

Assignment 2

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Github Link

<https://github.com/Tarandeep97/AI5030>

1 PROBLEM

(52) Suppose $X \sim \text{Cauchy}(0,1)$. Then the distribution of $\frac{1-X}{1+X}$ is?

2 SOLUTION

A continuous random variable X follows **Cauchy distribution** with parameters μ and λ if its pdf is given by

$$f(x) = \begin{cases} \frac{\lambda}{\pi} \cdot \frac{1}{\lambda^2 + (x-\mu)^2}, & -\infty < x < \infty; \\ 0, & -\infty < \mu < \infty, \lambda > 0; \\ \text{Otherwise.} \end{cases}$$

The parameter μ and λ are location and scale parameters respectively.

When $\mu=0$ and $\lambda=1$, then the distribution is called **Standard Cauchy Distribution**. The pdf of standard Cauchy distribution is

$$f(x) = \begin{cases} \frac{1}{\pi} \cdot \frac{1}{1+x^2}, & -\infty < x < \infty; \\ 0, & \text{Otherwise.} \end{cases}$$

Let,

$$Y = \frac{1-X}{1+X} \quad (2.0.1)$$

Then, cdf of Y is

$$F_Y(y) = P(Y \leq y) \quad (2.0.2)$$

$$= P\left(\frac{1-X}{1+X} \leq y\right) \quad (2.0.3)$$

$$= P((1-X) \leq y(1+X)) \quad (2.0.4)$$

$$= P((1-X) \leq (y+y.X)) \quad (2.0.5)$$

$$= P((1-y) \leq (X+y.X)) \quad (2.0.6)$$

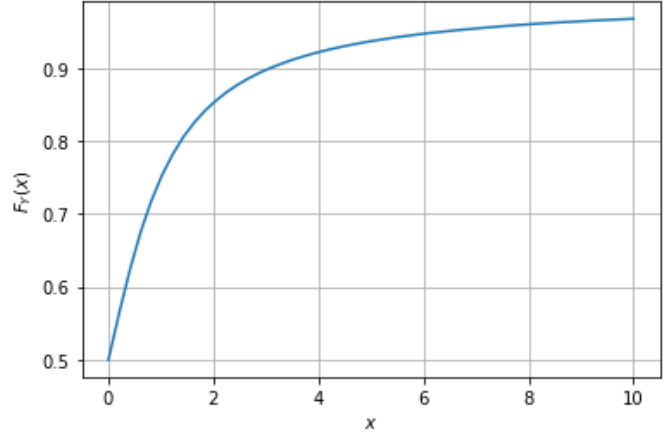


Fig 1. CDF of Y

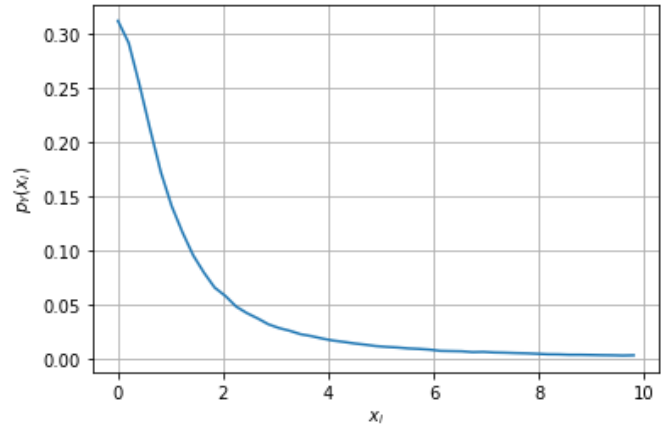


Fig 2. PDF of Y

$$= P\left(\frac{1-y}{1+y} \leq X\right) \quad (2.0.7)$$

$$= 1 - P\left(X < \frac{1-y}{1+y}\right) \quad (2.0.8)$$

$$= 1 - \int_{-\infty}^{\frac{1-y}{1+y}} f(x) dx \quad (2.0.9)$$

$$= 1 - \int_{-\infty}^{\frac{1-y}{1+y}} \frac{1}{\pi} \cdot \frac{1}{1+x^2} dx \quad (2.0.10)$$

$$= 1 - \frac{1}{\pi} \cdot \left[\tan^{-1} x \right]_{-\infty}^{\frac{1-y}{1+y}} \quad (2.0.11)$$

$$= 1 - \frac{1}{\pi} \cdot \left[\left[\tan^{-1} x \right]_{-\infty}^0 + \left[\tan^{-1} x \right]_0^{\frac{1-y}{1+y}} \right] \quad (2.0.12)$$

$$= 1 - \frac{1}{\pi} \cdot \left[-\frac{\pi}{2} + \left[\tan^{-1} x \right]_0^{\frac{1-y}{1+y}} \right] \quad (2.0.13)$$

$$F_Y(y) = 1 - \frac{1}{\pi} \cdot \left[-\frac{\pi}{2} + \tan^{-1} \left(\frac{1-y}{1+y} \right) \right] \quad (2.0.14)$$

The pdf of Y is

$$f_Y(y) = \frac{dF_Y(y)}{dy} = -\frac{1}{\pi} \cdot \frac{d \left(\tan^{-1} \left(\frac{1-y}{1+y} \right) \right)}{dy} \quad (2.0.15)$$

$$= -\frac{1}{\pi} \cdot \frac{1}{1 + \left(\frac{1-y}{1+y} \right)^2} \cdot \frac{d \left(\frac{1-y}{1+y} \right)}{dy} \quad (2.0.16)$$

$$= -\frac{1}{\pi} \cdot \frac{1}{1 + \left(\frac{1-y}{1+y} \right)^2} \cdot \left(-\frac{2}{(y+1)^2} \right) \quad (2.0.17)$$

$$= \frac{1}{\pi} \cdot \frac{1}{1+y^2} \quad (2.0.18)$$

Hence, $Y \sim \text{Cauchy}(0,1)$.