Assignment 2

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Github Link

https://github.com/Tarandeep97/AI5030

1 Problem

(52) Suppose X∼ Cauchy(0,1). Then the distribution of $\frac{1-X}{1+X}$ is?

2 Solution

A continuous random variable X follows **Cauchy distribution** with parameters μ and λ if its pdf is given by

$$f(x) = \begin{cases} \frac{\lambda}{\pi} \cdot \frac{1}{\lambda^2+(x-\mu)^2}, & -\infty < x < \infty; \\ & -\infty < \mu < \infty,\ \lambda > 0; \\ 0, & \text{Otherwise.} \end{cases}$$

The parameter μ and λ are location and scale parameters respectively.
When μ=0 and λ=1, then the distribution is called **Standard Cauchy Distribution**. The pdf of standard Cauchy distribution is

$$f(x) = \begin{cases} \frac{1}{\pi} \cdot \frac{1}{1+x^2}, & -\infty < x < \infty; \\ 0, & \text{Otherwise.} \end{cases}$$

Let,

$$Y = \frac{1-X}{1+X} \tag{2.0.1}$$

Then, cdf of Y is

$$F_Y(y) = P(Y \le y) \tag{2.0.2}$$

$$= P\left(\frac{1-X}{1+X} \le y\right) \tag{2.0.3}$$

$$= P\left((1-X) \le y.(1+X)\right) \tag{2.0.4}$$

$$= P\left((1-X) \le (y+y.X)\right) \tag{2.0.5}$$

$$= P\left((1-y) \le (X+y.X)\right) \tag{2.0.6}$$

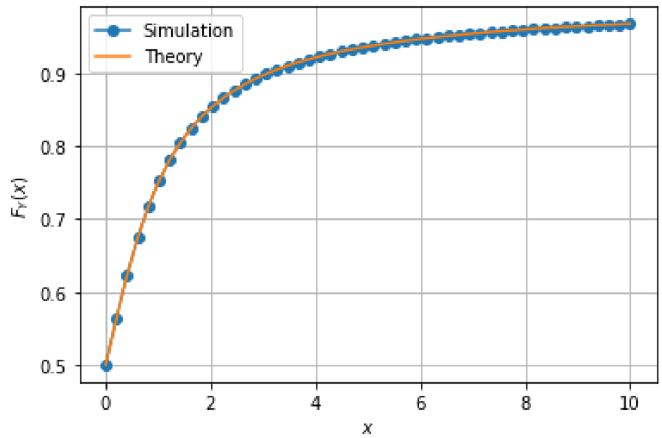

Fig 1. CDF of Y

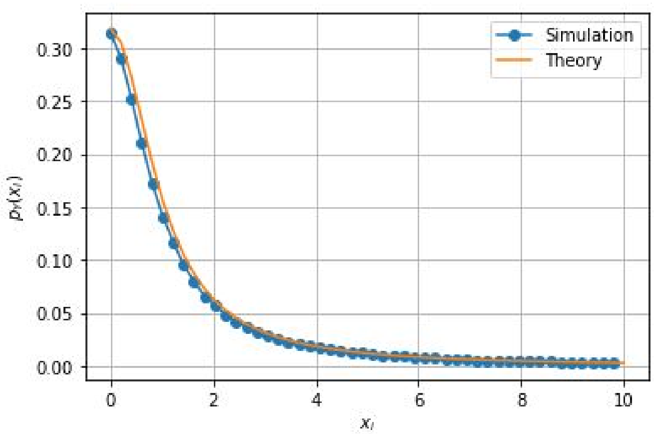

Fig 2. PDF of Y

$$= P\left(\frac{1-y}{1+y} \le X\right) \tag{2.0.7}$$

$$= 1 - P\left(X < \frac{1-y}{1+y}\right) \tag{2.0.8}$$

$$= 1 - \int_{-\infty}^{\frac{1-y}{1+y}} f(x)\,dx \tag{2.0.9}$$

$$= 1 - \int_{-\infty}^{\frac{1-y}{1+y}} \frac{1}{\pi} \cdot \frac{1}{1+x^2}\,dx \tag{2.0.10}$$

$$= 1 - \frac{1}{\pi} \cdot \left[tan^{-1} x \right]_{-\infty}^{\frac{1-y}{1+y}} \tag{2.0.11}$$

$$= 1 - \frac{1}{\pi} \cdot \left[\left[tan^{-1} x \right]_{-\infty}^{0} + \left[tan^{-1} x \right]_{0}^{\frac{1-y}{1+y}} \right] \tag{2.0.12}$$

$$= 1 - \frac{1}{\pi} \cdot \left[-\frac{\pi}{2} + \left[tan^{-1} x \right]_{0}^{\frac{1-y}{1+y}} \right] \tag{2.0.13}$$

$$F_Y(y) = 1 - \frac{1}{\pi} \cdot \left[-\frac{\pi}{2} + tan^{-1} \left(\frac{1-y}{1+y} \right) \right] \tag{2.0.14}$$

The pdf of Y is

$$f_Y(y) = \frac{dF_Y(y)}{dy} = -\frac{1}{\pi} \cdot \frac{d\left(tan^{-1} \left(\frac{1-y}{1+y} \right) \right)}{dy} \tag{2.0.15}$$

$$= -\frac{1}{\pi} \cdot \frac{1}{1 + \left(\frac{1-y}{1+y} \right)^2} \cdot \frac{d\left(\frac{1-y}{1+y} \right)}{dy} \tag{2.0.16}$$

$$= -\frac{1}{\pi} \cdot \frac{1}{1 + \left(\frac{1-y}{1+y} \right)^2} \cdot \left(-\frac{2}{(y+1)^2} \right) \tag{2.0.17}$$

$$= \frac{1}{\pi} \cdot \frac{1}{1 + y^2} \tag{2.0.18}$$

Hence, Y $\sim$ Cauchy(0,1).