Probability Axioms

Assignment 3 - Q49, Dec 2018

Tarandeep Singh

IIT Hyderabad

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Problem Statement

Let $X \geq 0$ be a random variable on $(\Omega, \mathcal{F}, P)$ with $\mathbb{E}(X) = 1$. Let $A \in \mathcal{F}$ be an event with $0 < P(A) < 1$. Which of the following defines another probability measure on $(\Omega, \mathcal{F})$?

1. $Q(B) = P(A \cap B) \ \forall B \in \mathcal{F}$
2. $Q(B) = P(A \cup B) \ \forall B \in \mathcal{F}$
3. $Q(B) = \mathbb{E}(XI_B) \ \forall B \in \mathcal{F}$
4. $Q(B) = \begin{cases} P(A/B) & P(B) > 0 \\ 0 & P(B) = 0 \end{cases}$

Properties of Probability Measure

A Probability measure on $(\Omega, \mathcal{F})$ is a function $P : \mathcal{F} \to [0,1]$ satisfying:

1. $P(\Omega) = 1, P(\phi) = 0$
2. If $A_1, A_2, A_3 ... \in \mathcal{F}$ is a collection of disjoint members in $\mathcal{F}$, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Solution

Option 1, if $B = \Omega$, then

$$Q(\Omega) = P(A \cap \Omega) = P(A) \neq 1 \tag{1}$$

Option 2, if $B = \phi$, then

$$Q(\phi) = P(A \cup \phi) = P(A) \neq 0 \tag{2}$$

Option 4, if $B = \Omega$, then

$$Q(\Omega) = P(A/\Omega) = \frac{P(A \cap \Omega)}{P(\Omega)} = P(A) \neq 1 \tag{3}$$

All these options doesn't satisfy the properties of Probability measure. Hence, option 3 is right option.

Solution

Option 3, The indicator function I_B defines a Bernoulli random variable. Then $\forall x \in \Omega$

$$I_B(x) = \begin{cases} 1 & \text{if } x \in B, \\ 0 & \text{if } x \notin B, \end{cases} \tag{4}$$

if $B = \phi$,

$$Q(\phi) = \mathbb{E}(XI_\phi) = \sum_{x\in\Omega}\sum_{x\in\Omega} x.I_B(x)P((X = x).(I_B(x))) \tag{5}$$

Here, $\forall x \in \Omega$, $I_B(x) = 0$, as $B = \phi$.
if $B = \Omega$,

$$Q(\Omega) = \mathbb{E}(XI_\Omega) = \sum_{x\in\Omega}\sum_{x\in\Omega} x.I_B(x)P((X = x).(I_B(x))) \tag{6}$$

$$= \mathbb{E}(X) = 1 \tag{7}$$

Here, $\forall x \in \Omega$, $I_B(x) = 1$, as $B = \Omega$.

Solution

Checking for Countable additivity of Q(property 2),

$$Q\left(\bigcup_{i=1}^{\infty} A_i\right) = \mathbb{E}(XI_{\bigcup_{i=1}^{\infty} A_i}) \tag{8}$$

Using Property of Countable additivity on indicator functions expectation can be re-written as below,

$$= \mathbb{E}\left(X\sum_{i=1}^{\infty} I_{A_i}\right) = \mathbb{E}\left(\sum_{i=1}^{\infty} XI_{A_i}\right) = \sum_{i=1}^{\infty} \mathbb{E}\left(XI_{A_i}\right) \tag{9}$$

$$\sum_{i=1}^{\infty} Q(A_i) \tag{10}$$