$$\mathcal{H}om_{\mathcal{O}_{S_0}}\big(\overline{H}_2, F(\overline{H}_1^{(p)})/\overline{H}_2\big) \otimes_{\mathcal{O}_{S_0}} I.$$

Similarly the possible lifts H_1 form a torsor under the group

$$\mathcal{H}om_{\mathcal{O}_{S_0}}\big(\overline{H}_1/V^{-1}(\overline{H}_2^{(p)}), \overline{H}_1'/\overline{H}_1\big) \otimes_{\mathcal{O}_{S_0}} I.$$

The possible lifts H_1' form a torsor under the group

$$\mathcal{H}om_{\mathcal{O}_{S_0}}\big(\overline{H}_1'/V^{-1}(\overline{H}_2^{'(p)}), H_1^{\mathrm{dR}}(B_0/S_0)_1^{\circ}/\overline{H}_1'\big) \otimes_{\mathcal{O}_{S_0}} I.$$

The possible lifts H_2' form a torsor under the group

$$\mathcal{H}om_{\mathcal{O}_{S_0}}\big(\overline{H}_2'/\overline{H}_2, F(\overline{H}_1^{',(p)})/\overline{H}_2'\big) \otimes_{\mathcal{O}_{S_0}} I.$$

To compute the tangent bundle $T_{Y_j'}$, we take $S = \mathrm{Spec}(\mathcal{O}_{S_0}[\epsilon]/\epsilon^2)$ and $I = \epsilon\mathcal{O}_S$. The morphism $g_0 : S_0 \to C_i'$ corresponds to an S_0-valued point of C_i', say c_0. Then the possible liftings g form the tangent space $T_{C_i'}$ at c_0, denote by T_{C_i', c_0}. The discussion above shows that

$$T_{Y_j', y_0} \cong \mathfrak{F} \oplus \mathfrak{G},$$

where $\mathfrak{F}, \mathfrak{G}$ satisfies:

- $0 \to \mathcal{H}om_{\mathcal{O}_{S_0}}\big(\overline{H}'_1/V^{-1}(\overline{H}_2^{'(p)}), H_1^{\mathrm{dR}}(B_0/S_0)_1^\circ/\overline{H}'_1\big) \to \mathfrak{F}$
  $\to \mathcal{H}om_{\mathcal{O}_{S_0}}\big(\overline{H}_1/V^{-1}(\overline{H}_2^{(p)}), \overline{H}'_1/\overline{H}_1\big) \to 0$ is exact.
- $0 \to \mathcal{H}om_{\mathcal{O}_{S_0}}\big(\overline{H}'_2/\overline{H}_2, F(\overline{H}_1^{',(p)})/\overline{H}'_2\big) \to \mathfrak{G} \to \mathcal{H}om_{\mathcal{O}_{S_0}}\big(\overline{H}_2, F(\overline{H}_1^{(p)})/\overline{H}_2\big) \to 0$ is exact.

$T_{C'_i}$ is certainly a vector bundle over S_0 of rank $i-1+1+(n-i-1)+1=n$. Applying this to the universal case when $g_0 : S_0 \to C'_i$ is the identity morphism, the formula of the tangent bundle follows. □

To construct a morphism from C_i to C'_i for $1 \le i \le n-1$, we need the following lemma:

Lemma 9.6. *Let $(A, \lambda, \eta, B, \lambda', \eta', \phi)$ be an S-point of Y_j. Then the image of $\phi_{*,1}$ contains both $\omega^\circ_{A^\vee/S,1}$ and $F\big(H_1^{\mathrm{dR}}(A/S)_2^{\circ,(p)}\big)$, and the image of $\phi_{*,2}$ is contained in $\omega^\circ_{A^\vee/S,2}$ and $F\big(H_1^{\mathrm{dR}}(A/S)_1^{\circ,(p)}\big)$.*

Proof. See [11, Lemma 4.6]. □

There is a natural morphism $\alpha : C_i \to C'_i$ for $1 \le j \le n-1$ defined as follows. For a locally noetherian $\mathbb{F}_{p^2}$-scheme S and an S-point $(A, \lambda, \eta, A', \lambda', \eta', \phi, B, \lambda'', \eta'', \psi, \psi')$ of C_i, we define

$$H_1 := \psi_{*,1}^{-1}(\omega^\circ_{A^\vee/S,1}) \subseteq H_1^{\mathrm{dR}}(B/S)_1^\circ, \quad \text{and} \quad H_2 := p\psi_{*,1}^{'-1}(\omega^\circ_{A^\vee/S,2}) \subseteq H_1^{\mathrm{dR}}(B/S)_2^\circ.$$

$$H'_1 := \psi_{*,2}^{'-1}(\omega^\circ_{A'^\vee/S,1}) \subseteq H_1^{\mathrm{dR}}(B/S)_1^\circ, \quad \text{and} \quad H'_2 := p\psi_{*,2}^{'-1}(\omega^\circ_{A'^\vee/S,2}) \subseteq H_1^{\mathrm{dR}}(B/S)_2^\circ.$$

In particular, H_1, H'_1, H_2 and H'_2 are $\mathcal{O}_S$-subbundles of rank $i, i+1, i-1$ and i, respectively. By 9.6, we can easily see α is well-defined.

Therefore, we have a well-defined map $\alpha\colon C_i \to C'_i$ given by

$$\alpha\colon (A, \lambda, \eta, A', \lambda', \eta', \phi, B, \lambda'', \eta'', \psi, \psi') \longmapsto (B, \lambda'', \eta'', H_1, H_2, H'_1, H'_2).$$

Moreover, it is clear from the definition that $\pi'_i \circ \alpha = \mathrm{pr}'_i$.

Proposition 9.7. *The morphism α is an isomorphism.*

Proof. Let k be a perfect field containing $\mathbb{F}_{p^2}$. We first prove that α induces a bijection of points $\alpha : C_i(k) \xrightarrow{\sim} C'_i(k)$. It suffices to show that there exists a morphism of sets $\beta : C'_i(k) \to C_i(k)$ inverse to α. Let $y = (B, \lambda'', \eta'', H_1, H_2, H'_1, H'_2) \in C'_i(k)$. Let $\beta(y) = (A, \lambda, \eta, A', \lambda', \eta', \phi, B, \lambda'', \eta'', \psi, \psi')$ as follows. We let $\tilde{\mathcal{E}}_1, \tilde{\mathcal{E}}'_1 \subseteq \tilde{\mathcal{D}}(B)_1^\circ$ and $\tilde{\mathcal{E}}_2, \tilde{\mathcal{E}}'_2 \subseteq \tilde{\mathcal{D}}(B)_2^\circ$ be respectively the inverse images of $V^{-1}(H_2^{(p)}), V^{-1}(H_2^{'(p)}) \subseteq H_1^{\mathrm{dR}}(B/k)_1^\circ$ and $F(H_1^{(p)}), F(H_1^{'(p)}) \subseteq H_1^{\mathrm{dR}}(B/k)_2^\circ$ under the natural reduction maps

$$\tilde{\mathcal{D}}(B)_i^\circ \to \tilde{\mathcal{D}}(B)_i^\circ/p\tilde{\mathcal{D}}(B)_i^\circ \cong H_1^{\mathrm{dR}}(B/k)_i^\circ \quad \text{for } i=1,2.$$

The condition (3.4) ensures that $F(\tilde{\mathcal{E}}_i) \subseteq \tilde{\mathcal{E}}_{3-i}$ and $V(\tilde{\mathcal{E}}_i) \subseteq \tilde{\mathcal{E}}_{3-i}$ for $i=1,2$. Applying Proposition 5.1.2 with $p^{-1}\tilde{\mathcal{E}}_1, p^{-1}\tilde{\mathcal{E}}_2$ and $p^{-1}\tilde{\mathcal{E}}'_1, p^{-1}\tilde{\mathcal{E}}'_2$, we get two triples $(A, \lambda, \eta), (A', \lambda', \eta')$ and two $\mathcal{O}_D$-equivariant isogenies $\psi : B \to A, \psi' : B \to A'$, where A, A' is abelian varieties over k with an action of $\mathcal{O}_D$, λ, λ' are prime-to-p polarizations on A, A' repectively, and η, η' are prime-to-p level structures on A, A' respectively, such that $\psi^\vee \circ \lambda \circ \psi = p\lambda''$ and $\psi'^\vee \circ \lambda' \circ \psi' = p\lambda''$. Moreover we have $\psi \circ \eta'' = \eta$ and $\psi' \circ \eta'' = \eta'$. Moreover, the dimension formula (5.1.3) implies that $\omega^\circ_{A^\vee/k,1}, \omega^\circ_{A'^\vee/k,1}$ has dimension 1, and $\omega^\circ_{A^\vee/k,2}, \omega^\circ_{A'^\vee/k,2}$ has dimension $n-1$. Therefore, $(A, \lambda, \eta), (A', \lambda', \eta')$ are two points of $\mathrm{Sh}_{1,n-1}$. and $(A, \lambda, \eta, B, \lambda'', \eta'', \psi) \in \mathrm{Y}_i, (A', \lambda', \eta', B, \lambda'', \eta'', \psi') \in \mathrm{Y}_{i+1}$. Take $\phi = \psi' \circ \psi^{-1}$ to be an quasi-isogeny from A to A'. Then we can check $(A, \lambda, \eta, A', \lambda', \eta', \phi, B, \lambda'', \eta'', \psi, \psi')$ is a point of $\mathrm{Sh}_{1,n-1}(K^1_{\mathfrak{p}})$. This finishes the construction of $\beta(y)$. It is direct to check that β is the set theoretic inverse to $\alpha : C_i(k) \to C'_i(k)$.

We show now that α induces an isomorphism on the tangent spaces at each closed point; as we have already shown that C'_i is smooth, it will then follow that α is an isomorphism. Let $x = (A, \lambda, \eta, A', \lambda', \eta', \phi, B, \lambda'', \eta'', \psi, \psi') \in C_i(k)$ be a closed point. Consider the infinitesimal deformation over $k[\epsilon] = k[t]/t^2$. Note that (B, λ', η') has a unique deformation $(\hat{B}, \hat{\lambda}', \hat{\eta}')$ to $k[\epsilon]$, namely the trivial deformation. By the Serre–Tate and Grothendieck–Messing deformation theory (cf. Theorem 5.2.1), deformation of $(\hat{A}, \hat{\lambda}, \hat{\eta}, \hat{A}', \hat{\lambda}', \hat{\eta}', \hat{\phi})$ of $(A, \lambda, \eta, A', \lambda', \eta', \phi)$ to $k[\epsilon]$ corresponds to the tangent space of $\mathrm{Sh}_{1,n-1}(K^1_{\mathfrak{p}})$ at $\mathrm{pr}_i(x)$.

By Lemma 9.6, we see that $V\tilde{\mathcal{D}}(A')^\circ_2 \subseteq \psi'_{*,1}\tilde{\mathcal{D}}(B)^\circ_1 \subseteq \phi_{*,1}\tilde{\mathcal{D}}(A)_1$ and $p\tilde{\mathcal{D}}(A')^\circ_2 \subseteq \psi'_{*,2}\tilde{\mathcal{D}}(B)^\circ_2 \subseteq \phi_{*,2}V\tilde{\mathcal{D}}(A)_1$. Thus $\omega^\circ_{A'^\vee,1} = \mathrm{Im}(\phi_{*,1})$ and $\mathrm{Im}(\phi_{*,2}) = \omega^\circ_{A'^\vee,2}$. The image of C_i under pr_i is in Y_{11}.

Since $\mathrm{pr}_i(x) \in \mathrm{Y}_{11}$, we can see it is exactly the description of the tangent space of C'_i at $\alpha(x)$,by direct calculation. This concludes the proof. □

In the sequel, we will always identify C_i with C'_i and pr'_i with π'_i for $1 \leq i \leq n$.

For C_n, we have a morphism from C_n to $\mathrm{Sh}_{1,n-1}$, denoted by pr_n, mapping any S-point $(A, \lambda, \eta, A', \lambda', \eta', \phi, B_1, \lambda''_1, \eta''_1, B_2, \lambda''_2, \eta''_2, \psi, \psi')$ to $(A, \lambda, \eta, A', \lambda', \eta', \phi)$. Since $(A, \lambda, \eta, B_1, \lambda''_1, \eta''_1, \psi) \in Y_n$ and $(A', \lambda', \eta', B_2, \lambda''_2, \eta''_2, \psi') \in Y_1$, we get $\psi_{*,1}(H^{dR}_1(B_1/S)^\circ_1) = \omega^\circ_{A^\vee,2}$, $\psi_{*,2}(H^{dR}_1(B_1/S)^\circ_2) = 0$, and $\psi'_{*,1}(H^{dR}_1(B_2/S)^\circ_1) = H^{dR}_1(A'/S)^\circ_1$ and $\psi'_{*,2}(H^{dR}_1(B_2/S)^\circ_2) = \omega^\circ_{A'^\vee,1}$. Since $B_1 \in S_{\mathfrak{p}}(B_2)$, we get $\omega^\circ_{A^\vee,1} = \mathrm{Ker}(\phi_{*,1})$ and $\mathrm{Ker}(\phi_{*,2}) = \omega^\circ_{A^\vee,2}$. Thus the image of C_n is contained in Y_{00}.

Proposition 9.8. *The morphism* $\mathrm{pr}_n : C_n \to \mathrm{Y}_{00}$ *is an isomorphism.*

Proof. Let k be a perfect field containing $\mathbb{F}_{p^2}$. We first prove that pr_n induces a bijection of points $\mathrm{pr}_n : C_n(k) \xrightarrow{\sim} \mathrm{Y}_{00}(k)$. It suffices to show that there exists a morphism of sets $\beta : \mathrm{Y}_{00}(k) \to C_n(k)$ inverse to pr_n. Let $y = (A, \lambda, \eta, A', \lambda', \eta', \phi) \in \mathrm{Y}_{00}(k)$. We define $\beta(y) = (A, \lambda, \eta, A', \lambda', \eta', \phi, B, \lambda'', \eta'', \psi, \psi')$ as follows. We let $\tilde{\mathcal{E}}_1 = p\tilde{\mathcal{D}}(A')_1 = \phi_{*,1}V\tilde{\mathcal{D}}(A)_2, \tilde{\mathcal{E}}'_1 = \tilde{\mathcal{D}}(A')_1$ and $\tilde{\mathcal{E}}_2 = p\phi_{*,2}\tilde{\mathcal{D}}(A)_2, \tilde{\mathcal{E}}'_2 = V\tilde{\mathcal{D}}(A')_2 = \phi_{*,2}\tilde{\mathcal{D}}(A)_2$. Applying PropositionProposition 5.1.1 with $m = 1$, we get an triple (B, λ'', η'') and an $\mathcal{O}_D$-equivariant isogeny $\psi : B \to A, \psi' : B \to A'$, where B is an abelian variety over k with an action of $\mathcal{O}_D$, λ'' is a prime-to-p polarization on B, and η'' is a prime-to-p level structure on B, such that $\psi^\vee \circ \lambda \circ \psi = p\lambda'', \psi'^\vee \circ \lambda' \circ \psi' = p\lambda'', \eta = \psi \circ \eta'', \eta' = \psi' \circ \eta''$ and such that $\psi_{*,i} : \tilde{\mathcal{D}}(B)^\circ_i \to \tilde{\mathcal{D}}(A)^\circ_i, \psi'_{*,i} : \tilde{\mathcal{D}}(B)^\circ_i \to \tilde{\mathcal{D}}(A')^\circ_i$ are naturally identified with the inclusion $\tilde{\mathcal{E}}_i \hookrightarrow \tilde{\mathcal{D}}(A)^\circ_i, \tilde{\mathcal{E}}'_i \hookrightarrow \tilde{\mathcal{D}}(A')^\circ_i$ for $i = 1, 2$. Moreover, the dimension formula (5.1.2) implies that $\omega^\circ_{B^\vee/k,1}$ has dimension 0, and $\omega^\circ_{B^\vee/k,2}$ has dimension n. Therefore, (B, λ'', η'') is a point of $\mathrm{Sh}_{0,n}$. This finishes the construction of $\beta(y)$. It is direct to check that β is the set theoretic inverse to $\mathrm{pr}_n : C_n(k) \to \mathrm{Y}_{00}(k)$.

By a simple argument on Serre–Tate and Grothendieck–Messing deformation theory, we can show that pr_n induces an isomorphism on the tangent spaces at each closed point; it follows that pr_n is an isomorphism. □

As a corollary, we can write the correspondence between $\mathrm{Sh}_{0,n}$ and $\mathrm{Sh}_{1,n-1}(K^1_{\mathfrak{p}})$ determined by C_i for $1 \leq i \leq n$ in the following diagrams:

Corollary 9.9. (1) *For* $1 \leq i \leq n-1$*, we have the following diagram:*

$$\begin{tikzcd}
\mathrm{Y}_{11} \arrow[d] & C_i \arrow[l, "\mathrm{pr}_i"'] \arrow[r, "\mathrm{pr}'_i"] \arrow[d] & \mathrm{Sh}_{0,n} \arrow[d, "{(1,1)}"] \\
\mathrm{Sh}_{1,n-1} \times \mathrm{Sh}_{1,n-1} & \mathrm{Y}_i \times \mathrm{Y}_{i+1} \arrow[l, "\overleftarrow{p}_i \times \overleftarrow{p}_{i+1}"'] \arrow[r, "\overrightarrow{p}_i \times \overrightarrow{p}_{i+1}"] & \mathrm{Sh}_{0,n} \times \mathrm{Sh}_{0,n}
\end{tikzcd}$$

where the first two vertical arrows are induced by natrual projections.

(2) *For C_n, we have the following diagram:*

$$\begin{tikzcd}
\mathrm{Y}_{00} \arrow[d] & C_n \arrow[l, "\mathrm{pr}_n, \cong"'] \arrow[r, "\mathrm{pr}'_n, \mathbb{P}^{n-1}\times\mathbb{P}^{n-1} bundle"] \arrow[d] & \mathrm{Sh}_{0,n} \arrow[d, "(1,S_\mathfrak{p})"] \\
\mathrm{Sh}_{1,n}\times\mathrm{Sh}_{1,n} & \mathrm{Y}_n\times\mathrm{Y}_1 \arrow[l, "\overleftarrow{p}_n\times\overleftarrow{p}_1"'] \arrow[r, "\overrightarrow{p}_2\times\overrightarrow{p}_1"] & \mathrm{Sh}_{0,n}\times\mathrm{Sh}_{0,n}
\end{tikzcd}$$

where the first two vertical arrows are induced by natrual projections.

Proof. What remains to show is that pr'_n makes C_n be a $\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}$ bundle over $\mathrm{Sh}_{0,n}$. To see this, we construct a morphism α from Y_{00} to $\mathbf{Gr}\big(H_1^{\mathrm{dR}}(\mathcal{B}/\mathrm{Sh}_{0,n})_1^\circ, n-1\big)\times\mathbf{Gr}\big(H_1^{\mathrm{dR}}(\mathcal{B}'/\mathrm{Sh}_{0,n})_2^\circ, 1\big)$ with $(\mathcal{B},\lambda'',\eta'')$ the universal object of $\mathrm{Sh}_{0,n}$. More explicitly, for any S-point $y=(A,\lambda,\eta,A',\lambda',\eta',\phi)\in\mathrm{Y}_{00}$, $\alpha(y)$ is defined to be the S-point $(\omega_{A^\vee,2},\omega_{A'^\vee,1})$.

By a similar argument as we done for $C_i, 1\le i\le n-1$ in Proposition 9.7, we can show α is an isomorphism. Hence we finish the proof. □

The correspondences $C_1,\dots,C_n$ can help us to analyze structures of $\mathrm{Y}_{00},\mathrm{Y}_{11}$. We are now going to analyze the strucures of $\mathrm{Y}_{01},\mathrm{Y}_{10}$. First, we need the following lemma:

Lemma 9.10. (1) *The S-point $(A,\lambda,\eta)\in\mathrm{Sh}_{1,n-1}$ is in $\overleftarrow{p}_n(\mathrm{Y}_n)$ if and only if $F\tilde{\mathcal{D}}(A)_2^\circ = V\tilde{\mathcal{D}}(A)_2^\circ$.*

(2) *The S-point $(A,\lambda,\eta)\in\mathrm{Sh}_{1,n-1}$ is in $\overleftarrow{p}_1(\mathrm{Y}_1)$ if and only if $F\tilde{\mathcal{D}}(A)_1^\circ = V\tilde{\mathcal{D}}(A)_1^\circ$.*

Proof. The proof of (1) and (2) are similar. For simplicity, we only give the proof of (1).

For the 'if' part, we take $\tilde{\mathcal{E}}_1 = V\tilde{\mathcal{D}}(A)_2$ and $\tilde{\mathcal{E}}_2 = p\tilde{\mathcal{D}}(A)_2$. Applying Proposition 5.1.1 with $m=1$, we get an triple (B,λ'',η'') and an $\mathcal{O}_D$-equivariant isogeny $\phi: B\to A$, where B is an abelian variety over k with an action of $\mathcal{O}_D$, λ'' is a prime-to-p polarization on B, and η'' is a prime-to-p level structure on B, such that $\phi^\vee\circ\lambda\circ\phi = p\lambda''$, $\eta=\phi\circ\eta''$ and such that $\phi_{*,i}:\tilde{\mathcal{D}}(B)_i^\circ\to\tilde{\mathcal{D}}(A)_i^\circ$ are naturally identified with the inclusion $\tilde{\mathcal{E}}_i\hookrightarrow\tilde{\mathcal{D}}(A)_i^\circ$ for $i=1,2$. Moreover, the dimension formula (5.1.2) implies that $\omega^\circ_{B^\vee/k,1}$ has dimension 0, and $\omega^\circ_{B^\vee/k,2}$ has dimension n. Therefore, (B,λ'',η'') is a point of $\mathrm{Sh}_{0,n}$ and $(A,\lambda,\eta,B,\lambda'',\eta'',\phi)$ is a point of Y_n.

Conversely, if$(A,\lambda,\eta,B,\lambda'',\eta'',\phi)$ is an S-point of Y_n, then by the defintion of Y_n, we see that $\phi_{*,2}\tilde{\mathcal{D}}(B)_2^\circ = p\tilde{\mathcal{D}}(A)_2^\circ$ and $\phi_{*,1}\tilde{\mathcal{D}}(B)_1^\circ$ has corank $n-1$ contained in $\tilde{\mathcal{D}}(A)_1^\circ$. By Lemma 9.6, we have $V\tilde{\mathcal{D}}(A)_2^\circ\subseteq\phi_{*,1}\tilde{\mathcal{D}}(B)_1^\circ$. It forces that $V\tilde{\mathcal{D}}(A)_2^\circ\subseteq\phi_{*,1}\tilde{\mathcal{D}}(B)_1^\circ$. By $F\tilde{\mathcal{D}}(B)_2^\circ = V\tilde{\mathcal{D}}(B)_2^\circ = p\tilde{\mathcal{D}}(B)_1^\circ$, we get $F\tilde{\mathcal{D}}(A)_2^\circ = V\tilde{\mathcal{D}}(A)_2^\circ$. Thus we finish the proof. □

In particular, we have the following propositions:

Proposition 9.11. (1) *There is a natural morphism $\pi_{10}:\mathrm{Y}_{10}\to\mathrm{Sh}_{1,n-1}$ by mapping any point $(A,\lambda,\eta,A',\lambda',\eta',\phi)$ to (A,λ,η). The closed subscheme Y_{10} of $\mathrm{Sh}_{1,n-1}(K_\mathfrak{p}^1)$ is isomorphic to $\mathrm{Bl}_{\overleftarrow{p}_n(\mathrm{Y}_n)}\mathrm{Sh}_{1,n-1}$ with π_{10} to be exacly the blowing-up map. The exceptional divisor is $\mathrm{Y}_{00}\bigcap\mathrm{Y}_{10}$.*

(2) *There is a natural morphism $\pi_{01}:\mathrm{Y}_{10}\to\mathrm{Sh}_{1,n-1}$ by mapping any point $(A,\lambda,\eta,A',\lambda',\eta',\phi)$ to (A',λ',η'). The closed subscheme Y_{01} of $\mathrm{Sh}_{1,n-1}(K_\mathfrak{p}^1)$ is isomorphic to $\mathrm{Bl}_{\overleftarrow{p}_1(\mathrm{Y}_1)}\mathrm{Sh}_{1,n-1}$ with π_{01} to be exacly the blowing-up map. The exceptional divisor is $\mathrm{Y}_{00}\bigcap\mathrm{Y}_{01}$.*

Proof. The proof of (1) and (2) are similar. For simplicity, we only give the proof of (1).

First, we introduce a new scheme C_{10} and show it is isomorphic both to $\mathrm{Sh}_{1,n-1}$ and to Y_{10}.

Let $\underline{C}_{10}$ be the moduli space over $\mathbb{F}_{p^2}$ that associates to each locally noetherian $\mathbb{F}_{p^2}$-scheme S, the set of isomorphism classes of tuples (A,λ,η,H), where

- $(A,\lambda,\eta)\in\mathrm{Sh}_{1,n-1}$,

- H is a subbundle contained in $V^{-1}\big(H_1^{dR}(A/S)_2^{\circ,(p)}\big)$ of rank 2.
- H satisfies $F\big(H_1^{dR}(A/S)_2^{\circ,(p)}\big)\bigcup\omega_{A^\vee/S,2}\subseteq H$.

With a similar argument as we done for $\underline{C}_i, 1\le i\le n-1$, we can show $\underline{C}_{10}$ is represented by a smooth, projective scheme over $\mathrm{Sh}_{1,n-1}$ of dimension $2(n-1)$. We denote it by C_{10}. There is natural morphism α from Y_{10} to C_{10} by mapping any point $(A,\lambda,\eta,A',\lambda',\eta',\phi)$ to $(A,\lambda,\eta,\omega^\circ_{A'^\vee,1})$. It can be checked α is well defined and is an isomorphism using a method similar to Proposition 9.7.

Given any S-point $(A,\lambda,\eta,H)\in C_{10}$, we can see by dimension counting that the subbundle of rank 2 $H=F\big(H_1^{dR}(A/S)_2^{\circ,(p)}\big)\bigcup\omega_{A^\vee/S,2}$ if $F\big(H_1^{dR}(A/S)_2^{\circ,(p)}\big)\neq\omega_{A^\vee/S,2}$. Otherwise if $F\big(H_1^{dR}(A/S)_2^{\circ,(p)}\big)=\omega_{A^\vee/S,2}$,that is $(A,\lambda,\eta)\in\mathrm{Sh}_{1,n-1}$ is in $\overleftarrow{p}_n(\mathrm{Y}_n)$, H correpsonds to a point in $\mathbf{Gr}(V^{-1}\big(H_1^{dR}(A/S)_2^{\circ,(p)}\big),1)$. Base on this we can see the morphism $\beta: C_{10}\to\mathrm{Sh}_{1,n-1}$ mapping (A,λ,η,H) to (A,λ,η) inducing the isomorhism $C_{10}\cong\mathrm{Bl}_{\overleftarrow{p}_n(\mathrm{Y}_n)}\mathrm{Sh}_{1,n-1}$. Under the isomorphism, the exceptional divisor consists of points $(A,\lambda,\eta,A',\lambda',\eta',\phi)\in\mathrm{Y}_{10}$ such that $(A,\lambda,\eta)\in\overleftarrow{p}_n(\mathrm{Y}_n)$, which is exactly $\mathrm{Y}_{00}\bigcap\mathrm{Y}_{10}$. Thus we finish the proof. $\square$

Now we begin the computation of cohomology groups. Most of the calculation based on Appendix C.2. For $\mathrm{Y}_{00}\bigcap\mathrm{Y}_{01},\mathrm{Y}_{00}\bigcap\mathrm{Y}_{10}$ and $\mathrm{Y}_{00}\bigcap\mathrm{Y}_{11}$, we have the following proposition:

Proposition 9.12. *The cohomology groups* $H^*_{ét}(\mathrm{Y}_{00}\bigcap\mathrm{Y}_{10})=H^*_{ét}(\mathrm{Y}_{00}\bigcap\mathrm{Y}_{01})=H^*_{ét}(\mathbb{P}^{n-1}/\mathrm{Sh}_{0,n})\otimes H^*_{ét}(\mathbb{P}^{n-2})$ *and* $H^{2n-4}_{ét}(\mathrm{Y}_{00}\bigcap\mathrm{Y}_{11})=H^0(\mathrm{Sh}_{0,n},\mathbb{F}_l)^{n-2}\oplus H^0_{ét}(\mathrm{Sh}_{0,n}(K^1_{\mathfrak{p}}),\mathbb{F}_l)$. *Moreover, we have* $H^{2n-2}_{ét}(\mathrm{Y}_{00}\bigcap\mathrm{Y}_{11})=H^0(\mathrm{Sh}_{0,n},\mathbb{F}_l)^{n-2}$ *and* $H^i_{ét}(\mathrm{Y}_{00}\bigcap\mathrm{Y}_{11})=0$, *for i odd. As a direct corollary, if* $l\nmid\frac{p^{2n-2}-1}{p^2-1}$, *we have* $H^{2n-3}_c(\mathrm{Y}_{00}\bigcap\mathrm{Y}_{01}-\mathrm{Y}_{00}\bigcap\mathrm{Y}_{11})=H^{2n-3}_c(\mathrm{Y}_{00}\bigcap\mathrm{Y}_{10}-\mathrm{Y}_{00}\bigcap\mathrm{Y}_{11})=H^0_{ét}(\mathrm{Sh}_{0,n},\rho_{n-1,1})$ *and* $H^{2n-2}_c(\mathrm{Y}_{00}\bigcap\mathrm{Y}_{01}-\mathrm{Y}_{00}\bigcap\mathrm{Y}_{11})=H^0_{ét}(\mathrm{Sh}_{0,n})$.

Proof. Under the isomorphism $\mathrm{Y}_{00}\cong\mathbf{Gr}\big(H_1^{\mathrm{dR}}(\mathcal{B}/\mathrm{Sh}_{0,n})_1^\circ,n-1\big)\times\mathbf{Gr}\big(H_1^{\mathrm{dR}}(\mathcal{B}'/\mathrm{Sh}_{0,n})_2^\circ,1\big)$ as in Proposition 9.9, we can easily check that

(1) The subscheme $\mathrm{Y}_{00}\bigcap\mathrm{Y}_{11}$ is isomorphic to a closed subscheme of $\mathbf{Gr}\big(H_1^{\mathrm{dR}}(\mathcal{B}/\mathrm{Sh}_{0,n})_1^\circ,n-1\big)\times\mathbf{Gr}\big(H_1^{\mathrm{dR}}(\mathcal{B}'/\mathrm{Sh}_{0,n})_2^\circ,1\big)$ over $\mathrm{Sh}_{0,n}$ satisfying that for any S-point (H,L), we have
$$L\subseteq F(H^{(p)})\subseteq H_1^{\mathrm{dR}}(\mathcal{B}'/\mathrm{Sh}_{0,n})_2^\circ,\quad V^{-1}(L^{(p)})\subseteq H\subseteq H_1^{\mathrm{dR}}(\mathcal{B}'/\mathrm{Sh}_{0,n})_1^\circ.$$

(2) The subscheme $\mathrm{Y}_{00}\bigcap\mathrm{Y}_{01}$ is isomorphic to a closed subscheme of $\mathbf{Gr}\big(H_1^{\mathrm{dR}}(\mathcal{B}/\mathrm{Sh}_{0,n})_1^\circ,n-1\big)\times\mathbf{Gr}\big(H_1^{\mathrm{dR}}(\mathcal{B}'/\mathrm{Sh}_{0,n})_2^\circ,1\big)$ over $\mathrm{Sh}_{0,n}$ satisfying that for any S-point (H,L), we have
$$L\subseteq F(H^{(p)})\subseteq H_1^{\mathrm{dR}}(\mathcal{B}'/\mathrm{Sh}_{0,n})_2^\circ.$$

(3) The subscheme $\mathrm{Y}_{00}\bigcap\mathrm{Y}_{11}$ is isomorphic to a closed subscheme of $\mathbf{Gr}\big(H_1^{\mathrm{dR}}(\mathcal{B}/\mathrm{Sh}_{0,n})_1^\circ,n-1\big)\times\mathbf{Gr}\big(H_1^{\mathrm{dR}}(\mathcal{B}'/\mathrm{Sh}_{0,n})_2^\circ,1\big)$ over $\mathrm{Sh}_{0,n}$ satisfying that for any S-point (H,L), we have
$$V^{-1}(L^{(p)})\subseteq H\subseteq H_1^{\mathrm{dR}}(\mathcal{B}'/\mathrm{Sh}_{0,n})_1^\circ.$$

Consider the partial Frobenius as in [11, Section 5,6], we define $\phi:\mathbf{Gr}\big(H_1^{\mathrm{dR}}(\mathcal{B}/\mathrm{Sh}_{0,n})_1^\circ,n-1\big)\times\mathbf{Gr}\big(H_1^{\mathrm{dR}}(\mathcal{B}'/\mathrm{Sh}_{0,n})_2^\circ,1\big)\to\mathbf{Gr}\big(H_1^{\mathrm{dR}}(\mathcal{B}/\mathrm{Sh}_{0,n})_2^\circ,n-1\big)\times\mathbf{Gr}\big(H_1^{\mathrm{dR}}(\mathcal{B}'/\mathrm{Sh}_{0,n})_2^\circ,1\big)$ mapping any S-point (H,L) to $(F(H^{(p)}),L)$, we have under the isomophism $\mathrm{Y}_{00}\bigcap\mathrm{Y}_{10}$ is mapped to the closed subscheme $\tilde{Y}$ of $\mathbf{Gr}\big(H_1^{\mathrm{dR}}(\mathcal{B}/\mathrm{Sh}_{0,n})_2^\circ,n-1\big)\times\mathbf{Gr}\big(H_1^{\mathrm{dR}}(\mathcal{B}'/\mathrm{Sh}_{0,n})_2^\circ,1\big)$ consisting of points (H,L) satisfying
$$L\subseteq H\subseteq H_1^{\mathrm{dR}}(\mathcal{B}/\mathrm{Sh}_{0,n})_2^\circ.$$
It is a $\mathbb{P}^{n-2}$-bundle over $\mathbf{Gr}\big(H_1^{\mathrm{dR}}(\mathcal{B}'/\mathrm{Sh}_{0,n})_2^\circ,1\big)=\mathbb{P}^{n-1}/\mathrm{Sh}_{0,n}$, which consists of points L such that $L\subseteq H_1^{\mathrm{dR}}(\mathcal{B}/\mathrm{Sh}_{0,n})_2^\circ$. We denote it by $X\cong\mathbb{P}^{n-1}$. Hence we get the first two equations via Kunnéth formula. In particular, we have $H^{2n-4}_{ét}(\mathrm{Y}_{00}\bigcap\mathrm{Y}_{10})=\bigoplus_{i=0}^{n-2}H^{2i}(X)\otimes H^{2(n-2-i)}(\mathbb{P}_{n-2})$. If we take η_1

be the class of $\mathcal{O}(1) = c_1\big(H_1^{\mathrm{dR}}(\mathcal{B}'/\mathrm{Sh}_{0,n})_2^\circ/\mathcal{H}\big)$, then $H_{\acute{e}t}^{2n-4}(\mathrm{Y}_{00}\bigcap \mathrm{Y}_{10}) = \bigoplus\limits_{i=0}^{n-2} H_{\acute{e}t}^{2i}(X)\eta_1^{n-2-i}$. Here we use $(\mathcal{H},\mathcal{L})$ to denote the universal vector bundles of $\tilde{Y}$.

Via ϕ, we see that $\mathrm{Y}_{00}\bigcap \mathrm{Y}_{11}$ is mapped to a closed subscheme W of $\mathbf{Gr}\big(H_1^{\mathrm{dR}}(\mathcal{B}/\mathrm{Sh}_{0,n})_2^\circ, n-1\big) \times \mathbf{Gr}\big(H_1^{\mathrm{dR}}(\mathcal{B}'/\mathrm{Sh}_{0,n})_2^\circ, 1\big)$ consisting of points (H,L) satisfying

$$L \subseteq H \subseteq H_1^{\mathrm{dR}}(\mathcal{B}/\mathrm{Sh}_{0,n})_2^\circ, \quad F(V^{-1}(L^{(p)})^{(p)}) \subseteq H \subseteq H_1^{\mathrm{dR}}(\mathcal{B}/\mathrm{Sh}_{0,n})_2^\circ.$$

Blowing up at points satisfying $L = F(V^{-1}(L^{(p)})^{(p)})$ where we denote the locus with T, we get a closed subscheme Z of $\mathbf{Gr}\big(H_1^{\mathrm{dR}}(\mathcal{B}/\mathrm{Sh}_{0,n})_2^\circ, n-1\big)\times\mathbf{Gr}\big(H_1^{\mathrm{dR}}(\mathcal{B}'/\mathrm{Sh}_{0,n})_2^\circ, 1\big)\times\mathbf{Gr}\big(H_1^{\mathrm{dR}}(\mathcal{B}'/\mathrm{Sh}_{0,n})_2^\circ, 2\big)$ with points (H,L,M) such that

$$L \cup F(V^{-1}(L^{(p)})^{(p)}) \subseteq M \subseteq H \subseteq H_1^{\mathrm{dR}}(\mathcal{B}/\mathrm{Sh}_{0,n})_2^\circ.$$

We denote the exceptional divisor by E, with points (H,L,M) such that $F(V^{-1}(L^{(p)})^{(p)}) = L$. It is a $\mathbb{P}^{n-3}$-bundle over the closed subscheme Y of $\mathbf{Gr}\big(H_1^{\mathrm{dR}}(\mathcal{B}/\mathrm{Sh}_{0,n})_2^\circ, 2\big) \times \mathbf{Gr}\big(H_1^{\mathrm{dR}}(\mathcal{B}'/\mathrm{Sh}_{0,n})_2^\circ, 1\big)$ consisting of points (M,L) such that

$$L \cup F(V^{-1}(L^{(p)})^{(p)}) \subseteq M \subseteq H_1^{\mathrm{dR}}(\mathcal{B}/\mathrm{Sh}_{0,n})_2^\circ,$$

which is blowing-up of $\mathbf{Gr}\big(H_1^{\mathrm{dR}}(\mathcal{B}'/\mathrm{Sh}_{0,n})_2^\circ, 1\big) = \mathbb{P}^{n-1}/\mathrm{Sh}_{0,n}$. It is just the scheme X we have defined above. We also denote the rational locus of X to be T'' which is isomorphic to $\mathbb{P}^{n-1}(\mathbb{F}_{p^2})$. We denote the exceptional divisor by T', which consists of points (M,L) such that $L = F(V^{-1}(L^{(p)})^{(p)})$. It is easy to see $T' \cong \coprod\limits_{\#\mathbb{P}^{n-1}(\mathbb{F}_{p^2})} \mathbb{P}^{n-2}/\mathrm{Sh}_{0,n}$.

By blowing-up sequence, we have for any $i \geq 0$,

$$\begin{array}{ccccc} & & & & \cdots \\ H_{\acute{e}t}^i(X) & \longrightarrow & H_{\acute{e}t}^i(Y) \oplus H_{\acute{e}t}^i(T'') & \longrightarrow & H_{\acute{e}t}^i(T') \\ \cdots & & & & \end{array}$$

Since $H_{\acute{e}t}^i(\mathbb{P}^n) = \mathbb{F}_l$ if i is even such that $0 \geq i \leq n$ and 0 otherwise, we have $H_{\acute{e}t}^i(Y) = 0$ if i is odd or $i \geq 2n-1$. Let $\begin{bmatrix} n \\ 1 \end{bmatrix}_{p^2} = \frac{p^n-1}{p-1} = \#\mathbb{P}^{n-1}(\mathbb{F}_{p^2})$. Consider the GL-action on $\coprod\limits_{\#\mathbb{P}^{n-1}(\mathbb{F}_{p^2})} \mathbb{P}^{n-2}/\mathrm{Sh}_{0,n}$ by switching the points in $\mathbb{P}^{n-1}(\mathbb{F}_{p^2})$, we get $H_{\acute{e}t}^i(\coprod\limits_{\#\mathbb{P}^{n-1}(\mathbb{F}_{p^2})} \mathbb{P}^{n-2}/\mathrm{Sh}_{0,n}) = H_{\acute{e}t}^0(\mathrm{Sh}_{0,n}, \mathrm{Ind}_{K_p}^{K_\mathfrak{p}^1}\mathbb{1}) = H_{\acute{e}t}^0(\mathrm{Sh}_{0,n}(K_\mathfrak{p}^1), \mathbb{F}_l)$, with the first equation coming from the Leray Spectral Sequence. Thus, we have $H^i(X) = H^0(\mathrm{Sh}_{0,n})$ for $0 \leq i \leq 2n-2, i$ is even and is equal to 0 otherwise. We also have $H_{\acute{e}t}^{2i}(T') = H_{\acute{e}t}^0(\mathrm{Sh}_{0,n}(K_\mathfrak{p}^1), \mathbb{F}_l)$ for $0 \leq i \leq 2n-4, i$ is even and is equal to 0 otherwise. Similar $H_{\acute{e}t}^i(T'')$ is not zero if and only if $i = 0$ and $H_{\acute{e}t}^0(T'') = H_{\acute{e}t}^0(\mathrm{Sh}_{0,n}(K_\mathfrak{p}^1), \mathbb{F}_l)$.

If $i = 0$, we have the exact sequence

$$0 \longrightarrow H_{\acute{e}t}^0(X) \longrightarrow H_{\acute{e}t}^0(Y) \oplus H_{\acute{e}t}^0(T'') \longrightarrow H_{\acute{e}t}^0(T') \longrightarrow 0 \ .$$

If $0 < i \leq 2n-4$, we have the exact sequence

$$0 \longrightarrow H_{\acute{e}t}^i(X) \longrightarrow H_{\acute{e}t}^i(Y) \longrightarrow H_{\acute{e}t}^i(T'') \longrightarrow 0$$

and if $i = 2n-2$, we have $H_{\acute{e}t}^{2n-2}(Y) = H_{\acute{e}t}^0(X)$. Thus we have $H_{\acute{e}t}^0(Y) = H_{\acute{e}t}^0(\mathrm{Sh}_{0,n})$ and $H_{\acute{e}t}^{2i}(Y) = H_{\acute{e}t}^{2i}(X) \oplus H_{\acute{e}t}^{2i}(T'') = H_{\acute{e}t}^0(\mathrm{Sh}_{0,n}) \oplus H_{\acute{e}t}^0(\mathrm{Sh}_{1,n-1}(K_\mathfrak{p}^1), \mathbb{F}_l)$ for $1 \leq i \leq n-2$.

Since Z is the $\mathbb{P}^{n-3}$-bundle over Y, we have $H^*_{\acute{e}t}(Z) = H^*_{\acute{e}t}(Y) \otimes H^*_{\acute{e}t}(\mathbb{P}^{n-3})$. In paritcular, $H^{2n-4}_{\acute{e}t}(Z) = \bigoplus_{i=1}^{n-2} H^{2i}(Y) \otimes H^{2(n-2-i)}_{\acute{e}t}(\mathbb{P}^{n-3})$. Since the $\mathbb{P}_{n-3}$ corresponds to the choice of $M \subseteq H$ for any point (M, L), we can express $H^{2n-4}_{\acute{e}t}(Z)$ as $\bigoplus_{i=1}^{n-2} H^{2i}_{\acute{e}t}(Y)\eta_1^{n-2-i}$, where η_1 corresponds to $c_1\big(H_1^{\mathrm{dR}}(\mathcal{B}'/\mathrm{Sh}_{0,n})_2^{\circ}/\mathcal{H}\big)$ and $(\mathcal{H}, \mathcal{L}, \mathfrak{m})$ is the universal bundle of Z.

Moreover, by the blowing-up exact sequence, we have

$$\begin{array}{ccccc} & & \cdots & \longrightarrow & H^{2n-5}_{\acute{e}t}(E) \\ H^{2n-4}_{\acute{e}t}(W) & \longrightarrow & H^{2n-4}_{\acute{e}t}(Z) \oplus H^{2n-4}_{\acute{e}t}(T) & \longrightarrow & H^{2n-4}_{\acute{e}t}(E) \\ H^{2n-3}_{\acute{e}t}(W) & \longrightarrow & H^{2n-3}_{\acute{e}t}(Z) \oplus H^{2n-3}_{\acute{e}t}(T) & \longrightarrow & \cdots \end{array}$$

where $H^{2n-5}_{\acute{e}t}(E) = 0$ and $H^{2n-3}_{\acute{e}t}(Z) \oplus H^{2n-3}_{\acute{e}t}(T)$. Since $T = \bigsqcup_{\#\mathbb{P}^{n-1}(\mathbb{F}_{p^2})} \mathbb{P}^{n-2}/\mathrm{Sh}_{0,n}$, the GL-action on T by switching the points in $\mathbb{P}^{n-1}(\mathbb{F}_{p^2})$ shows $H^{2n-4}_{\acute{e}t}(T) = H^0_{\acute{e}t}(\mathrm{Sh}_{0,n}(K^1_{\mathfrak{p}}), \mathbb{F}_l)$.

We have denoted the locus of Y which consists of points (M, L) such that $L = F(V^{-1}(L)^{(p)})^{(p)}$ by T'. Then E is the $\mathbb{P}_{n-3}$-bundle over T', we have $H^*_{\acute{e}t}(E) = H^*_{\acute{e}t}(T') \otimes H^*_{\acute{e}t}(\mathbb{P}^{n-3})$. In particular, we have $H^{2n-5}_{\acute{e}t}(E) = 0$ and $H^{2n-4}_{\acute{e}t}(E) = \bigoplus_{i=1}^{n-2} H^{2i}_{\acute{e}t}(T') \otimes H^{2(n-2-i)}_{\acute{e}t}(\mathbb{P}^{n-3})$. By the same reason as above, we can express $H^{2n-4}_{\acute{e}t}(E)$ as $\bigoplus_{i=1}^{n-2} H^{2i}(T')\eta_1^{n-2-i}$, where η_1 corresponds to $c_1\big(H_1^{\mathrm{dR}}(\mathcal{B}'/\mathrm{Sh}_{0,n})_2^{\circ}/\mathcal{H}\big)$ the same as above. We have $H^i_{\acute{e}t}(T') = H^i(\bigsqcup_{\#\mathbb{P}^{n-1}(\mathbb{F}_{p^2})} \mathbb{P}^{n-2}/\mathrm{Sh}_{0,n})$ and $H^{2i}_{\acute{e}t}(Y) = H^0_{\acute{e}t}(\mathrm{Sh}_{0,n}) \oplus H^{2i}(T')$. Thus the map $H^{2n-4}_{\acute{e}t}(Z) \to H^{2n-4}(E)$ corresponds to the map $H^{2i}_{\acute{e}t}(Y) \to H^{2i}_{\acute{e}t}(T')$ for each $1 \le i \le n-2$.

Similarly, E is the $\mathbb{P}^{n-3}$-bundle over T, we have $H^*_{\acute{e}t}(E) = H^*_{\acute{e}t}(T) \otimes H^*_{\acute{e}t}(\mathbb{P}^{n-3})$. In particular, we have $H^{2n-4}_{\acute{e}t}(E) = \bigoplus_{i=1}^{n-2} H^{2i}(T) \otimes H^{2(n-2-i)}_{\acute{e}t}(\mathbb{P}^{n-3})$. By the same reason as above, we can express $H^{2n-4}_{\acute{e}t}(E)$ as $\bigoplus_{i=1}^{n-2} H^{2i}_{\acute{e}t}(T)\eta_2^{n-2-i}$, where η_2 corresponds to $c_1((\frac{\mathfrak{m}}{\mathcal{L}})^\vee)$ with $(\mathcal{H}, \mathcal{L}, \mathfrak{m})$ the universal bundle of Z.

Furthermore, both T, T' can be viewed as $\mathbb{P}_{n-2}$-bundle over T''. Thus, we have for $1 \le i \le n-2$, $H^{2i}_{\acute{e}t}(T) = H^0_{\acute{e}t}(T'')\eta_1^i$ and $H^{2i}_{\acute{e}t}(T') = H^0_{\acute{e}t}(T'')\eta_2^i$. In this way, $H^{2n-4}_{\acute{e}t}(E) = \bigoplus_{i=1}^{n-2} H^0(T'')\eta_2^i\eta_1^{n-2-i}$. The map $H^{2n-4}_{\acute{e}t}(T) \to H^{2n-4}_{\acute{e}t}(E)$ corresponds to the map $H^0_{\acute{e}t}(T'')\eta_1^{n-2} \to H^{2n-4}_{\acute{e}t}(E)$. The map $H^{2n-4}_{\acute{e}t}(Z) \to H^{2n-4}_{\acute{e}t}(E)$ corresponds to the map $\bigoplus_{i=1}^{n-2}(H^{2i}_{\acute{e}t}(X) \oplus H^0_{\acute{e}t}(T'')\eta_2^i)\eta_1^{n-2-i} \to H^{2n-4}_{\acute{e}t}(E)$.

Thus we can see easily the map $H^{2n-4}_{\acute{e}t}(Z) \oplus H^{2n-4}_{\acute{e}t}(T) \to H^{2n-4}_{\acute{e}t}(E)$ is surjective and the cohomology group $H^{2n-4}_{\acute{e}t}(\mathrm{Y}_{00} \bigcap \mathrm{Y}_{11}) = \bigoplus_{i=1}^{n-2} H^{2i}_{\acute{e}t}(X)\eta_1^{n-2-i} \bigoplus H^0_{\acute{e}t}(T'')\eta_1^{n-2} = H^0_{\acute{e}t}(\mathrm{Sh}_{0,n}, \mathbb{F}_l)^{n-2} \oplus H^0_{\acute{e}t}(\mathrm{Sh}_{0,n}(K^1_{\mathfrak{p}}), \mathbb{F}_l)$. And by excision sequence we have $H^{2n-3}_c(\mathrm{Y}_{00} \bigcap \mathrm{Y}_{10} - \mathrm{Y}_{00} \bigcap \mathrm{Y}_{11})$ is the cokernel of the map $H^{2n-4}_{\acute{e}t}(\mathrm{Y}_{00} \bigcap \mathrm{Y}_{10}) \to H^{2n-4}_{\acute{e}t}(\mathrm{Y}_{00} \bigcap \mathrm{Y}_{11})$, which is the map $\bigoplus_{i=0}^{n-2} H^{2i}_{\acute{e}t}(X)\eta_1^{n-2-i} \to$

$\bigoplus_{i=1}^{n-2} H^{2i}_{\acute{e}t}(X)\eta_1^{n-2-i} \bigoplus H^0_{\acute{e}t}(T'')\eta_1^{n-2}$. Note that if $l \mid \#\mathbb{P}_{n-1}(\mathbb{F}_{p^2}) = \frac{p^{2n-2}-1}{p^2-1}$, $\mathrm{Ind}_{K_p}^{K^1_{\mathfrak{p}}} \mathbb{1} = \mathbb{1} \oplus \rho_{(n-1,1)}$. Hence it is equal to $H^0_{\acute{e}t}(\mathrm{Sh}_{0,n}, \rho_{(n-1,1)})$.

Similar as above, we have $H^{2n-2}(\mathrm{Y}_{00} \bigcap \mathrm{Y}_{11}) = \bigoplus_{i=2}^{n-1} H^{2i}_{\acute{e}t}(X)\eta_1^{n-1-i} = H^0_{\acute{e}t}(\mathrm{Sh}_{0,n})^{\oplus n-2}$. Moreover, $H^{2n-2}_{\acute{e}t}(\mathrm{Y}_{00} \bigcap \mathrm{Y}_{10}) = \bigoplus_{i=1}^{n-1} H^{2i}_{\acute{e}t}(X)\eta_1^{n-1-i}$. Thus be excision sequence, we have $H^{2n-2}_c(\mathrm{Y}_{00} \bigcap \mathrm{Y}_{10} - \mathrm{Y}_{00} \bigcap \mathrm{Y}_{11}) = H^2_{\acute{e}t}(X)\eta_1^{n-2} = H^0_{\acute{e}t}(\mathrm{Sh}_{0,n})$.

$\square$

Remark 9.13. We omit the Galois action during the computation. It should be noted that if we consider the Galois action, then if $l \nmid \frac{p^{2n-2}-1}{p^2-1}$, we have $H^{2n-3}_c(\mathrm{Y}_{00} \bigcap \mathrm{Y}_{01} - \mathrm{Y}_{00} \bigcap \mathrm{Y}_{11}) = H^{2n-3}_c(\mathrm{Y}_{00} \bigcap \mathrm{Y}_{10} - \mathrm{Y}_{00} \bigcap \mathrm{Y}_{11}) = H^0_{\acute{e}t}(\mathrm{Sh}_{0,n}, \rho_{n-1,1}(2-n))$.

We have shown that for any point $(A, \lambda, \eta, A', \lambda', \eta', \phi) \in \mathrm{Y}_{00}$, the abelian varieties A, A' are supersingular. Now we are going to give an equivalent condition of when points in $\mathrm{Sh}_{1,n-1}(K^1_{\mathfrak{p}})$ are supersingular First we give an equivalent condition of when points lie in $\mathrm{Y}_{11} \backslash (\mathrm{Y}_{00} \bigcup \mathrm{Y}_{10} \bigcup \mathrm{Y}_{01})$.

Lemma 9.14. *For any S-point $(A, \lambda, \eta, A', \lambda', \eta', \phi)$, it lies in $\mathrm{Y}_{11} \backslash (\mathrm{Y}_{00} \bigcup \mathrm{Y}_{10} \bigcup \mathrm{Y}_{01})$ if and only if there exists $\alpha \in p\tilde{\mathcal{D}}(A')^\circ_1$ such that:*

- $p\phi_{*,1}\tilde{\mathcal{D}}(A)^\circ_1 + W(\overline{\mathbb{F}}_p)\alpha = p\phi_{*,1}\tilde{\mathcal{D}}(A')^\circ_1$,
- $F\alpha - V\alpha \in p\phi_{*,2}\tilde{\mathcal{D}}(A)^\circ_2$,
- $\alpha \in \phi_{*,1}\tilde{\mathcal{D}}(A)^\circ_1 \bigcap \phi_{*,1}F^{-1}V\tilde{\mathcal{D}}(A)^\circ_1$
- $\alpha \notin \phi_{*,1}V\tilde{\mathcal{D}}(A)^\circ_2, \phi_{*,1}F\tilde{\mathcal{D}}(A)^\circ_2$.

Proof. For the 'if' part, if there exists $\alpha \in p\tilde{\mathcal{D}}(A')^\circ_1$ such that $p\phi_{*,1}\tilde{\mathcal{D}}(A)^\circ_1 + W(\overline{\mathbb{F}}_p)\alpha = p\phi_{*,1}\tilde{\mathcal{D}}(A')^\circ_1$, $F\alpha - V\alpha \in p\phi_{*,2}\tilde{\mathcal{D}}(A)^\circ_2$, $\alpha \in \phi_{*,1}\tilde{\mathcal{D}}(A)^\circ_1 \bigcap \phi_{*,1}F^{-1}V\tilde{\mathcal{D}}(A)^\circ_1$ $\alpha \notin \phi_{*,1}V\tilde{\mathcal{D}}(A)^\circ_2, \phi_{*,1}F\tilde{\mathcal{D}}(A)^\circ_2$. then we can take we take $\tilde{\mathcal{E}}_1 = \tilde{\mathcal{D}}(A)_1 + W(\overline{\mathbb{F}}_p)\phi^{-1}_{*,1}(p^{-1}\alpha)$ and $\tilde{\mathcal{E}}_2 = W(\overline{\mathbb{F}}_p)(V^{-1}(\phi^{-1}_{*,1}(\alpha))) + \tilde{\mathcal{D}}(A)_2$. Applying Proposition 5.1.2, we get an triple (A', λ', η') and an $\mathcal{O}_D$-equivariant p-quasi-isogeny $\phi : A \to A'$, where A' is an abelian variety over k with an action of $\mathcal{O}_D$, λ' is a prime-to-p polarization on A', and η' is a prime-to-p level structure on A', such that $\phi^\vee \circ \lambda' \circ \phi = \lambda$, $\eta' = \phi \circ \eta$ and such that $\phi_{*,i} : \tilde{\mathcal{D}}(A)^\circ_i \to \tilde{\mathcal{D}}(A')^\circ_i$ are naturally identified with the inclusion $\tilde{\mathcal{D}}(A)^\circ_i \to \tilde{\mathcal{E}}_i$ for $i = 1, 2$. Moreover, the dimension formula (5.1.3) implies that $\omega^\circ_{A'^\vee/k,1}$ has dimension 1, and $\omega^\circ_{A'^\vee/k,2}$ has dimension $n-1$. Therefore, (A', λ', η') is a point of $\mathrm{Sh}_{1,n-1}$ and $(A, \lambda, \eta, A', \lambda', \eta', \phi)$ is a point of $\mathrm{Y}_{11} \backslash (\mathrm{Y}_{00} \bigcup \mathrm{Y}_{10} \bigcup \mathrm{Y}_{01})$ since $\omega^\circ_{A^\vee,1} \neq \mathrm{Ker}(\phi_{*,1})$ and $\mathrm{Ker}(\phi_{*,2}) \neq \omega^\circ_{A^\vee,2}$.

Conversely, if $(A, \lambda, \eta, A', \lambda', \eta', \phi)$ is a point of $\mathrm{Y}_{11} \backslash (\mathrm{Y}_{00} \bigcup \mathrm{Y}_{10} \bigcup \mathrm{Y}_{01})$, we can take $\alpha \in p\tilde{\mathcal{D}}(A')^\circ_1$ such that $p\phi_{*,1}\tilde{\mathcal{D}}(A)^\circ_1 + W(\overline{\mathbb{F}}_p)\alpha = p\phi_{*,1}\tilde{\mathcal{D}}(A')^\circ_1$. Then since $\omega^\circ_{A^\vee,1} \neq \mathrm{Ker}(\phi_{*,1})$ and $\mathrm{Ker}(\phi_{*,2}) \neq \omega^\circ_{A^\vee,2}$, we have $p\tilde{\mathcal{D}}(A')_1 \neq \phi_{*,1}V\tilde{\mathcal{D}}(A)_2$ and $V\tilde{\mathcal{D}}(A')_1 \neq \phi_{*,2}\tilde{\mathcal{D}}(A)_2$. Thus $W(\overline{\mathbb{F}}_p)\alpha + \phi_{*,1}(V\tilde{\mathcal{D}}(A)_2) = V\tilde{\mathcal{D}}(A')^\circ_2, W(\overline{\mathbb{F}}_p)(p^{-1}\alpha) + \phi_{*,1}\tilde{\mathcal{D}}(A)_1 = \tilde{\mathcal{D}}(A')^\circ_1$ and $W(\overline{\mathbb{F}}_p)(F\alpha) + \phi_{*,2}(p\tilde{\mathcal{D}}(A)_2) = p\tilde{\mathcal{D}}(A')^\circ_2$. Moreover, we have $W(\overline{\mathbb{F}}_p)(F^{-1}\alpha) + \phi_{*,2}(V\tilde{\mathcal{D}}(A))_1 = V\tilde{\mathcal{D}}(A')^\circ_1$ and $W(\overline{\mathbb{F}}_p)(V^{-1}\alpha) + \phi_{*,2}\tilde{\mathcal{D}}(A)_2 = \tilde{\mathcal{D}}(A')^\circ_2$. Moreover, we have $F^{-1}\alpha - V^{-1}\alpha = xV^{-1}\alpha$ for some $x \in W(\overline{\mathbb{F}}_p)$ since $V\tilde{\mathcal{D}}(A')_1 \neq \phi_{*,2}\tilde{\mathcal{D}}(A)_2$ and $\phi_{*,2}\tilde{\mathcal{D}}(A)_2$ has corank 1 in $\tilde{\mathcal{D}}(A')_2$. If $x \notin pW(\overline{\mathbb{F}}_p)$, we can modify α by $y\alpha$ for some $y \in W(\overline{\mathbb{F}}_p)$ such that $F^{-1}(y\alpha) - V^{-1}(y\alpha) = y^{\sigma^{-1}}F^{-1}\alpha - y^\sigma V^{-1}\alpha = ((x+1)y^{\sigma^{-1}} - y^\sigma)V^{-1}\alpha$. Take y such that $(x+1)y^{\sigma^{-1}} - y^\sigma \in pW(\overline{\mathbb{F}}_p)$ we get $F^{-1}(y\alpha) - V^{-1}(y\alpha) \in p\phi_{*,2}\tilde{\mathcal{D}}(A)^\circ_2$. Substituting α with $y\alpha$, we finish the proof. $\square$

We have a proposition to help us to show when points in $\mathrm{Sh}_{1,n-1}(K^1_{\mathfrak{p}})$ is supersingular. First, we need some definitions.

Definition 9.15. Let k be a perfect field of characteristic p and $W(k)$ be the witt vector ring corresponding to k. Supoose (P, π) is a $Q(k) = Frac(W(k))$-isocrystal defined in 8.6. We say (P, π) is average of slope 0 if there exist(thus for every) some (full) lattice H in P such that $\ell(H/H \bigcap \pi(H)) = \ell(\pi(H)/H \bigcap \pi(H)) \leq 1$. We say (P, π) is pure of slope 0 if P admits a π-invariant (full) lattice.

Definition 9.16. Let k be a perfect field of characteristic p and $W(k)$ be the witt vector ring corresponding to k. Supoose (P, π) is a $Q(k) = Frac(W(k))$-isocrystal average of slope 0 and $H \subseteq P$ is a sublattice. Suppose $\ell(H/H \bigcap \pi(H)) = \ell(\pi(H)/H \bigcap \pi(H)) \leq 1$. Let $Lat_{\leq 1}(P)$ to be the set of H satisfies the above conditions.For $i \geq 0$, we define $S_i(H) = \sum_{j=0}^{i} \pi^j(H)$ and $T_i(H) = \bigcap_{j=0}^{i} \pi^j(H)$. Moreover, we define $S_\infty(H) = \lim_{i \to \infty} S_i(H)$ and $T_\infty(H) = \lim_{i \to \infty} T_i(H)$. We define $s(H) = \inf\{s | S_s(H) = S_\infty(H)\}$ and $t(H) = \inf\{t | T_t(H) = T_\infty(H)\}$.

With defintions above, we have the following lemma:

Lemma 9.17. *Assume (P, π) is a $Q(k) = Frac(W(k))$-isocrystal average of slope 0. Let $H \in Lat_{\leq}(P)$. Then we have:*

(1) $s(H) = 0 \iff t(H) = 0 \iff H = \pi(H)$,
(2) *S_i, T_i commute with π and mulitplication by p. So $s(H) = s(\pi(H)) = s(pH)$,*
(3) *For $0 \leq i, j < \infty$, $S_i(H), T_i(H) \in Lat_{\leq 1}(P)$ and $S_i(S_j(H)) = S_{i+j}(H), T_i(T_j(H)) = T_{i+j}(H)$,*
(4) *If (P, π) is pure of slope 0, then $s(H), t(H) \leq \operatorname{rank}(P) - 1$. Otherwise, $s(H) = t(H)$.*
(5) *Let $0 \leq i, j < \infty$. Then*

$$T_j(S_i(H)) = \begin{cases} S_\infty(H), & \text{if } i \geq s(H); \\ \pi^j(S_{i-j}(H)), & \text{if } j \leq i < s(H); \\ \pi^i(T_{j-i}(H)), & \text{if } i < s(H) \text{ and } i < j < i + t(H); \\ T_\infty(H), & \text{if } i < s(H) \text{ and } j \geq i + t(H); \end{cases}$$

So $t(S_i(H)) = t(H) + i$ if $0 \leq i < s(H)$.

Proof. It is easy to check (1)-(2). For (4), if (P, π) is pure of slope 0, then $S_{\operatorname{rank}(P)-1}(H)$ is π-invariant by [22, Proposition 2.17], so $s(H) \leq \operatorname{rank}(P) - 1$. Otherwise, by our definition, P has no π-invariant lattice, so $S_i(H) \subsetneq S_{i+1}(H)$ for every $i < \infty$, that is, $s(H) = \infty$.

For (3), first prove by induction on $0 \leq i < s(H)$ that $\ell(S_{i+1}(H)/S_i(H)) = 1$. For $i = 0$,it follows from $H \in Lat_{\leq 1}(P)$. For $0 < i < s(H)$, $S_i(H) \neq \pi(S_i(H))$ and $\pi(S_{i-1}(H)) \subseteq S_i(H) \bigcap \pi(S_{i-1}(H)) \subsetneq \pi(S_i(H))$. By the inductive bypothesis, $\ell(S_i(H)/S_{i-1}(H)) = 1$ and thus $\ell(\pi(S_i(H))/\pi(S_{i-1}(H))) = 1$. It forces that

$$\pi(S_{i-1}(H)) = S_i(H) \bigcap \pi(S_i(H)) \text{ and } \ell(\pi(S_i(H))/S_i(H) \bigcap (S_i(H))) = 1.$$

So $\ell(\pi(S_{i+1}(H))/\pi(S_i(H))) = \ell(\pi(S_i(H))/S_i(H) \bigcap (S_i(H))) = 1$. This completes the induction. It follows immediately that $S_i(H) \in Lat_{\leq 1}(P)$. The other assertion of (3) is clear.

We have seen $\pi(S_{i-1}(H)) = T_1(S_i(H))$ for $0 < i < s(H)$. So $T_i(S_j(H)) = \pi^j(S_{i-j}(H))$ for $j \leq i < s(H)$. In particular, $T_i(S_i(H)) = \pi^i(H)$ for $i < s(H)$. So for $i < j < \infty$, $T_j(S_i(H)) = T_{j-i}(T_i S_i(H)) = T_{i-j}(\pi^i(H)) = \pi^i(T_{j-i}(H))$. So we get (5). ☐

Now we state the proposition we need:

Proposition 9.18. *For any S-point $(A, \lambda, \eta, A', \lambda', \eta', \phi) \in \mathrm{Sh}_{1,n-1}(K_{\mathfrak{p}}^1)$, A, A' are supersingular if $S_\infty(V\tilde{\mathcal{D}}(A')_2^\circ) \subseteq T_\infty(\phi_{*,1}\tilde{\mathcal{D}}(A)_1^\circ)$ or $S_\infty(p\tilde{\mathcal{D}}(A')_2^\circ) \subseteq T_\infty(\phi_{*,2}(V\tilde{\mathcal{D}}(A)_1^\circ))$.*

Proof. Since A, A' are supersingular, the isocrystals $(\tilde{\mathcal{D}}(A)_i^\circ \otimes_{W(k)} Q(k), FV^{-1} \otimes 1), (\tilde{\mathcal{D}}(A')_i^\circ \otimes_{W(k)} Q(k), FV^{-1} \otimes 1)$ are pure of slope 0 for $i = 1, 2$. Let $\pi = FV^{-1}$. If $S_\infty(V\tilde{\mathcal{D}}(A')_2^\circ) \subseteq T_\infty(\phi_{*,1}\tilde{\mathcal{D}}(A)_1^\circ)$, we take $\tilde{\mathcal{E}}_1 \subseteq \tilde{\mathcal{D}}(A)_1^\circ$ to be an π-invariant lattice such that $S_\infty(V\tilde{\mathcal{D}}(A')_2^\circ) \subseteq \tilde{\mathcal{E}}_1 \subseteq T_\infty(\phi_{*,1}\tilde{\mathcal{D}}(A)_1^\circ)$, which always exists since $\tilde{\mathcal{D}}(A)_1^\circ$ has a π-invariant full lattice. Suppose $\ell(\phi_{*,1}(\tilde{\mathcal{D}}(A)_1^\circ)/\tilde{\mathcal{E}}) = i - 1$. Let $\tilde{\mathcal{E}}_2 = V\tilde{\mathcal{E}}_1$. Applying PropositionProposition 5.1.1 with $m = 1$, we get an triple (B, λ'', η'') and an $\mathcal{O}_D$-equivariant isogeny $\psi : B \to A, \psi' : B \to A'$, where B is an abelian variety over k with an action of $\mathcal{O}_D$, λ'' is a prime-to-p polarization on B, and η'' is a prime-to-p level structure on B, such that $\psi^\vee \circ \lambda \circ \psi = p\lambda'', \psi'^\vee \circ \lambda' \circ \psi' = p\lambda''$, $\eta = \psi \circ \eta'', \eta' = \psi' \circ \eta''$ and such that $\psi_{*,i} : \tilde{\mathcal{D}}(B)_i^\circ \to \tilde{\mathcal{D}}(A)_i^\circ, \psi'_{*,i} : \tilde{\mathcal{D}}(B)_i^\circ \to \tilde{\mathcal{D}}(A')_i^\circ$ are naturally identified with the inclusion $\tilde{\mathcal{E}}_i \hookrightarrow \tilde{\mathcal{D}}(A)_i^\circ, \tilde{\mathcal{E}}'_i \hookrightarrow \tilde{\mathcal{D}}(A')_i^\circ$ for $i = 1, 2$. Moreover, the dimension formula (5.1.2) implies that $\omega^\circ_{B^\vee/k,1}$ has dimension 0, and $\omega^\circ_{B^\vee/k,2}$ has dimension n. Therefore, (B, λ'', η'') is a point of $\mathrm{Sh}_{0,n}$. We have $(A, \lambda, \eta, B, \lambda'', \eta'') \in \mathrm{Y}_i, (A', \lambda', \eta', B, \lambda'', \eta'') \in \mathrm{Y}_{i+1}$. By Proposition 3.4.2, we get A, A' are supersingular.

The case $S_\infty(p\tilde{\mathcal{D}}(A')_2^\circ) \subseteq T_\infty(\phi_{*,2}(V\tilde{\mathcal{D}}(A)_1^\circ))$ is similar and we omit here. Hence we finish the proof.

□

Now we give the proposition decribing the supersingular locus $\mathrm{Sh}_{1,n-1}(K_{\mathfrak{p}}^1)^{ss}$.

Proposition 9.19. *With notations as above, we have* $\mathrm{Sh}_{1,n-1}(K_{\mathfrak{p}}^1)^{ss} = \mathrm{pr}_n(C_n) \bigcup \bigcup_{i=1}^{n-1} \mathrm{pr}_i(C_i) = \mathrm{Y}_{00} \bigcup \bigcup_{i=1}^{n-1} \mathrm{pr}_i(C_i)$.

Proof. We have seen $\mathrm{Y}_{00} \subseteq \mathrm{Sh}_{1,n-1}(K_{\mathfrak{p}}^1)^{ss}$. For any S-point $(A, \lambda, \eta, A', \lambda', \eta', \phi) \in \mathrm{Y}_{10}$ or Y_{01}, we show if A, A' are supersingular, $(A, \lambda, \eta, A', \lambda', \eta', \phi) \in \bigcup_{i=1}^{n-1} \mathrm{pr}_i(C_i)$. For simplicity, we only prove for Y_{10}. The case for Y_{01} is quite the same. If $(A, \lambda, \eta, A', \lambda', \eta', \phi) \in \mathrm{Y}_{10}$, then $\omega^\circ_{A'^\vee,1} = \mathrm{Im}(\phi_{*,1})$ and $\mathrm{Ker}(\phi_{*,2}) = \omega^\circ_{A^\vee,2}$. Thus $V\tilde{\mathcal{D}}(A')_2^\circ \subseteq \phi_{*,1}\tilde{\mathcal{D}}(A)_1^\circ$ and $V\tilde{\mathcal{D}}(A)_1^\circ = \phi_{*,2}(\tilde{\mathcal{D}}(A)_2^\circ)$. If $p\tilde{\mathcal{D}}(A')_1^\circ = \phi_{*,1}V\tilde{\mathcal{D}}(A)_2^\circ$, then $F\tilde{\mathcal{D}}(A)_2^\circ = V\tilde{\mathcal{D}}(A)_2^\circ$. That is, $A \in \mathrm{Y}_n$. Hence A, A' are supersingular. If $p\tilde{\mathcal{D}}(A')_1^\circ \neq \phi_{*,1}V\tilde{\mathcal{D}}(A)_2^\circ$, we have $\phi_{*,1}V\tilde{\mathcal{D}}(A)_2^\circ \in Lat_{\leq 1}(\tilde{\mathcal{D}}(A)_1^\circ \otimes_{W(k)} Q(k))$ and $\phi_{*,1}(p\tilde{\mathcal{D}}(A)_1^\circ) = \phi_{*,1}(V\tilde{\mathcal{D}}(A)_2^\circ) \bigcap \phi_{*,1}(F\tilde{\mathcal{D}}(A)_2^\circ) = \phi_{*,1}(V\tilde{\mathcal{D}}(A)_2^\circ) \bigcap \phi_{*,1}(\pi V\tilde{\mathcal{D}}(A)_2^\circ)$. Thus we can get $t(\phi_{*,1}(\tilde{\mathcal{D}}(A)_1^\circ)) = t(\phi_{*,1}(p\tilde{\mathcal{D}}(A)_1^\circ)) = t(\phi_{*,1}(V\tilde{\mathcal{D}}(A)_2^\circ)) - 1$. Hence $T_{t(\phi_{*,1}(\tilde{\mathcal{D}}(A)_1^\circ))}(\phi_{*,1}(V\tilde{\mathcal{D}}(A)_2^\circ)) \subseteq T_{t(\phi_{*,1}(\tilde{\mathcal{D}}(A)_1^\circ))}(\phi_{*,1}(\tilde{\mathcal{D}}(A)_1^\circ))$. Therefore $S_\infty(\phi_{*,1}(V\tilde{\mathcal{D}}(A)_2^\circ)) = S_\infty(T_{t(\phi_{*,1}(\tilde{\mathcal{D}}(A)_1^\circ))}(\phi_{*,1}(V\tilde{\mathcal{D}}(A)_2^\circ))) \subseteq S_\infty(T_{t(\phi_{*,1}(\tilde{\mathcal{D}}(A)_1^\circ))}(\phi_{*,1}(\tilde{\mathcal{D}}(A)_1^\circ))) = T_\infty(\phi_{*,1}(\tilde{\mathcal{D}}(A)_1^\circ))$ by Lemma 9.17. This shows $s(\phi_{*,1}(V\tilde{\mathcal{D}}(A)_2^\circ)) + t(\phi_{*,1}(\tilde{\mathcal{D}}(A)_1^\circ)) \leq n - 1$. Hence $s(\phi_{*,1}(V\tilde{\mathcal{D}}(A)_2^\circ)) + t(\phi_{*,1}(V\tilde{\mathcal{D}}(A)_2^\circ)) \leq n$. Since we have $V\tilde{\mathcal{D}}(A)_2^\circ = \phi_{*,1}(V\tilde{\mathcal{D}}(A)_2^\circ) \bigcup \phi_{*,1}(F\tilde{\mathcal{D}}(A)_2^\circ) = \phi_{*,1}(V\tilde{\mathcal{D}}(A)_2^\circ) \bigcup \phi_{*,1}(\pi V\tilde{\mathcal{D}}(A)_2^\circ)$, we can easily get $s(V\tilde{\mathcal{D}}(A')_2^\circ) = s(\phi_{*,1}(V\tilde{\mathcal{D}}(A)_2^\circ)) - 1$. Hence $s(V\tilde{\mathcal{D}}(A')_2^\circ) + t(\phi_{*,1}(\tilde{\mathcal{D}}(A)_1^\circ)) \neq n - 2$, which means $S_\infty(V\tilde{\mathcal{D}}(A')_2^\circ)) \subseteq T_\infty(\phi_{*,1}(\tilde{\mathcal{D}}(A)_1^\circ))$. By Proposition 9.18, we get A, A' are supersingular. Furthermore, there exists $1 \leq i \leq n - 1$ such that $(A, \lambda, \eta, A', \lambda', \eta', \phi) \in \mathrm{pr}_i(C_i)$ by the proof of Proposition 9.18.

For any S-point $(A, \lambda, \eta, A', \lambda', \eta', \phi) \in \mathrm{Y}_{11} \backslash (\mathrm{Y}_{00} \bigcup \mathrm{Y}_{10} \bigcup \mathrm{Y}_{01})$, we show $(A, \lambda, \eta, A', \lambda', \eta', \phi) \in \bigcup_{i=1}^{n-1} \mathrm{pr}_i(C_i)$, if A, A' are supersingular. There is a morphism $\delta : \mathrm{Sh}_{1,n-1}(K_{\mathfrak{p}}^1) \to \mathrm{Sh}_{1,n-1}$ mapping $(A, \lambda, \eta, A', \lambda', \eta', \phi)$ to (A, λ, η). Suppose the image $(A, \lambda, \eta) \in V^{(\omega_1, \omega_2)}$, with $(\omega_1, \omega_2) = (a, b)$ and $a \leq b$(We will show every supersingular Ekedahl-Oort stratum except $a = b = 1$ is in $\delta(\mathrm{Y}_{00} \bigcup \mathrm{Y}_{10} \bigcup \mathrm{Y}_{01})$ later.). Then there is a basis of $\tilde{\mathcal{D}}(A)_1^\circ \oplus \tilde{\mathcal{D}}(A)_2^\circ$, denoted by $\{e_{i,j} | i = 1, 2; 1 \leq$

$j \le n\}$ such that F, V act on $\tilde{\mathcal{D}}(A)_1^\circ \oplus \tilde{\mathcal{D}}(A)_2^\circ$ mod p by

$$F(e_{1,i}) = \begin{cases} e_{2,i} & \text{if } i \le 1 \le a-1; \\ 0 & \text{if } i = a; \\ e_{2,i-1} & \text{if } i \ge a+1. \end{cases} \quad F(e_{2,i}) = \begin{cases} 0 & \text{if } i \le 1 \le b-1; \\ e_{1,1} & \text{if } i = b; \\ 0 & \text{if } i \ge b+1. \end{cases}$$

$$V(e_{1,i}) = \begin{cases} 0 & \text{if } i = 1; \\ e_{2,i-1} & \text{if } 2 \le i \le b; \\ e_{2,i} & \text{if } b+1 \le i \le n; \end{cases} \quad V(e_{2,i}) = \begin{cases} 0 & \text{if } i \le 1 \le a-1; \\ 0 & \text{if } a \le i \le n-1; \\ e_{1,a} & \text{if } i = n. \end{cases}$$

Suppose α in Lemma 9.14 to be written as $\alpha = \sum_{i=1}^{n} x_i e_{1,i}$(Here we identify $\tilde{\mathcal{D}}(A)_i^\circ$ with its image in $\tilde{\mathcal{D}}(A')_i^\circ$ by $\phi_{*,i}$ for $i = 1, 2$.). Checking the conditions in Lemma 9.14 directly, we get $x_n, x_{b+1} \in pW(\overline{\mathbb{F}}_p)$ and there exists $i \ne 1, a$ such that $x_i \notin pW(\overline{\mathbb{F}}_p)$. Moreover, we have

$$\begin{cases} x_i^\sigma \equiv x_{i+1}^{\sigma^{-1}} \ mod\ p & \text{if } 1 \le i \le a-1; \\ x_i^\sigma \equiv x_i^{\sigma^{-1}} \ mod\ p & \text{if } a+1 \le i \le b; \\ x_{1,i} \equiv 0 \ mod\ p & \text{if } b+1 \le i \le n. \end{cases}$$

To show $S_\infty(V\tilde{\mathcal{D}}(A')_2^\circ) \subseteq T_\infty(\phi_{*,1}\tilde{\mathcal{D}}(A)_1^\circ)$, it suffices to check that $\alpha \in T_\infty(\phi_{*,1}\tilde{\mathcal{D}}(A)_1^\circ)$ and the inclusion $S_\infty(\phi_{*,1}(V\tilde{\mathcal{D}}(A)_2^\circ)) \subseteq T_\infty(\phi_{*,1}\tilde{\mathcal{D}}(A)_1^\circ)$, which has been shown as above.

By induction, we can show that $T_\infty(\phi_{*,1}\tilde{\mathcal{D}}(A)_1^\circ) = W(\overline{\mathbb{F}}_p)\{e_{1,i}, i \le b; pe_{1,i}, b+1 \le i \le n\}$. Thus we can see obviously that $\alpha \in T_\infty(\phi_{*,1}\tilde{\mathcal{D}}(A)_1^\circ)$. Hence we finish the proof. $\square$

For the morphisms from Y_{01} and Y_{10} to $\mathrm{Sh}_{1,n-1}$ induced by isomorphisms in Proposition 9.11 and their relations with stratification on $\mathrm{Sh}_{1,n-1}$, we have the following proposition:

Proposition 9.20. (1) *The morphism from* Y_{10} *to* $\mathrm{Sh}_{1,n-1}$ *is surjective and maps* $\mathrm{Y}_{10} \bigcap \mathrm{Y}_{11}$ *to the complement of union of Newton strata of first slope less than* $\frac{1}{4}$*, that is, equal to* 0*. We denote it by* N_{10}*. Moreover the morphism maps* $\mathrm{pr}_i(C_i) \bigcap \mathrm{Y}_{10}$ *to* Y_i *surjectively for* $1 \le i \le n$*.*

(2) *The morphism from* Y_{10} *to* $\mathrm{Sh}_{1,n-1}$ *is surjective and maps* $\mathrm{Y}_{10} \bigcap \mathrm{Y}_{11}$ *to the complement of union of Newton strata of last slope bigger than* $\frac{3}{4}$*, that is, equal to* 1*. We denote it by* N_{01}*. Moreover the morphism maps* $\mathrm{pr}_i(C_i) \bigcap \mathrm{Y}_{10}$ *to* Y_i *surjectively.*

Proof. By symmetry, we only prove (1).

Since $\mathrm{Y}_{10} \cong \mathrm{Sh}_{1,n-1}$ by Proposition 9.11, the morphism is surjective obviously.

For any S-point $(A, \lambda, \eta, A', \lambda', \eta', \phi) \in \mathrm{Y}_{10} \bigcap \mathrm{Y}_{11}$, we have $V\tilde{\mathcal{D}}(A')_2^\circ \subseteq \phi_{*,1}(\tilde{\mathcal{D}}(A)_1^\circ)$, $p\tilde{\mathcal{D}}(A')_2^\circ \subseteq \phi_{*,2}(V\tilde{\mathcal{D}}(A)_1^\circ)$ and $V\tilde{\mathcal{D}}(A')_1^\circ = \phi_{*,2}(\tilde{\mathcal{D}}(A)_2^\circ)$. Therefore $\phi_{*,1}(F\tilde{\mathcal{D}}(A)_2^\circ) = p\tilde{\mathcal{D}}(A')_1^\circ \subseteq V\tilde{\mathcal{D}}(A')_2^\circ \subseteq \phi_{*,1}(\tilde{\mathcal{D}}(A)_1^\circ) \bigcap \phi_*(F^{-1}V\tilde{\mathcal{D}}(A)_1^\circ)$. Moreover, we have $\phi_{*,1}(F\tilde{\mathcal{D}}(A)_2^\circ) \subseteq \phi_*(F^{-1}V\tilde{\mathcal{D}}(A)_1^\circ)$ which means $F^4\tilde{\mathcal{D}}(A)_2^\circ \subseteq p\tilde{\mathcal{D}}_2$. Hence we have shown the image of $\mathrm{Y}_{10} \bigcap \mathrm{Y}_{11}$ is contained in N_{10}.

Conversely, to show N_{10} is contained in the image of $\mathrm{Y}_{10} \bigcap \mathrm{Y}_{11}$, it suffices to show $V^{(\omega_1,\omega_2)}$ is contained in the image of $\mathrm{Y}_{10} \bigcap \mathrm{Y}_{11}$ where $(\omega_1, \omega_2) = (n, 2)$. Then we can get $N_{10} = \overline{V}^{(\omega_1,\omega_2)}$ from the morphism is proper. For any S-point $(A, \lambda, \eta) \in V^{(\omega_1,\omega_2)}$, by Proposition 8.4, $\tilde{\mathcal{D}}(A)_1^\circ \bigoplus \tilde{\mathcal{D}}(A)_2^\circ$ has a basis $\{e_{i,j} | i = 1, 2 \text{ and } 1 \le j \le n\}$ such that F, V act on $\tilde{\mathcal{D}}(A)_1^\circ \oplus \tilde{\mathcal{D}}(A)_2^\circ$ mod p by

$$F(e_{1,i}) = \begin{cases} e_{2,i} & \text{if } i \le 1 \le n-1; \\ 0 & \text{if } i = n; \end{cases} \quad F(e_{2,i}) = \begin{cases} 0 & \text{if } i = 1; \\ e_{1,1} & \text{if } i = 2; \\ 0 & \text{if } 3 \le i \le n. \end{cases}$$

$$V(e_{1,i}) = \begin{cases} 0 & \text{if } i = 1; \\ e_{2,1} & \text{if } i = 2; \\ e_{2,i} & \text{if } 3 \le i \le n; \end{cases} \quad V(e_{2,i}) = \begin{cases} 0 & \text{if } 1 \le i \le n-1; \\ e_{1,n} & \text{if } i = n. \end{cases}$$

We take $\tilde{\mathcal{E}}_1 = W(\overline{\mathbb{F}}_p)\{\frac{1}{p}e_{1,1}, e_{1,2}, \dots, e_{1,n}\}$ and $\tilde{\mathcal{E}}_2 = W(\overline{\mathbb{F}}_p)\{\frac{1}{p}e_{2,1}, e_{2,2}, \dots, e_{2,n}\}$. Applying Proposition 5.1.2, we get an triple (A', λ', η') and an $\mathcal{O}_D$-equivariant isogeny $\phi : A \to A'$, where A' is an abelian variety over k with an action of $\mathcal{O}_D$, λ' is a prime-to-p polarization on A', and η' is a prime-to-p level structure on A', such that $\phi'^{\vee} \circ \lambda' \circ \phi' = \lambda$, $\eta' = \phi' \circ \eta$. Moreover, the dimension formula (5.1.3) implies that $\omega^{\circ}_{A'^{\vee}/k,1}$ has dimension 1, and $\omega^{\circ}_{A'^{\vee}/k,2}$ has dimension $n-1$. Therefore, (A', λ', η') is a point of $\mathrm{Sh}_{1,n-1}$ and it can be checked that $(A, \lambda, \eta, A', \lambda', \eta', \phi)$ is a point of $\mathrm{Y}_{10} \bigcap \mathrm{Y}_{11}$.

To show the morphism maps $\mathrm{pr}_i(C_i) \bigcap \mathrm{Y}_{10}$ to Y_i surjectively, we note that $\mathrm{pr}_i(C_i) \bigcap \mathrm{Y}_{10}$ is contained in Y_i. By Proposition 9.19, we see that $V^{(\omega_1, \omega_2)}$ with $(\omega_1, \omega_2) = (n+1-i, n+1-i)$ is contained in $\mathrm{pr}_i(C_i) \bigcap \mathrm{Y}_{10}$. Hence Y_i, as its closure is contained in $\mathrm{pr}_i(C_i) \bigcap \mathrm{Y}_{10}$. We finish the proof. □

To illustrate the relation between Y_{10} and Y_{01}, we note there are two morphisms between Y_{10} and Y_{01}, which are called 'essential Frobenius' as in [28].

We first construct $Fr' : \mathrm{Y}_{10} \to \mathrm{Y}_{01}$. For any locally Noetherian $\mathbb{F}_{p^2}$-scheme S, any S-point $y = (A, \lambda, \eta, A', \lambda', \eta', \phi)$ satisfies $V\tilde{\mathcal{D}}(A')^{\circ}_1 = \phi_{*,2}(\tilde{\mathcal{D}}(A)^{\circ}_2)$ and $V\tilde{\mathcal{D}}(A')^{\circ}_2 \subseteq \phi_{*,1}(\tilde{\mathcal{D}}(A)^{\circ}_1)$. We take $\tilde{\mathcal{E}}_1 = FV^{-1}\tilde{\mathcal{D}}(A)_1$ and $\tilde{\mathcal{E}}_2 = FV^{-1}\tilde{\mathcal{D}}(A)^{\circ}_2$. Applying Proposition 5.1.2, we get an triple (A'', λ'', η'') and an $\mathcal{O}_D$-equivariant isogeny $\phi'' : A \to A''$, where A'' is an abelian variety over k with an action of $\mathcal{O}_D$, λ'' is a prime-to-p polarization on A'', and η'' is a prime-to-p level structure on A'', such that $\phi''^{\vee} \circ \lambda'' \circ \phi'' = \lambda$, $\eta'' = \phi'' \circ \eta$. Moreover, the dimension formula (5.1.3) implies that $\omega^{\circ}_{A''^{\vee}/k,1}$ has dimension 1, and $\omega^{\circ}_{A''^{\vee}/k,2}$ has dimension $n-1$. Therefore, (A'', λ'', η'') is a point of $\mathrm{Sh}_{1,n-1}$ and it can be checked that $(A'', \lambda'', \eta'', A', \lambda', \eta', \phi \circ \phi''^{-1})$ is a point of $\mathrm{Sh}_{1,n-1}(K^1_{\mathfrak{p}})$. It can be checked that $(A'', \lambda'', \eta, A', \lambda', \eta', \phi \circ \phi''^{-1}) \in \mathrm{Y}_{01}$. Thus we let $Fr'(y) = (A'', \lambda'', \eta, A', \lambda', \eta', \phi'')$ and finish constructing Fr'.

Next, we construct $Fr'' : \mathrm{Y}_{01} \to \mathrm{Y}_{10}$. For any locally Noetherian $\mathbb{F}_{p^2}$-scheme S, any S-point $y = (A, \lambda, \eta, A', \lambda', \eta', \phi)$ satisfies $p\tilde{\mathcal{D}}(A')^{\circ}_2 \subseteq V\tilde{\mathcal{D}}(A)^{\circ}_1$ and $p\tilde{\mathcal{D}}(A')^{\circ}_1 = V\tilde{\mathcal{D}}(A)^{\circ}_2$. We take $\tilde{\mathcal{E}}_1 = FV^{-1}\tilde{\mathcal{D}}(A')_1$ and $\tilde{\mathcal{E}}_2 = FV^{-1}\tilde{\mathcal{D}}(A')^{\circ}_2$. Applying Proposition 5.1.2, we get an triple (A'', λ'', η'') and an $\mathcal{O}_D$-equivariant isogeny $\phi'' : A' \to A''$, where A'' is an abelian variety over k with an action of $\mathcal{O}_D$, λ'' is a prime-to-p polarization on A'', and η'' is a prime-to-p level structure on A'', such that $\phi''^{\vee} \circ \lambda'' \circ \phi'' = \lambda'$, $\eta'' = \phi'' \circ \eta'$. Moreover, the dimension formula (5.1.3) implies that $\omega^{\circ}_{A''^{\vee}/k,1}$ has dimension 1, and $\omega^{\circ}_{A''^{\vee}/k,2}$ has dimension $n-1$. Therefore, (A'', λ'', η'') is a point of $\mathrm{Sh}_{1,n-1}$ and it can be checked that $(A, \lambda, \eta, A'', \lambda'', \eta'', \phi'' \circ \phi)$ is a point of $\mathrm{Sh}_{1,n-1}(K^1_{\mathfrak{p}})$. It can be checked that $(A, \lambda, \eta, A'', \lambda'', \eta'', \phi'' \circ \phi) \in \mathrm{Y}_{10}$. Thus we let $Fr''(y) = (A, \lambda, \eta, A'', \lambda'', \eta'', \phi'' \circ \phi)$ and finish constructing Fr''.

It can be checked directly that $Fr' \circ Fr''$ induces the p^2-Frobenius action on Y_{01} and $Fr'' \circ Fr'$ induces the p^2-Frobenius action on Y_{10}. Furthermore, for the action of Fr' and Fr'', we have the following proposition:

Proposition 9.21. (1) *For any $1 \le i \le n-1$, the morphism Fr' induces a morphism from $\mathrm{pr}_i(C_i) \bigcap \mathrm{Y}_{10}$ to $\mathrm{pr}_i(C_i) \bigcap \mathrm{Y}_{01}$. Moreover, it induces a morphism from $\mathrm{pr}_n(C_n) \bigcap \mathrm{Y}_{10} = \mathrm{Y}_{00} \bigcap \mathrm{Y}_{10}$ to $\mathrm{Y}_{00} \bigcap \mathrm{Y}_{01}$ and a morphism from $\mathrm{pr}_n(C_n) \bigcap \mathrm{Y}_{11} = \mathrm{Y}_{00} \bigcap \mathrm{Y}_{11}$ to itself.*

(2) *For any $1 \le i \le n-1$, the morphism Fr'' induces a morphism from $\mathrm{pr}_i(C_i) \bigcap \mathrm{Y}_{01}$ to $\mathrm{pr}_i(C_i) \bigcap \mathrm{Y}_{10}$. It also induces a morphism from $\mathrm{pr}_n(C_n) \bigcap \mathrm{Y}_{01} = \mathrm{Y}_{00} \bigcap \mathrm{Y}_{01}$ to $\mathrm{Y}_{00} \bigcap \mathrm{Y}_{10}$ and a morphism from $\mathrm{pr}_n(C_n) \bigcap \mathrm{Y}_{11} = \mathrm{Y}_{00} \bigcap \mathrm{Y}_{11}$ to itself.*

Proof. By symmetry, we only prove (1).

By Proposition 9.7, we have for any $1 \le i \le n-1$, $\mathrm{pr}_i(C_i) \bigcap \mathrm{Y}_{10} \cong \mathrm{pr}_i(C'_i) \bigcap \mathrm{Y}_{10}$. It can be calculated directly that for each locally noetherian $\mathbb{F}_{p^2}$-scheme S, the set of isomorphism classes of tuples $(B, \lambda'', \eta'', H_1, H_2, H'_1, H'_2)$ in $C'_i \bigcap \mathrm{Y}_{10}$ satisfies: $H'_2 = H_1^{(p)}$ and the set of isomorphism classes of tuples $(B, \lambda'', \eta'', H_1, H_2, H'_1, H'_2)$ in $C'_i \bigcap \mathrm{Y}_{01}$ satisfies: $H_2^{'(p)} = H_1$. Hence we have

$Fr'(B,\lambda'',\eta'',H_1,H_2,H_1',H_2') = (B,\lambda'',\eta'',F(V(H_1^{(p)})^{(p)}),F(V(H_2^{(p)})^{(p)}),H_1',H_2')$. It can be checked directly that the S-point $(B,\lambda'',\eta'',F(V^{-1}(H_1^{(p)})^{(p)}),F(V^{-1}(H_2^{(p)})^{(p)}),H_1',H_2') \in C_i' \bigcap \mathrm{Y}_{01}$. Therefore we have shown the morphism Fr' induces a morphism from $\mathrm{pr}_i(C_i)\bigcap \mathrm{Y}_{10}$ to $\mathrm{pr}_i(C_i)\bigcap \mathrm{Y}_{01}$.

For any point S-point $y = (A,\lambda,\eta,A',\lambda',\eta',\phi) \in \mathrm{Y}_{00}\bigcap \mathrm{Y}_{10}$, we have $V\tilde{\mathcal{D}}(A')_1^\circ = \phi_{*,2}(\tilde{\mathcal{D}}(A)_2^\circ)$ and $p\tilde{\mathcal{D}}(A')_1^\circ = \phi_{*,1}(V\tilde{\mathcal{D}}(A)_2^\circ)$. Therefore we have $F\tilde{\mathcal{D}}(A)_2^\circ = V\tilde{\mathcal{D}}(A)_2^\circ$ and $F\tilde{\mathcal{D}}(A')_1^\circ = V\tilde{\mathcal{D}}(A')_1^\circ$. Hence $V\tilde{\mathcal{D}}(A')_1^\circ = \phi_{*,2}(F(V^{-1}\tilde{\mathcal{D}}(A)_2^\circ))$ and $p\tilde{\mathcal{D}}(A')_1^\circ = \phi_{*,1}(V(F(V^{-1}\tilde{\mathcal{D}}(A)_2^\circ))$. Therefore $Fr(y) \in \mathrm{Y}_{00}$ and in $\mathrm{Y}_{00}\bigcap \mathrm{Y}_{01}$ furthermore. Hence Fr' induces a morphism from $\mathrm{pr}_n(C_n)\bigcap \mathrm{Y}_{10} = \mathrm{Y}_{00}\bigcap \mathrm{Y}_{10}$ to $\mathrm{Y}_{00}\bigcap \mathrm{Y}_{01}$.

If we assume $y \in \mathrm{Y}_{11}$, then $V\tilde{\mathcal{D}}(A')_2^\circ \subseteq \phi_{*,1}(\tilde{\mathcal{D}}(A)_1^\circ)$ and $p\tilde{\mathcal{D}}(A')_2^\circ \subseteq \phi_{*,1}(V\tilde{\mathcal{D}}(A)_1^\circ)$. Therefore, $V\tilde{\mathcal{D}}(A)_2^\circ \subseteq \phi_{*,1}(V(F(V^{-1}\tilde{\mathcal{D}}(A)_2^\circ)))$ and $p\tilde{\mathcal{D}}(A')_1^\circ \subseteq \phi_{*,1}(V(F(V^{-1}\tilde{\mathcal{D}}(A)_2^\circ)))$. This shows Fr' induces a morphism from $\mathrm{pr}_n(C_n)\bigcap \mathrm{Y}_{11} = \mathrm{Y}_{00}\bigcap \mathrm{Y}_{11}$ to itself. $\square$

10. Ihara lemma for $n \geq 3$

In this section we prove the Ihara lemma for $n \geq 3$. We state the theorem first:

Theorem 10.1. *Under the assumption of Hypothesis 1.4, we have:*

(1) *(Definite Ihara) The map*

$$\mathrm{H}^0_{\acute{e}t}(\mathrm{T},\mathbb{F}_l)_{\mathfrak{m}} \xrightarrow{\psi} \mathrm{H}^0_{\acute{e}t}(\mathrm{Sh}_{0,n},\mathbb{F}_l)_{\mathfrak{m}}^{\oplus n}$$

is surjective, where $\psi = (\overleftarrow{p}_*, \overrightarrow{p}_*, \overrightarrow{p}_*A, \overrightarrow{p}_*(A\circ A),\cdots,\overrightarrow{p}_*\underbrace{(A\circ\cdots\circ A)}_{n-2})$ *with* $(\overleftarrow{p}_*,\overrightarrow{p}_*)$ *given by the correpsondence above and A is a correpondence defined between different* T*'s. The composition $\circ$ of A is defined in Section 5.*

(2) *(Indefinite Ihara) The map [11]*

$$\mathrm{H}^{2(n-1)}_{\acute{e}t}(\overline{Sh}_{1,n-1}(K^1_{\mathfrak{p}}),\mathbb{F}_l(n))_{\mathfrak{m}} \xrightarrow{\psi} \mathrm{H}^{2(n-1)}_{\acute{e}t}(\overline{Sh}_{1,n-1},\mathbb{F}_l(n))_{\mathfrak{m}}^{\oplus n}$$

is surjective, where $\overline{Sh}_{1,n-1},\overline{Sh}_{1,n-1}(K^1_{\mathfrak{p}})$ are the generic fibers of $\mathcal{S}h_{1,n-1},\mathcal{S}h_{1,n-1}(K^1_{\mathfrak{p}})$, $\psi = (\overleftarrow{p}_*, \overrightarrow{p}_*, \overrightarrow{p}_*A, \overrightarrow{p}_*(A\circ A),\cdots,\overrightarrow{p}_*\underbrace{(A\circ\cdots\circ A)}_{n-2})$ *with all the maps induced from those in* (1) *and we use the same notation for simplicity.*

We have the following diagram:

$$\begin{tikzcd}
\mathrm{Y}_{00}\bigcap\mathrm{Y}_{10} \arrow[r, hook] \arrow[d] & \mathrm{Y}_{10} \arrow[d] & \mathrm{Y}_{10}\bigcap\mathrm{Y}_{11} \arrow[l, hook'] \arrow[d] & \mathrm{Y}_{00}\bigcap\mathrm{Y}_{11} \arrow[l, hook'] \arrow[d] \\
\mathrm{Y}_n \arrow[r, hook] & \mathrm{Sh}_{1,n-1} & N_{10} \arrow[l, hook'] & \mathrm{Y}_n \arrow[l, hook']
\end{tikzcd}$$

where the vertical morphisms are induced by the blowing-up under $\mathrm{Y}_{10} \cong \mathrm{Bl}_{\mathrm{Y}_n}\mathrm{Sh}_{1,n-1}$ in 9.11 and N_{10} is the complement of union of newton strata of first slope equal to 0 defined in 9.20.

Take $\mathcal{U}_{10} = \mathrm{Y}_{10} - \mathrm{Y}_{10}\bigcap\mathrm{Y}_{11}$, we have the following excision exact sequence:

$$\begin{tikzcd}
 & \cdots \arrow[r] & H^{2n-5}(\mathrm{Y}_{10}\bigcap\mathrm{Y}_{11}) \arrow[dll] \\
H^{2n-4}_c(\mathcal{U}_{10}) \arrow[r] & H^{2n-4}(\mathrm{Y}_{10}) \arrow[r, "Res"] & H^{2n-4}(\mathrm{Y}_{10}\bigcap\mathrm{Y}_{11}) \arrow[dll, "\partial"'] \\
H^{2n-3}_c(\mathcal{U}_{10}) \arrow[r] & \cdots &
\end{tikzcd}$$

Therefore the sequence $H^{2n-4}(\mathrm{Y}_{10}) \xrightarrow{Res} H^{2n-4}(\mathrm{Y}_{10}\bigcap \mathrm{Y}_{11}) \xrightarrow{\partial} H_c^{2n-3}(\mathcal{U}_{10})$ is exact. Moreover, the closed immersion of $\mathrm{Y}_{00}\bigcap \mathrm{Y}_{11}$ and $\mathrm{pr}_i(C_i)\bigcap \mathrm{Y}_{10}$ into $\mathrm{Y}_{10}\bigcap \mathrm{Y}_{11}$ induces Gysin maps

$$H^{(2n-6)}(\mathrm{Y}_{00}\bigcap \mathrm{Y}_{11}, \mathbb{F}_l(n-3)) \oplus \bigoplus_{i=1}^{n-1} H^0(C_i\bigcap \mathrm{Y}_{10}, \mathbb{F}_l) \xrightarrow{Gys} H^{2n-4}(\mathrm{Y}_{10}\bigcap \mathrm{Y}_{11}, \mathbb{F}_l(n-2)).$$

Putting together, we have the following diagram with the vertical sequence exact:

$$\begin{tikzcd}
H_c^{2n-3}(\mathcal{U}_{10}, \mathbb{F}_l(n-2)) & \\
H^{2n-4}(\mathrm{Y}_{10}\bigcap \mathrm{Y}_{11}, \mathbb{F}_l(n-2)) \arrow[u, "\partial"] & H^{(2n-6)}(\mathrm{Y}_{00}\bigcap \mathrm{Y}_{11}, \mathbb{F}_l(n-3)) \oplus \bigoplus_{i=1}^{n-1} H^0(C_i\bigcap \mathrm{Y}_{10}, \mathbb{F}_l) \arrow[l, "Gys{,}\mathrm{pr}_{i,!}"'] \\
H^{2n-4}(\mathrm{Y}_{10}, \mathbb{F}_l(n-2)) \arrow[u, "Res"] &
\end{tikzcd}$$

Remark 10.2. Since $2n-6<0$ when $n=2$, this method won't work for the case $n=2$.

By Tate Conjecture, we have the following proposition:

Proposition 10.3. *The map* $\partial \circ Gys : \bigoplus_{i=1}^{n-1} H^0_{\acute{e}t}(C_i\bigcap \mathrm{Y}_{10}, \mathbb{F}_l)_{\mathfrak{m}} \to H_c^{2n-3}(\mathcal{U}_{10}, \mathbb{F}_l(n-2))_{\mathfrak{m}}$ *is injective.*

Proof. By Proposition 9.12, we have $H^{2n-4}_{\acute{e}t}(\mathrm{Y}_{00}\bigcap \mathrm{Y}_{10}) = \bigcup_{i=0}^{n-2} H^{2i}_{\acute{e}t}(\mathbb{P}^{n-2}) \oplus H^{2(n-2-i)}_{\acute{e}t}(\mathbb{P}^{n-1})$. Considering the ring structure of $H^{*}_{\acute{e}t}(\mathrm{Y}_{00}\bigcap \mathrm{Y}_{10})$, We see that we can write $H^{2n-4}_{\acute{e}t}(\mathrm{Y}_{00}\bigcap \mathrm{Y}_{10})$ as homogenous polynomials of degree $n-2$ with two indeterminants ξ, η corresponds to $\mathcal{O}(1)$ in $H^2_{\acute{e}t}(\mathbb{P}^{n-1}, \mathbb{F}_l(1))$ and $\mathcal{O}_1$ in $H^2_{\acute{e}t}(\mathbb{P}^{n-2}, \mathbb{F}_l(1))$. Considering the self-intersection given by the closed immersion of $\mathrm{Y}_{00}\bigcap \mathrm{Y}_{10}$ into Y_{10}, we have the following diagram:

$$\begin{tikzcd}
H^{2n-4}_{\acute{e}t}(\mathrm{Y}_{10}, \mathbb{F}_l(n-2)) \arrow[r, "Res"] & H^{2n-4}_{\acute{e}t}(\mathrm{Y}_{00}\bigcap \mathrm{Y}_{10}, \mathbb{F}_l(n-2)) \\
H^{2n-6}_{\acute{e}t}(\mathrm{Y}_{00}\bigcap \mathrm{Y}_{10}, \mathbb{F}_l(n-3)) \arrow[u, "Gys"] \arrow[ur, "\cup c_1(\mathcal{N}_{\mathrm{Y}_{00}\bigcap \mathrm{Y}_{10}/\mathrm{Y}_{10}})"'] &
\end{tikzcd}$$

Since $\mathcal{N}_{\mathrm{Y}_{00}\bigcap \mathrm{Y}_{10}/\mathrm{Y}_{10}}$ is the normal bundle of the exceptional divisor, we have the cup product with $c(\mathcal{N}_{\mathrm{Y}_{00}\bigcap \mathrm{Y}_{10}/\mathrm{Y}_{10}})$ is just multiplication by -η. Moreover, we have blowing-up exact sequence

$$0 \longrightarrow H^{2n-4}_{\acute{e}t}(\mathrm{Sh}_{1,n-1}) \longrightarrow H^{2n-4}_{\acute{e}t}(\mathrm{Y}_{10}) \oplus H^{2n-4}_{\acute{e}t}(\mathrm{Y}_n) \longrightarrow H^{2n-4}_{\acute{e}t}(\mathrm{Y}_{00}\bigcap \mathrm{Y}_{11}) \longrightarrow 0\ .$$

After localizing at $\mathfrak{m}$, we have $H^{2n-4}(\mathrm{Y}_{10})_{\mathfrak{m}} \oplus H^{2n-4}_{\acute{e}t}(\mathrm{Y}_n)_{\mathfrak{m}} \cong H^{2n-4}_{\acute{e}t}(\mathrm{Y}_{00}\bigcap \mathrm{Y}_{11})_{\mathfrak{m}}$. Therefore the image of $H^{2n-4}_{\acute{e}t}(\mathrm{Y}_{10})_{\mathfrak{m}}$ in $H^{2n-4}_{\acute{e}t}(\mathrm{Y}_{00}\bigcap \mathrm{Y}_{11})$ is consisting of the homogenous polynomials of degree $2n-4$ with the degree of η is not zero, that is, there is an isomorphism $H^{2n-4}_{\acute{e}t}(\mathrm{Y}_{10}, \mathbb{F}_l(n-2))_{\mathfrak{m}} =$

$H_{ét}^{2n-6}(\mathrm{Y}_{00}\bigcap \mathrm{Y}_{10},\mathbb{F}_l(n-3))_{\mathfrak{m}}$. Thus by the commutativity of the following diagram:

$$\begin{tikzcd}
H_c^{2n-3}(\mathcal{U}_{10},\mathbb{F}_l(n-2)) & \\
H^{2n-4}(\mathrm{Y}_{10}\bigcap \mathrm{Y}_{11},\mathbb{F}_l(n-2)) \arrow[u, "\partial"] & H^{(2n-6)}(\mathrm{Y}_{00}\bigcap \mathrm{Y}_{11},\mathbb{F}_l(n-3))\oplus \bigoplus_{i=1}^{n-1} H^0(C_i\bigcap \mathrm{Y}_{10},\mathbb{F}_l) \arrow[l, "Gys"'] \\
H^{2n-4}(\mathrm{Y}_{10},\mathbb{F}_l(n-2)) \arrow[u, "Res"] & H^{2n-6}(\mathrm{Y}_{00}\bigcap \mathrm{Y}_{10},\mathbb{F}_l(n-3)) \arrow[l, "Gys"'] \arrow[u, "Res"]
\end{tikzcd}$$

we have the image of $H^{2n-4}(\mathrm{Y}_{10},\mathbb{F}_l(n-2))_{\mathfrak{m}}$ in $H^{2n-4}(\mathrm{Y}_{10}\bigcap \mathrm{Y}_{11},,\mathbb{F}_l(n-2))_{\mathfrak{m}}$ is the same as the image of $H^{(2n-6)}(\mathrm{Y}_{00}\bigcap \mathrm{Y}_{11},,\mathbb{F}_l(n-3))_{\mathfrak{m}}$.

Now, by Proposition 9.20 and Theorem 3.7.3, we have the following commutative diagram:

$$\begin{tikzcd}
H^{2n-2}(\mathrm{Sh}_{1,n-1},\mathbb{F}_l(n-1))_{\mathfrak{m}} & H^{2n-2}(\mathrm{Y}_{10},\mathbb{F}_l(n-1))_{\mathfrak{m}} \arrow[l, "\delta"'] \\
& H^{2n-4}(\mathrm{Y}_{00}\bigcap \mathrm{Y}_{10},\mathbb{F}_l(n-2))_{\mathfrak{m}} \arrow[u, "Gys"'] \\
H^0(\mathrm{Y}_n)_{\mathfrak{m}}\oplus \bigoplus_{i=1}^{n-1} H^0(\mathrm{Y}_i)_{\mathfrak{m}} \arrow[uu, "\overrightarrow{p}_{i,!}"] & H^{2n-6}(\mathrm{Y}_{00}\bigcap \mathrm{Y}_{11},\mathbb{F}_l(n-3))_{\mathfrak{m}}\oplus \bigoplus_{i=1}^{n-1} H^0(C_i\bigcap \mathrm{Y}_{10})_{\mathfrak{m}} \arrow[l, "(\alpha,\beta)"] \arrow[u, "(Gys{,}pr_{i,!})"']
\end{tikzcd}$$

where the vertical maps are gysin maps induced by the blowing-up. It is easy to see $H^0(C_i\bigcap \mathrm{Y}_{10})_{\mathfrak{m}} = H^0(\mathrm{Y}_i)_{\mathfrak{m}}$. If there is $(x,y)\in H^{2n-6}(\mathrm{Y}_{00}\bigcap \mathrm{Y}_{11},\mathbb{F}_l(n-3))_{\mathfrak{m}}\oplus \bigoplus_{i=1}^{n-1} H^0(C_i\bigcap \mathrm{Y}_{10},\mathbb{F}_l)_{\mathfrak{m}}$ such that the image of (x,y) through the horizontal map is zero, then $\overrightarrow{p}_{j,!}\circ(\alpha(x),\beta(y))=0$. By injectivity of $\overrightarrow{p}_{j,!}$, we see $\beta(y)=0$. Hence $y=0$. This shows the image of $\bigoplus_{i=1}^{n-1} H^0(\mathrm{Y}_i)_{\mathfrak{m}}$ in $H^{2n-4}(\mathrm{Y}_{10}\bigcap \mathrm{Y}_{11},\mathbb{F}_l(n-2))_{\mathfrak{m}}$ has trivial intersection with the image of $H^{2n-4}(\mathrm{Y}_{10})_{\mathfrak{m}}$. Thus we have finish the proof. $\square$

Symmetrically, we define $\mathcal{U}_{01}=\mathrm{Y}_{01}-\mathrm{Y}_{01}\bigcap \mathrm{Y}_{11}$. By Proposition 9.21, we have $Fr''(\mathcal{U}_{01})$ contains $C_i\bigcap \mathrm{Y}_{10}$ for $1\le i\le n$. Thus we have a simlar diagram as above:

$$\begin{tikzcd}
\mathrm{Y}_{00}\bigcap \mathrm{Y}_{10} \arrow[r, hook] \arrow[d] & \mathrm{Y}_{10} \arrow[d] & \mathrm{Y}_{10}\backslash Fr''(\mathcal{U}_{01}) \arrow[l, hook'] \arrow[d] & \mathrm{Y}_{00}\bigcap \mathrm{Y}_{11} \arrow[l, hook'] \arrow[d] \\
\mathrm{Y}_n \arrow[r, hook] & \mathrm{Sh}_{1,n-1} & N_{01} \arrow[l, hook'] & \mathrm{Y}_n \arrow[l, hook']
\end{tikzcd}.$$

we also have the following diagram with the vertical sequence exact:

$$\begin{tikzcd}
H_c^{2n-3}(Fr''(\mathcal{U}_{01}),\mathbb{F}_l(n-2)) & \\
H^{2n-4}(\mathrm{Y}_{10}\backslash Fr''(\mathcal{U}_{01}),\mathbb{F}_l(n-2)) \arrow[u, "\partial"] & H^{(2n-6)}(\mathrm{Y}_{00}\bigcap \mathrm{Y}_{11},\mathbb{F}_l(n-3))\oplus \bigoplus_{i=1}^{n-1} H^0(C_i\bigcap \mathrm{Y}_{10},\mathbb{F}_l) \arrow[l, "Gys{,}pr_{i,!}"'] \\
H^{2n-4}(\mathrm{Y}_{10},\mathbb{F}_l(n-2)) \arrow[u, "Res"] &
\end{tikzcd}$$

By a similar argument as above, we have the following proposition:

Proposition 10.4. *The map* $\partial \circ Gys : \bigoplus_{i=1}^{n-1} H^0_{\acute{e}t}(C_i \bigcap \mathrm{Y}_{10})_{\mathfrak{m}} \to H^{2n-3}_c(Fr''(\mathcal{U}_{01}), \mathbb{F}_l(n-2))_{\mathfrak{m}}$ *is injective.*

Consider $Fr''(\mathcal{U}_{01}) \bigcup \mathcal{U}_{10}$ with the Mayer-Vietoris sequence, we have the following diagram:

$$\begin{tikzcd}
 & H^{2n-3}_c(Fr''(\mathcal{U}_{01}) \bigcup \mathcal{U}_{10}, \mathbb{F}_l(n-2))_{\mathfrak{m}} \\
\bigoplus_{i=1}^{n-1} H^0_{\acute{e}t}(C_i \bigcap \mathrm{Y}_{10}, \mathbb{F}_l)_{\mathfrak{m}} \arrow[r, "\Phi"] \arrow[dr, dashed, "\phi"'] & H^{2n-3}_c(Fr''(\mathcal{U}_{01}), \mathbb{F}_l(n-2))_{\mathfrak{m}} \bigoplus H^{2n-3}_c(\mathcal{U}_{10}, \mathbb{F}_l(n-2))_{\mathfrak{m}} \arrow[u] \\
 & H^{2n-3}_c(Fr''(\mathcal{U}_{01}) \bigcap \mathcal{U}_{10}, \mathbb{F}_l(n-2))_{\mathfrak{m}} \arrow[u, "i"]
\end{tikzcd}$$

Since two parts of the last horizontal map has different signs, the composition of the last horizontal map with Φ is zero. Hence Φ factors through the fisrt horizontal map: $\Phi = i \circ \phi$. Thus we have ϕ is injective.

We claim that ϕ gives us the desired map appeared in the definite Ihara lemma. First, we note that the morphism from C_i to Y_i induced by the blowing-up for $1 \le i \le n-1$ gives the isomorphism $H^0_{\acute{e}t}(C_i \bigcap \mathrm{Y}_{10}) = H^0(\mathrm{Y}_i)$. Since $H^0_{\acute{e}t}(\mathrm{Y}_i)_{\mathfrak{m}} = H^0_{\acute{e}t}(\mathrm{Sh}_{0,n})_{\mathfrak{m}}$, we have $H^0_{\acute{e}t}(C_i \bigcap \mathrm{Y}_{10})_{\mathfrak{m}} = H^0_{\acute{e}t}(\mathrm{Sh}_{0,n})_{\mathfrak{m}}$. Moreover, $Fr''(\mathcal{U}_{01}) \bigcup \mathcal{U}_{10} = V^{(\omega_1,\omega_2)} \bigsqcup (\mathrm{Y}_{00} \bigcap \mathrm{Y}_{10} - \mathrm{Y}_{00} \bigcap \mathrm{Y}_{11})$, with $(\omega_1, \omega_2) = (n, 1)$, that is, $V^{(\omega_1,\omega_2)}$ is the μ-ordinary locus of $\mathrm{Sh}_{1,n-1}$. By Proposition 8.10, we have $V^{(\omega_1,\omega_2)}$ is affine and thus $H^{2n-3}_c(V^{(\omega_1,\omega_2)}) = 0$. By excision sequence, we get an injection $i : H^{2n-3}_c(Fr''(\mathcal{U}_{01}) \bigcap \mathcal{U}_{10})_{\mathfrak{m}} \to H^{2n-3}_c(\mathrm{Y}_{00} \bigcap \mathrm{Y}_{10} - \mathrm{Y}_{00} \bigcap \mathrm{Y}_{11})_{\mathfrak{m}}$. Composition these maps together, we get the following map is injecitve:

$$\begin{tikzcd}
\bigoplus_{i=1}^{n-1} H^0_{\acute{e}t}(\mathrm{Sh}_{0,n})_{\mathfrak{m}} \arrow[r] & \bigoplus_{i=1}^{n-1} H^0_{\acute{e}t}(C_i \bigcap \mathrm{Y}_{10})_{\mathfrak{m}} \arrow[r, "\phi"] & H^{2n-3}_c(Fr''(\mathcal{U}_{01}) \bigcap \mathcal{U}_{10}, \mathbb{F}_l(n-2))_{\mathfrak{m}} \arrow[d, "i"] \\
 & & H^{2n-3}_c(\mathrm{Y}_{00} \bigcap \mathrm{Y}_{10} - \mathrm{Y}_{00} \bigcap \mathrm{Y}_{11}, \mathbb{F}_l(n-2))_{\mathfrak{m}} \arrow[d] \\
 & & H^0_{\acute{e}t}(\mathrm{Sh}_{0,n}, \rho_{(n-1,1)})_{\mathfrak{m}}
\end{tikzcd}$$

where the last map is an identity following Proposition 9.12. Considering the direct sum of this map with the identity map $H^0_{\acute{e}t}(\mathrm{Sh}_{0,n})_{\mathfrak{m}} \to H^0_{\acute{e}t}(\mathrm{Sh}_{0,n})_{\mathfrak{m}}$, we get

$$\bigoplus_{i=1}^{n} H^0_{\acute{e}t}(\mathrm{Sh}_{0,n}, \mathbb{F}_l)_{\mathfrak{m}} \longrightarrow H^0_{\acute{e}t}(\mathrm{Sh}_{0,n}(K^1_{\mathfrak{p}}), \mathbb{F}_l)_{\mathfrak{m}} .$$

We denote the map by Ψ. Now it suffices to check Ψ coincides with the map in the definite Ihara lemma. We need the following lemma:

Lemma 10.5. *For* $1 \le i \le n-1$, *the Hecke correspondence* $T^{n-i}_{\mathfrak{p}}$ *gives a correspondence between* $\mathrm{pr}'_i(C_i)$ *and* $\mathrm{pr}'_n(C_n)$, *that is, for any* $\overline{\mathbb{F}}_p$*-point* $(A, \lambda, \eta, \hat{A}, \hat{\lambda}, \hat{\eta}, \hat{\phi}, B, \lambda'', \eta'', \psi) \in C_i(\overline{\mathbb{F}}_p)$ *and any* $\overline{\mathbb{F}}_p$*-point* $(A, \lambda, \eta, \tilde{A}, \tilde{\lambda}, \tilde{\eta}, \tilde{\phi}, B_1, \lambda_1, \eta_1, B_2, \lambda_2, \eta_2, \psi_1, \psi_2) \in C_n(\overline{\mathbb{F}}_p)$, *then* $B_1 \in T^{n-i}_{\mathfrak{p}}(B)$.

Proof. By definition, if for any $\overline{\mathbb{F}}_p$-point $(A, \lambda, \eta, \hat{A}, \hat{\lambda}, \hat{\eta}, \hat{\phi}, B, \lambda'', \eta'', \psi) \in C_i(\overline{\mathbb{F}}_p)$ and any $\overline{\mathbb{F}}_p$-point $(A, \lambda, \eta, \tilde{A}, \tilde{\lambda}, \tilde{\eta}, \tilde{\phi}, B_1, \lambda_1, \eta_1, B_2, \lambda_2, \eta_2, \psi_1, \psi_2) \in C_n(\overline{\mathbb{F}}_p)$, then $(A, \lambda, \eta) \in \mathrm{Y}_{i,B} \bigcap \mathrm{Y}_{n,B_1}$.

By [11, Proposition 6.4], we have $\mathrm{Y}_{i,B}\bigcap \mathrm{Y}_{n,B_1}$ is not empty if and only there exists $\delta \leq \min\{n-n, i-1\}$, that is, $\delta=0$ such that $B_1 \in R_{\mathfrak{p}}^{(n-i+\delta,n-\delta)}S_{\mathfrak{p}}^{-1}(B)=T_{\mathfrak{p}}^{n-i}(B)$. This concludes the proof. $\square$

It can be checked directly that the n-th summation of Ψ correpsonds to $\overrightarrow{p}^* : H^0_{\acute{e}t}(\mathrm{Sh}_{0,n})_{\mathfrak{m}} \to H^0_{\acute{e}t}(\mathrm{Sh}_{0,n}(K^1_{\mathfrak{p}}),\mathbb{F}_l)_{\mathfrak{m}}$. Moreover, for the $n-i$-th summation of Ψ, i.e., the summation corresponds to $H^0_{\acute{e}t}(\mathrm{Sh}_{0,n})_{\mathfrak{m}} \xrightarrow{\mathrm{pr}'^*_{n-i}} H^0_{\acute{e}t}(C_{n-i}\bigcap \mathrm{Y}_{10})_{\mathfrak{m}} \xrightarrow{\mathrm{pr}_{n-i,*}} H^{2n-3}_c(\mathrm{Y}_{00}\bigcap \mathrm{Y}_{10}-\mathrm{Y}_{00}\bigcap \mathrm{Y}_{11})_{\mathfrak{m}}$, we get the map $H^0_{\acute{e}t}(\mathrm{Sh}_{0,n})_{\mathfrak{m}} \to H^0_{\acute{e}t}(\mathrm{Sh}_{0,n}(K^1_{\mathfrak{p}}),\mathbb{F}_l)_{\mathfrak{m}}$ is $\overrightarrow{p}^* \circ T^{(i)}_{\mathfrak{p}}$ and it can be checked by definition it is exactly the map given by $\overrightarrow{p}_* \underbrace{(A\circ\cdots\circ A)}_{i-1}$. Hence we finish the proof of the definite Ihara lemma.

Now we give the proof of the Indefinite Ihara lemma.

First, we recall the following definition as in [24, Definition 5.2]:

Definition 10.6. Let $(R,\mathfrak{m}_R)$ be a noetherian local ring, G some group, and

$$\sigma_R : G \to \mathrm{GL}_n(R)$$

an n-dimensional representation such that $\overline{\sigma}_R=\sigma_R \mod \mathfrak{m}_R$ is absolutely irreducible, i.e., irreducible over the algebra closure of R. Let M be an $R[G]$-module. Then M is said to be σ_R-typic if one can write M as a tensor product

$$M=\sigma_R\otimes_R M_0,$$

where M_0 is an R-module, and G acts only through its action on σ_R.

By [24, Proposition 5.3], we have the following proposition:

Proposition 10.7. *In the situation of Definition 10.6, if M is σ-typic, then*

$$M_0=Hom_{R[G]}(\sigma_R,M).$$

The functor $M_0\mapsto \sigma_R\otimes_R M_0$, $M\mapsto Hom_{R[G]}(\sigma_R,M)$ induce an equivalence of categories between the category of σ_R-typic $R[G]$-modules and the category of R-modules.

To prove the indefinite Ihara lemma, we need to introduce some elementary primes.

Definition 10.8. We call a prime $p'\neq p,l$ in $\mathbb{Q}$ is *good*, if $K_{p'}$ is hyperspecial and Hypothesis 1.4 holds for p'.

For any *good* prime p', we have canonical decompositions of $\mathscr{H}_K\times \mathrm{Gal}(\overline{\mathbb{F}}_{p'}/\mathbb{F}_{p'^2})$-modules as in (3.6.1):

$$H^0_{\acute{e}t}(\overline{\mathrm{Sh}}_{0,n,p'},\mathbb{Q}_l)_{\mathfrak{m}}=\pi^K\otimes R_{(0,n),l}(\pi)$$
$$H^{2(n-1)}_{\acute{e}t}(\overline{\mathrm{Sh}}_{1,n-1,p'},\mathbb{Q}_l)_{\mathfrak{m}}=\pi^K\otimes R_{(1,n-1),l}(\pi).$$

Here we use subscript p' to express Shimura varieties defined at p'. Then up to semisimplification, we have

$$\big[R_{(0,n),\ell}(\pi)\big]=\#\ker^1(\mathbb{Q},G_{0,n})m_{0,n}(\pi)\big[\wedge^n\rho_{\pi_{\mathfrak{p}'}}\otimes\chi^{-1}_{\pi_{p'},0}\otimes\overline{\mathbb{Q}}_l(\tfrac{n(n-1)}{2})\big], \tag{10.8.1}$$

$$\big[R_{(1,n-1),\ell}(\pi)\big]=\#\ker^1(\mathbb{Q},G_{1,n-1})m_{0,n}(\pi)\big[\rho_{\pi_{\mathfrak{p}'}}\otimes\wedge^{n-1}\rho_{\pi_{\mathfrak{p}'}}\otimes\chi^{-1}_{\pi_{p'},0}\otimes\overline{\mathbb{Q}}_l(\tfrac{(n-1)(n-2)}{2})\big]. \tag{10.8.2}$$

Note that $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})$ acts on the two cohomology groups simply on the second factor. More explicitly, by $\phi^{\pi}_R(S_{\mathfrak{p}'})=1$ and the proof of Theorem 3.7.3 in [11], the group $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})$ acts on $\bigoplus_{j=1}^n H^0_{\acute{e}t}(\overline{\mathrm{Sh}}_{0,n,p'},\mathbb{Q}_l)_{\mathfrak{m}}$ and $H^{2(n-1)}_{\acute{e}t}(\overline{\mathrm{Sh}}_{1,n-1,p'},\mathbb{Q}_l)^{\mathrm{fin}}_{\mathfrak{m}}$ by $(\chi^{-1}_{\pi_{p'},0})^{\oplus m}$. Here $m=n\#\ker^1(\mathbb{Q},G_{0,n})m_{0,n}(\pi)=\#\ker^1(\mathbb{Q},G_{1,n-1})m_{0,n}(\pi)$. we write $\bigoplus_{j=1}^n H^0_{\acute{e}t}(\overline{\mathrm{Sh}}_{0,n,p'},\mathbb{Q}_l)_{\mathfrak{m}}=\pi^K\otimes\chi^{-m}_{\pi_{p'},0}$.

Moreover, let K' be an open compact subgroup of $G(\mathbb{A}^\infty)$ satisfies $K'^p = K^p$ and $K'_p = K^1_p$. Since $K'_{p'} = K_{p'}$ is hyperspecial, we get $\bigoplus_{j=1}^{n} H^0_{ét}(\overline{\mathrm{Sh}}_{0,n,p'}(K'), \mathbb{Q}_l)_{\mathfrak{m}} = \pi^{K'} \otimes \chi^{-m}_{\pi_{p'},0}$ similarly as above. By definite Ihara lemma and proper base change theorem, we have a surjection

$$H^0_{ét}(\overline{Sh}_{0,n}(K')_{\overline{\mathbb{Q}}_p}, \mathbb{F}_l)_{\mathfrak{m}} \to H^0_{ét}(\overline{Sh}_{0,n}(K)_{\overline{\mathbb{Q}}_p}, \mathbb{F}_l)^n_{\mathfrak{m}}.$$

Under the isomorphism $\overline{\mathbb{Q}}_p \cong \mathbb{C} \cong \overline{\mathbb{Q}}_{p'}$, we have a surjection

$$H^0_{ét}(\overline{Sh}_{0,n}(K')_{\overline{\mathbb{Q}}_{p'}}, \mathbb{F}_l)_{\mathfrak{m}} \to H^0_{ét}(\overline{Sh}_{0,n}(K)_{\overline{\mathbb{Q}}_{p'}}, \mathbb{F}_l)^n_{\mathfrak{m}}.$$

Again by torion-freeness and proper base change, we get

$$H^0_{ét}(\overline{\mathrm{Sh}}_{0,n,p'}(K'), \mathbb{Q}_l)_{\mathfrak{m}} \to H^0_{ét}(\overline{Sh}_{0,n,p'}(K), \mathbb{Q}_l)^n_{\mathfrak{m}}.$$

Then by torsion-freeness and Proposition 10.7, we get a surjection $\pi^{K'} \to (\pi^K)^{\oplus n}$. Hence $\pi^{K'} \otimes R_{(1,n-1),l}(\pi) \to (\pi^K \otimes R_{(1,n-1),l}(\pi))^{\oplus n}$ is surjective. After proper base change, we get the indefinite Ihara lemma since the generic fibers at p and p' coincide.

11. Arithmetic level raising theorem for $n = 2$

We first construct the Arithmetic level raising map for general n.

Recall in Proposition 6.2 we have $\mathrm{Ch}^n(\mathrm{Sh}_{1,n-1}, 1, \mathbb{F}_l) = \mathrm{H}^{2n-1}_{\mathfrak{m}}(X, \mathbb{F}_l(n))$. We have a canonical map from $\mathrm{H}^{2n-1}_{\mathfrak{m}}(X, \mathbb{F}_l(n))$ to $\mathrm{H}^{2n-1}_{ét}(\mathrm{Sh}_{1,n-1}, \mathbb{F}_l(n)))$, hence we have a map from $\mathrm{Ch}^n(\mathrm{Sh}_{1,n-1}, 1, \mathbb{F}_l)$ to $\mathrm{H}^{2n-1}_{ét}(\mathrm{Sh}_{1,n-1}, \mathbb{F}_l(n)))$. Composing with the map $\mathrm{Ch}^n(\mathrm{Sh}_{1,n-1}, 1, \mathbb{F}_l) \to \mathrm{Ch}^n(\mathrm{Sh}_{1,n-1}, 1, \mathbb{F}_l)$ induced by the closed immersion of $\mathrm{Sh}^{ss}_{1,n-1}$ into $\mathrm{Sh}_{1,n-1}$, we can get the cycle class map

$$\mathrm{Ch}^1(\mathrm{Sh}_{1,n-1}, 1, \mathbb{F}_l) \to \mathrm{H}^{2n-1}_{ét}(\mathrm{Sh}_{1,n-1}, \mathbb{F}_l(n)))$$

.

On the other hand, considering the Galois action on the special fiber $\mathrm{Sh}_{1,n-1}$, we get a short exact sequence:

$$0 \to \mathrm{H}^1(\mathbb{F}_{p^2}, \mathrm{H}^{2n-2}_{ét}(\overline{\mathrm{Sh}}_{1,n-1}, \mathbb{F}_l(n))) \to \mathrm{H}^{2n-1}_{ét}(\mathrm{Sh}_{1,n-1}, \mathbb{F}_l(n))) \to \mathrm{H}^0(\mathbb{F}_{p^2}, \mathrm{H}^{2n-1}_{ét}(\overline{\mathrm{Sh}}_{1,n-1}, \mathbb{F}_l(n)))$$

.

By localizaing at a maximal ideal which is 'generic' and non-Eisenstein as in [4], we can get that $\mathrm{H}^i(\overline{\mathrm{Sh}}_{1,n-1}, 1, \mathbb{F}_l(n))_{\mathfrak{m}}$ is nonzero if and only if $i = 2n-2$. Hence after localizing at such an m, we get the third term of the short exact sequence is zero. And we get two maps by lifting:

$$\mathrm{Ch}^n(\mathrm{Sh}_{1,n-1}, 1, \mathbb{F}_l)_{\mathfrak{m}} \to \mathrm{H}^1(\mathbb{F}_{p^2}, \mathrm{H}^{2n-2}_{ét}(\overline{\mathrm{Sh}}_{1,n-1}, \mathbb{F}_l(n))_{\mathfrak{m}}$$

, which is the so-called Abel-Jacobi map; and

$$\mathrm{Ch}^1(\mathrm{Sh}_{1,n-1}, 1, \mathbb{F}_l)_{\mathfrak{m}} \to \mathrm{H}^1(\mathbb{F}_{p^2}, \mathrm{H}^{2n-2}_{ét}(\overline{\mathrm{Sh}}_{1,n-1}, \mathbb{F}_l(n))_{\mathfrak{m}}$$

, which is the so-called level raising map.

Summing up, we have the diagram as below:

$$\begin{tikzcd}
 & & 0 = \mathrm{H}^0(\mathbb{F}_{p^2}, \mathrm{H}^{2n-1}_{ét}(\overline{\mathrm{Sh}}_{1,n-1}, \mathbb{F}_l(n))_{\mathfrak{m}}) \\
\mathrm{Ch}^n(\mathrm{Sh}_{1,n-1}, 1, \mathbb{F}_l)_{\mathfrak{m}} \arrow[r, equal] & \mathrm{H}^{2n-1}_{\mathcal{M}}(\mathrm{Sh}_{1,n-1}, \mathbb{F}_l(n))_{\mathfrak{m}} \arrow[r] \arrow[dr, "Abel-Jacobi\ map"'] & \mathrm{H}^{2n-1}_{ét}(\mathrm{Sh}_{1,n-1}, \mathbb{F}_l(n))_{\mathfrak{m}} \arrow[u, two heads] \\
\mathrm{Ch}^1(\mathrm{Sh}^{\mathrm{ss}}_{1,n-1}, 1, \mathbb{F}_l)_{\mathfrak{m}} \arrow[u] \arrow[rr, "level-raising\ map"] & & \mathrm{H}^1(\mathbb{F}_{p^2}, \mathrm{H}^{2n-2}_{ét}(\overline{\mathrm{Sh}}_{1,n-1}, \mathbb{F}_l(n))_{\mathfrak{m}}) \arrow[u, hook]
\end{tikzcd}$$

As to the level raising map, we have the following theorem for $n = 2$:

Theorem 11.1. *Under Hypothesis 1.4, the level raising map*

$$\mathrm{Ch}^1(\mathrm{Sh}_{1,1},1,\mathbb{F}_l)_m \to \mathrm{H}^1(\mathbb{F}_{p^2},\mathrm{H}^2(\overline{\mathrm{Sh}}_{1,1},1,\mathbb{F}_l(2))_{\mathfrak{m}})$$

is surjective.

To prove the theorem, we need the following lemma:

Lemma 11.2. *For any filtration* $\mathrm{Fil}.A$ *of a group* A *with* $\mathrm{Fil}_{i-1}A \subseteq \mathrm{Fil}_iA$ *for any* i*, we define a sequence* $(gr_iA)_{i\in\mathbb{Z}}$ *as* $gr_iA := \frac{\mathrm{Fil}_iA}{\mathrm{Fil}_{i-1}A}$.

Now suppose we are given two groups A *and* B *and their filtrations* $\mathrm{Fil}_{\bullet,A}$ *and* $\mathrm{Fil}_{\bullet,B}$ *satisfying:*

(1) $0 = \mathrm{Fil}_{-3}A \subseteq \mathrm{Fil}_{-2}A = \mathrm{Fil}_{-1}A \subseteq \mathrm{Fil}_0A = \mathrm{Fil}_1A \subseteq \mathrm{Fil}_2A = A$.
(2) $0 = \mathrm{Fil}_{-1}B \subseteq \mathrm{Fil}_0B = B$.

Then we have $(gr_iA)_{i\in\mathbb{Z}}$ *and* $(gr_iB)_{i\in\mathbb{Z}}$ *satisfying:*

(1) $gr_iA \neq 0$ *if and only if* $i = -2, 0, 2$.
(2) $gr_iB \neq 0$ *if and only if* $i = 0$.

Moreover for any map from A *to* B *preserving filtrations, we have* $gr_2A \twoheadrightarrow \mathrm{Coker}(gr_0A \to gr_0B)$.

Proof. In fact, we have

$$gr_2(A) = \frac{\mathrm{Fil}_2A}{\mathrm{Fil}_1A} = \frac{A}{\mathrm{Fil}_0A} \twoheadrightarrow \frac{B}{\mathrm{Im}(\mathrm{Fil}_0A \to B)} = \frac{gr_0B}{\mathrm{Im}(gr_0A \to gr_0B)} = \mathrm{Coker}(gr_0A \to gr_0B)$$

. □

By indefinite Ihara lemma, we get the map $\mathrm{H}^2(\overline{Sh}_{1,1}(K^1_{\mathfrak{p}}),\mathbb{F}_l(2))_{\mathfrak{m}} \twoheadrightarrow \mathrm{H}^2(\overline{Sh}_{1,1},\mathbb{F}_l(2))^{\oplus 2}_{\mathfrak{m}}$ is surjective. Applying Proposition B.4 to $\mathcal{S}h_{1,1}$, we can also get a spectral sequence. Since $\mathcal{S}h_{1,1}$ is smooth and irreducible, the monodromy filtration of $\mathcal{S}h_{1,1}$ concentrates on itself. Moreover the surjection in the indefinite Ihara lemma preserves monodromy filtrations by Proposition B.5.2 and Proposition B.5.1.

Proposition 11.3. *For the weight spectral sequence obtained in B.4, it satisfies* $H^1(Y^{(1)}_{\overline{\mathbb{F}}_p})(-1) = 0$ *in the* E_1*-page and after localizaing at* m*, the* E_2*-page can be expressed as the following diagram:*

$$\begin{array}{ccccc}
E_{2,m}^{-2,4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & E_{2,m}^{0,2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & E_{2,m}^{2,0}
\end{array}$$

, where we use index m *to denote the localization. Moreover, we have* $E_{2,m}^{-2,4} = \mathrm{Ch}^1(\mathrm{Sh}^{ss}_{1,1},1,\mathbb{F}_l)_{\mathfrak{m}}$.

Proof. We first show $H^1_{\acute{e}t}(Y^{(1)}_{\overline{\mathbb{F}}_p})(-1) = 0$ in E^1-page. For simplicity, we omit the twist '-1' here, as it does not affect the group structure.

It can be checked that

$$H^1_{\acute{e}t}(\mathrm{Y}^{(1)}_{\overline{\mathbb{F}}_p}) = H^1_{\acute{e}t}(\overline{\tilde{\mathrm{Y}}}_{00}\bigcap\overline{\mathrm{Y}}_{01})\bigoplus H^1_{\acute{e}t}(\overline{\tilde{\mathrm{Y}}}_{00}\bigcap\overline{\mathrm{Y}}_{10})\bigoplus \mathrm{H}^1_{\acute{e}t}(\overline{\tilde{\mathrm{Y}}}_{11}\bigcap\overline{\mathrm{Y}}_{01})$$
$$\bigoplus H^1_{\acute{e}t}(\overline{\tilde{\mathrm{Y}}}_{11}\bigcap\overline{\mathrm{Y}}_{10})\bigoplus H^1_{\acute{e}t}(\overline{\tilde{\mathrm{Y}}}_{11}\bigcap\overline{\tilde{\mathrm{Y}}}_{00}).$$

Here we use a line overhead to express the geoemtric special fiber.

By 7.6, we have $H^1_{\acute{e}t}(\overline{\mathrm{Y}}^{(1)}_{\overline{\mathbb{F}}_p}) = H^1_{\acute{e}t}(\overline{\mathrm{Y}}_1)^{\oplus 2}\bigoplus H^1_{\acute{e}t}(\overline{\mathrm{Y}}_2)\bigoplus H^1_{\acute{e}t}(\overline{\mathrm{T}}\times\mathbb{P}^1)$. Both $\overline{\mathrm{Y}}_1$ and $\overline{\mathrm{Y}}_2$ can be expressed as $\mathbb{P}_1$ -bundle over $\overline{\mathrm{Sh}}_{0,2}$. Since $\overline{\mathrm{Sh}}_{0,2}$ and $\overline{\mathrm{T}}$ are both of dimension 0, we have $H^1_{\acute{e}t}(\overline{\mathrm{Y}}_1) = H^1_{\acute{e}t}(\overline{\mathrm{Y}}_2) = H^1_{\acute{e}t}(\overline{\mathrm{T}}\times\mathbb{P}^1) = 0$ following from $H^1_{\acute{e}t}(\mathbb{P}^1) = 0$.

Secondly, we calculate the E_2-page.

It is easy to check by definition that the E_2-page degenerates at every index (p,q) except $(p,q) = (-1,2),(1,2)$ and $p+q=2$. Hence other than the exceptional indices (p,q), $E_2^{p,q} = E_\infty^{p,q}$ is a subquotient of $H^{p+q}_{\acute{e}t}(X_{\overline{\mathbb{Q}_l}},\mathbb{F}_l)_m$, which is zero. Hence it suffices to check $E_{2,m}^{-1,2} = E_{2,m}^{1,2} = 0$. By duality, we only need to check $E_{2,m}^{-1,2} = 0$, i.e $d_1^{-1,2}$ is injective.

To calculate $\mathrm{H}^2_{\acute{e}t}(Y^{(0)}_{\overline{\mathbb{F}}_p})$, we need to calculate $\mathrm{H}^2_{\acute{e}t}(\tilde{\mathrm{Y}}_{00}) = \mathrm{H}^2_{\acute{e}t}(\widetilde{\mathbb{P}^1\times\mathbb{P}^1}/\overline{\mathrm{Sh}}_{0,2})$.By the blowing-up exact sequence, we have the following exact sequence

$$0\to H^2_{\acute{e}t}(\mathbb{P}^1\times\mathbb{P}^1/\overline{\mathrm{Sh}}_{0,2})\to \mathrm{H}^2_{\acute{e}t}(\widetilde{\mathbb{P}^1\times\mathbb{P}^1}/\overline{\mathrm{Sh}}_{0,2})\bigoplus \mathrm{H}^2_{\acute{e}t}(\overline{\mathrm{T}})\to H^2_{\acute{e}t}(\overline{\mathrm{T}}\times\mathbb{P}^1)\to 0,$$

where the first and the last zero term comes from $H^1_{\acute{e}t}(\overline{\mathrm{T}}\times\mathbb{P}^1) = 0$ and $H^3_{\acute{e}t}(\mathbb{P}^1\times\mathbb{P}^1/\mathrm{Sh}_{0,2}) = 0$. Moreover, we have $\mathrm{H}^2_{\acute{e}t}(\overline{\mathrm{T}}) = 0$ since T is of dimension zero. By Kunneth formula, we have $H^2_{\acute{e}t}(\mathbb{P}^1\times\mathbb{P}^1/\overline{\mathrm{Sh}}_{0,2}) = H^0_{\acute{e}t}(\overline{\mathrm{Sh}}_{0,2})^{\oplus 2}$.[12]

Thus, we have $E_1^{0,2} = \mathrm{H}^2_{\acute{e}t}(Y^{(0)}_{\overline{\mathbb{F}}_p})\oplus \mathrm{H}^0_{\acute{e}t}(Y^{(2)}_{\overline{\mathbb{F}}_p})(-1) = H^0_{\acute{e}t}(\overline{\mathrm{Sh}}_{0,2})^{\oplus 4}\bigoplus H^0_{\acute{e}t}(\overline{\mathrm{T}})^{\oplus 4}\bigoplus H^2_{\acute{e}t}(\overline{\mathrm{Sh}}_{1,1})^{\oplus 2}$. Moreover, $E_1^{-1,2} = \mathrm{H}^0_{\acute{e}t}(Y^{(1)}_{\overline{\mathbb{F}}_p})(-1) = H^0_{\acute{e}t}(\overline{\mathrm{Sh}}_{0,2})^{\oplus 4}\bigoplus H^0_{\acute{e}t}(\overline{\mathrm{T}})$.

[12]Rigorously, we should write $H^0_{\acute{e}t}(\overline{\mathrm{Sh}}_{0,2}, H^2_{\acute{e}t}(\mathbb{P}^1\times\mathbb{P}^1))$, which comes from the Leray Spectral Sequence.

With identification as above, we can express $d_1^{-1,2}$ as a matrix:

$$
\begin{array}{c|ccccc}
 & H^0_{\acute{e}t}(\overline{\mathrm{Sh}}_{0,2}) & H^0_{\acute{e}t}(\overline{\mathrm{Sh}}_{0,2}) & H^0_{\acute{e}t}(\overline{\mathrm{Sh}}_{0,2}) & H^0_{\acute{e}t}(\overline{\mathrm{Sh}}_{0,2}) & H^0_{\acute{e}t}(\overline{\mathrm{T}}) \\
H^0_{\acute{e}t}(\overline{\mathrm{Sh}}_{0,2}) & -Fr^{-1}S_{\mathfrak{p}} & -1 & 0 & 0 & -\overleftarrow{p}_* \\
H^0_{\acute{e}t}(\overline{\mathrm{Sh}}_{0,2}) & -S_{\mathfrak{p}} & -Fr & 0 & 0 & -\overleftarrow{p}_* \\
H^0_{\acute{e}t}(\overline{\mathrm{T}}) & \overleftarrow{p}^* S_{\mathfrak{p}} & \overleftarrow{p}^* & 0 & 0 & -1 \\
H^2_{\acute{e}t}(\overline{\mathrm{Sh}}_{1,1}) & -\overleftarrow{p}_{1,*} & 0 & \overleftarrow{p}_{1,*} & 0 & 0 \\
H^2_{\acute{e}t}(\overline{\mathrm{Sh}}_{1,1}) & 0 & \overleftarrow{p}_{1,*} & 0 & \overleftarrow{p}_{1,*} & 0 \\
H^0_{\acute{e}t}(\overline{\mathrm{Sh}}_{0,2}) & 0 & 0 & -1 & -Fr^{-1} & \overrightarrow{p}_* \\
H^0_{\acute{e}t}(\overline{\mathrm{Sh}}_{0,2}) & 0 & 0 & -Fr & -1 & \overrightarrow{p}_* \\
H^0_{\acute{e}t}(\overline{\mathrm{T}}) & 0 & 0 & -\overrightarrow{p}^* & -\overrightarrow{p}^* & 1 \\
H^0_{\acute{e}t}(\overline{\mathrm{T}}) & \overleftarrow{p}^* & 0 & \overrightarrow{p}^* & 0 & -TS_{\mathfrak{p}} \\
H^0_{\acute{e}t}(\overline{\mathrm{T}}) & 0 & \overleftarrow{p}^* & 0 & \overrightarrow{p}^* & -1
\end{array}
$$

The five columns each corresponding to $\overline{\mathrm{Y}}_{00} \bigcap \overline{\mathrm{Y}}_{10}, \overline{\mathrm{Y}}_{00} \bigcap \overline{\mathrm{Y}}_{01}, \overline{\mathrm{Y}}_{11} \bigcap \overline{\mathrm{Y}}_{10}, \overline{\mathrm{Y}}_{11} \bigcap \overline{\mathrm{Y}}_{01}$ and $\overline{\mathrm{Y}}_{00} \bigcap \overline{\mathrm{Y}}_{11}$. The first three rows correspond to $\overline{\mathrm{Y}}_{00}$. The fourth and fifth rows correspond to $\overline{\mathrm{Y}}_{10}$ and $\overline{\mathrm{Y}}_{01}$. The sixth to eighth rows correspond to $\overline{\mathrm{Y}}_{11}$. The ninth and tenth rows correspond to $\overline{\mathrm{Y}}_{00} \bigcap \overline{\mathrm{Y}}_{10} \bigcap \overline{\mathrm{Y}}_{11}$ and $\overline{\mathrm{Y}}_{00} \bigcap \overline{\mathrm{Y}}_{01} \bigcap \overline{\mathrm{Y}}_{11}$.

To show the injectivity after localization, we first consider the image of $d_{1,\mathfrak{m}}^{-1,2}$ projecting to the last two $H^0(\overline{\mathrm{T}})_{\mathfrak{m}}$. The map is exactly the β in the proof of the definite Ihara lemma. Hence we can suppose elements in $\mathrm{Ker} d_{1,\mathfrak{m}}^{1,2}$ as $(-S_{\mathfrak{p}}s, -s, t, t, , -\overleftarrow{p}^* s + \overrightarrow{p}^* t)$, where $(s,t) \in H^0_{\acute{e}t}(\overline{\mathrm{Sh}}_{0,2})_m^{\oplus 2}$. Then by taking this form into the eighth factor, we get the $\overleftarrow{p}^* s + \overrightarrow{p}^* t = 0$. Hence by the proof of the definite Ihara lemma, we have $s = t = 0$. And we get the injectivity. $\square$

Thus we get a filtration of $\mathrm{H}^2(\mathrm{Sh}_{1,1}(K^1_{\mathfrak{p}}), \mathbb{F}_l)_m$. After twisting every cohomology group with the Frobenius twist '(2)', by the 11.2, we have $E_{2,m}^{-2,4}(2) \twoheadrightarrow \mathrm{Coker}(E_{2,m}^{0,2}(2) \xrightarrow{\alpha} \mathrm{H}^2_{\acute{e}t}(\overline{\mathrm{Sh}}_{1,1}, \mathbb{F}_l(2))_{\mathfrak{m}})$.

Now we give the proof of theorem 7.1.

Proof of Theorem 7.1. First, we claim that $\mathrm{Coker}\alpha = \mathrm{Coker}(H^2_{ét}(\overline{\mathrm{Sh}}_{1,1})^{\oplus 2}(2)_{\mathfrak{m}} \to H^2_{ét}(\overline{\mathrm{Sh}}_{1,1})^{\oplus 2}(2)_{\mathfrak{m}})$ with the second map induced by the composition of immersions of $\overline{\mathrm{Y}}_{01}$ and $\overline{\mathrm{Y}}_{10}$ into $\overline{\mathrm{Sh}}_{1,1}(K^1_{\mathfrak{p}})$ and the projection from $\overline{\mathrm{Sh}}_{1,1}(K^1_{\mathfrak{p}})$ onto $\overline{\mathrm{Sh}}_{1,1}$ induced by the inclusion of the level groups.

If the claim holds, then by Proposition 7.5, we get α corresponds to the map $H^2_{ét}(\overline{\mathrm{Sh}}_{1,1})^{\oplus 2}(2)_{\mathfrak{m}} \to H^2_{ét}(\overline{\mathrm{Sh}}_{1,1})^{\oplus 2}(2)_{\mathfrak{m}}$ which is induced by $\begin{pmatrix} 1 & F \\ S_{\mathfrak{p}}^{-1}F & 1 \end{pmatrix}$. Then by basic linear algebra, we get the level raising map is surjective.

In fact, we only need to show the maps

$$\begin{aligned}\phi_1 :&H^2_{ét}(\overline{\tilde{\mathrm{Y}}}_{00}, \mathbb{F}_l(2))_{\mathfrak{m}} \to H^2_{ét}(\overline{\mathrm{Sh}}_{1,1}, \mathbb{F}_l(2))^{\oplus 2}_{\mathfrak{m}} \\ \phi_2 :&H^2_{ét}(\overline{\tilde{\mathrm{Y}}}_{11}, \mathbb{F}_l(2))_{\mathfrak{m}} \to H^2_{ét}(\overline{\mathrm{Sh}}_{1,1}, \mathbb{F}_l(2))^{\oplus 2}_{\mathfrak{m}}\end{aligned}$$

all have images contained in $(1 - Fr^2)H^2_{ét}(\overline{\mathrm{Sh}}_{1,1}, \mathbb{F}_l(2))^{\oplus 2}_{\mathfrak{m}}$, where ϕ_i are canonical maps induced by the maps $(\overleftarrow{p}_*, \overrightarrow{p}_*)$.

As to the map ϕ_1, we have have $\overrightarrow{p}_* : H^2_{ét}(\overline{\tilde{\mathrm{Y}}}_{00}, \mathbb{F}_l(2))_{\mathfrak{m}} \to H^2_{ét}(\overline{\mathrm{Sh}}_{1,1}, \mathbb{F}_l(2))_{\mathfrak{m}}$ can be expressed as $H^2_{ét}(\overline{\tilde{\mathrm{Y}}}_{00}, \mathbb{F}_l(2))_{\mathfrak{m}} \xrightarrow{\overrightarrow{p}_*} H^2_{ét}(\overline{\mathrm{Y}}_2, \mathbb{F}_l(1))_{\mathfrak{m}} \xrightarrow{i_{2,*}} H^2_{ét}(\overline{\mathrm{Sh}}_{1,1}, \mathbb{F}_l(2))_{\mathfrak{m}}$ with the second map $i_{2,*}$ to be the Gysin map induced by natural inclusion $i_2 : \overline{\mathrm{Y}}_2 \to \overline{\mathrm{Sh}}_{1,1}$. For any $x \in H^2_{ét}(\overline{\mathrm{Y}}_2, \mathbb{F}_l)_{\mathfrak{m}}$, we have $Fr_{p^2}(i_{2,*}(x)) = i_{2,*}(Fr_{p^2}(x)) = p^{-2} i_{2,*}(Fr'_{p^2}(x))$. Here we use Fr'_{p^2} to express the Frobenius action on $\mathrm{Sh}_{1,1}$. Therefore, since $l \nmid p^2 - 1$, if we take $x' = (1 - p^{-2})^{-1} Fr'^{-1}_{p^2} x$, then we have $i_{2,*}(x) = (1 - Fr_{p^2}) i_{2,*}(x')$. The proof for $\overleftarrow{p}_*$ is similar and we omit here.

The proof of ϕ_2 is also the same as ϕ_1. Then with an argument similar as above, we finish our proof. ∎

It remains to check that the map we constructed here is exact the Abel-Jacobi map. Under the same notation as in Section B.3, let $\Lambda = \mathbb{F}_\ell$ and $R\psi\Lambda$ be the sheaf of nearby cycles. Let $\mathrm{Sh}_{1,1}(K^1_{\mathfrak{p}})^{\mathrm{ord}}$ be the ordinary locus of $\mathrm{Sh}_{1,1}(K^1_{\mathfrak{p}})$ and $\mathrm{Sh}_{1,1}(K^1_{\mathfrak{p}})^{\mathrm{ss}}$ be the supersingular locus of $\mathrm{Sh}_{1,1}(K^1_{\mathfrak{p}})$. We also use a similar notation for $\mathrm{Sh}_{1,n-1}$. Consider the monodromy filtration $M_\bullet : 0 = M_{-2}R\psi\Lambda \subseteq M_{-1}R\psi\Lambda \subseteq M_0 R\psi\Lambda \subseteq M_1 R\psi\Lambda \subseteq M_2 R\psi\Lambda = R\psi\Lambda$, we take $M_{\geq i}R\psi\Lambda = M_2 R\psi\Lambda / M_{i-1}R\psi\Lambda$ and $gr^M_i R\psi\Lambda = M_i R\psi\Lambda / M_{i-1} R\psi\Lambda$. As in the proof of Proposition 11.3, we can see easily that $M_{\geq 1} = M_{\geq 2}$.

Lemma 11.4. *For $gr^M_i R\psi\Lambda$ and $M_{\geq 0}R\psi\Lambda$, we have $M_{\geq 0}R\psi\Lambda|_{\mathrm{Sh}_{1,1}(K^1_{\mathfrak{p}})^{\mathrm{ord}}} = gr^M_0 R\psi\Lambda|_{\mathrm{Sh}_{1,1}(K^1_{\mathfrak{p}})^{\mathrm{ord}}}$ when restricting to $\mathrm{Sh}_{1,1}(K^1_{\mathfrak{p}})^{\mathrm{ord}}$.*

Proof. We have the exact sequence in $\mathrm{Perv}(Y_{\overline{F}})[-2]$:

$$0 \to gr^M_0 R\psi\Lambda \to M_{\geq 0} R\psi\Lambda \to gr^M_2 R\psi\Lambda \to 0.$$

. By Proposition B.4, we have $gr^M_2 R\psi\Lambda$ is isomorphic to $a_{2*}\Lambda(-2)[-2]$, which is zero restricted to the ordinary locus. Hence we get

$$M_{\geq 0} R\psi\Lambda|_{\mathrm{Sh}_{1,1}(K^1_{\mathfrak{p}})^{\mathrm{ord}}} = gr^M_0 R\psi\Lambda|_{\mathrm{Sh}_{1,1}(K^1_{\mathfrak{p}})^{\mathrm{ord}}}.$$

Here we finish the proof. ∎

Proposition 11.5. *The surjective map obtained from the weight spectral sequences coincides with the Arithmetic level raising map in Theorem 11.1.*

Proof. We have the following commutative diagram:

$$\begin{tikzcd}
H^2_{\acute{e}t}(\overline{\widetilde{\mathrm{Sh}}}_{1,1}(K^1_{\mathfrak{p}}), gr^M_0 R\psi\Lambda(2))_{\mathfrak{m}} \arrow[r, "="] \arrow[d, "\alpha_1"] & H^2_{\acute{e}t}(\overline{\widetilde{\mathrm{Sh}}}_{1,1}(K^1_{\mathfrak{p}}), gr^M_0 R\psi\Lambda(2))_{\mathfrak{m}} \arrow[r, "\sigma_1"] \arrow[d, "\alpha_2"] & H^2_{\acute{e}t}(\overline{\mathrm{Sh}}_{1,1}, \Lambda(2))_{\mathfrak{m}} \arrow[d, "\alpha_3"] \\
H^2_{\acute{e}t}(\overline{\widetilde{\mathrm{Sh}}}_{1,1}(K^1_{\mathfrak{p}}), M_{\geq 0} R\psi\Lambda(2))_{\mathfrak{m}} \arrow[r, "\rho_1"] \arrow[d, "\beta_1"] & H^2_{\acute{e}t}(\overline{\widetilde{\mathrm{Sh}}}_{1,1}(K^1_{\mathfrak{p}})^{\mathrm{ord}}, gr^M_0 R\psi\Lambda(2))_{\mathfrak{m}} \arrow[r, "\sigma_2"] \arrow[d, "\beta_2"] & H^2_{\acute{e}t}(\overline{\mathrm{Sh}}^{\mathrm{ord}}_{1,1}, \Lambda(2))_{\mathfrak{m}} \arrow[d, "\beta_3"] \\
H^2_{\acute{e}t}(\overline{\widetilde{\mathrm{Sh}}}_{1,1}(K^1_{\mathfrak{p}}), gr^M_2 R\psi\Lambda(2))_{\mathfrak{m}} \arrow[r, "\rho_2"] & H^3_{\acute{e}t}(\overline{\widetilde{\mathrm{Sh}}}_{1,1}(K^3_{\mathfrak{p}})^{\mathrm{ss}}, i'^! gr^M_0 R\psi\Lambda(2))_{\mathfrak{m}} \arrow[r, "\sigma_3"] & H^3_{\acute{e}t}(\overline{\mathrm{Sh}}^{\mathrm{ss}}_{1,1}, i^!\Lambda(2))_{\mathfrak{m}}
\end{tikzcd}$$

Here σ_1, σ_2 and σ_3 are induced by the map in the definite Ihara lemma. The map ρ_1 is induced by the composition map and the isomorphism in Lemma 11.4. The maps α_1 and β_1 are induced by the exact sequence

$$0 \to gr^M_0 R\psi\Lambda \to M_{\geq 0} R\psi\Lambda \to gr^M_2 R\psi\Lambda \to 0.$$

The maps α_i, β_i for $i = 2, 3$ come from the Gysin sequence with $i'^!$ and $i^!$ to be closed immersions. It can be checked directly that α_1 is injective and β_1 is surjective. Hence ρ_2 is well-defined.

Furthermore, the map obtained from the weight spectral sequence satisfies the following commutative diagram:

$$\begin{tikzcd}
H^2_{\acute{e}t}(\overline{\widetilde{\mathrm{Sh}}}_{1,1}(K^1_{\mathfrak{p}}), gr^M_2 R\psi\Lambda(2))_{\mathfrak{m}} \arrow[d] & H^2_{\acute{e}t}(\overline{\widetilde{\mathrm{Sh}}}_{1,1}(K^1_{\mathfrak{p}}), M_{\geq 0} R\psi\Lambda(2))_{\mathfrak{m}} \arrow[l] \arrow[d] & H^2_{\acute{e}t}(\overline{\widetilde{\mathrm{Sh}}}_{1,1}(K^1_{\mathfrak{p}}), gr^M_0 R\psi\Lambda(2))_{\mathfrak{m}} \arrow[l] \arrow[d] \\
H^3_{\acute{e}t}(\overline{\mathrm{Sh}}^{\mathrm{ss}}_{1,1}, i^!\Lambda(2))_{\mathfrak{m}} \arrow[d] & H^2_{\acute{e}t}(\overline{\mathrm{Sh}}^{\mathrm{ord}}_{1,1}, \Lambda(2))_{\mathfrak{m}} \arrow[l] & H^2_{\acute{e}t}(\overline{\mathrm{Sh}}_{1,1}, \Lambda(2))_{\mathfrak{m}} \arrow[l] \arrow[dl] \\
H^3_{\acute{e}t}(\mathrm{Sh}_{1,1}, \mathbb{F}_\ell(2))_{\mathfrak{m}} & H^1(\mathbb{F}_{p^2}, H^2_{\acute{e}t}(\overline{\mathrm{Sh}}_{1,1}, \mathbb{F}_\ell)_{\mathfrak{m}}) \arrow[l, "\cong"'] &
\end{tikzcd}$$

Here the lower left vertical map is the Gysin map.

By the definition of $\mathrm{Ch}^1(\mathrm{Sh}_{1,1}, 1, \mathbb{F}_l)_m$, the following diagram is commutative.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ch}^1(\mathrm{Sh}_{1,1}, 1, \mathbb{F}_l)_{\mathfrak{m}} \arrow[r] \arrow[d] & (H^0_{\acute{e}t}(\mathrm{Y}^\circ_1, \mathbb{G}_m)_{\mathfrak{m}} \oplus H^0_{\acute{e}t}(\mathrm{Y}^\circ_2, \mathbb{G}_m)_{\mathfrak{m}}) \otimes \mathbb{F}_\ell \arrow[r, "div"] \arrow[d] & H^0_{\acute{e}t}(\mathrm{Y}_1 \cap \mathrm{Y}_2)_{\mathfrak{m}} \arrow[d, "="] \\
0 \arrow[r] & H^1_{\acute{e}t}(\mathrm{Sh}^{ss}_{1,1}, \mathbb{F}_\ell(1))_{\mathfrak{m}} \arrow[r] \arrow[d] & (H^0_{\acute{e}t}(\mathrm{Y}^\circ_1, \mathbb{F}_\ell(1))_{\mathfrak{m}} \oplus H^0_{\acute{e}t}(\mathrm{Y}^\circ_2, \mathbb{F}_\ell(1))_{\mathfrak{m}}) \arrow[r] & H^0_{\acute{e}t}(\mathrm{Y}_1 \cap \mathrm{Y}_2)_{\mathfrak{m}} \\
& H^3_{\acute{e}t}(\mathrm{Sh}_{1,1}, \mathbb{F}_\ell(2))_{\mathfrak{m}} & &
\end{tikzcd}$$

Here the left vertical maps are cycle class map and Gysin map. The middle vertical map comes from the Kummer sequence.

After identifying $\mathrm{Ch}^1(\mathrm{Sh}_{1,1}, 1, \mathbb{F}_l)_{\mathfrak{m}}$ with a subgroup of $H^0_{\acute{e}t}(\mathrm{Y}_1 \cap \mathrm{Y}_2)_{\mathfrak{m}} = H^0_{\acute{e}t}(\mathrm{T})_{\mathfrak{m}}$, we can see the arithmetic level raising map coincides with the map obtained from the weight spectral sequence directly. Hence we finish the proof. $\square$

For $n \geq 3$, we have a similar conjecture:

Conjecture 11.6. *Under the Hypothesis 1.4, the level raising map*

$$\mathrm{Ch}^1(\mathrm{Sh}^{\mathrm{ss}}_{1,n-1}, 1, \mathbb{F}_l)_{\mathfrak{m}} \to \mathrm{H}^1(\mathbb{F}_{p^2}, \mathrm{H}^{2n-2}_{\acute{e}t}(\overline{\mathrm{Sh}}_{1,n-1}, \mathbb{F}_l(n))_{\mathfrak{m}})$$

is surjective.

12. Nonvanishing of the higher Chow group

In this section we define when the higher chow group is nonvanishing for $n = 2, 3$. By the Ihara lemma 1.5, it is equivalent to when the arithmetic level raising theorem is nontrivial. In fact we have the following theorem:

Theorem 12.1. *When $n = 2, 3$, the higher Chow group $\mathrm{Ch}^1(\mathrm{Sh}^{\mathrm{ss}}_{1,n-1}, 1, \mathbb{F}_l)_{\mathfrak{m}}$ is nonzero if and only if there exist two Satake parameters α_i, α_j such that $\alpha_i = p^2\alpha_j$. If $n = 2$, we can further show $\Pi_{\mathfrak{p}}$ is isomorphic to a twisted Steinberg representation and if $n = 3$ and futher assume there exists only one pair such (α_i, α_j) we can show $\Pi_{\mathfrak{p}}$ is isomorphic to the isobaric sum of a 2 dimensional twisted Steinberg representation and a 1 dimensional reprentation, denoted by $St_2(\gamma) \boxplus \beta$.*

For $0 \le i \le n-1$, let K_i be the open compact subgroup of $G(\mathbb{A}^\infty)$ with hyperspecial level at p such that $K_{i,\mathfrak{p}} = \mathrm{diag}\{p^{-1}\mathrm{I}_i, \mathrm{I}_{n-i}\}\mathrm{GL}_n(\mathbb{Z}_{p^2})\mathrm{diag}\{p\mathrm{I}_i, \mathrm{I}_{n-i}\}$ For $0 \le i \le n-2$, let K_i^1 be the intersection of K_i and K_{i-1} (If $i = 0$, let $i-1$ be $n-1$.) For $0 \le i \le n-2$, $\mathcal{S}h_{0,n}(K_i^1)$ gives a correspondence between $\mathcal{S}h_{0,n}(K_i)$ and $\mathcal{S}h_{0,n}(K_{i-1})$ and A functors as an interwining operator between each $\mathcal{S}h_{0,n}(K_i^1)$ and $\mathcal{S}h_{0,n}(K_{i+1}^1)$ as depicted in Section 6. Since A's between different i are not the same, we denote the A from $\mathcal{S}h_{0,n}(K_i^1)$ to $\mathcal{S}h_{0,n}(K_{i-1}^1)$ by A_i. We also use $\overrightarrow{p}_i$ and $\overleftarrow{p}_i$ to denote the projection from $\mathcal{S}h_{0,n}(K_i^1)$ to $\mathcal{S}h_{0,n}(K_{i-1})$ and $\mathcal{S}h_{0,n}(K_i)$.

By Section 3.6, we have for any K, $H^0_{\acute{e}t}(\mathrm{Sh}_{0,n}(K_i)) = \bigoplus_{\pi \in \mathrm{Irr}(G_{a_\bullet}(\mathbb{A}^\infty))} m(\pi)\iota_\ell(\pi^K)$. Here $m(\pi)$ is short for the multiplicity of π, which is not dependent on K. $H^0_{\acute{e}t}(\mathrm{Sh}_{0,n}(K_i))$ can be viewed as a $\mathscr{H}_K \times \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})$-module with the Galois action trivial. Thus we only need to consider the Hecke action. If we further assume the component away from $\mathfrak{p}$, denoted by $K_i^{\mathfrak{p}}$, are the same for different i, we only need to consider $\pi_{\mathfrak{p}}^{K_{\mathfrak{p}}}$. After base change to a $\mathrm{GL}_n(\mathbb{Q}_{p^2})$ representation $\Pi_{\mathfrak{p}}$, we only need to consider $\Pi_{\mathfrak{p}}^{K_{\mathfrak{p}}}$. We use ι_i^+ and ι_i^- to denote the map from $\Pi_{\mathfrak{p}}^{K_{i,\mathfrak{p}}}$ to $\Pi_{\mathfrak{p}}^{K^1_{i,\mathfrak{p}}}$ and $\Pi_{\mathfrak{p}}^{K^1_{i+1,\mathfrak{p}}}$ induced by $\overrightarrow{p}_i$ and $\overleftarrow{p}_{i+1}$. Let B be the Iwahori subgroup of $\mathrm{GL}_n(\mathbb{Q}_{p^2})$, we can view all the Hecke algebra correponds to different K_i and K_i^1 as subalgebra of $\mathscr{H}_B$. The triviality of the higher Chow group is equivalent to the surjectivity of the map $\bigoplus_{i=0}^{n-1} \Pi_{\mathfrak{p}}^{K_{i,\mathfrak{p}}} \xrightarrow{\iota^-_{n-1}, \iota_0^+, A_1\circ\iota_1^+, \cdots, A_1\circ A_2\cdots\circ A_{n-2}\circ\iota^+_{n-2}} \Pi_{\mathfrak{p}}^{K^1_{0,\mathfrak{p}}}$ for all possible Π.

Moreover, we have the following proposition.

Proposition 12.2. *An irreducible admissible representation π with non-zero B-fixed vector imbeds into an unramified principal series representation.*

Proof. We refer to [16] here. □

Therefore, we can embed $\Pi_{\mathfrak{p}}$ to an unramified principal series represeentation $I(\chi)$ for some unramified character χ from the diagonal torus to $\mathbb{C}$, which is determined by n-tuple $\{\alpha_1, \cdots, \alpha_n\} \in \mathbb{C}^n$. Hence We only need to consider when the map $\bigoplus_{i=0}^{n-1} I(\chi)^{K_{i,\mathfrak{p}}} \xrightarrow{\iota^-_{n-1}, \iota_0^+, A_1\circ\iota_1^+, \cdots, A_1\circ A_2\cdots\circ A_{n-2}\circ\iota^+_{n-2}} I(\chi)^{K^1_{0,\mathfrak{p}}}$. For $I(\chi)^B$ and the Hecke action on it, we first recall some results in [13].

Let $W = S_n$ be the Weyl group of GL_n and W_{aff} be the affine Weyl group. Let $\{s_1, \cdots, s_{n-1}\}$ be the simple roots of the Weyl group and $s_0, s_1, \cdots, s_{n-1}$ be the simple roots of the affine Weyl group. Let $\mathcal{H}(GL_n, B)$ be the affine Hecke algebra with generators $R_0, \cdots, R_{n-1}$ correpsond to $s_0, \cdots, s_{n-1}$. The B-fixed vectors in the unramified principal series $I(\chi)^B$ have a basis $\{\phi_w \mid w \in W\}$. such that $\phi_w(pw'b) = \chi\delta^{1/2}(p)$ with $pw'b$ correpsonding to the decomposition $GL_n = PWB$[13] and δ to be the

[13]Here we use P to denote the canonical Borel subgroup of GL_n.

modular of GL_n. The action of $\mathcal{H}(\mathrm{GL}_n, B)$ on $I(\chi)^B$ satisfies: If $1 \leq i \leq n-1$,

$$R_i\phi_w = \begin{cases} \phi_{ws_i} & \text{if } w(i) < w(i+1); \\ p^2\phi_{ws_i} + (p^2-1)\phi_w & \text{if } w(i) = w(i+1); \end{cases}$$

And

$$R_0\phi_w = \begin{cases} p^{2(w(1)-w(n))}\alpha_{w(1)}\alpha_{w(n)}^{-1}\phi_{w(1n)} & \text{if } w(1) > w(n); \\ p^{2(1+w(1)-w(n))}\alpha_{w(1)}\alpha_{w(n)}^{-1}\phi_{w(1n)} + (p^2-1)\phi_w & \text{if } w(1) < w(n); \end{cases}$$

In particular, $I(\chi)^{<B,s_i>}$ is spanned by $\phi_w + \phi_{ws_i}$ for $1 \leq i \leq n-1$ and $w \in S_n$ and $I(\chi)^{<B,s_0>}$ is spanned by $\phi_w + p^{2(w(1)-w(n))}\alpha_{w(1)}\alpha_{w(n)}^{-1}\phi_{w(1n)}$ for $w \in S_n$.

Thus we have the following proposition:

Proposition 12.3. *For $1 \leq i \leq n$. Let $\Theta_i = \sum\limits_{w(1)=i} \phi_w$ and $\Xi_i = \sum\limits_{w(n)=i} \phi_w$. Then $I(\chi)^{K^1_{1,\mathfrak{p}}}$ is spanned by $\Theta_1, \cdots, \Theta_n$ and $I(\chi)^{K^0_{n-1,\mathfrak{p}}}$ is spanned by $\Xi_1, \cdots, \Xi_n$. $I(\chi)^{K_{0,\mathfrak{p}}}$ is spanned by $\sum\limits_{i=1}^n \Theta_i = \sum\limits_{i=1}^n \Xi_i$, $I(\chi)^{K_{1,\mathfrak{p}}}$ is spanned by $\sum\limits_{i=1}^n p^{2(1-i)}\alpha_1\alpha_i^{-1}\Theta_i$ and $I(\chi)^{K_n}$ is spanned by $\sum\limits_{i=1}^n p^{2(i-n)}\alpha_i\alpha_n^{-1}\Xi_i$.*

Proof. It is easy to see that $I(\chi)^{K_{0,\mathfrak{p}}} = I(\chi)^{<B,s_1,\cdots,s_{n-1}>}$, $I(\chi)^{K_{1,\mathfrak{p}}} = I(\chi)^{<B,s_0,s_2,\cdots,s_{n-1}>}$ and $I(\chi)^{K_{n,\mathfrak{p}}} = I(\chi)^{<B,s_0,s_1,\cdots,s_{n-2}>}$. Moreover, we have $I(\chi)^{K^1_{1,\mathfrak{p}}} = I(\chi)^{<B,s_2,\cdots,s_{n-1}>}$ and $I(\chi)^{K^1_{0,\mathfrak{p}}} = I(\chi)^{<B,s_1,,s_2,\cdots,s_{n-2}>}$. Then we can get the result by direct calculation. ☐

Now we are going to depict the action of A from $I(\chi)^{K^1_{1,\mathfrak{p}}}$ to $I(\chi)^{K^1_{0,\mathfrak{p}}}$. It is depicted by the following linear-algebra model.

The intertwining operator A from $I(\chi)^{K^1_{1,\mathfrak{p}}}$ to $I(\chi)^{K^1_{0,\mathfrak{p}}}$. corresponds to the morphism from $\mathrm{Ind}_{K^1_{1,\mathfrak{p}}}^{\mathrm{GL}_n(\mathbb{F}_{p^2})}\mathbb{1}$ to $\mathrm{Ind}_{K^1_{0,\mathfrak{p}}}^{\mathrm{GL}_n(\mathbb{F}_{p^2})}\mathbb{1}$. After identifying $\mathrm{Ind}_{K^1_{1,\mathfrak{p}}}^{\mathrm{GL}_n(\mathbb{F}_{p^2})}\mathbb{1}$ with the vector space spanned by $\{[L] : L \text{ lines in } \mathbb{F}_{p^2}^n\}$ and $\mathrm{Ind}_{K^1_{0,\mathfrak{p}}}^{\mathrm{GL}_n(\mathbb{F}_{p^2})}\mathbb{1}$ with $\{[H] : H \text{ hyperplanes in } \mathbb{F}_{p^2}^n\}$, A acts by mapping $[L]$ to $\sum\limits_{L\subseteq H}[H]$.

Now we consider the Borel-fixed vectors. Given a sequence of vector spaces: $0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{n-1} \subseteq V_n = V$, for $1 \leq i \leq n$, Θ_i we constructed above corresponds to $\sum\{[L] : L \not\subseteq V_{i-1},\ L \subseteq V_i\}$ and Ξ_i we constructed above corresponds to $\sum\{[H] : V_{i-1} \subseteq H,\ V_i \subseteq H\}$. Therefore, we have

$$\begin{aligned} A\Theta_i &= \sum_{L+V_{i-1}=V_i}\sum_{L\subseteq H}[H] = \sum_H \#\{L : L \subseteq V_i \cap H,\ L \not\subseteq V_{i-1}\cap H\}[H] \\ &= \sum_{j>i} p^{2(n-i)}\Xi_j + \sum_{j<i} p^{2(i-2)}\Xi_j. \end{aligned}$$

By the above analysis, a sufficient condition for Theorem 12.1 is that

$$A \circ \iota_1^+(I(\chi)^{K_{1,\mathfrak{p}}}) \subseteq (\iota_0^+(I(\chi)^{K_{0,\mathfrak{p}}}) + \iota_{n-1}^-(I(\chi)^{K_{n-1,\mathfrak{p}}})),$$

since $A \circ \iota_0^-(I(\chi)^{K_{0,\mathfrak{p}}}) = A\sum\limits_{i=1}^n \Theta_i = \frac{p^{2n-2}-1}{p^2-1}\sum\limits_{i=1}^n \Xi_i = \iota_0^+(I(\chi)^{K_{0,\mathfrak{p}}})$ and $\frac{p^{2n-2}-1}{p^2-1}$ is prime to ℓ. Under the basis of $\Xi_1, \cdots, \Xi_n$ and after rescaling, $I(\chi)^{K_{0,\mathfrak{p}}})$ is spanned by a coordinate vector

$$(1, 1, \cdots, 1),$$

$\iota_{n-1}^-(I(\chi)^{K_{n-1,\mathfrak{p}}}$ is spanned by a coordinate vector

$$(\alpha_1, p^2\alpha_2, \cdots, p^{2(n-1)\alpha_n})$$

and $A \circ \iota_1^+(I(\chi)^{K_{1,\mathfrak{p}}})$ is spanned by a vector of coordinate

$$(p^{-2}(\alpha_2^{-1} + \cdots + \alpha_n^{-1}), \alpha_1^{-1} + p^{-2}(\alpha_3^{-1} + \cdots + \alpha_n^{-1}), \cdots, \alpha_1^{-1} + \cdots + \alpha_{n-1}^{-1}).$$

Then it is sufficient if the three vectors are linear dependent.

When $n = 2, 3$, this condition is also necessary. Thus we can calculate easily that this requires $\alpha_1 = p^2\alpha_2$ or $\alpha_2 = p^2\alpha_3$ or $\alpha_3 = p^2\alpha_1$. If we assume there exists only one pair of Satake parameters differ by p^2, then by the Bernstein-Zelevinsky classification, the representation $\Pi_{\mathfrak{p}} \cong St_2(\gamma) \boxplus \beta$ for some 1-dimension representations γ and β.

Remark 12.4. In fact for general n, if we assume there exists only one pair of Satake parameters differ by p^2, we can also get from Bernstein-Zelevinsky classification that $\Pi_{\mathfrak{p}} \cong St_2(\gamma)\boxplus\beta_1\boxplus\cdots\boxplus\beta_{n-2}$ for some 1-dimension representations γ and $\beta_1, \cdots, \beta_{n-2}$. In this by Langlands correpsondence, the monodromy operator of the Weil-Deligne representation correpsonds to it is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \oplus \mathbb{1}^{n-2}$.

Appendix A. Appendix on Coxeter groups, reductive group schemes

In this appendix we fix some conventions and recall results on Coxeter groups and on reductive group schemes as in [27, Appendix A].

A.1. **Coset representatives of Coxeter groups.** Let W be a Coxeter group and I its generating set of simple reflections. Let ℓ denote the length function on W.

Let J be a subset of I. We denote by W_J the subgroup of W generated by J and by W^J (respectively ${}^JW$) the set of elements w of W which have minimal length in their coset wW_J (respectively W_Jw). Then every $w \in W$ can be written uniquely as $w = w^J w_J = w'_J {}^J w$ with $w_J, w'_J \in W_J$, $w^J \in W^J$ and ${}^Jw \in {}^JW$, and $\ell(w) = \ell(w_J) + \ell(w^J) = \ell(w'_J) + \ell({}^Jw)$. In particular, W^J and ${}^JW$ are systems of representatives for W/W_J and $W_J\backslash W$ respectively.

Furthermore, if K is a second subset of I, let ${}^JW^K$ be the set of $w \in W$ which have minimal length in the double coset W_JwW_K. Then ${}^JW^K ={}^J W \cap W^K$ and ${}^JW^K$ is a system of representatives for $W_J\backslash W/W_K$.

A.2. **Bruhat order.** We let $\leq$ denote the Bruhat order on W. This natural partial order is characterized by the following property: For $x, w \in W$ we have $x \leq w$ if and only if for some (or, equivalently, any) reduced expression $w = s_{i_1} \cdots s_{i_n}$ as a product of simple reflections $s_i \in I$, one gets a reduced expression for x by removing certain s_{i_j} from this product. The set ${}^JW$ can be described as

$${}^JW = \{\, w \in W \;;\; w < sw \text{ for all } s \in J \,\}.$$

A.3. **Reductive group schemes, maximal tori, and Borel subgroups.** Let S be a scheme. A *reductive group scheme over* S is a smooth affine group scheme G over S such that for every geometric point $s \in S$ the geometric fiber $G_{\bar{s}}$ is a connected reductive algebraic group over $\kappa(\bar{s})$.

Let G be a reductive group scheme over S. A *maximal torus of* G is a closed subtorus T of G such that $T_{\bar{s}}$ is a maximal element in the set of subtori of $G_{\bar{s}}$ for all $s \in S$. A *Borel subgroup* of G is a closed smooth subgroup scheme B of G such that for all $s \in S$ the geometric fiber $B_{\bar{s}}$ is a Borel subgroup of $G_{\bar{s}}$ in the usual sense (i.e., a maximal smooth connected solvable subgroup). A reductive group scheme over S is called *split* if there exists a maximal torus T of G such that $T \cong \mathbb{G}_{m,S}^r$ for some integer $r \geq 0$. If S is local, G is called *quasi-split* if there exists a Borel subgroup of G. Every split reductive group scheme is quasi-split.

A.4. **Parabolic subgroups and Levi subgroups.** A smooth closed subgroup scheme P of G is called *parabolic subgroup of* G if the fppf quotient G/P is representable by a smooth projective scheme or, equivalently if $G_{\bar{s}}/P_{\bar{s}}$ is proper for all $s \in S$. Every Borel subgroup of G is a parabolic subgroup. The *unipotent radical* of P, denoted by U_P, is the largest smooth normal closed subgroup scheme with unipotent and connected fibers. If P contains a maximal torus T of G, there exists a unique reductive closed subgroup scheme L of P containing T such that the canonical homomorphism $L \to P/U_P$ is an isomorphism. Any such subgroup L is called a *Levi subgroup of* P.

The functor that sends an S-scheme T to the set of Borel (resp. parabolic) subgroups of $G \times_S T$ is representable by a smooth projective S-scheme. We call the representing scheme Bor_G (resp. Par_G). The functor that attaches to an S-scheme T the set of pairs (P, L), where P is a parabolic subgroup of $G \times_S T$ and L is a Levi subgroup of P is representable by a smooth quasi-projective S-scheme.

A.5. **Weyl groups and types of parabolic subgroups over connected base schemes.** Let G be a reductive group over an algebraically closed field, let B be a Borel subgroup of G, and let T be a maximal torus of B. Let $W(T) := \mathrm{Norm}_G(T)/T$ denote the associated Weyl group, and let $I(B,T) \subset W(T)$ denote the set of simple reflections defined by B. Then $W(T)$ is a Coxeter group with respect to the subset $I(B,T)$.

A priori this data depends on the pair (B,T). However, any other such pair (B', T') is obtained by conjugating (B,T) by some element $g \in G$ which is unique up to right multiplication by T. Thus conjugation by g induces isomorphisms $W(T) \xrightarrow{\sim} W(T')$ and $I(B,T) \xrightarrow{\sim} I(B',T')$ that are independent of g. Moreover, the isomorphisms associated to any three such pairs are compatible with each other. Thus $W := W_G := W(T)$ and $I := I(B,T)$ for any choice of (B,T) can be viewed as instances of "the" Weyl group and "the" set of simple reflections of G, in the sense that up to unique isomorphisms they depend only on G.

Now let G be a quasi-split reductive group scheme over a connected scheme S. Then we obtain for any geometric point $\bar{s} \to S$ the Weyl group and the set of simple reflections $(W_{\bar{s}}, I_{\bar{s}})$ of $G_{\bar{s}}$. The algebraic fundamental group $\pi_1(S, \bar{s})$ acts naturally on $W_{\bar{s}}$ preserving $I_{\bar{s}}$ (because G is quasi-split), and every étale path γ from $\bar{s}$ to another geometric point $\bar{s}'$ of S yields an isomorphism of $(W_{\bar{s}}, I_{\bar{s}}) \xrightarrow{\sim} (W_{\bar{s}'}, I_{\bar{s}'})$ that is equivariant with respect to the isomorphism $\pi_1(S,\bar{s}) \xrightarrow{\sim} \pi_1(S, \bar{s}')$ induced by γ. In particular $(W_{\bar{s}}, I_{\bar{s}})$ together with its action by $\pi_1(S,\bar{s})$ is independent of the choice of $\bar{s}$ up to isomorphism. We denote it by (W, I) and call it the *Weyl system of* G.

If P is a parabolic subgroup of G and $s \in S$, the type $J_{\bar{s}} \subset I$ of the parabolic subgroup $P_{\bar{s}}$ of $G_{\bar{s}}$ is independent of $s \in S$ and we call $J := J_{\bar{s}}$ the *type of* P. For a subset J of I we denote by Par_J the open and closed subscheme of Par parameterizing parabolic subgroups of type J. If S is a semi-local scheme, J and Par_J are defined over a finite étale covering of S.

For simplicity assume that S is local. Let $J, K \subseteq I$ be subsets and let $S_1 \to S$ be the finite étale extension over which J and K are defined. Let $w \in {}^JW^K$. For every S_1-scheme S' and for every parabolic subgroup P of $G_{S'}$ of type J and every parabolic subgroup Q of $G_{S'}$ of type K we write

$$\mathrm{relpos}(P, Q) = w$$

if there exists an fppf-covering on $S'' \to S'$, a Borel subgroup B of $G_{S''}$ and a split maximal torus T of B such that $P_{S''}$ contains B and $Q_{S''}$ contains ${}^{\dot{w}}B$, where $\dot{w} \in \mathrm{Norm}_{G_{S''}}(T)(S'')$ is a representative of $w \in W = \mathrm{Norm}_{G_{S''}}(T)(S'')/T(S'')$.

If $S' = \operatorname{Spec} k$ for an algebraically closed field, then $(P, Q) \mapsto \mathrm{relpos}(P,Q)$ yields a bijection between $G(k)$-orbits on $\mathrm{Par}_J(k) \times \mathrm{Par}_K(k)$ and the set ${}^JW^K$.

Appendix B. Weight Spectral Sequences

In this appendix we recall some results on strictly semi-stable schemes and weight spectral seqeuences following [23].

B.1. **Strictly semi-stable schemes.** Let K be a finite extention of $\mathbb{Q}_p$ with residue field F. The specturm of the integer ring O_K will be denoted by S. A scheme X locally of finite presentation over $S = \operatorname{Spec} O_K$ is strictly semi-stable purely of relative dimension n if and only if the following conditions are satisfied.

(1) X is regular and flat over S.
(2) The generic fibre X_K is smooth purely of relative dimension n.
(3) The speical fibre X_F is a divisor of X with simple normal crossings, i.e., the irreducible components of X_F, denoted by $Y_1, \cdots Y_m$ for some m, satisfying for any $I \subseteq 1, \cdots, m$, $Y_I := \bigcap_{i \in I} Y_i$ is a closed subshcheme smooth of dimension $n + 1 - |I|$.

Zariski locally, X is étale over $\operatorname{Spec} O_K[T_0, \cdots, T_n]/(T_0 \cdots T_r - \pi)$ for a prime element π of K and an integer $0 \leq r \leq n$.

B.2. **Nearby cycles.** Let K, F, S be the same as above. Let X be a strictly semi-stable scheme over O_K purely of relative dimension n and $Y = X_F$ denotes the closed fibre of X. Let $\overline{K}$ be a separable closure of K and K^{ur} be the maximum unramified extension of K in $\overline{K}$. Let $\overline{F}$ be the residue field of K^{ur}. Let $I_K = \operatorname{Gal}(\overline{K}/K^{ur})$ be the inertia group. For a prime number ℓ invertible in F, let $t_\ell : I_K \to \mathbb{Z}_\ell(1)$ be the canonical surjection defined by $\sigma \mapsto (\sigma(\pi^{1/\ell^m})/\pi^{1/\ell^m})_m$ for a prime element π of K. Let S^{ur} denote the spectrum of the integer ring $O_{K^{ur}}$. Let $i : Y = X_F \to X, j : X_K \to X, \bar{i} : Y_{\overline{F}} \to X_{S^{ur}}$ and $\bar{j} : X_{\overline{K}} \to X_{S^{ur}}$ be the canonical maps.

Let ℓ be a prime number invertible on O_K and let Λ denote either of $\mathbb{Z}/\ell\mathbb{Z}, \mathbb{Z}_\ell$ and $\mathbb{Q}_\ell$. For $p \geq 0$, we define the *sheaf of nearby cycles* to be $R^p\psi\Lambda = \bar{i}^* R^p \bar{j}_* \Lambda$. It is a sheaf on $Y_{\overline{F}}$ with a continuous action of G_K compatible with the action of the quotient G_F on $Y_{\overline{F}}$. It can be shown $R\psi\Lambda$ is in the category $\operatorname{Perv}(Y_{\overline{F}})[-n]$ of $-n$-shifted perverse sheaves as in [23, Lemma 2.5].

B.3. **Monodromy filtration and Weight spectral sequences.** Let X be a scheme strictly semi-stable purely of relative dimension n over S. Let $Y = X_F$. Suppose $Y_1, \cdots, Y_m$ are irreducible components of Y. For a non-empty subset $I \subseteq \{1, \ldots, m\}$, we put $Y_I = \bigcap_{i \in I} Y_i$, which is smooth of dimension $n-p$ over F if Card $I = p+1$. For an integer $p \geq 0$, we put $Y^{(p)} = \coprod_{I \subseteq \{1,\ldots,m\}, \text{Card } I = p+1} Y_I$. Let $a_p : Y^{(p)} \to Y$ be the natural map.

Let T be an element in I_K such that $t_\ell(T)$ is a generator of $\mathbb{Z}_\ell(1)$ and $\nu = T - 1$. It can be checked that ν is an nilpotent element. Now we consider $R\psi\Lambda$ as an element in $\operatorname{Perv}(Y_{\overline{F}})[-n]$ with an action of I_K defined as above. The *monodromy filtration* of $R\psi\Lambda$ with repsect to ν is constructed as follows. Let $F_\bullet$ be an increasing filtration on $R\psi\Lambda$ satisfying $F_p R\psi\Lambda = \operatorname{Ker}(\nu^{p+1} : R\psi\Lambda \to R\psi\Lambda)$ for $p \geq 0$ and $F_p A = 0$ for $p < 0$. We also define a decreasing filtration $G^\bullet$ by $G^q R\psi\Lambda = \operatorname{Im}(\nu^q : R\psi\Lambda \to R\psi\Lambda)$ for $q > 0$ and $G^q R\psi\Lambda = R\psi\Lambda$ for $q \leq 0$. We call $F_\bullet$ the *kernel filtration* and $G_\bullet$ the *image filtration*. The *monodromy filtration* $M_\bullet$ is defined by $M_r R\psi\Lambda = \sum_{p-q=r} F_p R\psi\Lambda \cap G^q R\psi\Lambda$. The graded pieces are denoted by $Gr^M_r R\psi\Lambda = M_r R\psi\Lambda / M_{r-1} R\psi\Lambda$.

Then we have the following proposition as in [23, Proposition 2.7, Corollary 2.8]:

Proposition B.4. *There is an isomorphism*

$$\bigoplus_{p-q=r} a_{(p+q)*}\Lambda(-p)[-(p+q)] \to Gr^M_r R\psi\Lambda$$

compatible with the action of G_F. Furthermore if we assume X is proper, we have the following spectral sequence:

$$E_1^{p,q} = \bigoplus_{i \geq max(0,-p)} H^{q-2i}_{\acute{e}t}(Y^{(p+2i)}_{\overline{F}}, \mathbb{F}_l(-i)) \Longrightarrow H^{p+q}_{\acute{e}t}(X_{\overline{K}}, \mathbb{F}_l),$$

compatible with the Galois action.

The boundary map $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$ of the weight spectral sequence is defined as follows:

For subsets $J \subseteq I \subseteq \{1, \dots, m\}$ such that Card I = Card $J + 1$, let $i_{IJ} : Y_I \to Y_J$ denote the closed immersion. If $I = \{i_0, \dots, i_p\}$ with $0 \le i_0 < \cdots < i_p \le m$ and $J = i_0, \dots, i_{j-1}, i_{j+1}, \dots, i_p$, we put $\epsilon(J, I) = (-1)^j$. We define $\delta_p^* : \mathrm{H}^q(Y_{\overline{\mathbb{F}}_p}^{(p)}, \mathbb{F}_l) \to \mathrm{H}^q(Y_{\overline{\mathbb{F}}_p}^{(p+1)}, \mathbb{F}_l)$ to be the alternating sum $\sum_{I \subseteq J, CardI = CardJ - 1 = p+1} \epsilon(I, J) i_{IJ}^*$ of the pull-back maps. Similarly, let $\delta_{p^*} : \mathrm{H}^q(Y_{\overline{\mathbb{F}}_p}^{(p)}, \mathbb{F}_l) \to \mathrm{H}^{q+2}(Y_{\overline{\mathbb{F}}_p}^{(p-1)}, \mathbb{F}_l(1))$ be the alternating sum $\sum_{J \subseteq I, CardI = CardJ + 1 = p+1} \epsilon(J, I) i_{JI*}$ of the Gysin maps. Then $d_1^{p,q} = \sum_{i \ge max(0,-p)} (\delta_{p+2i}^* + \delta_{p+2i^*})$.

B.5. **Pull-back and push-forward.** In this subsection, we recall the functoriality of the weight spectral sequence.

Let X and X' be strictly semi-stable schemes over S purely of relative dimension n and n'. Let $f : X \to X'$ be a morphism over S. Let $Y_1, \cdots, Y_m$ be the irreducible components of $Y = X_F$ and $Y_1', \cdots, Y_m'$ be the irreducible components of $Y' = X_F'$. We define $Y'^{(p)} = \bigcup_{I \subseteq \{1, \cdots, m'\}, \text{Card } I = p+1} Y_I'$ and a_p, a_p' be the natural embeddings of $Y^{(p)}$ and $Y'^{(p)}$ into $Y = X_F$ and $Y' = X_F'$ for $p \ge 0$. Since $\sum_{i'=1}^{m'} f^* Y_{i'}' = \sum_{i=1}^{m} Y_i$ as divisors, there exists a unique $i' \in \{1, \cdots, m'\}$ such that $f(Y_i) \subseteq Y_{i'}'$ for each $i \in \{1, \cdots, m\}$. We define a function $\phi : \{1, \cdots, m\} \to \{1, \cdots, m'\}$ be requiring $f(D_i) \subseteq D_{\phi(i)}'$. Renumbering if necessary, we assume that ϕ is increasing. Let $p \ge 0$ be an integer. We put

$$\mathcal{I}_{f,p} = \{I \subseteq \{1, \cdots, m\} \mid \text{Card } I = \text{Card } \phi(I) = p+1\} \text{ and } Y_f^p = \coprod_{I \in \mathcal{I}_{f,p}} Y_I.$$

For $I \in \mathcal{I}_{f,p}$ and $I' = \phi(I)$, let $f_{I'I} : Y_I \to Y_{I'}'$ be the restriction of f and put $f^{(p)} = \coprod f_{\phi(I)I} : Y_f^{(p)} \to Y'^{(p)}$.

First we study the pull-back. For $p \ge 0$, we define maps $f^{(p)*} : f^* a_{p*}' \Lambda \to a_{p*} \Lambda$ and $f^{(p)*} : H_{\acute{e}t}^q(Y_{\overline{F}}'^{(p)}, \Lambda) \to H_{\acute{e}t}^q(Y_{\overline{F}}^{(p)}, \Lambda)$ to be the sum $\sum_{I \in \mathcal{I}_{f,p}} f_{\phi(I)I}^*$. Then we have the following proposition as in [23, Corollary 2.12]:

Proposition B.5.1. Assume that X and X' are proper over S. Then we have a map of weight spectral sequences:

$$\begin{array}{ccc}
E_1'^{p,q} = \bigoplus_{i \ge \max(0,-p)} H_{\acute{e}t}^{q-2i}(Y_{\overline{F}}'^{(p+2i)}, \Lambda(-i)) & \Longrightarrow & H_{\acute{e}t}^{p+q}(X_{\overline{K}}', \Lambda) \\
\Big\downarrow {\scriptstyle \oplus f^{(p+2i)*}} & & \Big\downarrow {\scriptstyle f_{\overline{K}}^*} \\
E_1^{p,q} = \bigoplus_{i \ge \max(0,-p)} H_{\acute{e}t}^{q-2i}(Y_{\overline{F}}^{(p+2i)}, \Lambda(-i)) & \Longrightarrow & H_{\acute{e}t}^{p+q}(X_{\overline{K}}, \Lambda).
\end{array}$$

Next, we consider the push-forward. Let $f_* : \Lambda \to Rf^! \Lambda(-d)[-2d]$ be the adjoint of the trace map with $d = n - n'$. If X and X' are proper, it induces the push-forward map $H_{et}^{q+2d}(X_{\overline{K}}, \Lambda(d)) \to H_{et}^q(X_{\overline{K}}', \Lambda)$ as the composition:

$$H_{et}^{q+2d}(X_{\overline{K}}, \Lambda(d)) = H_{et}^q(X_{\overline{K}}', Rf_! \Lambda(d)[2d]) \to H_{\acute{e}t}^q(X_{\overline{K}}', Rf_! Rf^! \Lambda(d)) \to H_{et}^q(X_{\overline{K}}', \Lambda).$$

For $p \ge 0$, we define $\Lambda \to Rf^{(p)!} \Lambda(-d)[-2d]$ of the trace map is an isomorphism. Let $I \in \mathcal{I}_{f,p}$ be a subset $I \subseteq \{1, \cdots, m\}$ such that Card $I = p+1$ and that the restriction $I \to I' = \phi(I)$ is a bijection. Let $f_{I'I*} : \Lambda \to Rf^!_{I'I}\Lambda(-d)[-2d]$ of be adjoint of the trace map. We put

$$Rf^{(p)!} = \bigoplus_{I \in \mathcal{I}_{f,p}} Rf^!_{\phi(I)I}.$$

Similar as above, if X and X' are proper, it induces the push-forward map $f_*^{(p)} : H^{q+2d}_{ét}(Y^p_{\overline{F}}, \Lambda(d)) \to H^q_{ét}(Y_{I',\overline{F}}, \Lambda)$. Then we have the following proposition as in [23, Corollary 2.14]:

Proposition B.5.2. Assume further that X and X' are proper. Then we have a map of spectral sequences:

$$\begin{tikzcd} E_1^{p,q+2d} = \displaystyle\bigoplus_{i \geq \max(0,-p)} H^{q+2d-2i}_{ét}(Y^{(p+2i)}_{\overline{F}}, \Lambda(-i+d)) \arrow[r, Rightarrow] \arrow[d, "\oplus f_*^{(p+2i)}"'] & H^{p+q+2d}_{ét}(X_{\overline{K}}, \Lambda(d)) \arrow[d, "f_*"] \\ E'^{p,q}_1 = \displaystyle\bigoplus_{i \geq \max(0,-p)} H^{q-2i}_{ét}(Y^{(p+2i)}_{\overline{F}}, \Lambda(-i)) \arrow[r, Rightarrow] & H^{p+q}_{ét}(X'_{\overline{K}}, \Lambda). \end{tikzcd}$$

Appendix C. Étale cohomology

In the appendix we recall some results for étale cohomology.

C.1. **Topological property of étale cohomology.** Étale cohomology is similar to singular cohomology in many aspects. Here we list some we used.

Proposition C.1.1 (Mayer-Vietoris Sequence)**.** Let X be a scheme and $X = U_0 \cup U_1$ be the union of two Zariski open subsets. For any sheaf $\mathcal{F}$ on $X_{ét}$, there is an infinite exact sequence:

$$\cdots \to H^s_{ét}(X, \mathcal{F}) \to H^s_{ét}(U_0, \mathcal{F}) \oplus H^s_{ét}(U_1, \mathcal{F}) \to H^s_{ét}(U_0 \cap U_1, \mathcal{F}) \to H^{s+1}_{ét}(X, \mathcal{F}) \to \cdots.$$

for any integer $s \geq 0$.

Proof. We refer to [17] here. □

Proposition C.1.2 (Blow up exact sequence)**.** Let X be a scheme and let $Z \subseteq X$ be a closed subscheme cut out by a quasi-coherent ideal of finite type. Consider the corresponding blow up square:

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "\pi"'] & X' \arrow[d, "b"] \\ Z \arrow[r, "i"'] & X \end{tikzcd}$$

Let Λ be a sheaf with torsion cohomology, we have the following long exact sequence:

$$\cdots \to H^s_{ét}(X, \Lambda) \to H^s_{ét}(X', \Lambda) \oplus H^s_{ét}(Z, \Lambda) \to H^s_{ét}(E, \Lambda) \to H^{s+1}_{ét}(X, \Lambda) \to \cdots$$

Proof. We refer to [26, 0EW4] for the proof. □

Proposition C.1.3 (Excision sequence)**.** Let X be a scheme. Let $X = X_1 \bigsqcup X_2$ be a partition of X into two subschemes with X_1 open and X_2 closed. We have a long exact sequence:

$$\cdots \to H^i_c(X_1) \to H^i_c(X) \to H^i_c(X_2) \to H^{i+1}_c(X_1) \to \cdots.$$

Here we use $H^i_c(X)$ to denote the cohomology with compact support.

Proof. We refer to [7, XVII, 5.1.16.3] for the proof. □

C.2. **Grassmannian and Chern classes.** Following [17, Chapter VI, Section 10], let X be a nonsingular projective variety. Let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules of rank $m+1$ on a nonsingular varity X(for the Zariski topology). Then there is a projective space bundle $\mathbb{P}(\mathcal{E})$ associated with $\mathcal{E}$. This is a nonsingular variety equipped with a regular map $\pi : \mathbb{P}(\mathcal{E}) \to X$ and a canonical invertible sheaf $\mathcal{O}(1)$. The fibre $\pi^{-1}(x) = \mathbb{P}(\mathcal{E}_x)$ where $\mathcal{E}_x$ is the fiber of $\mathcal{E}$ at x. For the line bundle $\mathcal{O}(-1)$, which is called the tautological subbundle, there is a more explicit description: for the point of $\mathbb{P}(\mathcal{E})$ correspond to pairs (x, ξ) with $x \in X$ and ξ a one-dimensional subspace $\xi \subseteq \mathcal{E}_x$, the fiber of the tautological subbundle $\mathcal{O}(-1)$ at the point is $\xi \subseteq \mathcal{E}_x$. This gives $\mathcal{O}(-1) \subseteq \pi^*\mathcal{E}$ with $\pi : \mathbb{P}(\mathcal{E}) \to X$ is the canonical projection. Dually, we have a surjection $\pi^*\mathcal{E}^* \to \mathcal{O}(1)$.

Moreover, we have the following proposition from [17, Chapter VI, Proposition 10.1]:

Proposition C.2.1. Let $\mathcal{E}$ be a locally free sheaf rank $m+1$ on X_{Zar} and let $\pi : \mathbb{P}(\mathcal{E}) \to X$ be the associated projective bundle. Let ξ be the class of $\mathcal{O}(1)$ in $H^2_{\acute{e}t}(\mathbb{P}(\mathcal{E}), \Lambda(1))$. Then π^* makes $H^*_{\acute{e}t}(\mathbb{P}(\mathcal{E}))$ into a free $H^*(X)$-module with basis $1, \xi, \ldots, \xi^m$.

We omit the general definition and representability of Grassmannians here, but we only recall the concepts of universal subbundles and quotient bundles.

Definition C.2.2. For any nonsingular variety X, let $\mathcal{V}$ be a locally free sheaf of rank n and $\mathbf{Gr}(\mathcal{V}, k)$ be the Grassmannian of k-subbundle of $\mathcal{V}$. Let $\mathcal{O}^n$ be the trivial vector bundle of rank n on $\mathbf{Gr}(\mathcal{V}, k)$. We write $\mathcal{S}$ for the rank-k subbundle of $\mathcal{O}^n$ whose fiber at a point $[\Lambda] \in \mathbf{Gr}(\mathcal{V}, k)$ is the subbundle Λ itself. The subbundle $\mathcal{S}$ is called the universal subbundle on $\mathbf{Gr}(\mathcal{V}, k)$, and $\mathcal{Q} = \mathcal{O}^n/\mathcal{S}$ is called the quotient bundle. In particular, in the case $k = 1$, the universal subbundle $\mathcal{S}$ is $\mathcal{O}(-1)$ over $\mathbf{Gr}(\mathcal{V}, 1) = \mathbb{P}^{n-1}$. In the case $k = n-1$, the universal quotient bundle $\mathcal{Q}$ is the line bundle $\mathcal{O}(-1)$.

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