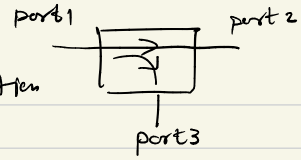
Power Divider & Directional Coupler [Chap 7]

3 or 4 or more ports element.
they may have equal or unequal power distribution



⇒ Three-port network

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

- Possible property:
  ① Matched system : $S_{11}=S_{22}=S_{33}=0$
  ② lossless $S^{\dagger}S = I$
  ③ Reciprocal : $S^T = S$

It is impossible to construct a lossless reciprocal time-invariant 3-port system that is matched at every ports.

Apr 7

Matched & Reciprocal network:

$$S=\begin{bmatrix} 0 & S_{12} & S_{13} \\ S_{12} & 0 & S_{23} \\ S_{13} & S_{23} & 0 \end{bmatrix} \rightsquigarrow S^{\dagger}\begin{bmatrix} 0 & S_{12}^* & S_{13}^* \\ S_{12}^* & 0 & S_{23}^* \\ S_{13}^* & S_{23}^* & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & S_{12}^* & S_{13}^* \\ S_{12}^* & 0 & S_{23}^* \\ S_{13}^* & S_{23}^* & 0 \end{bmatrix}\begin{bmatrix} 0 & S_{12} & S_{13} \\ S_{12} & 0 & S_{23} \\ S_{13} & S_{23} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

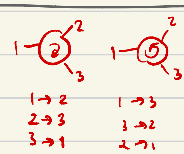
→ To be lossless

Diag: $|S_{12}|^2+|S_{13}|^2=1$, $|S_{12}|^2+|S_{23}|^2=1$, $|S_{13}|^2+|S_{23}|^2=1$ ‖ $S_{13}^* S_{23}=0$, $S_{23}^* S_{12}=0$, $S_{12}^* \cdot S_{13}=0$

(S_{13}, S_{23}, S_{12}) need at least two zeros, which will violate diagonal product condition.

∴ Matched & Reciprocal 3-port network cannot be lossless.

Lossless & Matched network. e.g. Ferrite circulator



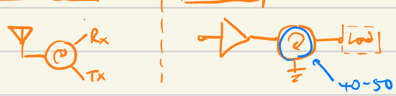
$$1\text{-}2\text{-}3 \Rightarrow \overset{S}{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}} \Rightarrow \overset{S^{\dagger}}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}\overset{S}{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(Pozar 9)

"non-reciprocal"

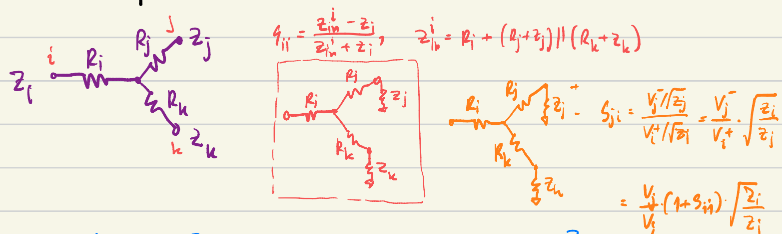
$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Full duplex | Isolator



Matched, lossy, reciprocal. (resistor divider)

$S_{ii} = \dfrac{Z_{in}^i - Z_i}{Z_{in}^i + Z_i}$, $Z_{in}^i = R_i + (R_j+Z_j)\,\|\,(R_k+Z_k)$



$S_{ji} = \dfrac{V_j^-/\sqrt{Z_j}}{V_i^+/\sqrt{Z_i}} = \dfrac{V_j^-}{V_i^+}\sqrt{\dfrac{Z_i}{Z_j}}$

$= \dfrac{V_j}{V_i}(1+S_{ii})\sqrt{\dfrac{Z_i}{Z_j}}$

$V_j = V_j^{+} + V_j^-$ (with $V_j^+ = 0$)

$V_i = V_i^+ + V_i^- = V_i^+(1+S_{ii})$

$\therefore V_i^+ = \dfrac{V_i}{1+S_{ii}}$

Assume $Z_i = Z_j = Z_k = Z_0$ and $R_i = R_j = R_k = \dfrac{Z_0}{3}$

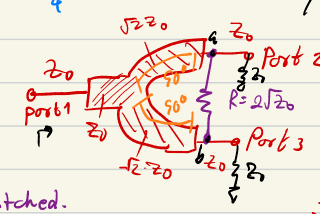
we get

$$S = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

⟶ lossy, equally split power.
↳ lost $\frac{1}{2}$ power & split into $\frac{1}{4}$

Wilkinson Power Divider (Pozar 7.3)

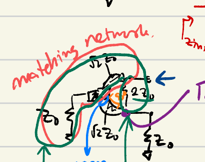
↳ Matched at all 3-ports,
↳ Good isolation between ports
↳ Lossy network but lossless when the output ports are matched.
↑ reflected power is dissipated.



For incident on port 1
a, b could have the same potential, suggesting no current flow through R.

→ If ports 2 & 3 are matched, terminated with Z_0:

$Z_{in} = Z_{in}^a \,\|\, Z_{in}^b = Z_0$ (matched)



This point is a virtual gnd (to be shown).

half power goes here
half power dissipated in $R = 2Z_0$

S-parameter of Wilkinson divider.

$$\begin{bmatrix} 0 & -j/\sqrt{2} & -j/\sqrt{2} \\ -j/\sqrt{2} & 0 & 0 \\ -j/\sqrt{2} & 0 & 0 \end{bmatrix} \Rightarrow -\frac{j}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

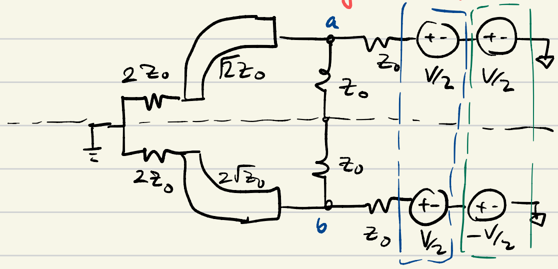
matched

isolation

Even-odd mode analysis (systematically analyze)

↳ Many couplers exhibit high level of symmetry

⇒ can be analyzed using odd-even mode analysis.

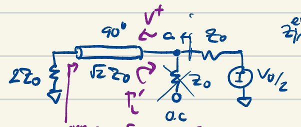


Excitation @ port 2 → even sources at ports 2 & 3
+odd sources —"—

} separately analyze them and then superposition.

① Even-mode analysis:

⇒ Points a and b are at the same potential → no current. → sym plane is "open"



$Z_{in,a}^{even} = \frac{(\sqrt{2}Z_0)^2}{2Z_0} = Z_0$ → "matched" → no scattering at ports 2 & 3 in even mode.

$\Gamma_L^{even} = \frac{2Z_0 - \sqrt{2}Z_0}{2Z_0+\sqrt{2}Z_0} = \frac{2-\sqrt{2}}{2+\sqrt{2}}$

⇒ $\boxed{V_2^{even} = V_3^{even} = \frac{V_0}{2}\cdot\frac{Z_0}{Z_0+Z_0} = \frac{V_0}{4}}$

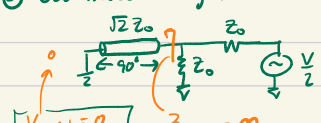
⇒ $V_1^{even} = V^+ e^{-j\theta}(1+\Gamma_L) =$

$V_2^{even} = V^+(1+\Gamma_L') = V^+(1+\Gamma_L e^{-j2\theta}) = V^+(1-\Gamma_L)$ $\Big\}$ $\frac{V_1^{even}}{V_2^{even}} = \frac{e^{-j\theta}(1+\Gamma_L)}{1-\Gamma_L} = (-j)\frac{1+\Gamma_L}{1-\Gamma_L} = -j\frac{4}{2\sqrt{2}}$ ($\theta = 90°$)

$\therefore \boxed{V_1^{even} = -j\frac{4}{2\sqrt{2}}\cdot\frac{V_0}{4} = -j\frac{\sqrt{2}}{4}V_0}$

Apr 10 :

② Odd-mode analysis



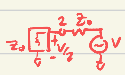
$\boxed{V_{1,odd} = 0}$ $Z_{in,2} = \infty$

Port 2 (& 3) are matched

⇒ no scattering wave for odd mode.

$\boxed{V_1^{odd} = 0, \quad V_2^{odd} = \frac{V}{2}\cdot\frac{Z_0}{Z_0+Z_0} = \frac{V}{4}, \quad V_3^{odd} = -V_2^{odd} = -\frac{V}{4}}$

Linear system : superposition



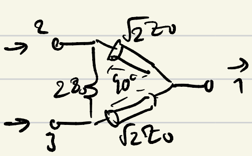
$V_2 = V_2^{even} + V_2^{odd} = V/4 + V/4 = V/2$ ⇝ Matched input ⇒ $S_{22} = 0$

$V_3 = V_3^{even} + V_3^{odd} = V/4 + (-\frac{V}{4}) = 0$ ⇝ $S_{32} = 0$

$V_1 = V_1^{even} + V_1^{odd} = -j\frac{\sqrt{2}}{4}V + 0 = -j\frac{\sqrt{2}}{4}V$ ⇝ $S_{12} = \frac{V_1^-}{V_2^+} = \frac{V_1}{V_2}(1+S_{22}) = -\frac{j}{\sqrt{2}}$

} ∴ symmetry: $\begin{bmatrix} 0 & -j/\sqrt{2} & -j/\sqrt{2} \\ -j/\sqrt{2} & 0 & 0 \\ -j/\sqrt{2} & 0 & 0 \end{bmatrix}$

We can use Wilkinson divider as a combiner.

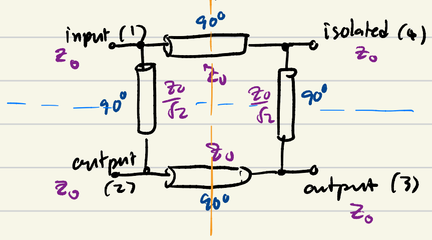


$b_1 = -j/\sqrt{2}\,(a_2+a_3)$

if $a_2 = a_3 = a$ ⇒ $b_1 = -j\sqrt{2}a$

$|b_1|^2 = 2|a_1|^2$

Branchline Coupler (Pozar 7.5)

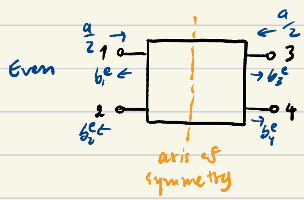


$$[S] = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & j & 1 & 0 \\ j & 0 & 0 & 1 \\ 1 & 0 & 0 & j \\ 0 & 1 & j & 0 \end{bmatrix}$$

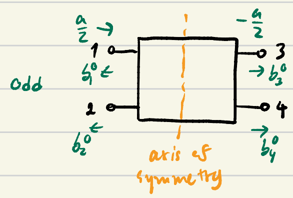
▸ Analysis of symmetric coupler using even-odd mode analysis

Consider excitation at port as superposition of even and odd mode excitation.

Even



Odd



Due to symmetry:

$b_3^e = b_1^e$, $b_2^e = b_4^e$

$b_3^o = -b_1^o$, $b_2^o = -b_4^o$

$$V_1^+ = \underbrace{\tfrac{a}{2}\sqrt{Z_0}}_{\text{even}} + \underbrace{\tfrac{a}{2}\sqrt{Z_0}}_{\text{odd}} = a\sqrt{Z_0}$$

$$V_3^+ = \tfrac{a}{2}\sqrt{Z_0} - \tfrac{a}{2}\sqrt{Z_0} = 0$$

$$V_1^- = (b_1^e + b_1^o)\sqrt{Z_0}$$

$$V_2^- = (b_2^e + b_2^o)\sqrt{Z_0}$$

$$V_3^- = (b_3^e + b_3^o)\sqrt{Z_0} = (b_1^e - b_1^o)\sqrt{Z_0}$$

$$V_4^- = (b_4^e + b_4^o)\sqrt{Z_0} = (b_2^e - b_2^o)\sqrt{Z_0}$$

$$S_{11} = \frac{V_1^-}{V_1^+} = \frac{(b_1^e + b_1^o)\sqrt{Z_0}}{a\sqrt{Z_0}} = \frac{b_1^e + b_1^o}{a} = \frac{1}{2}\left\{\frac{b_1^e}{a/2} + \frac{b_1^o}{a/2}\right\} = \frac{1}{2}(S_{11}^e + S_{11}^o)$$

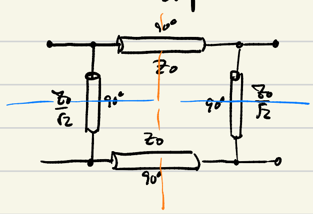
$$\therefore S_{11} = \frac{1}{2}(S_{11}^e + S_{11}^o)$$

$$S_{21} = \frac{V_2^-}{V_1^+} = \frac{(b_2^e + b_2^o)\sqrt{Z_0}}{a\sqrt{Z_0}} = \frac{1}{2}(S_{21}^e + S_{21}^o) \Rightarrow S_{21} = \frac{1}{2}(S_{21}^e + S_{21}^o)$$

$$S_{31} = \frac{V_3^-}{V_1^+} = \frac{(b_1^e - b_1^o)\sqrt{Z_0}}{a\sqrt{Z_0}} = \frac{1}{2}(S_{11}^e - S_{11}^o) \Rightarrow S_{31} = \frac{1}{2}(S_{11}^e - S_{11}^o)$$

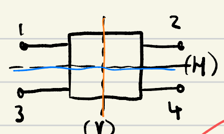
$$S_{41} = \frac{V_4^-}{V_1^+} = \frac{(b_2^e - b_2^o)\sqrt{Z_0}}{a\sqrt{Z_0}} \Rightarrow S_{41} = \frac{1}{2}(S_{21}^e - S_{21}^o)$$

Branchline coupler analysis

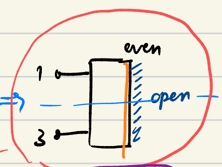


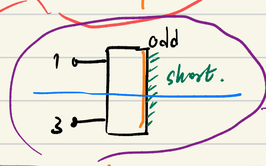
Branchline coupler has two-fold symmetry:

⇒ Break the circuit twice using even-odd analysis.



⇒



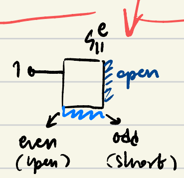


$$S_{11} = \frac{1}{2}(S_{11}^e + S_{11}^o)$$

$$S_{21} = \frac{1}{2}(S_{11}^e - S_{11}^o)$$

$$S_{31} = \frac{1}{2}(S_{31}^e + S_{31}^o)$$

$$S_{41} = \frac{1}{2}(S_{31}^e - S_{31}^o)$$



vertical — horizontal

$$S_{11}^e = \frac{1}{2}(S_{11}^{ee} + S_{11}^{eo})$$

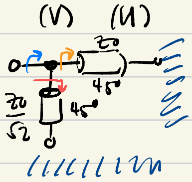
$$S_{31}^e = \frac{1}{2}(S_{11}^{ee} - S_{11}^{eo})$$

$$S_{11}^o = \frac{1}{2}(S_{11}^{oe} + S_{11}^{oo})$$

$$S_{31}^o = \frac{1}{2}(S_{11}^{oe} - S_{11}^{oo})$$

1) Even-even mode

(V) (H)



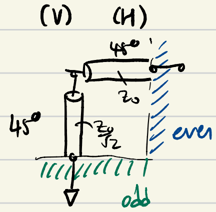
$$Z_{in}^1 = \frac{Z_0}{j\tan\beta l} = \frac{Z_0}{j}, \quad Z_{in}^2 = \frac{Z_0/\sqrt{2}}{j} \Rightarrow Z_{in}^{ee} = -jZ_0 \,\|\, -j\frac{Z_0}{\sqrt{2}} = -jZ_0\left(1 \,\|\, \frac{1}{\sqrt{2}}\right) = -jZ_0\frac{1}{1+\sqrt{2}}$$

$$S_{11}^{ee} = \frac{Z_{in}^{ee} - Z_0}{Z_{in}^{ee} + Z_0} = \frac{-j\frac{1}{\sqrt{2}+1} - 1}{-j\frac{1}{\sqrt{2}+1} + 1} = \frac{\sqrt{2}}{2}(1-j)$$

⋮

2) Even-Odd mode

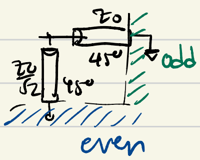
(V) (H)



$$Z_{in}^{eo} = (-jZ_0)\,\|\,(j\tfrac{Z_0}{\sqrt2})$$
$$= jZ_0\,(-1)\,\|\,\tfrac{1}{\sqrt2} = jZ_0\frac{1}{-1+\sqrt2}$$

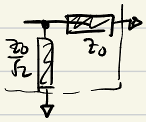
$$S_{11}^{eo} = \frac{Z_{in}^{eo}-Z_0}{Z_{in}^{eo}+Z_0} = \frac{jZ_0-(\sqrt2-1)Z_0}{jZ_0+(\sqrt2+1)Z_0} = \frac{\sqrt2}{2}(j+1) = S_{11}^{ee*}$$

3) Odd-even mode

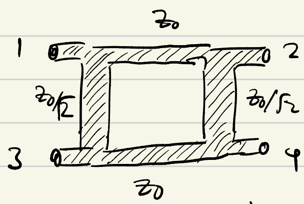


$$Z_{in}^{oe} = (jZ_0)\,\|\,(-j\tfrac{Z_0}{\sqrt2})$$
$$= jZ_0\,(1\,\|\,-\tfrac{1}{\sqrt2}) = jZ_0\frac{1}{1-\sqrt2}$$
$$\therefore S_{11}^{oe} = \frac{\sqrt2}{2}(1-j) = S_{11}^{ee}$$

(4) Odd-odd mode



$$Z_{in}^{oo} = jZ_0\,\|\,j\tfrac{Z_0}{\sqrt2} = jZ_0\frac{1}{1+\sqrt2}$$
$$\therefore S_{11}^{oo} = \frac{\sqrt2}{2}(-1+j) = -S_{11}^{ee}$$



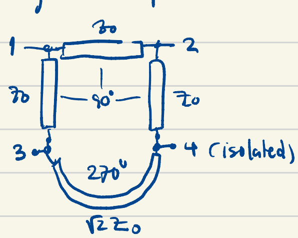
$$S_{11}^{e} = \frac12\left(S_{11}^{ee}+S_{11}^{eo}\right) =$$
$$S_{31}^{e} = \frac12\left(S_{11}^{ee}-S_{11}^{eo}\right) =$$
$$S_{11}^{o} = \frac12\left(S_{11}^{oe}+S_{11}^{oo}\right) =$$
$$S_{31}^{o} = \frac12\left(S_{11}^{oe}-S_{11}^{oo}\right) =$$

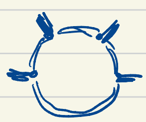
$$S_{11} = \frac12\left(S_{11}^{e}+S_{11}^{o}\right) = 0$$
$$S_{21} = \frac12\left(S_{11}^{e}-S_{11}^{o}\right) = 0$$
$$S_{31} = \frac12\left(S_{31}^{e}+S_{31}^{o}\right) = -\frac{j}{\sqrt2}$$
$$S_{41} = \frac12\left(S_{31}^{e}-S_{31}^{o}\right) = -\frac{1}{\sqrt2}$$

$$S = -\frac{1}{\sqrt2}\begin{pmatrix} 0 & 0 & j & 1 \\ 0 & 0 & 1 & j \\ j & 1 & 0 & 0 \\ 1 & j & 0 & 0 \end{pmatrix}$$

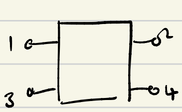
90-deg hybrid coupler

180° hybrid coupler → rat-race coupler (Pozar 7.8)



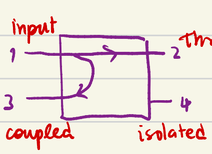


Coupler applications



$$S = -\frac{1}{\sqrt2}\begin{pmatrix} 0 & 0 & j & 1 \\ 0 & 0 & 1 & j \\ j & 1 & 0 & 0 \\ 1 & j & 0 & 0 \end{pmatrix}$$

Symbols and terminology for general 4-port coupler



Return loss (RL) : $RL = -20\log|S_{11}|$

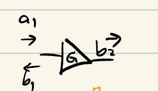
Insertion loss (IL) : $IL = -20\log|S_{21}|$

Coupling (C) : $C = -20\log|S_{31}|$

Directivity (D) : $D = 10\log(P_3/P_4)$

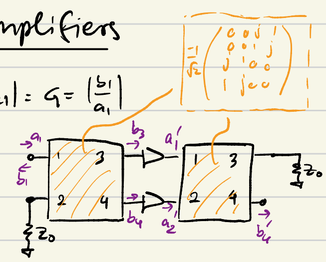
$= -20\log|S_{41}/S_{31}|$

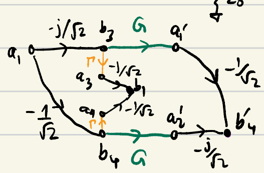
Couplers & Amplifiers



$|S_{21}| = G = \left|\frac{b_2}{a_1}\right|$

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$





- Reflection :

$$\frac{b_1}{a_1} = \left(\frac{-j}{\sqrt2}\right)\Gamma\left(\frac{-j}{\sqrt2}\right) + \left(-\frac{1}{\sqrt2}\right)\Gamma\left(-\frac{1}{\sqrt2}\right) = 0$$

- Transmission :

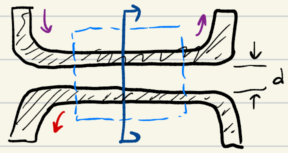
$$\frac{b_4'}{a_1} = \left(\frac{-j}{\sqrt2}\right)G\left(\frac{-1}{\sqrt2}\right) + \left(\frac{-j}{\sqrt2}\right)G\left(\frac{-1}{\sqrt2}\right) = jS_{21}$$

Get ① same power gain as a single amplifier

② provide match over the BW of hybrid coupler

③ enlarge power handling (split input powers into two amplifiers)

Coupled transmission line coupler (Pozar 7.6)



Constructed from two TLs

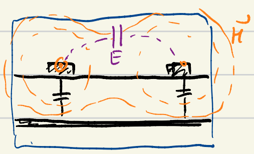
with enough coupling E/H fields

$\Rightarrow$ tapped line?

Advantages 1) Broadbrand

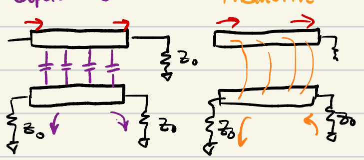
2) Wide range of coupling (tuning the distance

capacitive vs. inductive couplings

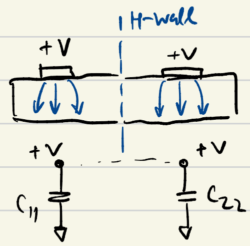


capacitive

Inductive

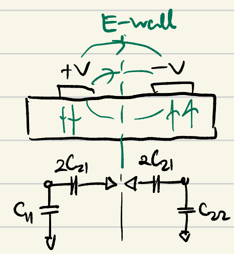


Even-Odd mode analysis



$C^e = C_{11} = C_{22}$

$$Z_0^e = \sqrt{\frac{L^e}{C^e}} = \frac{\sqrt{L^e C^e}}{C^e} = \frac{1}{v_p C^e}$$

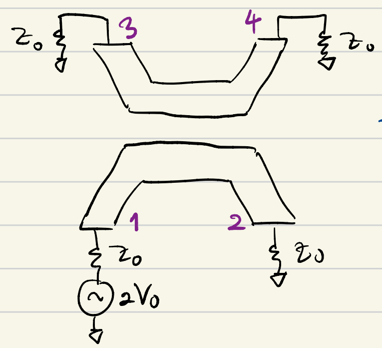


$\therefore \; C^o = C_{11} + 2C_{21}$

$$Z_0^o = \sqrt{\frac{L^o}{C^o}} = \frac{\sqrt{L^o C^o}}{C^o} = \frac{1}{v_p C^o}$$

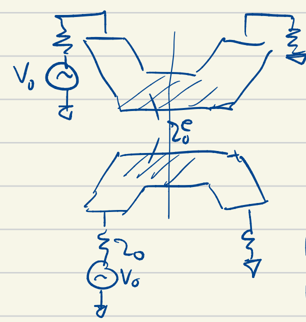
$\boxed{\text{TEM waveguide} \rightarrow \text{same } v_p = \frac{1}{\sqrt{L^e C^e}} = \frac{1}{\sqrt{L^o C^o}}}$

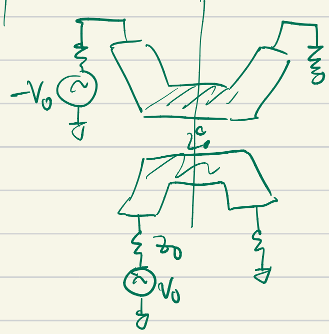
Design of coupled line coupler



$\Rightarrow$

Even mode





$$Z_{in}^e = Z_0^e \frac{Z_0 + jZ_0^e \tan\beta l}{Z_0^e + jZ_0 \tan\beta l}$$

$$\left\{ \begin{aligned} V_1^e &= V_0 \frac{Z_{in}^e}{Z_{in}^e + Z_0} \\ I_1^e &= V_0 \frac{1}{Z_{in}^e + Z_0} \end{aligned} \right.$$

$$\left. \begin{aligned} V_1^o &= V_0 \frac{Z_{in}^o}{Z_{in}^o + Z_0} \\ I_1^o &= V_0 \frac{1}{Z_{in}^o + Z_0} \end{aligned} \right\} Z_{in}^o = Z_0^o \frac{Z_0 + jZ_0^o \tan\beta l}{Z_0^o + jZ_0 \tan\beta l}$$

$$Z_{in} = \frac{V_1}{I_1} = \frac{V_1^e + V_1^o}{I_1^e + I_1^o} = \frac{V_0 \frac{Z_{in}^e}{Z_{in}^e + Z_0} + V_0 \frac{Z_{in}^o}{Z_{in}^o + Z_0}}{V_0 \frac{1}{Z_{in}^e + Z_0} + V_0 \frac{1}{Z_{in}^o + Z_0}} = Z_0 + \frac{2\left(Z_{in}^e Z_{in}^o - Z_0^2\right)}{Z_{in}^e Z_{in}^o + 2Z_0}$$

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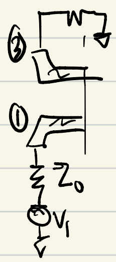
Matching condition $\Rightarrow Z_{in} = Z_0$

$\therefore \; \boxed{Z_0^2 = Z_{in}^e Z_{in}^o}$! $\rightarrow$ matching condition. (1)

Let $Z_0 = \sqrt{Z_0^e Z_0^o}$,

$$\left. \begin{aligned} Z_{in}^e &= Z_0^e \frac{\sqrt{Z_0^o} + j\sqrt{Z_0^e} \tan\beta l}{\sqrt{Z_0^e} + j\sqrt{Z_0^o} \tan\beta l} \\ Z_{in}^o &= Z_0^o \frac{\sqrt{Z_0^e} + j\sqrt{Z_0^o} \tan\beta l}{\sqrt{Z_0^o} + j\sqrt{Z_0^e} \tan\beta l} \end{aligned} \right\} \boxed{Z_{in}^e Z_{in}^o = Z_0^e Z_0^o = Z_0^2}$$



Want to determine the coupling:

$$V_3 = V_3^o + V_3^e = V_1^e - V_1^o = V_0 \frac{Z_{in}^e}{Z_{in}^e + Z_0} - V_0 \frac{Z_{in}^o}{Z_{in}^o + Z_0}$$

$$= V_0 \frac{j(Z_0^e - Z_0^o)\cdot \tan\theta}{2Z_0 + j(Z_0^e + Z_0^o)\tan\theta} = V_0 \frac{j(Z_0^e - Z_0^o)}{\underbrace{\frac{2Z_0}{\tan\theta}}_{0 \text{ if } \theta = 90°} + j(Z_0^e + Z_0^o)}$$

$$\boxed{V_3 = V_0 \frac{Z_0^e - Z_0^o}{Z_0^e + Z_0^e}} = cV_0$$

Define as C: coupler factor

Energy conservation.

Super long calculation:

$$V_4 = V_4^e + V_4^o = V_2^e - V_2^o = 0$$

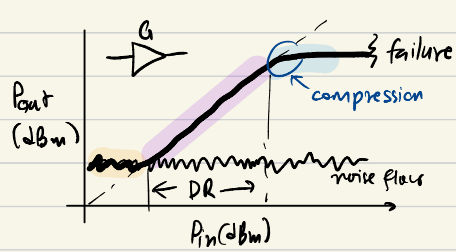
$$V_2 = V_2^e + V_2^o = V_0 \frac{\sqrt{1-C^2}}{\sqrt{1-C^2}\cdot \underbrace{\cos\theta}_{0 \text{ if } \theta = \frac{\pi}{2}} + j\sin\theta} = \boxed{-jV_0\sqrt{1-C^2}}$$

So far, we get

$$\star \quad \boxed{Z_0^e = Z_0\sqrt{\frac{1+C}{1-C}}} \quad \boxed{Z_0^o = Z_0\sqrt{\frac{1-C}{1+C}}} \quad \star \quad (2)$$

Noise & Nonlinearity in Microwave System

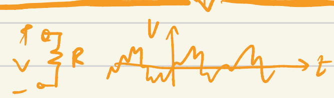
- Dynamic Range & Sources of noise



Sources of noises

- Thermal noise : Johnson's --→ Brownian motion
- Shot noise
- Generation-Recombination → Quantum noise
- Flicker (1/f)

Thermal noise



Rayleigh-Jeans approx: $V_n = \sqrt{4kTBR}$

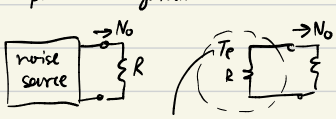
Resistance (Ω)

Bandwidth (Hz)

Boltzmann 1.38×10^{-23} J/K

Temp (K)

Noise power & Equivalent noise temp.



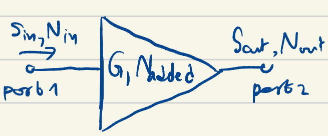
equivalent noise temp, T_e of an arbitrary white noise source

$$\boxed{T_e = \frac{N_0}{GkB}}$$

How to measure T_e?

⇒ Y-method (Pozar 10)

★ Noise figure : F



$$\boxed{F = \frac{SNR_{out}}{SNR_{in}} = \frac{S_{in}N_{out}}{S_{out}N_{in}}}$$

$S_{out} = G\, S_{in}$ amplified noise

$N_{out} = G N_{in} + N_{added}$

$$\boxed{F = 1 + \frac{N_{added}}{G N_{in}}}$$

F always > 1

System

need a definition, i.e., R @ 290 K

$$\boxed{F = 1 + \frac{N_{added}}{G\cdot 4kTBR}}$$