Markov Chains

Using Eigen Value Decomposition

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Problem Statement

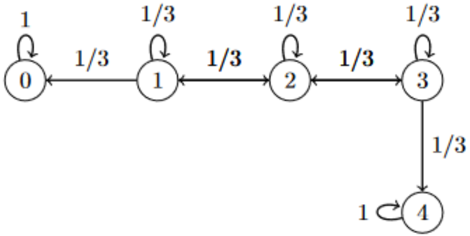
Consider a Markov Chain with state space {0,1,2,3,4} and transition matrix

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \tag{1}$$

Then $\lim_{n\to\infty} p_{23}^{(n)} equals$?

What is a Markov Chain?

Markov process: Random process for which the next step depends only on the present state; it has no memory of how the present state was reached.



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Solution

From the given transition matrix,

$$p_{23}^{(1)} = P\{X_1 = 2 | X_0 = 3\} = 1/3 \tag{3}$$

This is the initial probability of reaching state 2 from state 3 as per transition matrix P.

Solution

Here, we are expected to find below,

$$p_{23}^{(n)} = P\{X_n = 2 | X_0 = 3\} \tag{4}$$

i.e. probability of reaching state 2 after n steps from state 3.
In order to find $p_{ij}^{(n)}$, corresponding entry of P^n matrix is required. Matrix P^n can be found easily from diagonalized form of P.

Solution

Using characteristic equation,

$$|P - \lambda I| = 0 \tag{5}$$

$$\begin{vmatrix} 1-\lambda & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0 \tag{6}$$

Solution

$$(1-\lambda)^2\left(-\lambda^3+\lambda^2-\frac{\lambda}{9}-\frac{1}{27}\right)=0 \tag{7}$$

On solving, below are eigen values and corresponding eigen vectors of P

$$\lambda_1=\frac{1}{3},\begin{bmatrix}0\\-1\\0\\1\\0\end{bmatrix}\lambda_2=1,\begin{bmatrix}4\\3\\2\\1\\0\end{bmatrix}\lambda_3=1,\begin{bmatrix}-3\\-2\\-1\\0\\1\end{bmatrix} \tag{8}$$

$$\lambda_4 = -\frac{-1+\sqrt{2}}{3}, \begin{bmatrix} 0 \\ 1 \\ -\sqrt{2} \\ 1 \\ 0 \end{bmatrix} \lambda_5 = \frac{1+\sqrt{2}}{3}, \begin{bmatrix} 0 \\ 1 \\ \sqrt{2} \\ 1 \\ 0 \end{bmatrix} \tag{9}$$

Diagnolizing P from obtained eigen values and vectors,

$$P = XDX^{-1} \tag{10}$$

Solution

where,

$$D = \begin{bmatrix} \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{-1+\sqrt{2}}{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{1+\sqrt{2}}{3} \end{bmatrix} \tag{11}$$

$$X = \begin{bmatrix} 0 & 4 & -3 & 0 & 0 \\ -1 & 3 & -2 & 1 & 1 \\ 0 & 2 & -1 & -\sqrt{2} & \sqrt{2} \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{12}$$

Solution

$$X^{-1} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{4} & 0 & 0 & 0 & \frac{3}{4} \\ 0 & 0 & 0 & 0 & 1 \\ \frac{-2+\sqrt{2}}{8} & \frac{1}{4} & -\frac{\sqrt{2}}{4} & \frac{1}{4} & \frac{-2+\sqrt{2}}{8} \\ -\frac{\sqrt{2}+2}{8} & \frac{1}{4} & \frac{\sqrt{2}}{4} & \frac{1}{4} & -\frac{\sqrt{2}+2}{8} \end{bmatrix} \tag{13}$$

Now,

$$P^n = XDX^{-1} \cdot XDX^{-1} \cdot XDX^{-1} .. \; n \; times. \tag{14}$$

$$P^n = XD^nX^{-1} \tag{15}$$

Solution

After obtaining XD^nX^{-1}, the required entry comes out to be

$$P^n[3][2] = \frac{\sqrt{2}\left(\left(1-\sqrt{2}\right)^n - \left(1+\sqrt{2}\right)^n\right)}{4 \cdot 3^n} \tag{16}$$

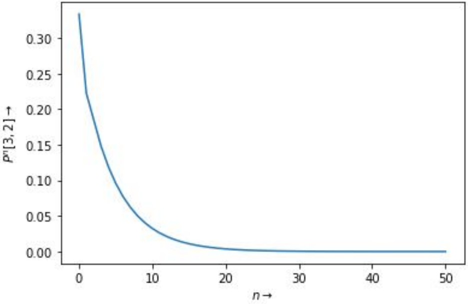
As n $\rightarrow \infty$,

$$P^n[3][2] = 0 \tag{17}$$

Hence, $\lim_{n\rightarrow\infty} p_{23}^{(n)} = 0$

Solution

Below figure shows change in probability wrt to n



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