Machine Learning

Uniform Convergence

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When is an Hypothesis Class PAC Learnable?

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any finite hypothesis class is PAC learnable by ERM.

What about the more general PAC learning model we have seen last? Recall the (agnostic) PAC learnability for general loss:

Definition

A hypothesis class $\mathcal{H}$ is *agnostic PAC learnable* with respect to a set Z and a loss function $\ell : \mathcal{H} \times Z \rightarrow \mathbb{R}_+$ if there exist a function $m_{\mathcal{H}}$: $(0,1)^2 \rightarrow \mathbb{N}$ and a learning algorithm such that for every $\delta, \varepsilon \in (0,1)$, for every distribution $\mathcal{D}$ over Z, when running the learning algorithm on $m \geq m_{\mathcal{H}}(\varepsilon, \delta)$ i.i.d. examples generated by $\mathcal{D}$ the algorithm returns a hypothesis h such that, with probability $\geq 1 - \delta$ (over the choice of the m training examples):

$$L_{\mathcal{D}}(h) \leq \min_{h' \in \mathcal{H}} L_{\mathcal{D}}(h') + \varepsilon$$

where $L_{\mathcal{D}}(h) = \mathbb{E}_{z \sim \mathcal{D}}[\ell(h, z)]$

Uniform Convergence and Learnability

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Definition

A training set S is called ε*-representative* (w.r.t. domain Z, hypothesis class $\mathcal{H}$, loss function ℓ, and distribution $\mathcal{D}$) if

$$\forall h \in \mathcal{H}, |L_S(h) - L_{\mathcal{D}}(h)| \leq \varepsilon$$

Proposition

Assume that training set S is $\frac{\varepsilon}{2}$-representative (w.r.t. domain Z, hypothesis class $\mathcal{H}$, loss function ℓ, and distribution $\mathcal{D}$). Then, any output of $\mathrm{ERM}_{\mathcal{H}}(S)$ (i.e., any $h_S \in \arg\min_{h\in\mathcal{H}} L_S(h)$) satisfies

$$L_{\mathcal{D}}(h_S) \leq \min_{h\in\mathcal{H}} L_{\mathcal{D}}(h) + \varepsilon$$

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Proof.

For every $h \in \mathcal{H}$:

$$\begin{aligned} L_{\mathcal{D}}(h_S) &\leq L_S(h_S) + \frac{\varepsilon}{2} \\ &\leq L_S(h) + \frac{\varepsilon}{2} \\ &\leq L_{\mathcal{D}}(h) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= L_{\mathcal{D}}(h) + \varepsilon \end{aligned}$$

□

Uniform convergence depends on training set: when do we have uniform convergence?

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Definition

A hypothesis class $\mathcal{H}$ has the *uniform convergence property* (w.r.t. a domain Z and a loss function ℓ) if there exists a function $m_{\mathcal{H}}^{UC} : (0,1)^2 \to \mathbb{N}$ such that for every $\varepsilon, \delta \in (0,1)$ and for every probability distribution $\mathcal{D}$ over Z, if S is a sample of $m \geq m_{\mathcal{H}}^{UC}(\varepsilon, \delta)$ i.i.d. examples drawn from $\mathcal{D}$, then with probability $\geq 1 - \delta$, S is ε-representative.

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Proposition

If a class $\mathcal{H}$ has the uniform convergence property with a function $m_{\mathcal{H}}^{UC}$ then the class is agnostically PAC learnable with the sample complexity $m_{\mathcal{H}}(\varepsilon, \delta) \leq m_{\mathcal{H}}^{UC}(\varepsilon/2, \delta)$. Furthermore, in that case the $\mathrm{ERM}_{\mathcal{H}}$ paradigm is a successful agnostic PAC learner for $\mathcal{H}$.

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What classes of hypotheses have uniform convergence?

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We prove that finite sets of hypotheses are agnostic PAC learnable under some restriction for the loss.

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Proposition

Let $\mathcal{H}$ be a finite hypothesis class, let Z be a domain, and let $\ell : \mathcal{H} \times Z \to [0, 1]$ be a loss function. Then:

- $\mathcal{H}$ enjoys the uniform convergence property with sample complexity
$$m_{\mathcal{H}}^{UC}(\varepsilon, \delta) \leq \left\lceil \frac{\log(2|\mathcal{H}|/\delta)}{2\varepsilon^2} \right\rceil$$
- $\mathcal{H}$ is agnostically PAC learnable using the ERM algorithm with sample complexity
$$m_{\mathcal{H}}(\varepsilon, \delta) \leq m_{\mathcal{H}}^{UC}(\varepsilon/2, \delta) \leq \left\lceil \frac{2\log(2|\mathcal{H}|/\delta)}{\varepsilon^2} \right\rceil$$

Idea of the proof:

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Useful tool: Hoeffding's Inequality

Hoeffding's Inequality

Let $\theta_1, \ldots, \theta_m$ be a sequence of i.i.d. random variables and assume that for all i, $\mathbb{E}[\theta_i] = \mu$ and $\mathbb{P}[a \leq \theta_i \leq b] = 1$. Then, for any $\varepsilon > 0$

$$\mathbb{P}\left[\left|\frac{1}{m}\sum_{i=1}^{m}\theta_i - \mu\right| > \varepsilon\right] \leq 2e^{-\frac{2m\varepsilon^2}{(b-a)^2}}$$

Proof (see also the book)

Bibliography

[UML] Chapter 4