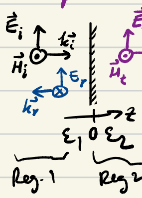
Reflection & Transmission

Consider plane wave perpendicular incidence onto plane boundaries



In absence of surface current: tangential fields are continuous at boundary.

$$\hat{n}\times(\vec{E}_+-\vec{E}_-)=0$$
$$\hat{n}\times(\vec{H}_+-\vec{H}_-)=\vec{J}_s=0$$

Region I

$$E_{tot}=E_i+E_r=E_0e^{-jk_1z}+E_re^{+jk_1z}$$
$$=E_0e^{-jk_1z}\left(1+\boxed{\frac{E_r}{E_0}}e^{j2k_1z}\right)$$

$\Gamma_0=\frac{E_r}{E_0}$ (reflection coefficient)

$$H_{tot}=H_i+H_r=H_0e^{-jk_1z}+H_re^{jk_1z}$$
$$=\frac{E_0}{\eta_1}e^{-jk_1z}-\frac{E_0}{\eta_2}e^{jk_1z}$$
$$=\frac{E_0}{\eta_1}e^{-jkz}(1-\Gamma_0e^{j2k_1z})$$

Due to continuity of tangential fields

$$E^1_{tot}\big|_{z=0}=E^2_{tot}\big|_{z=0}$$
$$H^1_{tot}\big|_{z=0}=H^2_{tot}\big|_{z=0}$$

$$\boxed{\frac{E^1_{tot}}{H^1_{tot}}\bigg|_{z=0}=\frac{E^2_{tot}}{H^2_{tot}}\bigg|_{z=0}}$$

"wave impedance" $\neq$ "intrinsic impedance"

$$\frac{E_0e^{-jkz}(1+\Gamma_0e^{j2kz})}{\frac{E_0}{\eta_1}e^{-jkz}(1-\Gamma_0e^{j2kz})}\Bigg|_{z=0}=\eta_2\Big|_{z=0}$$

$$\eta_1\cdot\frac{1+\Gamma_0}{1-\Gamma_0}=\eta_2\Rightarrow\boxed{\Gamma_0=\frac{\eta_2-\eta_1}{\eta_2+\eta_1}}$$ reflection coefficient

transmission:

$$\boxed{T_0=\frac{E^2_+}{E^1_0}\bigg|_{z=0}=\frac{E_0+E_r}{E_0}=1+\Gamma_0=\frac{2\eta_2}{\eta_2+\eta_1}}$$

Special case: perfect conductor: no E-field

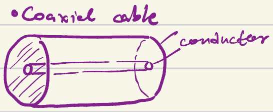
$T_0=0$, $\Gamma=-1$

shorting out generated sheet current $\vec{J}_s=\hat{n}\times\vec{H}=\hat{x}\frac{2}{\eta_0}E_0$

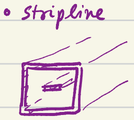
1D Transmission line

Example of lumped element

• Coaxial cable



• stripline



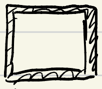
Idealization

(1) Structure is uniform and "small" in two direction (compared to wavelength)

but extending in other direction

⇒ lumped approx. good for two directions
not third (propagation dir.)

(2) The structure has two distinct conductors



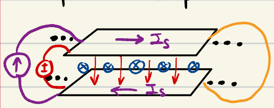
↳ one conductor

(note that not considering rectangular / circular waveguide yet)

↳ Pozar ch.3

Jan 27, 2025

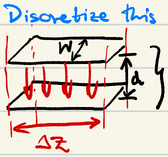
A parallel plate transmission

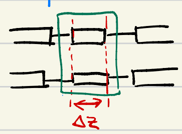


This is true for low freq

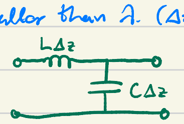
Assume $\vec{E}$ & $\vec{H}$ has no $\hat{z}$ components, i.e. $E_z=0$, $H_z=0$. ⇒ TEM mode

→ Discretize this into pieces with a size way smaller than λ. ($\Delta z\ll\lambda$)





Equivalent circuit:

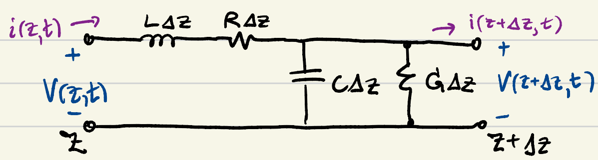


(lossless model)

$L=\frac{\mu d}{W}$

$C=\varepsilon\frac{W}{d}$

If resistive loss (R) and dielectric loss are considered:



R: Ω/m
L: H/m
G: S/m
C: F/m

KVL:

$$V(z,t)-L\Delta z\frac{d}{dt}i(z,t)-R\Delta z\, i(z,t)-V(z+\Delta z,t)=0$$

$$\frac{V(z,t)-V(z+\Delta z,t)}{\Delta z}=L\frac{d}{dt}i(z,t)+Ri(z,t)$$

$\Delta z\to0$

$$\boxed{-\frac{\partial}{\partial z}V(z,t)=L\frac{d}{dt}i(z,t)+Ri(z,t)}\quad(1)$$

KCL:

$$i(z,t)-i(z+\Delta z,t)-C\Delta z\frac{d}{dt}V(z+\Delta z,t)-G\Delta z\,V(z+\Delta z,t)=0$$

$$\frac{i(z,t)-i(z+\Delta z,t)}{\Delta z}=C\frac{d}{dt}V(z+\Delta z,t)+GV(z+\Delta z,t)$$

$\Delta z\to0$:

$$\boxed{-\frac{\partial}{\partial z}i(z,t)=C\frac{d}{dt}V(z,t)+GV(z,t)}\quad(2)$$

(1), (2) ⇒ Telegrapher's equation:

For sinusoid time-dependent $V(z,t) = Re[\underline{V(z)}e^{j\omega t}]$

$I(z,t) = Re[\underline{I(z)}e^{j\omega t}]$

(1) ⇒ $-\frac{d}{dz}\underline{V(z)} = j\omega L I(z) + R I(z)$

$-\frac{d}{dz}\underline{V(z)} = (j\omega L + R)\underline{I(z)} \Rightarrow \underline{I(z)} = -\frac{1}{(j\omega L+R)}\frac{d}{dz}\underline{V(z)}$

(2) ⇒ $-\frac{d}{dz}\underline{I(z)} = (j\omega C + G)\underline{V(z)}$

$\left.\right\} \frac{1}{(j\omega L+R)}\cdot\frac{d^2}{dz^2}\underline{V(z)} = (j\omega C + G)\underline{V(z)}$

$\Downarrow$

$$\boxed{\begin{array}{c}\frac{d^2}{dz^2}V(z) - \gamma^2\underline{V}(z) = 0 \\ \hline \frac{d^2}{dz^2}\underline{I(z)} - \gamma^2\underline{I}(z) = 0\end{array}}$$

and — Helmholtz eqs.

where $\gamma = \alpha + j\beta = \sqrt{(R+j\omega L)(G+j\omega C)}$

<complex propagation constant>

Solve wave eq.

$V(z) = V_0^+e^{-\gamma z} + V_0^-e^{\gamma z}$

$I(z) = I_0^+e^{-\gamma z} + I_0^-e^{\gamma z}$

⇝ $I(z) = -\frac{1}{R+j\omega L}\frac{dV(z)}{dz} = -\frac{1}{R+j\omega L}(-\gamma V_0^+e^{-\gamma z} + \gamma V_0^-e^{\gamma z})$

$= \frac{1}{R+j\omega L}\{\gamma V_0^+e^{-\gamma z} - \gamma V_0^-e^{\gamma z}\}$

$I(z) = \underbrace{\sqrt{\frac{R+j\omega C}{G+j\omega L}}}_{1/Z_0}(V_0^+e^{-\gamma z} - V_0^-e^{+\gamma z})$

↪ Characteristic impedance : $\boxed{Z_0 = \pm\sqrt{\frac{G+j\omega L}{R+j\omega C}}} = \frac{R+j\omega L}{\gamma}$

depends on propagation direction

$\lambda = \frac{2\pi}{\beta}$

$v_p = \frac{\omega}{\beta}$

→ Lossless line : $R=0,\ G=0 \ \therefore \gamma = \alpha + j\beta = j\omega\sqrt{LC}$

$\Rightarrow \alpha = 0\ \&\ \beta = \omega\sqrt{LC},\ Z_0 = \sqrt{\frac{L}{C}}$

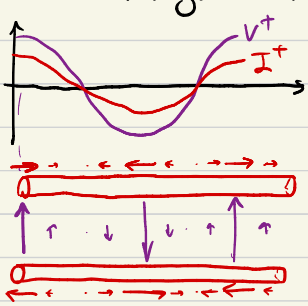
$$\boxed{\begin{array}{l}V(z) = V_0^+e^{-j\beta z} + V_0^-e^{+j\beta z} \\ I(z) = \frac{V_0^+}{Z_0}e^{-j\beta z} - \frac{V_0^-}{Z_0}e^{j\beta z}\end{array}}$$

$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{LC}}$

$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$

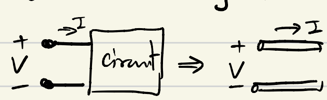
Comments on transmission line (T-line) propagation

(1) Current and voltage are traveling waves



$\frac{\underline{V}}{\underline{I}} = Z_0$ (purely real)

$\underline{V}$ and $\underline{I}$ on T-line follow same sign convention as circuit theory ⇒ current is "positive", flows into "+" terminal

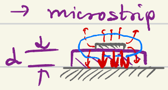


(2) Lossless line : Real Z_0 ⇒ only means $\underline{V}$ and $\underline{I}$ in phase.

No power dissipation

Field Analysis of transmission ⇒ Pozar Ch.3

→ microstrip (Pozar ch.3)



— E

— H

The substrate is electrically thin ($d \ll \lambda$),

so the fields are quasi-TEM

- propagation constant $\beta = k_0\sqrt{\varepsilon_e}$
- phase velocity : $v_p = \frac{c}{\sqrt{\varepsilon_e}}$

effective dielectric constant

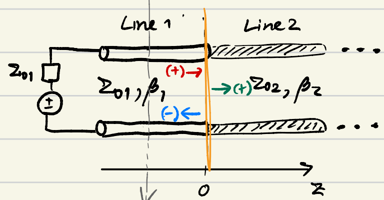
Jan 29

Reflection

In general, along T-line we have both $+z$, $-z$-waves: $+z$ incident wave, $-z$ reflected wave

traveling waves superposition → standing wave

Transmission line discontinuity



Line 2: infinitely long (no reflection from the other end.)

Apply boundary conditions

① $V_1|_{z=0} = V_2|_{z=0} \Rightarrow V_1^+ + V_1^- = V_2^+ \Big|_{z=0}$

② $I_1|_{z=0} = I_2|_{z=0} \Rightarrow \dfrac{V_1^+}{Z_{01}} - \dfrac{V_1^-}{Z_{01}} = \dfrac{V_2^+}{Z_{02}}\Big|_{z=0}$

$V_1^- = V_1^+ \dfrac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} \Rightarrow \boxed{\Gamma = \dfrac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}}}$ voltage reflection coef.

$V_2^+ = V_1^+ \dfrac{2Z_{02}}{Z_{02} + Z_{01}} \Rightarrow \boxed{\tau = \dfrac{2Z_{02}}{Z_{02} + Z_{01}}}$ voltage trans. coeff.

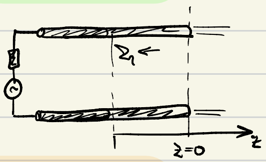
$\tau = 1 + \Gamma$

voltage: $\boxed{\Gamma_V(z) = \dfrac{V_1^-(0)\, e^{j\beta_1 z}}{V_1^+(0)\, e^{-j\beta_1 z}} = \Gamma_V(0)\, e^{+j2\beta_1 z}}$

current: $\Gamma_I(z) = \dfrac{I_1^-}{I_1^+} = \dfrac{-\frac{V_0^-}{Z_{01}} e^{j\beta_1 z}}{+\frac{V_0^+}{Z_{01}} e^{-j\beta_1 z}} = -\Gamma_V(0)\, e^{+j2\beta_1 z}$

Wave impedance

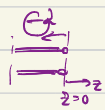
$Z(z) = \dfrac{V(z)}{I(z)} = \dfrac{V_1^+ + V_1^-}{I_1^+ + I_1^-} = \dfrac{V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}}{\frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{j\beta z}} = Z_0 \dfrac{1 + \frac{V_0^-}{V_0^+} e^{j2\beta z}}{1 - \frac{V_0^-}{V_0^+} e^{j2\beta z}}$



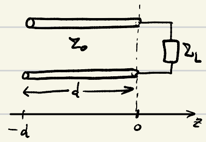
$Z(z) = Z_0 \dfrac{1 + \Gamma_V(z)}{1 - \Gamma_V(z)}$

$\boxed{\Gamma_V(z) = \dfrac{Z(z) - Z_0}{Z(z) + Z_0}}$

$\Gamma = \dfrac{V^-}{V^+} = \Gamma_0 e^{+j2\beta z}$



Loaded line



At $z=0$, $Z(0) = Z_L$

$\Gamma_0 = \dfrac{Z_L - Z_0}{Z_L + Z_0}$

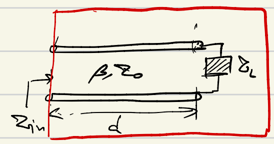
$Z(0) = Z_L = Z_0 \dfrac{1 + \Gamma_0}{1 - \Gamma_0}$

$Z(-d) = Z_0 \dfrac{1 + \Gamma_0(-d)}{1 - \Gamma_0(-d)} = Z_0 \dfrac{1 + \Gamma_0 e^{+j2\beta d}}{1 - \Gamma_0 e^{+j2\beta d}}$

$Z(-d) = Z_0 \dfrac{(Z_L + Z_0) e^{j\beta d} + (Z_L - Z_0) e^{-j\beta d}}{(Z_L + Z_0) e^{j\beta d} - (Z_L - Z_0) e^{-j\beta d}}$

$= Z_0 \dfrac{Z_L \cos\beta d + jZ_0 \sin\beta d}{Z_0 \cos\beta d + jZ_L \sin\beta d}$

$\Rightarrow \boxed{Z(-d) = Z_0 \dfrac{Z_L + jZ_0 \tan\beta d}{Z_0 + jZ_L \tan\beta d}}$



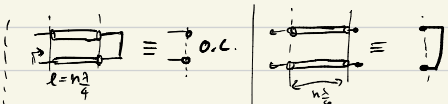
Examples

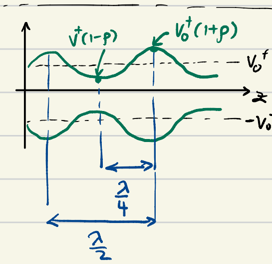
① Matched line: $Z_L = Z_0$: $\Gamma_0 = \dfrac{Z_L - Z_0}{Z_L + Z_0} = 0$; $Z(d) = Z_0$

② Shorted line: $Z_L = 0$: $\Gamma_0 = -1$; $Z(d) = Z_0\, j\tan\beta d$

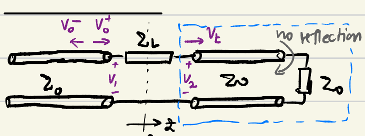
③ Open line: $Z_L = \infty$: $\Gamma_0 = 1$; $Z(d) = \dfrac{Z_0}{j\tan\beta d} = Z_0(-j\cot\beta d)$

④ $\boxed{d = \dfrac{\lambda}{4}} \rightarrow \beta d = \left(\dfrac{2\pi}{\lambda}\right)\left(\dfrac{\lambda}{4}\right) = \dfrac{\pi}{2} \rightarrow Z(d) = \dfrac{Z_0^2}{Z_L}$ Quarter-wave transformer.

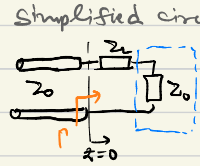




$\rho = |\Gamma|$



Simplified circuit.



$\Gamma = \dfrac{(Z_L + Z_0) - Z_0}{(Z_L + Z_0) + Z_0} = \dfrac{Z_L}{Z_L + 2Z_0}$

$V_1 = V_0^+ + V_0^- = V_0^+(1 + \Gamma)$

$V_2 = V_t$ (no reflection)

$I_1 = I_0^+ + I_0^- = \dfrac{V_0^+}{Z_0} - \dfrac{V_0^-}{Z_0} = \dfrac{V_0^+}{Z_0}(1 - \Gamma)$

$I_2 = \dfrac{V_t}{Z_0}$

KVL: $V_1 - V_2 = V_L \rightarrow V_0^+(1+\Gamma) - V_t = V_L$

KCL: $I_1 = I_2 = I_L \rightarrow \dfrac{V_0^+}{Z_0}(1 - \Gamma) = \dfrac{V_t}{Z_0} = \dfrac{V_L}{Z_L}$

$V_0^+(1+\Gamma) - \dfrac{Z_0}{Z_L} V_L = V_L \Rightarrow V_0^+ = \dfrac{\left(1 - \frac{Z_0}{Z_L}\right)}{1 + \Gamma} V_L$

$\therefore \dfrac{1+\Gamma}{1-\Gamma} = 1 + \dfrac{Z_L}{Z_0} = 1 + z_L$ ← normalized impedance

$\rightarrow \Gamma = \dfrac{z_L}{z_L + 2} = \dfrac{Z_L}{Z_L + 2Z_0}$

- Reflection coefficient: Γ
- Return loss: $RL = -20\log_{10}|\Gamma|$
- Insertion loss: $IL = -20\log_{10}|\tau|$, $\tau = 1 + \Gamma$

0.1 ⇒ 20 dB

0.01 ⇒ 40 dB

0.001 ⇒ 60 dB