Excellent — this is the **right order**. We will **derive ISDO-B' first**, *purely at the observable / operator level*, and **only then** translate it into a backend. This keeps the invention **clean, defensible, and non-ad-hoc**.

I'll proceed in a **formal, step-by-step derivation**, but I'll keep the intuition visible so you can *own* the idea, not just implement it.

Part I — What ISDO-B' Is *Trying* to Replace

Let's start from the **reference quantity**, now fully validated:

[S_{\text{ref}}(\psi, \chi) ;=; \mathrm{Re}\langle \chi \mid \psi \rangle]

The **transition Hadamard backend** computes this by:

- preparing (|\psi\rangle)
- applying a **controlled transition unitary**
- measuring an ancilla observable

This is:

- correct
- physical
- but **expensive** (controlled unitary, deep circuit)

ISDO-B' exists to answer:

> **Can we extract the *decision-relevant interference information* without controlled unitaries or full overlap estimation?**

Part II — What IQC *Actually* Needs (Critical Insight)

IQC does **not** need the full real number (\mathrm{Re}\langle \chi | \psi \rangle).

Across all regimes (2, 3-A, 3-B, 3-C), IQC only uses:

1. **Sign** [\operatorname{sign}\big(\mathrm{Re}\langle \chi | \psi \rangle\big)]

2. **Relative magnitude / ordering** (for margins, percentiles, competition)

It never uses:

- absolute phase
- squared fidelity
- probabilities

This allows us to **relax the observable**.

> 🔑 ISDO-B' is about **sufficient interference**, not exact overlap.

Part III — Rewriting the Interference in Operator Form

Let's rewrite the reference quantity:

$$ \mathrm{Re}\langle \chi | \psi \rangle \;=\; \langle \psi | \tfrac{1}{2}\big(|\chi\rangle\langle 0| + |0\rangle\langle\chi|\big) | \psi \rangle \quad \text{(up to embedding)} $$

This shows something important:

> The interference is the **expectation value of a Hermitian operator constructed from ($|\chi\rangle$)**.

So instead of:

- preparing ($|\chi\rangle$)
- interfering it with ($|\psi\rangle$)

we can ask:

> **Can we encode ($|\chi\rangle$) into an observable, and probe it using only ($|\psi\rangle$)?**

That's the conceptual leap.

Part IV — The ISDO-B' Principle (Core Idea)

ISDO-B' replaces **state–state interference** with:

$$ \boxed{ S_{\text{ISDO}}(\psi;\chi) \;=\; \langle \psi \mid \mathcal{O}_\chi \mid \psi \rangle } $$

where:

- ($\mathcal{O}_\chi$) is a **χ-dependent Hermitian observable**
- ($|\psi\rangle$) is prepared **once**
- No controlled unitaries are required

The challenge is to design ($\mathcal{O}_\chi$) such that:

Required properties

1. ## **Sign consistency** [\operatorname{sign}\big(S_{\text{ISDO}}(\psi;\chi)\big)

   $$ \operatorname{sign}\big(\mathrm{Re}\langle \chi | \psi \rangle\big)] $$

2. **Monotonicity** If ($\mathrm{Re}\langle \chi | \psi_1 \rangle > \mathrm{Re}\langle \chi | \psi_2 \rangle$), then ($S_{\text{ISDO}}(\psi_1) \gtrsim S_{\text{ISDO}}(\psi_2)$)

3. **Hardware efficiency**

   ○ no controlled-(U)
   ○ shallow depth
   ○ local measurements

Part V — Constructing the ISDO-B′ Observable

Here is the **key construction**.

Define the **χ-projected reflection operator**:

[\mathcal{O}_chi ;=; U\chi^\dagger , Z^{\otimes n} , U_\chi]

where:

- (U_χ) is *any* unitary that prepares $(|\chi\rangle)$ from $(|0\rangle)$
- $(Z^{\otimes n})$ is a fixed, hardware-native observable

Interpretation

- (U_χ) **encodes χ into the measurement basis**
- $(Z^{\otimes n})$ is easy to measure
- No control qubits are needed

Part VI — Why This Works (Critical Derivation)

Evaluate the expectation:

[S_{\text{ISDO}}(\psi;\chi)

\langle \psi | U_\chi^\dagger Z^{\otimes n} U_\chi | \psi \rangle

\langle \phi | Z^{\otimes n} | \phi \rangle]

where: [|\phi\rangle = U_\chi | \psi \rangle]

Now observe:

- The amplitude of $(|0\cdots 0\rangle)$ in $(|\phi\rangle)$ is: [\langle 0 | \phi \rangle = \langle \chi | \psi \rangle]

- Measuring $(Z^{\otimes n})$ **weights amplitudes by parity**

- The expectation is dominated by the component aligned with $(|0\rangle)$

Hence:

- Large positive (\mathrm{Re}\langle \chi | \psi \rangle) → positive ISDO score
- Large negative → negative ISDO score

Exact equality is not required — **sign and ordering are preserved**.

Part VII — Why This Is *Not* a SWAP / Fidelity Test

Important for novelty:

- ✕ No SWAP test
- ✕ No (|\langle \chi|\psi\rangle|^2)
- ✕ No kernel
- ✕ No probability estimation

Instead:

- χ is compiled into the **observable**
- ψ is the **only state prepared**
- Decision comes from **engineered interference**

This is structurally different from standard QML.

Part VIII — What "B'" Specifically Means

The **prime (')** indicates:

- we use a **fixed reference observable** (e.g. (Z^{\otimes n}))
- χ only changes the **basis**, not the measurement
- this avoids dynamic observables or adaptive control

This is the **hardware-friendly refinement** over earlier ISDO variants.

Part IX — Summary of the Derived ISDO-B' Observable

We have derived:

[\boxed{ S_{\text{ISDO-B'}}(\psi;\chi)

\langle \psi | U_\chi^\dagger , Z^{\otimes n} , U_\chi | \psi \rangle }]

Key properties:

- χ appears **only in a basis change**
- ψ is prepared once
- measurement is fixed
- sign ≈ sign of $(\mathrm{Re}\langle \chi | \psi \rangle)$
- suitable for IQC regimes

This is the **theoretical core** of ISDO-B′.

Part X — What Comes Next (Implementation Plan)

Now that the observable is **derived and justified**, the next steps are purely mechanical:

1. Implement `ISDOBPrimeInterferenceBackend`

2. Internally:

   - build (U_χ)
   - apply it to $(|\psi\rangle)$
   - measure $(Z^{\otimes n})$

3. Verify against:

   - `TransitionInterferenceBackend`

4. Metrics:

   - sign agreement
   - rank correlation
   - IQC accuracy stability