Assignment 2

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Github Link

https://github.com/Tarandeep97/AI5030

1 Problem

(52) Suppose X∼ Cauchy(0,1). Then the distribution of $\frac{1-X}{1+X}$ is?

2 Solution - Method I

A continuous random variable X follows **Cauchy distribution** with parameters μ and λ if its pdf is given by

$$f(x) = \begin{cases} \frac{\lambda}{\pi} \cdot \frac{1}{\lambda^2+(x-\mu)^2}, & -\infty < x < \infty; \\ & -\infty < \mu < \infty,\ \lambda > 0; \\ 0, & \text{Otherwise.} \end{cases}$$

The parameter μ and λ are location and scale parameters respectively.
When μ=0 and λ=1, then the distribution is called **Standard Cauchy Distribution**. The pdf of standard Cauchy distribution is

$$f(x) = \begin{cases} \frac{1}{\pi} \cdot \frac{1}{1+x^2}, & -\infty < x < \infty; \\ 0, & \text{Otherwise.} \end{cases}$$

Let,

$$Y = g(X) = \frac{1-X}{1+X} \tag{2.0.1}$$

Range of Y

Since g(X) is self-inverse of itself,it is bijective in nature. Self-inverse nature of g(X) is verified below,

$$g\left(\frac{1-X}{1+X}\right) = \frac{1-\frac{1-X}{1+X}}{1+\frac{1-X}{1+X}} = X \tag{2.0.2}$$

So, g(X) $\in (-\infty, \infty)$.

Expression for CDF of Y

$$F_Y(y) = P(Y \le y) \tag{2.0.3}$$

$$= P\left(\frac{1-X}{1+X} \le y\right) \tag{2.0.4}$$

If X $\in (-\infty, -1)$, $(1+X) < 0$. So, inequality should be reversed.

$$= P\left((1-X) \ge y.(1+X)\right) \tag{2.0.5}$$

$$= P\left((1-X) \ge (y+y.X)\right) \tag{2.0.6}$$

$$= P\left((1-y) \ge (X+y.X)\right) \tag{2.0.7}$$

$$= P\left(\frac{1-y}{1+y} \ge X\right) = P\left[-\infty < X \le \left(\frac{1-y}{1+y}\right)\right] \tag{2.0.8}$$

$$= F_X\left(\frac{1-y}{1+y}\right) - F_X(-\infty) \tag{2.0.9}$$

If X $\in (-1, \infty)$, $(1+X) > 0$. The expression can be rewritten as

$$= P\left(\frac{1-y}{1+y} \le X\right) \tag{2.0.10}$$

$$= P\left[\left(\frac{1-y}{1+y}\right) \le X < \infty\right] \tag{2.0.11}$$

$$= F_X(\infty) - F_X\left(\frac{1-y}{1+y}\right) \tag{2.0.12}$$

Now, using above equation CDF of Y will be,

$$F_Y(y) = P\left[-\infty < X \le -1\right] + P\left[\left(\frac{1-y}{1+y}\right) \le X < \infty\right] \tag{2.0.13}$$

$$= \left(F_X(-1) - F_X(-\infty)\right) + \left(F_X(\infty) - F_X\left(\frac{1-y}{1+y}\right)\right) \tag{2.0.14}$$

$$= \frac{3}{4} - \frac{1}{\pi}.tan^{-1}\left(\frac{1-y}{1+y}\right) \tag{2.0.15}$$

Expression for PDF of Y

$$f_Y(y) = \frac{dF_Y(y)}{dy} = -\frac{1}{\pi} \cdot \frac{d\left(tan^{-1}\left(\frac{1-y}{1+y}\right)\right)}{dy} \tag{2.0.16}$$

$$= -\frac{1}{\pi} \cdot \frac{1}{1+\left(\frac{1-y}{1+y}\right)^2} \cdot \frac{d\left(\frac{1-y}{1+y}\right)}{dy} \tag{2.0.17}$$

$$= -\frac{1}{\pi} \cdot \frac{1}{1+\left(\frac{1-y}{1+y}\right)^2} \cdot \left(-\frac{2}{(y+1)^2}\right) \tag{2.0.18}$$

$$= \frac{1}{\pi} \cdot \frac{1}{1+y^2} \tag{2.0.19}$$

Hence, Y $\sim$ Cauchy(0,1).

3 Solution - Method II

Theorem: If U$\sim$ Normal(0,1),V$\sim$ Normal(0,1) are independent then $X = \frac{U}{V}$ is a Cauchy Distribution.

$$Y = \frac{1-X}{1+X} = \frac{1-\frac{U}{V}}{1+\frac{U}{V}} = \frac{U-V}{U+V} \tag{3.0.1}$$

U-V and U+V are also independent and normally distributed.
Hence, Y$\sim$Cauchy(0,1).

4 Appendix

Theorem: If U$\sim$ Normal(0,1),V$\sim$ Normal(0,1) are independent then $X = \frac{U}{V}$ is a Cauchy Distribution.

Proof: PDF of U and V is given as,

$$f_U(u) = \frac{1}{\sqrt{2\pi}}.e^{-\frac{u^2}{2}}, f_V(v) = \frac{1}{\sqrt{2\pi}}.e^{-\frac{v^2}{2}} \tag{4.0.1}$$

where $-\infty <$u,v $< \infty$
Since U and V are independent, their joint PDF is given by,

$$f_{UV}(u,v) = \frac{1}{2\pi}.e^{-\frac{u^2+v^2}{2}} \tag{4.0.2}$$

$$F_X(x) = P(X \le x) = P\left(\frac{U}{V} \le x\right) \tag{4.0.3}$$

$$= P(U \le V.x, V \ge 0) + P(U \ge V.x, V < 0) \tag{4.0.4}$$

$$= \int_0^\infty \int_{-\infty}^{v.x} f_{uv}(u,v)\,du\,dv + \int_{-\infty}^0 \int_{-\infty}^{v.x} f_{uv}(u,v)\,du\,dv \tag{4.0.5}$$

Differentiating $F_X(x)$ to obtain PDF of X, $f_X(x)$ (Using Leibniz's Integral Rule)

$$f_X(x) = \int_0^\infty v.f_{uv}(vx,v)\,dv + \int_{-\infty}^0 (-v).f_{uv}(vx,v)\,dv \tag{4.0.6}$$

$$f_X(x) = \int_0^\infty v.\frac{1}{2\pi}.e^{-\frac{(1+x^2)v^2}{2}}\,dv + \int_{-\infty}^0 (-v).\frac{1}{2\pi}.e^{-\frac{(1+x^2)v^2}{2}}\,dv \tag{4.0.7}$$

Let,

$$u = \frac{(1+x^2)v^2}{2} \tag{4.0.8}$$

$$\frac{du}{dv} = (1+x^2)v \Rightarrow dv = \frac{du}{(1+x^2)v} \tag{4.0.9}$$

$$f_X(x) = \int_0^\infty \frac{1}{2\pi(1+x^2)}.e^{-u}\,du - \int_{-\infty}^0 \frac{1}{2\pi(1+x^2)}.e^{-u}\,du \tag{4.0.10}$$

$$f_X(x) = \frac{1}{\pi(1+x^2)} \tag{4.0.11}$$

Hence, X$\sim$ Cauchy(0,1)
Theorem: If U$\sim$ Normal(0,1) and V$\sim$ Normal(0,1), then (U-V) and (U+V) are independent.
Proof: For (U+V) and (U-V) to be independent their co-variance must be 0.

$$= E[(U+V)(U-V)] - E[U+V]E[U-V] \tag{4.0.12}$$

$$= E[U^2] + E[V^2] - (E[U]+E[V])(E[U]-E[V]) \tag{4.0.13}$$

Hence,

$$cov(U+V, U-V) = 0 \tag{4.0.14}$$

Leibniz Integral Rule: For an integral form

$$\int_{a(x)}^{b(x)} f(x,t)\,dt$$

where $-\infty < a(x), b(x) < \infty$, the derivative of this integral is expressed as,

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)\,dt\right) \tag{4.0.15}$$

$$= f(x, b(x)).\frac{d}{dx}b(x) - f(x, a(x)).\frac{d}{dx}a(x) \tag{4.0.16}$$

$$+ \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,dt \tag{4.0.17}$$

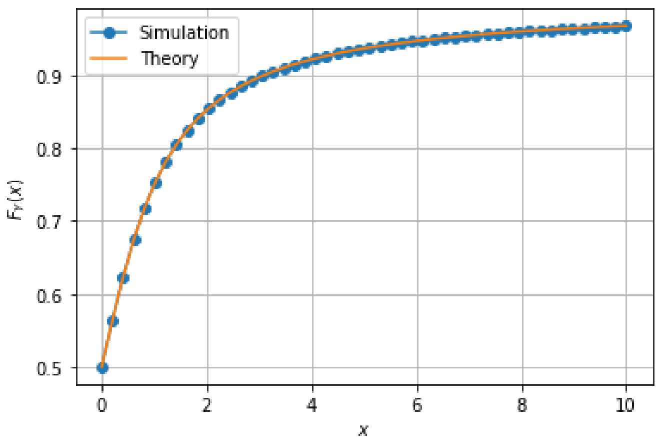

Fig 1. CDF of Y

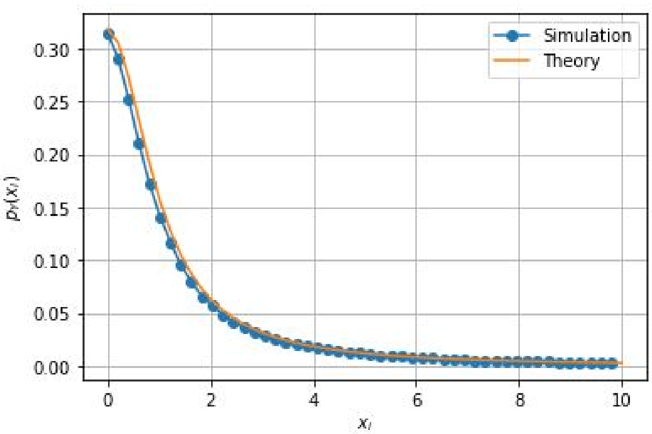

Fig 2. PDF of Y