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Computer Vision – HW4

Question 1: Prove the simple, but extremely useful, result that the perpendicular distance from a point (u, v) to a line (a, b, c) is given by abs(au+bv+c) if a^2 + b^2 = 1. (Notation: x^2 means x squared)

The equation of a line can be written as: ax + by + c = 0

The shortest distance from a point P (u, v) to a line is the perpendicular and can be expressed as: $d = \frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$

If we suppose that $a^2 + b^2$= 1, then $\sqrt{a^2 + b^2}$ equals also 1, therefore d = |au + bv + c|.

Question 2: Show that the curve in the 2D plane:

x(t) = (1-t^2)/(1+t^2)

y(t) = 2t/(1+t^2)

is a circular arc (the length of the arc depends on the interval for which the parameter t is defined).

The equation of a circle is: $(x - h)^2 + (y - k)^2 = r^2$, where (h, k) is the center and r is the radius. Given $x(t) = (1 - t^2)/(1 + t^2)$ and $y(t) = 2t/(1 + t^2)$, $x^2 + y^2 = \frac{\left((1-t^2)^2+4t^2\right)}{(1+t^2)^2}$ and in the end, $x^2 + y^2 = \frac{(1+2t^2+t^4)}{(1+t^2)^2} = 1$.

This means that the given formulas for x(t) and y(t) satisfy $x^2 + y^2 = 1$. This describes a circle centered in (0,0) with a radius of 1.

Question 3: Write out the equation in t for the closest point on the arc (from Problem 2) to some data point (x, y); what is the degree of this equation? How many solutions in t could there be?

The distance between any 2 points can be written as: $D^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$. So, for our given formulas for x(t) and y(t) from the second question, the formula for the squared distance is: $D(t) = \left(x - \frac{(1-t^2)}{(1+t^2)}\right)^2 + \left(y - \frac{2t}{(1+t^2)}\right)^2$. To determine the minimum distance, we derive the equation and make it equal to 0: $\frac{dD}{dt} = 0$, therefore, we will have $4t\left((x+1)t^3 + (x-1)t + y(1-t^2)\right) = 0$, which will give us 2 possible solutions:

1. t = 0
2. $(x+1)t^3 + (x-1)t + y(1-t^2) = 0$

The degree of this equation is 3, as the maximum power of t is 3. The maximum number of possible solutions is 3.

Question 4: Show that rotation matrices are characterized by the following properties: (1) the inverse of a rotation matrix is equal to its transpose, and (2) its determinant is equal to 1. Solve this for the 2D case first. Can you solve it for the 3D case also? How about the general case?

2D case:

R = [cos -sin θ] $\quad$ R^T = [cos θ sin θ]

[sin θ cos θ] $\quad$ [-sinθ cos θ]

To verify that R is equal to R^T, their product must produce the identity matrix:

R x R^T = [$\cos^2\theta + \sin^2\theta$ 0] $\quad$ R x R^T = [1 0]

[0 $\cos^2\theta + \sin^2\theta$] $\quad$ [0 1]

This is true, as $\cos^2\theta + \sin^2\theta$ = 1.

The determinant can be calculated as: $\det(R) = cos\theta \cdot cos\theta - (-sin\theta \cdot sin\theta) = \cos^2\theta + \sin^2\theta = 1$

3D case:

$R_z(\theta)$= [cos θ -sin θ 0]

[sin θ cos θ 0]

[0 0 1]

$R_z(\theta)^T$= [cos θ sin θ 0]

[-sin θ cos θ 0]

[0 0 1]

$R_z(\theta) \cdot R_z(\theta)^T$ = $[\cos^2 \theta + \sin^2 \theta \;\; 0 \;\; 0]$

$[0 \;\; \cos^2 \theta + \sin^2 \theta \;\; 0]$

$[0 \;\; 0 \;\; \cos^2 \theta + \sin^2 \theta]$

Which also equals to the identity matrix I^3, as $\cos^2 \theta + \sin^2 \theta = 1.$

For the determinant: using the expansion rule of the last column: $\det(R) = 0 \cdot (M_{13}) + 0 \cdot (-M_{23}) + 1 \cdot (M_{33})$, where $M_{i,j}$ are the minors. And this becomes: $\det(R) = 1 \cdot \det([cos\theta \; - sin\theta]) = [sin\theta \; cos\theta]$. For the 2x2 matrix, the determinant becomes: $det = (cos\theta \cdot cos\theta) - (-sin\theta \cdot sin\theta) = \cos^2 \theta + \sin^2 \theta = 1.$

General case:

From the above examples, we can draw the conclusion that regardless of θ, the determinant is always 1 and the rotation matrices are orthogonal(R x R^T=I).

Question 6: Assume that we wish to remove salt-and-pepper noise from a uniform (constant) background. Show that up to half of the elements in the neighborhood could be noise values and a median filter would still give the same (correct) answer.

It means that every pixel in the background has the same intensity value and random pixels are corrupted either to the minimum value(salt) or the maximum value(pepper). The median filter replaces each pixel's value with the middle of the values of its neighbours.

If we sort pixels from a neighborhood, in the beginning we will find the salt values (0), in the end the salt values (255) and the rest, in the middle, the correct background values. If the background value can be found in the middle of a sorted array of the neighborhood's values, the filter will return the correct background value.

To reach this statement, we need at least half of the pixels to be correct background pixels, because even less than half of the total of pixels are corrupt, after sorting, the middle value will still be a background value.

e.g.: Less than half of the pixels are noise (4) so 9-4 = 5 are correct and this is how the sorted array of a 3x3 neighborhood looks like: [0,0,P,P,P,P,P,1,1]. The median can be found in the 5th position, which is a correct one.

In the other case, when we have more than half of the pixels that are corrupted, there is no guarantee that the middle pixel is a correct one.

e.g.: More than half of the pixels are noise (5) so 9-5 = 4 are correct and this is how the sorted array of a 3x3 neighborhood looks like: [P,P,P,P,1,1,1,1,1]. The median can be found in the 5th position, which is a corrupt one (1).

Therefore, the median filter selects the correct pixel after sorting if the correct values are more numerous than the noise values.

Question 7: Derive the solution w that minimizes $J(w) = (Aw - b)^T(Aw - b)$

The expansion gives: $J(w) = (Aw - b)^T(Aw - b) = A^T w^T Aw - 2w^T A^T b + b^T b$

The gradient with respect to w: $\nabla_w J(w) = 2A^T Aw - 2A^T b$

To find the critical points we set the equation equal to 0: $2A^T Aw - 2A^T b = 0$, which will result to: $A^T Aw = A^T b$

So, assuming $A^T A$ is non-singular (invertible, this requires the matrix A to have a full rank, its columns are linearly independent), we multiply both sides with $(A^T A)^{-1}$:

$$w = (A^T A)^{-1} A^T b$$