Assignment 1

Tarandeep Singh

Github Link

https://github.com/Tarandeep97/AI5030

1 Problem

(51) Consider a Markov Chain with state space {0,1,2,3,4} and transition matrix

$$P = \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \tag{1.0.1}$$

Then $\lim_{n\to\infty} p_{23}^{(n)} equals$?

2 Solution

Here $p_{23}^{(n)}$ can be written as follows,

$$p_{23}^{(n)} = P\{X_n = 2|X_0 = 3\} \tag{2.0.1}$$

i.e. probability of reaching state 2 after n steps from state 3.
From the given transition matrix,

$$p_{23}^{(1)} = 1/3 \tag{2.0.2}$$

This is the initial probability as per transition matrix P.
In order to find $p_{ij}^{(n)}$, corresponding entry of P^n matrix is required. To find P^n, diagonalized form of P is required.
Using characteristic equation,

$$|P - \lambda I| = 0 \tag{2.0.3}$$

$$\begin{vmatrix} 1-\lambda & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0 \tag{2.0.4}$$

$$(1-\lambda)^2\left(-\lambda^3 + \lambda^2 - \frac{\lambda}{9} - \frac{1}{27}\right) = 0 \tag{2.0.5}$$

On solving, below are eigen values and corresponding eigen vectors of P

$$\lambda_1 = \frac{1}{3}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{2.0.6}$$

$$\lambda_2 = 1, \lambda_3 = 1, \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \tag{2.0.7}$$

$$\lambda_4 = -\frac{-1+\sqrt{2}}{3}, \begin{bmatrix} 0 \\ 1 \\ -\sqrt{2} \\ 1 \\ 0 \end{bmatrix} \tag{2.0.8}$$

$$\lambda_5 = \frac{1+\sqrt{2}}{3}, \begin{bmatrix} 0 \\ 1 \\ \sqrt{2} \\ 1 \\ 0 \end{bmatrix} \tag{2.0.9}$$

Diagnolizing P from obtained eigen values and vectors,

$$P = XDX^{-1} \tag{2.0.10}$$

where,

$$D = \begin{bmatrix} \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{-1+\sqrt{2}}{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{1+\sqrt{2}}{3} \end{bmatrix} \tag{2.0.11}$$

$$X = \begin{bmatrix} 0 & 4 & -3 & 0 & 0 \\ -1 & 3 & -2 & 1 & 1 \\ 0 & 2 & -1 & -\sqrt{2} & \sqrt{2} \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{2.0.12}$$

$$X^{-1} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{4} & 0 & 0 & 0 & \frac{3}{4} \\ 0 & 0 & 0 & 0 & 1 \\ \frac{-2+\sqrt{2}}{8} & \frac{1}{4} & -\frac{\sqrt{2}}{4} & \frac{1}{4} & \frac{-2+\sqrt{2}}{8} \\ -\frac{\sqrt{2}+2}{8} & \frac{1}{4} & \frac{\sqrt{2}}{4} & \frac{1}{4} & -\frac{\sqrt{2}+2}{8} \end{bmatrix} \tag{2.0.13}$$

$$P^n = XD^nX^{-1} \tag{2.0.14}$$

After obtaining XD^nX^{-1}, the required entry comes out to be

$$P^n[3][2] = \frac{\sqrt{2}\left(\left(1-\sqrt{2}\right)^n - \left(1+\sqrt{2}\right)^n\right)}{4 \cdot 3^n} \tag{2.0.15}$$

As n $\rightarrow \infty$,

$$P^n[3][2] = 0 \tag{2.0.16}$$

Hence, $\lim_{n\to\infty} p_{23}^{(n)} = 0$

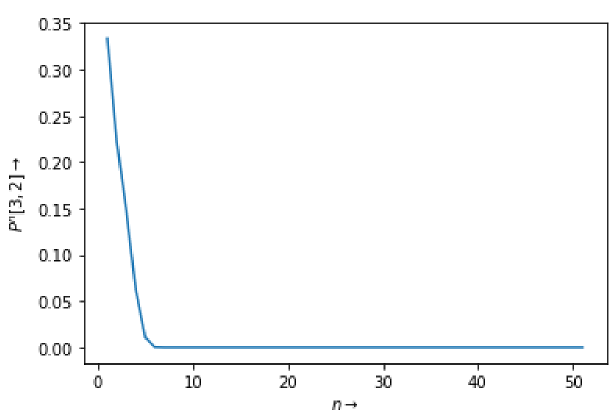

Fig. 0: Change in probability wrt n

3 Appendix

Calculation of Eigen Values and Eigen Vectors

$$\begin{vmatrix} 1-\lambda & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0 \tag{3.0.1}$$

Expanding along row 1:

$$(1-\lambda)\begin{vmatrix} \frac{1}{3}-\lambda & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{3} \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} \tag{3.0.2}$$

Expanding along row 4:

$$(1-\lambda)^2\begin{vmatrix} \frac{1}{3}-\lambda & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3}-\lambda \end{vmatrix} \tag{3.0.3}$$

Performing $C_1 = C_1 - (1-3\lambda)\, C_2$

$$(1-\lambda)^2\begin{vmatrix} 0 & \frac{1}{3} & 0 \\ \lambda(2-3\lambda) & \frac{1}{3}-\lambda & \frac{1}{3} \\ \lambda-\frac{1}{3} & \frac{1}{3} & \frac{1}{3}-\lambda \end{vmatrix} \tag{3.0.4}$$

Expanding along row 1:

$$(1-\lambda)^2 \cdot \left(-\frac{1}{3}\right)\left((\lambda(2-3\lambda)) \cdot \left(\frac{1}{3}-\lambda\right) - \left(\frac{1}{3}\right) \cdot \left(\lambda - \frac{1}{3}\right)\right) \tag{3.0.5}$$

$$(1-\lambda)^2\left(-\lambda^3+\lambda^2-\frac{\lambda}{9}-\frac{1}{27}\right) = 0 \tag{3.0.6}$$

Solving this equation,

$$\lambda = \frac{1}{3}, 1, 1, -\frac{-1+\sqrt{2}}{3}, \frac{1+\sqrt{2}}{3} \tag{3.0.7}$$

For $\lambda = \frac{1}{3}$, we can find eigen vector by putting its values in characterstic equation

$$\begin{bmatrix} \frac{2}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{2}{3} \end{bmatrix} \tag{3.0.8}$$

Finding row reduced echelon form of matrix and finding solution,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.0.9}$$

Here,

$$x_2 + x_4 = 0 \tag{3.0.10}$$

$$x_1 = x_3 = x_5 = 0 \tag{3.0.11}$$

Let $x_2 = -t$

From this, below is the eigen vector,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} t \tag{3.0.12}$$

Similary, we can obtain for other Eigen Values.

$$\lambda_2 = 1, \lambda_3 = 1, \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \tag{3.0.13}$$

$$\lambda_4 = -\frac{-1 + \sqrt{2}}{3}, \begin{bmatrix} 0 \\ 1 \\ -\sqrt{2} \\ 1 \\ 0 \end{bmatrix} \tag{3.0.14}$$

$$\lambda_5 = \frac{1 + \sqrt{2}}{3}, \begin{bmatrix} 0 \\ 1 \\ \sqrt{2} \\ 1 \\ 0 \end{bmatrix} \tag{3.0.15}$$