① Electromagnetics & Circuits

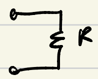
Jan 13, 2025

- <u>Low-freq circuits</u>: Analyze and design using "lumped-element" approach
  ⇒ field not every extend over extent of circuit element

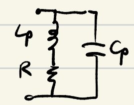
Advantages:

(1) cause-effect relations are easily determined functions of circuit elements are easy to understand.

low-freq model of R



high-freq model of R



(2) Fields associated with circuit elements are "quasi-static".

⇒ can use concepts/techniques from electrostatic to determine the circuit.

→ can describe elements in terms of voltage/current.

(3) Use powerful/simple technique to analyze lumped-element circuits: KCL, KVL, Thevenin/Norton eq.

⇒ Relations between Kirchhoff's laws and Maxwell's equations

- <u>Review of Maxwell equations</u>

$\vec{E}$: electric field (V/m)

$\vec{D}$: electrical flux density (C/m²)

$\vec{H}$: magnetic field (A/m)

$\vec{B}$: magnetic flux density (T: Wb/m²)

$\vec{J}$: electrical current density (A/m²)

ρ: electrical charge density (C/m³)

Gauss' law: $\nabla\cdot\vec{D} = \rho \iff \int_S \vec{D}\cdot d\vec{s} = \int_V \rho\, dV$

Gauss' law for magnetism: $\nabla\cdot\vec{B} = 0 \quad \int_S \vec{B}\cdot d\vec{s} = 0$

Faraday's law: $\nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \int_C \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{s}$

Ampere's law: $\nabla\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \int_C \vec{H}\cdot d\vec{l} = \int_S \vec{J}\cdot d\vec{s} + \frac{d}{dt}\int_S \vec{D}\cdot d\vec{s}$

Constitutive relations

Free space (microscopic): $\vec{B} = \mu_0 \vec{H}$ (permeability), $\vec{D} = \varepsilon_0 \vec{E}$ (permittivity)

Macroscopic: $\vec{B} = \underline{\mu}\vec{H}$, $\mu = \mu_r\mu_0$

$\vec{D} = \underline{\varepsilon}\vec{E}$, $\varepsilon = \varepsilon_r\varepsilon_0$

- <u>Continuity equations</u> (Gauss's + Ampere's)

$$\nabla\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0 \iff \oint \vec{J}\cdot d\vec{s} = -\frac{d}{dt}\overbrace{\iiint \rho\, dV}^{\text{total charge}}$$

conservation of charge.

- Origin of Kirchhoff's law (How it breaks at high freq)
  - <u>KCL</u>: algebraic sum of current flow into a node is zero.

$$\sum I_n = 0 \qquad \underbrace{\oint \vec{J}\cdot d\vec{s}}_{\sum I_k} = -\frac{d}{dt}\iiint_V \rho\, dV = 0$$

∴ KCL is valid for static field ($\frac{d}{dt}(\dots) = 0$)

For time-varying fields: KCL is only approximately true, provided that the RHS is negligible.

  - <u>KVL</u>: algebraic sum of voltage drop around the loop is zero.

Faraday's law

$$\oint_C \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\iint_S \vec{B}\cdot d\vec{s}$$

$$\sum V_k = 0$$ ← for static field only

Recall: $V_b - V_a = \int_b^a \vec{E}\cdot d\vec{l}$

<u>Conclusion</u>: KCL & KVL strictly valid for static circuits (DC) and approximately valid for AC of low freq.

At "microwave", it breaks down.

Goal of Section I:

To reformulate circuit theory to account for electromagnetic phenomena that become important at high frequency.

e.g. (1) "parasitic" for lumped elements

(2) distributed circuit/interconnection

Plane wave:

Assume medium is

(1) uniform: ε, μ are not functions of location

(2) isotropic: ε, μ are not functions of direction/fields

(3) time-invariant: ε, μ are not functions of time.

(4) source-free: $\vec{J} = 0$, $\rho = 0$.

Maxwell eq. simplified to

$$\nabla \cdot \vec{E} = 0$$
$$\nabla \cdot \vec{H} = 0$$
$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$
$$\nabla \times \vec{H} = \varepsilon \cdot \frac{\partial \vec{E}}{\partial t}$$

Here we are interested in "time-harmonic": AC quantities

$\Rightarrow$ convenient to use phasors

Benefits: for "linear" system (freq is constant) $\rightarrow$ only need to account for magnitude and phase of signal

Recall phasors.

ex: $f(t) = \sqrt{2}\,F\cos(\omega t + \theta)$ "instantaneous form"

RMS value; angular freq; phase angle

$$e^{j(\omega t+\theta)} = \cos(\omega t+\theta) + j\sin(\omega t+\theta)$$

$$f(t) = \sqrt{2}\,F\,\mathrm{Re}\{e^{j(\omega t+\theta)}\} = \sqrt{2}\,\mathrm{Re}\{F\cdot e^{j(\omega t+\theta)}\}$$
$$= \sqrt{2}\,\mathrm{Re}\{F e^{j\omega t} e^{j\theta}\}$$
$$= \sqrt{2}\,\mathrm{Re}\{\boxed{F e^{j\theta}}\, e^{j\omega t}\}$$

$\underline{F} = F \cdot e^{j\theta}$ — θ: phase angle; F: RMS amplitude

For vector fields with 3 components

$$\vec{F} = \hat{x}\cdot F_x + \hat{y}F_y + \hat{z}F_z \Rightarrow \vec{\underline{F}} = \hat{x}\underline{F_x} + \hat{y}\underline{F_y} + \hat{z}\underline{F_z}$$

Unique properties $\quad \frac{\partial}{\partial t} f(t) \rightarrow j\omega \underline{F}$

time-domain $\quad$ phasor domain

$$f(t) = \sqrt{2}\,F\cos(\omega t+\theta) = \sqrt{2}\,\mathrm{Re}\{\underline{F} e^{j\theta}\}$$

$$\frac{\partial f(t)}{\partial t} = \sqrt{2}\frac{\partial}{\partial t}\mathrm{Re}\{\underline{F} e^{j\omega t}\} \underset{\text{it can be proved}}{=} \sqrt{2}\,\mathrm{Re}\{\underline{F}\frac{d}{dt}e^{j\omega t}\} = \sqrt{2}\,\mathrm{Re}\{\underbrace{\underline{F}\cdot j\omega}_{\text{resulting phasor.}}\, e^{j\omega t}\}$$

Simplified Maxwell Eq. in phasor domain.

$$\nabla \cdot \underline{\vec{E}} = 0$$
$$\nabla \cdot \underline{\vec{H}} = 0$$
$$\nabla \times \underline{\vec{E}} = -j\omega\mu \underline{\vec{H}}$$
$$\nabla \times \underline{\vec{H}} = +j\omega\varepsilon \vec{E}$$

$$\nabla \times (\nabla \times \vec{E}) = \nabla \times (-j\omega\mu \vec{H}) = -j\omega\mu \cdot \nabla \times \vec{H}$$
$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$$

$$\nabla^2 \vec{E} = j\omega\mu\,(\cancel{\nabla \times \vec{H}}) \quad j\omega\varepsilon\vec{E}$$
$$\nabla^2 \vec{E} = -\omega^2\mu\varepsilon\vec{E} \Rightarrow \nabla^2\vec{E} + \omega^2\mu\varepsilon\vec{E} = 0$$

$k^2 = \omega^2\mu\varepsilon$

$$\boxed{\begin{aligned}\nabla^2\vec{E} + k^2\vec{E} = 0 \\ \nabla^2\vec{H} + k^2\vec{H} = 0\end{aligned}}$$ "Helmholtz" equation

To solve this, boundary condition is required.

Assumptions: (1) $\vec{E}$ ($\vec{H}$) has only one component, e.g., $\hat{x}$

(2) $\vec{E}$ ($\vec{H}$) vary along one direction, e.g., z

(1) $\Rightarrow \frac{d^2E_x}{dz^2} + k^2E_x = 0$ "plane wave"

Solution: $E_x = Ae^{-jkz} + Be^{+jkz}$ (phasor)

Comments (1) plane wave are good approximation in many cases.

(2) basic wave phenomena/behaviors easily understood.

(3) More complex (realistic) solutions can be constructed as superposition of plane waves.

Instantaneous solution (time-domain)

$$\vec{E}(z,t) = \sqrt{2}\,\mathrm{Re}\{E_x e^{j\omega t}\}\hat{x} = \sqrt{2}\,\mathrm{Re}\{(Ae^{-jkz} + Be^{jkz})e^{j\omega t}\}\hat{x}$$
$$= \sqrt{2}\{A\cos(\omega t - kz) + B\cos(\omega t + kz)\}\hat{x}$$

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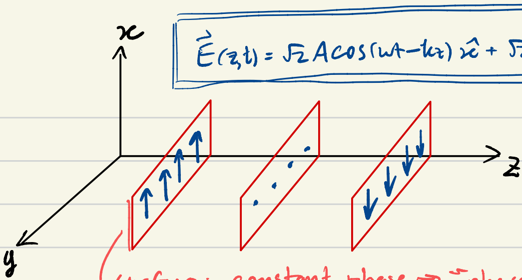


$$\boxed{\vec{E}(z,t) = \sqrt{2}A\cos(\omega t - kz)\,\hat{x} + \sqrt{2}B\cos(\omega t + kz)\,\hat{y}}$$

surface: constant phase ⇒ "phase front" / "wave front"

phase velocity: velocity of phase front

(1) $\omega t - kz = \text{constant}$, $k^2 = \omega^2\mu\varepsilon$

$$\omega - k\frac{dz}{dt} = 0 \Rightarrow \frac{dz}{dt} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} = v_p^+$$

(2) $\omega t + kz = \text{constant}$

$$\omega + k\frac{dz}{dt} = 0 \Rightarrow \frac{dz}{dt} = -\frac{\omega}{k} = -\frac{1}{\sqrt{\mu\varepsilon}} = v_p^-$$

Use Faraday's law:

$$\nabla\times\vec{E} = -j\omega\mu\vec{H}$$

$$= \nabla\times(Ae^{-jkz} + Be^{+jkz})\hat{x} = \hat{y}\cdot\frac{\partial}{\partial z}(Ae^{-jkz} + Be^{+jkz})$$

$$-j\omega\mu\vec{H} = (-jkAe^{-jkz} + jkBe^{+jkz})\hat{y}$$

$$\vec{H} = \left[\frac{k}{\omega\mu}Ae^{-jkz} - \frac{k}{\omega\mu}Be^{+jkz}\right]\hat{y}$$

$$= \left[\sqrt{\frac{\varepsilon}{\mu}}Ae^{-jkz} - \sqrt{\frac{\varepsilon}{\mu}}Be^{+jkz}\right]\hat{y}$$

$$\boxed{\vec{H}(z,t) = \sqrt{2}\cdot\sqrt{\frac{\varepsilon}{\mu}}\cdot A\cos(\omega t - kz)\hat{y} - \sqrt{2}\sqrt{\frac{\varepsilon}{\mu}}B\cos(\omega t + kz)\hat{y}}$$

wavelength: distance over which the phase of the wave changes by 2π at given time instant,

i.e., the distance it takes to repeat itself

$$k\cdot\lambda = 2\pi \Rightarrow \lambda = \frac{2\pi}{k} = \frac{2\pi v_p}{\omega} = \frac{v_p}{f}$$

Note

(1) $\vec{E}$ and $\vec{H}$ are orthogonal to each other, and to the propagation direction

(2) $\vec{E}$ and $\vec{H}$ are in phase

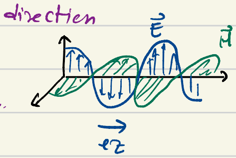
(3) Have '−' sign associated wave negative traveling H-wave in '−z' direction.

(4) Ratio of $\vec{E}$ to $\vec{H}$, termed "wave impedance", $\boxed{\eta = \sqrt{\frac{\mu}{\varepsilon}}\ [\Omega]}$

"intrinsic wave impedance"

Free-space intrinsic wave impedance:

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\,\Omega = 120\pi\,\Omega.$$



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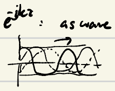
Wave in lossy media

- In lossless materials: purely real ε, μ ⇒ real $k = \omega\sqrt{\mu\varepsilon}$

  e^{-jkz}: as wave propagate, only phase change (no amplitude change)



- Lossy materials

  (1) dielectric loss

  (2) conductors

General lossy medium

$$\boxed{\vec{J} = \sigma\vec{E}} \text{ conductivity}$$

$$\nabla\times\vec{E} = -j\omega\mu\vec{H}$$

$$\nabla\times\vec{H} = (\sigma + j\omega\varepsilon)\vec{E} = j\omega\varepsilon\boxed{\left(1 - \frac{j\sigma}{\omega\varepsilon}\right)}\vec{E}$$

$$\nabla^2\vec{E} + k^2\vec{E} = 0$$

$$k^2 = \omega^2\mu\varepsilon\left(1 - \frac{j\sigma}{\omega\varepsilon}\right)$$

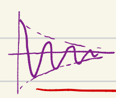
$$k = \beta - j\alpha$$

propagation constant; wave number; attenuation constant

phasor waveform $e^{-jkz} = e^{-j(\beta - j\alpha)z}$

$$= e^{-j\beta z}\cdot e^{-j(-j\alpha)z}$$

$$= e^{-j\beta z}\cdot e^{-\alpha z}$$



Intrinsic wave impedance

$$\eta = \sqrt{\frac{\mu}{\varepsilon\left(1 - \frac{j\sigma}{\omega\varepsilon}\right)}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$$

▸ Good conductor: $\sigma \gg \omega\varepsilon$ (easier to achieve at low freq.)

$$k^2 = \omega^2\mu\varepsilon\left(\cancel{1} - j\frac{\sigma}{\omega\varepsilon}\right) = -j\omega^2\mu\varepsilon\frac{\sigma}{\omega\varepsilon} = -j\omega\mu\sigma$$

↳ $k = \sqrt{\omega\mu\sigma}\cdot\frac{1}{\sqrt{2}}(1 - j) = \beta - j\alpha$

Determine distance over which the field decay by $1/e$, (5δ rule)

$$\beta = \alpha = \sqrt{\frac{\omega\mu\sigma}{2}} \Rightarrow \text{skin depth } \delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}}$$

▸ lossy dielectric: $\varepsilon = \varepsilon' - j\varepsilon''$ (complex ε / loss tangent: $\tan\delta = \varepsilon''/\varepsilon'$)

$$k = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu(\varepsilon' - j\varepsilon'')}$$

$$= \omega\sqrt{\mu\varepsilon'}\,(1 - j\tan\delta)^{1/2}$$

∘ Good dielectric: $\varepsilon'' \ll \varepsilon \Rightarrow \varepsilon''/\varepsilon' = \tan\delta \ll 1 \Rightarrow k = \omega\sqrt{\mu\varepsilon'}\left(1 - \frac{1}{2}j\tan\delta\right) = \beta - j\alpha$