

Circuits and Transforms

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Abstract—This manual provides a simple introduction to Transforms

1 DEFINITIONS

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \quad (1.1)$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st} dt \quad (1.2)$$

2 LAPLACE TRANSFORM

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1 \mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2 \mu C$.

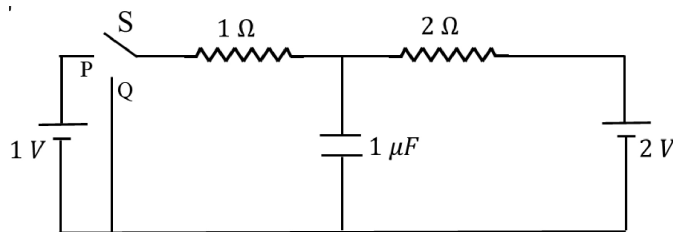


Fig. 2.1

2. Find q_1 . **Solution:** The equivalent circuit at steady-state when the switch is at P is shown alongside. Assuming the circuit to be grounded

at G and the relative potential at point X to be V , we use KCL at X and get

$$\frac{V-1}{1} + \frac{V-2}{2} = 0 \quad (2.1)$$

$$\Rightarrow V = \frac{4}{3} V \quad (2.2)$$

Hence,

$$q_1 = CV = \frac{4}{3} \mu C \quad (2.3)$$

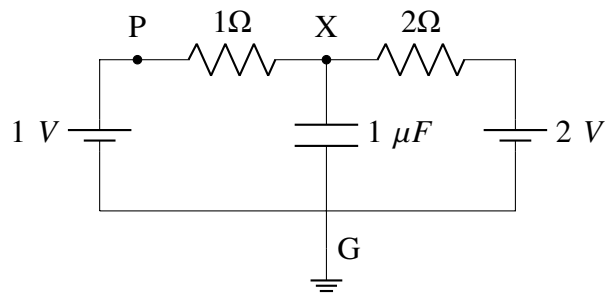


Fig. 2.2

3. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC. **Solution:** We have,

$$u(t) \xleftrightarrow{\mathcal{L}} \int_0^{\infty} u(t)e^{-st} dt \quad (2.4)$$

$$= \int_0^0 \frac{1}{2} e^{-st} dt + \int_0^{\infty} e^{-st} dt \quad (2.5)$$

$$= \frac{1}{s}, \quad \Re(s) > 0 \quad (2.6)$$

4. Show that

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad a > 0 \quad (2.7)$$

and find the ROC. **Solution:** Note that by substituting $s := s+a$ in (2.6), and considering $a \in \mathbb{R}$,

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \int_0^{\infty} u(t)e^{-(s+a)t} dt \quad (2.8)$$

$$= \frac{1}{s+a}, \quad \Re(s) > -a \quad (2.9)$$

5. Now consider the following resistive circuit transformed from Fig. 2.1 where

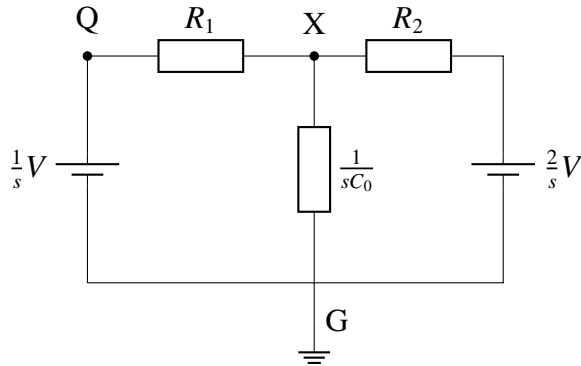


Fig. 2.3

$$u(t) \xrightarrow{\mathcal{L}} V_1(s) \quad (2.10)$$

$$2u(t) \xrightarrow{\mathcal{L}} V_2(s) \quad (2.11)$$

Find the voltage across the capacitor $V_{C_0}(s)$.

Solution: We see that

$$V_1(s) = \frac{1}{s} V_2(s) = \frac{2}{s} \quad (2.12)$$

Now, labelling points G and X as in Fig. 2.2, we use KCL at X.

$$\frac{V - \frac{1}{s}}{R_1} + \frac{V - \frac{2}{s}}{R_2} + sC_0 V = 0 \quad (2.13)$$

$$V \left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0 \right) = \frac{1}{s} \left(\frac{1}{R_1} + \frac{2}{R_2} \right) \quad (2.14)$$

$$V(s) = \frac{\frac{1}{R_1} + \frac{2}{R_2}}{s \left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0 \right)} \quad (2.15)$$

$$= \frac{\frac{1}{R_1} + \frac{2}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2}} \left(\frac{1}{s} - \frac{1}{\frac{1}{C_0} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) + s} \right) \quad (2.16)$$

6. Find $v_{C_0}(t)$. Plot using python. **Solution:** Taking the inverse Laplace transform in (2.16),

$$V(s) \xrightarrow{\mathcal{L}} \frac{2R_1 + R_2}{R_1 + R_2} u(t) \left(1 - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2} \right) \frac{t}{C_0}} \right) \quad (2.17)$$

$$= \frac{4}{3} \left(1 - e^{-(1.5 \times 10^6)t} \right) u(t) \quad (2.18)$$

The python code `codes/2_6.py` plots the graph below.

7. Verify your result using ngspice. **Solution:** The ngspice script `codes/2_7.cir` simulates

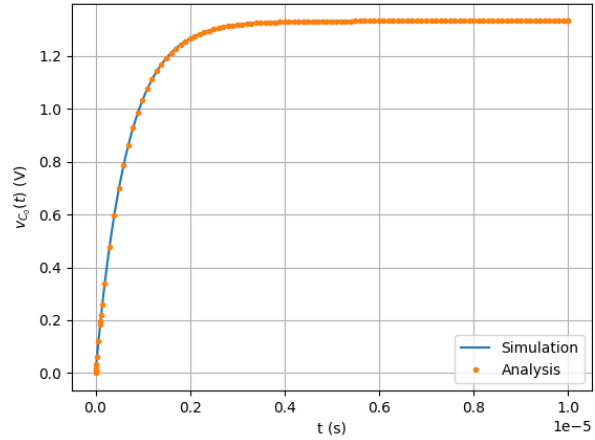


Fig. 2.4: $v_{C_0}(t)$ before the switch is flipped

the given circuit and the generated output is depicted in Fig. (2.4)

3 INITIAL CONDITIONS

1. Find q_2 in Fig. 2.1. **Solution:** The equivalent circuit at steady state when the switch is at Q is shown below. Since capacitor behaves as an

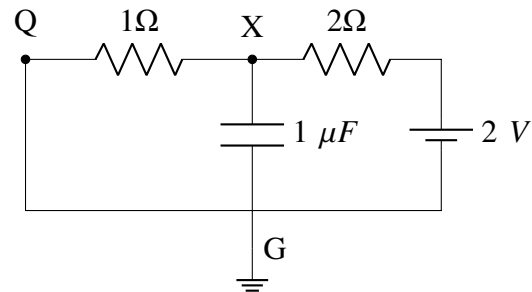


Fig. 3.1

open circuit, we use KCL at X.

$$\frac{V - 0}{1} + \frac{V - 2}{2} = 0 \implies V = \frac{2}{3} \text{ V} \quad (3.1)$$

and hence, $q_2 = \frac{2}{3} \mu\text{C}$.

2. Draw the equivalent s -domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements.
3. $V_{C_0}(s) = ?$ **Solution:** Using KCL at node X in

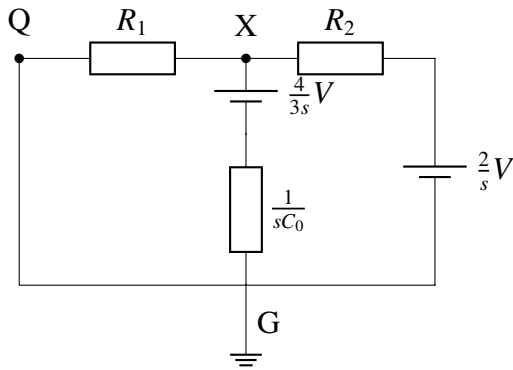


Fig. 3.2

Fig. 3.2

$$\frac{V-0}{R_1} + \frac{V-\frac{4}{s}}{R_2} + sC_0 \left(V - \frac{4}{s} \right) = 0 \quad (3.2)$$

$$\Rightarrow V_{C_0}(s) = \frac{\frac{2}{sR_2} + \frac{4C_0}{3}}{\frac{1}{R_1} + \frac{2}{R_2} + sC_0} \quad (3.3)$$

4. $v_{C_0}(t) = ?$ Plot using python. **Solution:** From (3.3),

$$V_{C_0}(s) = \frac{4}{3} \left(\frac{1}{\frac{1}{C_0} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) + s} \right) + \frac{2}{R_2 \left(\frac{1}{R_1} + \frac{1}{R_2} \right)} \left(\frac{1}{s} - \frac{1}{\frac{1}{C_0} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) + s} \right) \quad (3.4)$$

Taking an inverse Laplace Transform,

$$v_{C_0}(t) = \frac{4}{3} e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \frac{t}{C_0}} u(t) + \frac{2}{R_2 \left(\frac{1}{R_1} + \frac{1}{R_2} \right)} \left(1 - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \frac{t}{C_0}} \right) u(t) \quad (3.5)$$

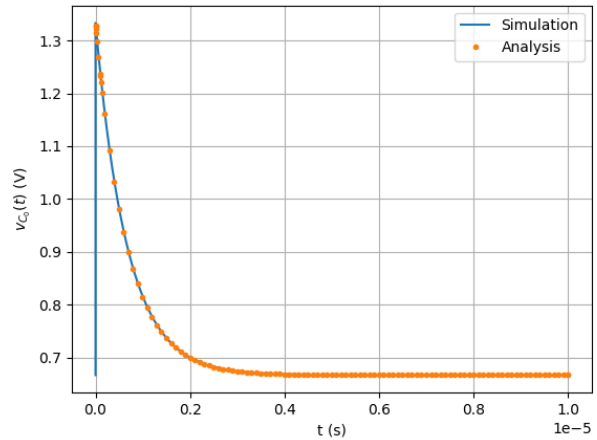
Substituting values gives

$$v_{C_0}(t) = \frac{2}{3} \left(1 + e^{-(1.5 \times 10^6)t} \right) u(t) \quad (3.6)$$

The Python code `codes/3_4.py` plots the graph below.

5. Verify your result using ngspice. **Solution:** The ngspice script `codes/3_5.cir` simulates the given circuit and the generated output is depicted in Fig. (3.3).
6. Find $v_{C_0}(0-)$, $v_{C_0}(0+)$ and $v_{C_0}(\infty)$. **Solution:** From the initial conditions,

$$v_{C_0}(0-) = \frac{q_1}{C_0} = \frac{4}{3} \text{ V} \quad (3.7)$$

Fig. 3.3: $v_{C_0}(t)$ after the switch is flipped

Using (3.6),

$$v_{C_0}(0+) = \lim_{t \rightarrow 0+} v_{C_0}(t) = \frac{4}{3} \text{ V} \quad (3.8)$$

$$v_{C_0}(\infty) = \lim_{t \rightarrow \infty} v_{C_0}(t) = \frac{2}{3} \text{ V} \quad (3.9)$$

7. Obtain the Fig. in problem 3.2 using the equivalent differential equations. **Solution:** The equivalent circuit in the t -domain is shown below.

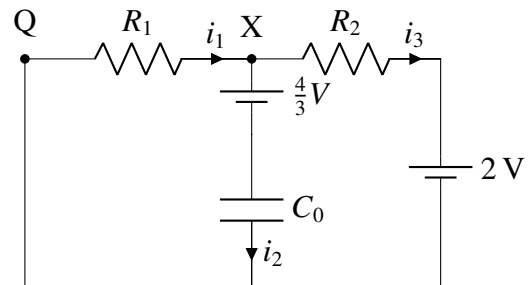


Fig. 3.4

From KCL and KVL,

$$i_1 = i_2 + i_3 \quad (3.10)$$

$$i_1 R_1 + \frac{4}{3} + \frac{1}{C_0} \int_0^t i_2 dt = 0 \quad (3.11)$$

$$\frac{4}{3} + \frac{1}{C_0} \int_0^t i_2 dt - i_3 R_2 - 2 = 0 \quad (3.12)$$

Taking Laplace Transforms on both sides and

using the properties of Laplace Transforms,

$$I_1 = I_2 + I_3 \quad (3.13)$$

$$I_1 R_1 + \frac{4}{3} + \frac{1}{sC_0} I_2 = 0 \quad (3.14)$$

$$\frac{4}{3} + \frac{1}{sC_0} I_2 - I_3 R_2 - 2 = 0 \quad (3.15)$$

where $i(t) \xleftrightarrow{\mathcal{L}} I(s)$. Note that the capacitor is equivalent to a resistive element of resistance $R_C = \frac{1}{sC_0}$ in the s -domain. Equations (3.13) - (3.15) precisely describe Fig. 3.2.

4 BILINEAR TRANSFORM

1. In Fig. 2.1, consider the case when S is switched to Q right in the beginning. Formulate the differential equation. **Solution:** The equivalent circuit in the t -domain is shown below.

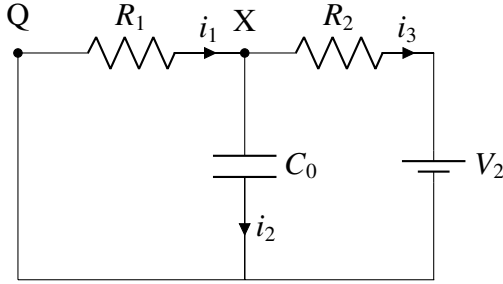


Fig. 4.1

Applying KCL and KVL,

$$i_1 = i_2 + i_3 \quad (4.1)$$

$$i_1 R_1 + \frac{1}{C_0} \int_0^t i_2 dt = 0 \quad (4.2)$$

$$i_3 R_2 + 2 - \frac{1}{C_0} \int_0^t i_2 dt = 0 \quad (4.3)$$

Differentiating the above equations,

$$\frac{di_1}{dt} = \frac{di_2}{dt} + \frac{di_3}{dt} \quad (4.4)$$

$$R_1 \frac{di_1}{dt} + \frac{i_2}{C_0} = 0 \quad (4.5)$$

$$R_2 \frac{di_3}{dt} - \frac{i_2}{C_0} = 0 \quad (4.6)$$

Using (4.4) and (4.6) in (4.5),

$$R_1 \left(\frac{di_2}{dt} + \frac{di_3}{dt} \right) + \frac{i_2}{C_0} = 0 \quad (4.7)$$

$$R_1 \frac{di_2}{dt} + \left(1 + \frac{R_1}{R_2} \right) \frac{i_2}{C_0} = 0 \quad (4.8)$$

$$\frac{di_2}{dt} + \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \frac{i_2}{C_0} = 0 \quad (4.9)$$

$$\frac{di_2}{dt} + \frac{i_2}{\tau} = 0 \quad (4.10)$$

where $\tau = \frac{C_0 R_1 R_2}{R_1 + R_2}$ is the RC time constant of the circuit. Note that $i_2(0) = \frac{V_2}{R_2}$ A and $i_2 = C_0 \frac{dV}{dt}$, where V is the voltage of the capacitor. Hence, integrating (4.10),

$$C_0 \frac{dV}{dt} - \frac{V_2}{R_2} + \frac{C_0 V}{\tau} = 0 \quad (4.11)$$

$$\Rightarrow \frac{dV}{dt} + \frac{V}{\tau} = \frac{V_2}{C_0 R_2} \quad (4.12)$$

2. Find $H(s)$ considering the output voltage at the capacitor. **Solution:** Transforming Fig. 4.1 to the s -domain, Applying nodal analysis at X,

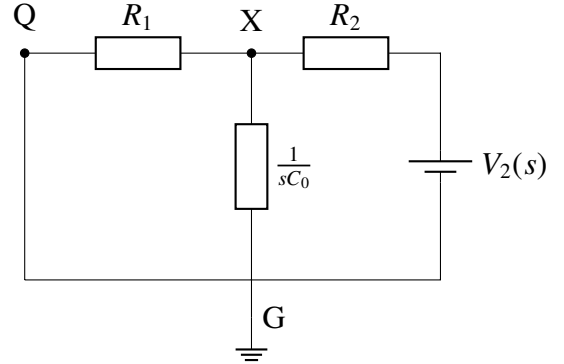


Fig. 4.2

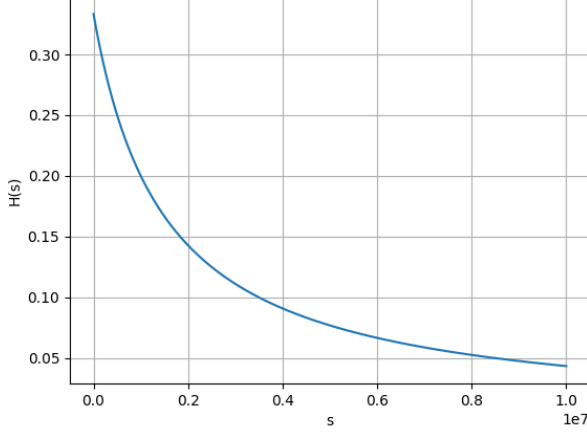
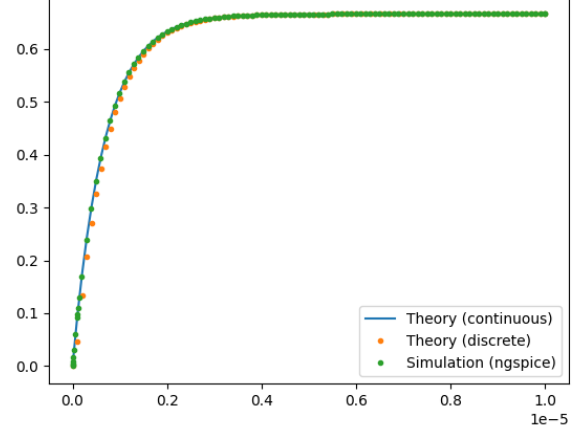
and noting that $H(s) = \frac{V(s)}{V_2(s)}$,

$$\frac{V}{R_1} + \frac{V}{\frac{1}{sC_0}} + \frac{V - V_2}{R_2} = 0 \quad (4.13)$$

$$H(s) \left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0 \right) = \frac{1}{R_2} \quad (4.14)$$

$$H(s) = \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \quad (4.15)$$

3. Plot $H(s)$. What kind of filter is it? **Solution:** The Python code `codes/4_3.py` plots $H(s)$. Clearly, $H(s)$ is a low-pass filter.

Fig. 4.3: Plot of $H(s)$.Fig. 4.4: Representation of output across C_0 .

4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n) = y(t)|_{t=n} \quad (4.16)$$

Solution: Integrating (4.12) between limits n to $n + 1$ and applying the trapezoidal formula,

$$v(n+1) - v(n) + \frac{v(n) + v(n+1)}{2\tau} = \frac{V_2(u(n) + u(n+1))}{C_0 R_2} \quad (4.17)$$

$$v(n)(2\tau + 1) + v(n-1)(2\tau - 1) = \frac{V_2\tau(u(n) + u(n-1))}{C_0 R_2} \quad (4.18)$$

for $n > 0$, where $v(0) = 0$.

5. Find $H(z)$. **Solution:** Note that for the input voltage, $v_i(n) = 2u(n)$ and so, $V_i(z) = \frac{2}{1-z^{-1}}$. Applying the Z-transform on both sides of (4.18),

$$\begin{aligned} V(z) \left[(2\tau + 1) - z^{-1}(2\tau - 1) \right] \\ = \frac{\tau(1 + z^{-1}) V_i(z)}{C_0 R_2} \end{aligned} \quad (4.19)$$

Hence,

$$H(z) = \frac{\tau(1 + z^{-1})}{C_0 R_2 ((2\tau + 1) - (2\tau - 1)z^{-1})} \quad (4.20)$$

since $\left| \frac{2\tau-1}{2\tau+1} \right| < 1$, the ROC is $|z| > 1$.

6. How can you obtain $H(z)$ from $H(s)$? **Solution:**

We use the bilinear transformation. Setting

$$s := \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}} \quad (4.21)$$

we get

$$H(z) = \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{2C_0}{T} \frac{1-z^{-1}}{1+z^{-1}}} \quad (4.22)$$

$$= \frac{T\tau(1 + z^{-1})}{C_0 R_2 ((2\tau + T) - (2\tau - T)z^{-1})} \quad (4.23)$$

Setting $T = 1$ gives (4.20).

7. Find $v(n)$. Verify using ngspice and the differential equation. **Solution:** We have,

$$V(z) = H(z)V_i(z) \quad (4.24)$$

$$\begin{aligned} &= \frac{TV_2\tau(1 + z^{-1})}{C_0 R_2 (1 - z^{-1})((2\tau + T) - (2\tau - T)z^{-1})} \\ &= \frac{V_2\tau(z + 1)}{2C_0 R_2} \sum_{k=-\infty}^{\infty} (1 - p^k) u(k) z^{-k} \end{aligned} \quad (4.25)$$

$$= \frac{V_2\tau(z + 1)}{2C_0 R_2} \sum_{k=-\infty}^{\infty} (1 - p^k) u(k) z^{-k} \quad (4.26)$$

where $p := \frac{2\tau-T}{2\tau+T}$. Thus,

$$v(n) = \frac{V_2\tau}{C_0 R_2} \left[u(n)(1 - p^n) + u(n+1)(1 - p^{n+1}) \right] \quad (4.27)$$

where $p := \frac{2\tau-1}{2\tau+1}$. We take $T = 10^{-7}$ as the sampling interval. The python code codes/4_7.py verifies these equalities.