Feb 10

Microwave Network (Pozar Ch. 4)

Solving complicated Maxwell equation at RF

⇒ Extending network theory at lower freq. to handle microwave network

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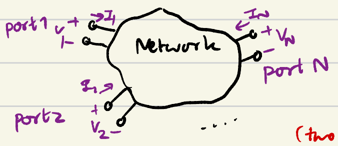
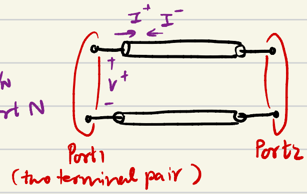
Impedance Matrice





Terminal voltage and terminal current

$V_n = V_n^+ + V_n^-$

$I_n = I_n^+ - I_n^-$

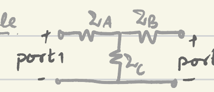
related by N×N impedance matrix

$$\begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} & \cdots & Z_{1N} \\ Z_{21} & Z_{22} & \cdots & Z_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{N1} & Z_{N2} & \cdots & Z_{NN} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{pmatrix} \Rightarrow [V] = [Z][I]$$

$$Z_{ij} = \frac{V_i}{I_j}\bigg|_{I_k = 0 \text{ for } k \neq j}$$

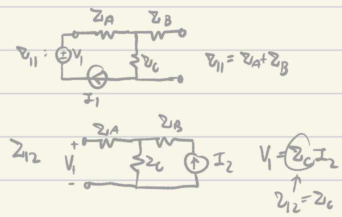
Driving port j with I_j, measure voltage at port i (V_i) when other ports are open-circuited.

Example



$$[Z] = \begin{bmatrix} Z_A + Z_C & Z_C \\ Z_C & Z_B + Z_C \end{bmatrix}$$

(reciprocal network)



Admittance matrix: $[I] = [Y][V] \Rightarrow [Y] = [Z]^{-1}$

$$\begin{pmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{pmatrix} = \begin{pmatrix} Y_{11} & Y_{12} & \cdots & Y_{1N} \\ Y_{21} & Y_{22} & \cdots & Y_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{N1} & Y_{N2} & \cdots & Y_{NN} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{pmatrix}$$

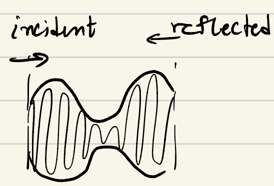
$$Y_{ij} = \frac{I_i}{V_j}\bigg|_{V_k = 0 \text{ for } k \neq j}$$

Driving port j with voltage V_j, measure current I_i at port i while short circuit other ports.

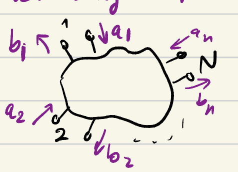
Scattering matrix (Chap 4.3)

☆

Difficult in measuring voltage and current as they are likely function of position



Scattering matrix: generalization of the notion of reflection / transmission coefficient.



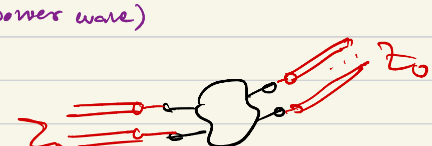
- Task: relate voltage wave incident into ports to those reflected

↳ a_i, b_i wave amplitude (power wave)

a_i: incident wave into port i

b_i: reflected wave out of port j

$$a_i = \frac{V_i^+}{\sqrt{Z_i}}, \quad b_i = \frac{V_i^-}{\sqrt{Z_i}}$$



Z_i, Z_j defined as normalized impedance for port i and port j usually (not necessarily) equal to Z_0 of the T-line attached to the port.

Z_i is the impedance of the network looking into port i, which is equivalent to the Z_0 of the transmission line at port i, assuming infinitely long T-lines (no reflection).

Discussion

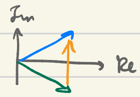
$$V_i = V_i^+ + V_i^- = a_i\sqrt{Z_i} + b_i\sqrt{Z_i} = (a_i + b_i)\sqrt{Z_i}$$

$$I_i = I_i^+ - I_i^- = \frac{V_i^+}{Z_i} - \frac{V_i^-}{Z_i} = (a_i - b_i)/\sqrt{Z_i}$$

$$\boxed{a_i = \frac{V_i + Z_i I_i}{2\sqrt{Z_i}}, \quad b_i = \frac{V_i - Z_i I_i}{2\sqrt{Z_i}}}$$

Net power into port i

$$P_i = \mathrm{Re}\{V \cdot I^*\}$$
$$= \mathrm{Re}\left\{(a_i+b_i)\cdot\sqrt{Z_i}\,(a_i^*-b_i^*)\frac{1}{\sqrt{Z_i}}\right\}$$
$$= \mathrm{Re}\left\{|a_i|^2-|b_i|^2 \underbrace{- a_i b_i^* + (a_i b_i^*)^*}_{\text{purely imaginary}}\right\}$$



$P_i = |a_i|^2-|b_i|^2$ → wave amplitude (power wave) are defined in a way that

$|a_i|^2 = |V_i^+|^2/Z_i$: incident power

$|b_i|^2 = |V_i^-|^2/Z_i$: scattered power

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix}, \quad P = \sum_{i=1}^{N} P_i = \vec{a}^{\dagger}\vec{a} - \vec{b}^{\dagger}\vec{b}$$

"$\dagger$" (dagger) denotes conjugate transpose "Hermitian" conjugate

$\vec{a}^{\dagger} = (a_1^* \; a_2^* \; \ldots \; a_N^*)$

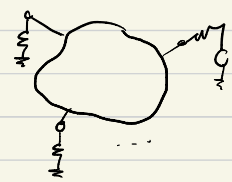
Define scattering matrix incident and scattered wave.

$$[b] = [S][a] : \quad \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & \cdots & S_{1N} \\ S_{21} & S_{22} & \cdots & S_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ S_{N1} & S_{N2} & & S_{NN} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}$$

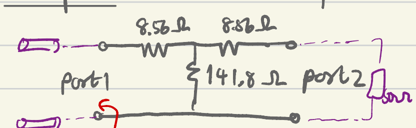
$$S_{ij} = \left.\frac{b_i}{a_j}\right|_{a_k=0,\ k\neq i}$$

(i: output, j: input)

Measure the scattered wave at port i, make sure that all others are matched to the reference impedance of S-parameter.

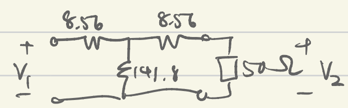


Example 1: evaluate S-param for 50Ω ref.



$Z_{in1} = 8.56 + 141.8 \| (8.56+50) = 50\Omega \rightarrow S_{11} = 0$

$$S_{11} = \left.\frac{V_1^-}{V_1^+}\right|_{V_2^+=0} = \Gamma_1 \Big|_{50\Omega \text{ at port 2}}$$

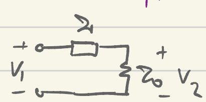


$V_2 = V_2^+ + V_2^-$ ($V_2^+ = 0$)

$$S_{21} = \left.\frac{V_2^-}{V_1^+}\right|_{V_2^+=0} = \frac{V_2}{V_1^+} = \frac{V_2}{V_1} = \boxed{0.707} \longrightarrow \text{3dB attenuator} : \frac{|V_2^-|^2}{|V_1^+|^2} = 0.5$$

$V_1^+ + V_1^- S_{11} = V_1$ ($S_{11} = 0$)

$V_2 = V_2^+ + V_2^-$ ($V_2^+ = 0$)



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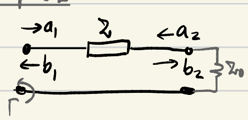
Example 2:



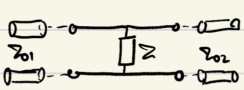
$$S_{11} = \frac{b_1}{a_1} = \left.\frac{V_1^-}{V_1^+}\right|_{V_2^+=0} = \Gamma_1$$
$$= \frac{Z_0+Z_1-Z_0}{Z_0+Z_1+Z_0} = \frac{Z_1}{2Z_0+Z_1}$$

$$S_{21} = \frac{b_2}{a_1} = \left.\frac{V_2^-}{V_1^+}\right|_{V_2^+=0} = \frac{V_2}{V_1^+} = \frac{V_2}{V_1}(1+S_{11}) = \frac{Z_0}{Z_0+Z_1}\cdot\left(1+\frac{Z_1}{Z_1+2Z_0}\right) = \frac{2Z_0}{2Z_0+Z_1}$$

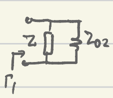
$V_1 = V_1^+(1+\Gamma)$

$\therefore V_1^+ = \frac{V_1}{1+S_{11}}$

Example 3:

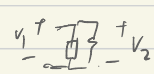


① Find S_{11}:



$$S_{11} = \Gamma_1 = \frac{(Z \| Z_{02}) - Z_{01}}{(Z \| Z_{02}) - Z_{01}} = \frac{\frac{ZZ_{02}}{Z+Z_{02}} - Z_{01}}{\frac{ZZ_{02}}{Z+Z_{02}} + Z_{01}} = \frac{Z(Z_{02}-Z_{01}) - Z_{01}Z_{02}}{Z(Z_{02}+Z_{01}) + Z_{01}Z_{02}}$$

Automatically: $S_{22} = \frac{Z(Z_{01}-Z_{02}) - Z_{01}Z_{02}}{Z(Z_{01}+Z_{02}) + Z_{01}Z_{02}}$

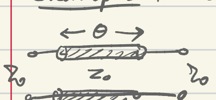


② Find S_{21}:

$$S_{21} = \frac{V_2^-/\sqrt{Z_{02}}}{V_1^+/\sqrt{Z_{01}}} = \sqrt{\frac{Z_{01}}{Z_{02}}}\cdot\frac{V_2^-}{V_1^+} = \sqrt{\frac{Z_{01}}{Z_{02}}}\cdot\frac{V_2}{V_1/(1+\Gamma_1)} = \sqrt{\frac{Z_{01}}{Z_{02}}}\cdot(1+S_{11}) = \sqrt{\frac{Z_{01}}{Z_{02}}}\cdot\left(1+\frac{Z(Z_{02}-Z_{01})-Z_{01}Z_{02}}{Z(Z_{02}+Z_{01})+Z_{01}Z_{02}}\right)$$

$$S_{21} = \sqrt{\frac{Z_{01}}{Z_{02}}}\cdot\left(\frac{}{Z(Z_{01}+Z_{02})+Z_{01}Z_{02}}\right.$$

Example 4: Section of T-line

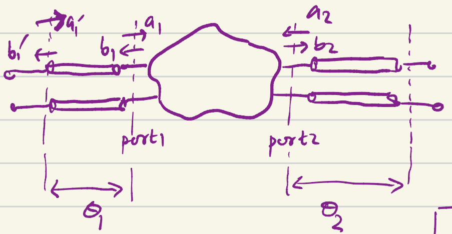


$S_{11} = 0, \; S_{22} = 0$

$$S_{21} = \frac{b_2}{a_1} = \frac{V_2^-}{V_1^+} = \frac{V_1^+ e^{-j\theta}}{V_1^+} = e^{-j\theta} = S_{12}$$

$$\boxed{[S] = \begin{pmatrix} 0 & e^{-j\theta} \\ e^{-j\theta} & 0 \end{pmatrix}}$$

Reference plane



$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} b_1' \\ b_2' \end{pmatrix} = \begin{pmatrix} S_{11}' & S_{12}' \\ S_{21}' & S_{22}' \end{pmatrix} \begin{pmatrix} a_1' \\ a_2' \end{pmatrix}$$

$$\boxed{\therefore [S'] = \begin{pmatrix} e^{-j2\theta_1} S_{11} & e^{-j(\theta_1+\theta_2)} S_{12} \\ e^{-j(\theta_1+\theta_2)} S_{21} & e^{-j2\theta_2} S_{22} \end{pmatrix}}$$

$a_1 = a_1' \cdot e^{-j\theta_1}$ | $b_1' = b_1 e^{-j\theta_1}$

$a_2 = a_2' e^{-j\theta_2}$ | $b_2' = b_2' e^{-j\theta_2}$

$$\Downarrow$$

$$\begin{pmatrix} b_1' \\ b_2' \end{pmatrix} = \underbrace{\begin{pmatrix} e^{-j\theta_1} & 0 \\ 0 & e^{-j\theta_2} \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{21} \end{pmatrix} \begin{pmatrix} e^{-j\theta_1} & 0 \\ 0 & e^{-j\theta_2} \end{pmatrix}}_{\begin{pmatrix} S_{11}' & S_{12}' \\ S_{21}' & S_{22}' \end{pmatrix}} \begin{pmatrix} a_1' \\ a_2' \end{pmatrix}$$

Properties of S-parameter

① **Reciprocity** : can exchange excitation & measurement power without changing relationship between them.

$$\begin{bmatrix} \triangle & \bigcirc \\ \bigcirc & \square \end{bmatrix}$$

For reciprocity to hold: the circuit must be

① linear

② time-invariant

③ contains no V/I source e.g. amplifier

④ made of reciprocal material

• examples : transmission line, filter, impedance matching, antenna → reciprocal

amplifier, circulator, transistor → nonreciprocal

② **Lossless network** : dissipates no power, i.e., scattered powers = incident power

$[S]$ is lossless iff $[S]$ = unitary : $[S]^{-1} = [S^*]$

$$\sum_{k=1}^{N} S_{ki} S_{kj}^* = \delta_{ij} \leftarrow \text{Kronecker delta function} \quad \delta_{ij} = \begin{cases} 1 & ; i=j \\ 0 & , \text{otherwise} \end{cases}$$

	S	Z	Y	$ABCD$
S_{11}	S_{11}	$\frac{(Z_{11}-Z_0)(Z_{22}+Z_0)-Z_{12}Z_{21}}{\Delta Z}$	$\frac{(Y_0-Y_{11})(Y_0+Y_{22})+Y_{12}Y_{21}}{\Delta Y}$	$\frac{A+B/Z_0-CZ_0-D}{A+B/Z_0+CZ_0+D}$
S_{12}	S_{12}	$\frac{2Z_{12}Z_0}{\Delta Z}$	$\frac{-2Y_{12}Y_0}{\Delta Y}$	$\frac{2(AD-BC)}{A+B/Z_0+CZ_0+D}$
S_{21}	S_{21}	$\frac{2Z_{21}Z_0}{\Delta Z}$	$\frac{-2Y_{21}Y_0}{\Delta Y}$	$\frac{2}{A+B/Z_0+CZ_0+D}$
S_{22}	S_{22}	$\frac{(Z_{11}+Z_0)(Z_{22}-Z_0)-Z_{12}Z_{21}}{\Delta Z}$	$\frac{(Y_0+Y_{11})(Y_0-Y_{22})+Y_{12}Y_{21}}{\Delta Y}$	$\frac{-A+B/Z_0-CZ_0+D}{A+B/Z_0+CZ_0+D}$
Z_{11}	$Z_0\frac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{11}	$\frac{Y_{22}}{\|Y\|}$	$\frac{A}{C}$
Z_{12}	$Z_0\frac{2S_{12}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{12}	$\frac{-Y_{12}}{\|Y\|}$	$\frac{AD-BC}{C}$
Z_{21}	$Z_0\frac{2S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{21}	$\frac{-Y_{21}}{\|Y\|}$	$\frac{1}{C}$
Z_{22}	$Z_0\frac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{22}	$\frac{Y_{11}}{\|Y\|}$	$\frac{D}{C}$
Y_{11}	$Y_0\frac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{Z_{22}}{\|Z\|}$	Y_{11}	$\frac{D}{B}$
Y_{12}	$Y_0\frac{-2S_{12}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{-Z_{12}}{\|Z\|}$	Y_{12}	$\frac{BC-AD}{B}$
Y_{21}	$Y_0\frac{-2S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{-Z_{21}}{\|Z\|}$	Y_{21}	$\frac{-1}{B}$
Y_{22}	$Y_0\frac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{Z_{11}}{\|Z\|}$	Y_{22}	$\frac{A}{B}$
A	$\frac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{2S_{21}}$	$\frac{Z_{11}}{Z_{21}}$	$\frac{-Y_{22}}{Y_{21}}$	A
B	$Z_0\frac{(1+S_{11})(1+S_{22})-S_{12}S_{21}}{2S_{21}}$	$\frac{\|Z\|}{Z_{21}}$	$\frac{-1}{Y_{21}}$	B
C	$\frac{1}{Z_0}\frac{(1-S_{11})(1-S_{22})-S_{12}S_{21}}{2S_{21}}$	$\frac{1}{Z_{21}}$	$\frac{-\|Y\|}{Y_{21}}$	C
D	$\frac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{2S_{21}}$	$\frac{Z_{22}}{Z_{21}}$	$\frac{-Y_{11}}{Y_{21}}$	D

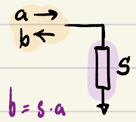
$|Z| = Z_{11}Z_{22} - Z_{12}Z_{21}$; $|Y| = Y_{11}Y_{22} - Y_{12}Y_{21}$; $\Delta Y = (Y_{11}+Y_0)(Y_{22}+Y_0) - Y_{12}Y_{21}$; $\Delta Z = (Z_{11}+Z_0)(Z_{22}+Z_0) - Z_{12}Z_{21}$; $Y_0 = 1/Z_0$.

Feb 17

How to handle S-parameters for complicated networks?

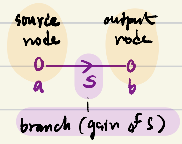
⇒ Use **signal flow graph** as an alternative to circuit diagrams.

(can be described by linear equation from inspecting a graph).



In signal flow graph (sfg), variables (wave amplitude) as nodes, and coefficient (S-parameters) as a directional branches/paths with a path gain.

⇓



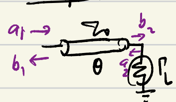
Remarks: 1) The arrow points from the source node to the output node.

2) A node may have multiple incoming or outgoing paths.

However, the source node of the overall sfg can only have outgoing path.

3) The value of any variable/node is equal to the sum of nodes incoming to that node multiply respective path gain.

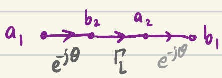
Example:



(–[S]–[Γ_L])

① Sketch all wave amplitude.

② For sfg, indicate all wave amplitudes and nodes



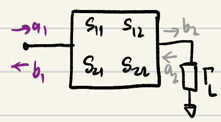
$b_2 = a_1 e^{-j\theta}$

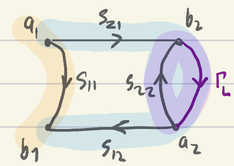
$a_2 = \Gamma_L b_1$

$b_1 = a_2 e^{-j\theta}$

$a_{11}' = \frac{b_1}{a_1} = \Gamma_L e^{-j2\theta}$

Example 2:





$b_2 = S_{21}a_1 + S_{22}a_2$ —(1)

$a_2 = b_2\Gamma_L$ —(2)

$b_1 = S_{11}a_1 + S_{12}a_2$ —(3)

$a_2 = \Gamma_L(S_{21}a_1 + S_{22}a_2)$

$a_2 = \frac{\Gamma_L S_{21}}{1 - S_{22}\Gamma_L} a_1$

$S_{11}' = \frac{b_1}{a_1} = \frac{S_{11}a_1 + S_{12}a_2}{a_1} = S_{11} + S_{12}\frac{a_2}{a_1} = S_{11} + \frac{S_{12}S_{21}\Gamma_L}{1 - S_{22}\Gamma_L}$

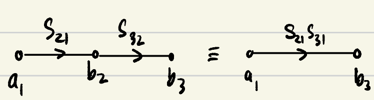
open-loop gain

S_{11}'

1-loop gain

Decomposition Rule (Reminder: node represents wave amplitude)

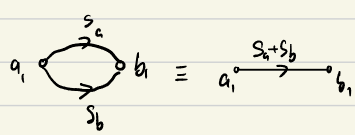
1)



replace two series branches with common node with single branch with gain

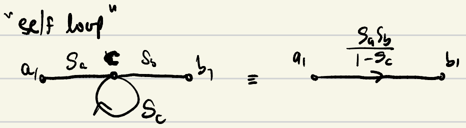
(product of gains of original branches)

2)



(sum of original branches)

3) "self loop"

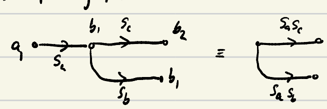


$c = S_a a_1 + S_c c \rightarrow a_1 = \frac{S_a}{1 - S_c} c$

$b_1 = S_b c$

$\frac{b_1}{a_1} = \frac{S_a S_b}{1 - S_c}$

4) "Splitting rule"



Mason's rule: Solving linear equation by inspecting sfg and involving Mason's rule.

- Definitions:

1) path: series of branches, in the direction of arrows along which no node is crossed more than once.

2) loop: 2.1) 1st-order-loop: product of branches encountered in a journey starting from a node moving along which

2.2) 2nd-order-loop product of 2 non-touching loops

⋮

2.3) nth-order-loop product of n non-touching loops.

2.4) non-touching loop(s) in the system that not touch each other(s).

Mason's Rule:

ratio of interest

$$\boxed{T = \sum_k G_k \frac{\Delta k}{\Delta}}$$

gain of path k

Δ_k: cofactor of path k : $\Delta_k = 1 - \Sigma L^{k}(1) + \Sigma L^{k}(2) - \ldots$

↳ $\Sigma L^{k}(n)$: Sum of all n^{th} open loops not touching path k.

Δ: gain determinant

$= 1 - \Sigma L(1) + \Sigma L(2) - \Sigma L(3) + \ldots$

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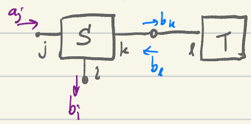
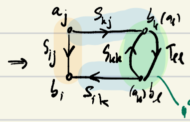
$\sum_k G_k \frac{\Delta_k}{\Delta}$

Example 1 Find b_i/a_j



$\Rightarrow$

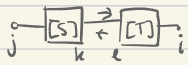


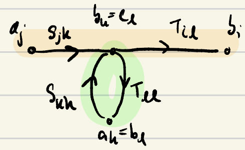
$$\frac{b_i}{a_j} = S_{ij}\frac{1-S_{kk}T_{ll}}{1-S_{kk}T_{ll}} + S_{kj}T_{ll}S_{ik}\frac{1}{1-S_{kk}T_{ll}} \Rightarrow S_{ij} + \frac{S_{kj}T_{ll}S_{ik}}{1-S_{kk}T_{ll}}$$

1st-order-loop : $\Delta = 1-S_{kk}T_{ll}$

non-touching loop : $\Delta_1 = 1-S_{kk}T_{ll}$, $\Delta_2 = 1$

Example 2





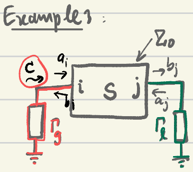
$$\frac{b_i}{a_j} = S_{jk}T_{il}\cdot\frac{1}{1-S_{kk}T_{ll}}$$

non-touching: 1

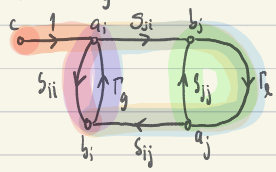
delta: $1-S_{kk}T_{ll}$

Example 3 :

b_i/c (not b_i/a_i)



$a_i = c + b_i\Gamma_g$



1) Path : S_{ii}, $S_{ji}\Gamma_l S_{ij}$

2) loops :

- 1st order loops : $S_{ii}\Gamma_g$, $S_{jj}\Gamma_l$, $S_{ji}\Gamma_l S_{ij}\Gamma_g$
- a product of any 2 non-touching 1st order loop
  - $\hookrightarrow$ $S_{ii}\Gamma_g S_{jj}\Gamma_l$

$$\Delta = 1-(S_{ii}\Gamma_g + S_{jj}\Gamma_l + S_{ji}\Gamma_l S_{ij}\Gamma_g) + S_{ii}\Gamma_g S_{jj}\Gamma_l$$

$$\boxed{\Delta = (1-\Gamma_g S_{ii})(1-\Gamma_l S_{jj}) - S_{ji}\Gamma_l S_{ij}\Gamma_g}$$

$G_1 : S_{ii} \Rightarrow \boxed{\Delta_1 = 1 - S_{jj}\Gamma_l}$

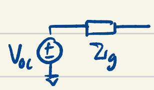
$G_2 : S_{ji}\Gamma_l S_{ij} \Rightarrow \boxed{\Delta_2 = 1}$

$$\frac{b_i}{c} = \frac{G_1\Delta_1 + G_2\Delta_2}{\Delta}$$

Wave source

$\hookrightarrow$ source in lower-freq circuits

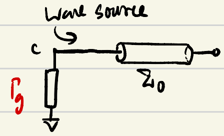
- Thevenin equivalent



Z_g : Thevenin input impedance

V_{oc} : open circuit gain

Wave source



Γ_g : generator reflection coefficient

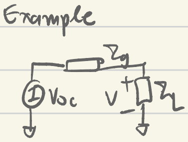
C : incident wave amplitude generator. (when connect to a matched load)

Γ_g : need to specify Z_0 (normalized impedance)

$$\Gamma_g = \frac{Z_g - Z_0}{Z_g + Z_0}, \quad C = \frac{V_0^+}{\sqrt{Z_0}} = \frac{V_{oc}\frac{Z_0}{Z_0+Z_g}}{\sqrt{Z_0}} = \frac{V_{oc}\sqrt{Z_0}}{Z_0+Z_g}$$

$\hookrightarrow$ function Z_0

Example

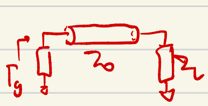


$$V = V_{oc}\frac{Z_L}{Z_L+Z_g}$$

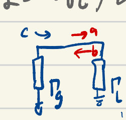
$$I = V_{oc}/Z_L+Z_g$$

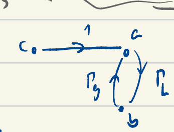
$$P_L = Re\{VI^*\} = |V_{oc}|^2\frac{R_L}{|Z_L+Z_g|^2}$$

$Z_L + Z_L^* = R_L$



$\Rightarrow$





$$\frac{a}{c} = \frac{1}{1-\Gamma_g\Gamma_L}$$

$G_1 = 1$, $\Delta_1 = 1$

$\Delta_2 = 1 - \frac{1}{1-\Gamma_L\Gamma_g} + \ldots$

Power delivered to load

$$P_L = |a|^2 - |b|^2 = |a|^2(1-|\Gamma_L|^2)$$

$$P_L = \frac{|c|^2(1-|\Gamma_L|^2)}{|1-\Gamma_g\Gamma_L|^2}, \quad C = \frac{V_{oc}\sqrt{Z_0}}{Z_0+Z_g}, \quad \Gamma_L = \frac{Z_L - Z_0}{Z_L+Z_0}, \quad \Gamma_g = \frac{Z_g - Z_0}{Z_g+Z_0}$$

$$1-|\Gamma_L|^2 = 1-\Gamma_L\Gamma_L^* = \frac{2Z_0(Z_L+Z_L^*)}{(Z_L+Z_0)^2}$$

$$1-\Gamma_g\Gamma_L = \frac{2Z_0(Z_g+Z_L)}{(Z_L+Z_0)(Z_g+Z_0)}$$

$$P_L = \frac{|V_{oc}|^2 Z_0}{|Z_0+Z_g|}\cdot\frac{2Z_0(Z_L+Z_L^*)}{\cancel{|Z_L+Z_0|^2}}\cdot\frac{\cancel{|Z_L+Z_0|^2}\cancel{|Z_g+Z_0|^2}}{4Z_0|Z_g+Z_L|^2} \Rightarrow \boxed{P_L = \frac{|V_{oc}|^2(Z_L+Z_L^*)}{4Z_0|Z_0+Z_L|^2}}$$