理论作业三　量子纠错

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1. 写出 9 量子比特 Shor 编码的稳定子，并证明其稳定 $\{|0_L\rangle, |1_L\rangle\}$ 张成的向量空间，其中

$$|0_L\rangle = \frac{(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)}{2\sqrt{2}}$$
$$|1_L\rangle = \frac{(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)}{2\sqrt{2}}$$

解：对 Shor 编码的某个稳定子 S_i，要证它稳定 $\{|0_L\rangle, |1_L\rangle\}$ 张成的向量空间，即 $\forall |\psi\rangle = \alpha|0_L\rangle + \beta|1_L\rangle, S_i|\psi\rangle = |\psi\rangle$

只需证 $S_i|0_L\rangle = |0_L\rangle, S_i|1_L\rangle = |1_L\rangle$，就有 $S_i|\psi\rangle = S_i(\alpha|0_L\rangle + \beta|1_L\rangle) = \alpha S_i|0_L\rangle + \beta S_i|1_L\rangle = \alpha|0_L\rangle + \beta|1_L\rangle = |\psi\rangle$

(1)$S_1 = Z_1Z_2$

$Z_i = I^{\otimes i-1}ZI^{\otimes n-i}, Z_i|0\rangle = |0\rangle, Z_i|1\rangle = -|1\rangle$

$$\begin{aligned}
S_1|0_L\rangle &= \frac{1}{2\sqrt{2}}[Z_1Z_2(|000\rangle + |111\rangle)](|000\rangle + |111\rangle)(|000\rangle + |111\rangle) \\
&= \frac{1}{2\sqrt{2}}(|000\rangle + (-1)(-1)|111\rangle))(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) \\
&= \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle))(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) = |0_L\rangle \\
S_1|1_L\rangle &= \frac{1}{2\sqrt{2}}[Z_1Z_2(|000\rangle - |111\rangle)](|000\rangle - |111\rangle)(|000\rangle - |111\rangle) \\
&= \frac{1}{2\sqrt{2}}(|000\rangle - (-1)(-1)|111\rangle))(|000\rangle - |111\rangle)(|000\rangle - |111\rangle) \\
&= \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle))(|000\rangle - |111\rangle)(|000\rangle - |111\rangle) = |1_L\rangle
\end{aligned}$$

(2)$S_2 = Z_1Z_3$

(3)$S_3 = Z_4Z_5$

$(4)S_4 = Z_4Z_6$
$(5)S_5 = Z_7Z_8$
$(6)S_6 = Z_7Z_9$
(2)-(6) 的证明类似 (1)
$(7)S_7 = X_1X_2X_3X_4X_5X_6$
$X_i = I^{\otimes i-1}XI^{\otimes n-i}, X_i\ket{0} = \ket{1}, X_i\ket{1} = \ket{0}$

$$\begin{aligned}
S_7\ket{0_L} &= \frac{1}{2\sqrt{2}}[X_1X_2X_3(\ket{000}+\ket{111})][X_4X_5X_6(\ket{000}+\ket{111})](\ket{000}+\ket{111}) \\
&= \frac{1}{2\sqrt{2}}(\ket{111}+\ket{000}))(\ket{111}+\ket{000})(\ket{000}+\ket{111}) = \ket{0_L} \\
S_7\ket{1_L} &= \frac{1}{2\sqrt{2}}[X_1X_2X_3(\ket{000}-\ket{111})][X_4X_5X_6(\ket{000}-\ket{111})](\ket{000}-\ket{111}) \\
&= \frac{1}{2\sqrt{2}}(\ket{111}-\ket{000}))(\ket{111}-\ket{000})(\ket{000}-\ket{111}) = \\
&= \frac{1}{2\sqrt{2}}(\ket{000}-\ket{111}))(\ket{000}-\ket{111})(\ket{000}-\ket{111}) = \ket{1_L}
\end{aligned}$$

$(8)S_8 = X_1X_2X_3X_7X_8X_9$, 证明类似 (7)

2. 证明操作 $\bar{Z} = X_1X_2X_3X_4X_5X_6X_7X_8X_9$ 和 $\bar{X} = Z_1Z_2Z_3Z_4Z_5Z_6Z_7Z_8Z_9$ 可以充当 Shor 编码中逻辑量子比特上的逻辑 Z 操作和逻辑 X 操作。(提示：推导 $\bar{Z}$ 和 $\bar{X}$ 作用在 $\ket{0_L}$ 和 $\ket{1_L}$ 上的形式)

证明：即证 $\bar{Z}\ket{0_L} = \ket{0_L}, \bar{Z}\ket{1_L} = -\ket{1_L}$,$\bar{X}\ket{0_L} = \ket{1_L}, \bar{X}\ket{1_L} = -\ket{0_L}$

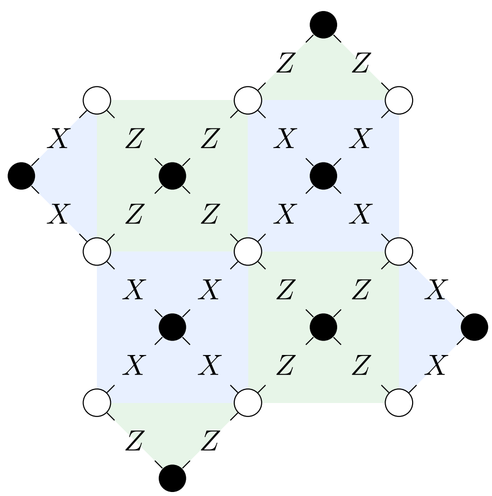
$$\begin{aligned}
\bar{Z}\ket{0_L} &= X_1X_2X_3X_4X_5X_6X_7X_8X_9\ket{0_L} \\
&= \frac{1}{2\sqrt{2}}[X_1X_2X_3(\ket{000}+\ket{111})][X_4X_5X_6(\ket{000}+\ket{111})][X_7X_8X_9(\ket{000}+\ket{111})] \\
&= \frac{1}{2\sqrt{2}}(\ket{111}+\ket{000})(\ket{111}+\ket{000})(\ket{111}+\ket{000}) = \ket{0_L} \\
\bar{Z}\ket{1_L} &= X_1X_2X_3X_4X_5X_6X_7X_8X_9\ket{1_L} \\
&= \frac{1}{2\sqrt{2}}[X_1X_2X_3(\ket{000}-\ket{111})][X_4X_5X_6(\ket{000}-\ket{111})][X_7X_8X_9(\ket{000}-\ket{111})] \\
&= \frac{1}{2\sqrt{2}}(\ket{111}-\ket{000})(\ket{111}-\ket{000})(\ket{111}-\ket{000}) \\
&= \frac{-1}{2\sqrt{2}}(\ket{000}-\ket{111})(\ket{000}-\ket{111})(\ket{000}-\ket{111}) = -\ket{1_L}
\end{aligned}$$

$$\therefore \bar{Z}\ket{0_L} = \ket{0_L}, \bar{Z}\ket{1_L} = -\ket{1_L}$$

$$
\begin{aligned}
\bar{X}\left|0_L\right\rangle &= Z_1Z_2Z_3Z_4Z_5Z_6Z_7Z_8Z_9\left|0_L\right\rangle \\
&= \frac{1}{2\sqrt{2}}[Z_1Z_2Z_3(|000\rangle+|111\rangle)][Z_4Z_5Z_6(|000\rangle+|111\rangle)][Z_7Z_8Z_9(|000\rangle+|111\rangle)] \\
&= \frac{1}{2\sqrt{2}}(|000\rangle+(-1)^3|111\rangle)(|000\rangle+(-1)^3|111\rangle)(|000\rangle+(-1)^3|111\rangle) \\
&= \frac{1}{2\sqrt{2}}(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)=\left|1_L\right\rangle \\
\bar{X}\left|1_L\right\rangle &= Z_1Z_2Z_3Z_4Z_5Z_6Z_7Z_8Z_9\left|1_L\right\rangle \\
&= \frac{1}{2\sqrt{2}}[Z_1Z_2Z_3(|000\rangle-|111\rangle)][Z_4Z_5Z_6(|000\rangle-|111\rangle)][Z_7Z_8Z_9(|000\rangle-|111\rangle)] \\
&= \frac{1}{2\sqrt{2}}(|000\rangle-(-1)^3|111\rangle)(|000\rangle-(-1)^3|111\rangle)(|000\rangle-(-1)^3|111\rangle) \\
&= \frac{1}{2\sqrt{2}}(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)=\left|0_L\right\rangle
\end{aligned}
$$

$$
\therefore \bar{X}\left|0_L\right\rangle=\left|1_L\right\rangle, \bar{X}\left|1_L\right\rangle=\left|0_L\right\rangle
$$

3. 下图展示了一个二维网格上码距为 3 的表面码。其中，空心圆点表示数据量子比特，实心圆点表示辅助量子比特，每个辅助量子比特对其相邻的数据量子比特执行征状测量。写出该纠错码的稳定子，并推导逻辑态 $\left|0_L\right\rangle$ 和 $\left|1_L\right\rangle$ 的形式。



解：设 9 个数据量子比特从左上到右下依次为 1-9.
根据图中的连接情况可得 8 个稳定子依次为

$$
\begin{aligned}
S_{Z_1} &= Z_1 Z_2 Z_4 Z_5 \\
S_{Z_2} &= Z_2 Z_3 \\
S_{Z_3} &= Z_5 Z_6 Z_8 Z_9 \\
S_{Z_4} &= Z_7 Z_8 \\
S_{X_1} &= X_1 X_4 \\
S_{X_2} &= X_2 X_3 X_5 X_6 \\
S_{X_3} &= X_4 X_5 X_7 X_8 \\
S_{X_4} &= X_6 X_9
\end{aligned}
$$

再由稳定子推出逻辑态 $|0_L\rangle$ 和 $|1_L\rangle$ 的形式:
对 $|000000000\rangle$ 处理 X 型稳定子（已经是 Z 型稳定子的 +1 本征态）

$$
\begin{aligned}
&(I + S_{X_1})(I + S_{X_2})(I + S_{X_3})(I + S_{X_4}) |000000000\rangle \\
&= (I + S_{X_1})(I + S_{X_2})(I + S_{X_3})(|000000000\rangle + |000001001\rangle) \\
&= (I + S_{X_1})(I + S_{X_2})(|000000000\rangle + |000001001\rangle + |000110110\rangle + |000111111\rangle) \\
&= (I + S_{X_1})(|000000000\rangle + |000001001\rangle + |000110110\rangle + |000111111\rangle \\
&+ |011011000\rangle + |011010001\rangle + |011101110\rangle + |011100111\rangle) \\
&= |000000000\rangle + |000001001\rangle + |000110110\rangle + |000111111\rangle \\
&+ |011011000\rangle + |011010001\rangle + |011101110\rangle + |011100111\rangle \\
&+ |100100000\rangle + |100101001\rangle + |100010110\rangle + |100011111\rangle \\
&+ |111111000\rangle + |111110001\rangle + |111001110\rangle + |111000111\rangle
\end{aligned}
$$

归一化有:

$$
\begin{aligned}
|0_L\rangle =& \frac{1}{4}(|000000000\rangle + |000001001\rangle + |000110110\rangle + |000111111\rangle \\
&+ |011011000\rangle + |011010001\rangle + |011101110\rangle + |011100111\rangle \\
&+ |100100000\rangle + |100101001\rangle + |100010110\rangle + |100011111\rangle \\
&+ |111111000\rangle + |111110001\rangle + |111001110\rangle + |111000111\rangle)
\end{aligned}
$$

对 $|0_L\rangle$ 应用 $X^{\otimes 9}$, 显然 $X^{\otimes 9}$ 与稳定子都对易

$$
\begin{aligned}
|1_L\rangle &= X^{\otimes 9} |0_L\rangle \\
=& \frac{1}{4}(|111111111\rangle + |111110110\rangle + |111001001\rangle + |111000000\rangle \\
&+ |100100111\rangle + |100101110\rangle + |100010001\rangle + |100011000\rangle \\
&+ |011011111\rangle + |011010110\rangle + |011101001\rangle + |011100000\rangle \\
&+ |000000111\rangle + |000001110\rangle + |000110001\rangle + |000111000\rangle)
\end{aligned}
$$