Assignment 7 - Q7, GATE 2021

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Github Link

https://github.com/Tarandeep97/AI5030

1 Problem

(Q7, GATE 2021) Let the joint distribution of (X,Y) be bivariate normal with the mean vector $\begin{pmatrix}0\\0\end{pmatrix}$ and variance-covariance matrix $\begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix}$, where -1 $< \rho < 1$. Let $\phi_\rho(0,0) = P(X \leq 0, Y \leq 0)$. Then the Kendall's τ coefficient between X and Y equals

1) $4.\phi_\rho(0,0) - 1$
2) $4.\phi_\rho(0,0)$
3) $4.\phi_\rho(0,0) + 1$
4) $\phi_\rho(0,0)$

2 Solution

Kendall's τ coefficient between X and Y equals is defined as

$$P((X - X')(Y - Y') > 0) - P((X - X')(Y - Y') < 0) \tag{2.0.1}$$

where (X', Y') is bivariate normal independent of X and Y.

$$\begin{pmatrix}X'\\Y'\end{pmatrix} \sim N_2\left[\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix}\right] \tag{2.0.2}$$

Let $Z_1 = X - X'$ and $Z_2 = Y - Y'$
In equation 2.0.1,

$$\tau_\rho(X, Y) = P(Z_1Z_2 > 0) - P(Z_1Z_2 < 0) \tag{2.0.3}$$

$$\tau_\rho(X, Y) = P(Z_1Z_2 > 0) - [1 - P(Z_1Z_2 \geq 0)] \tag{2.0.4}$$

$$= 2.[P(Z_1Z_2 > 0) + P(Z_1Z_2 = 0)] - 1 \tag{2.0.5}$$

$$= 2.[P(Z_1Z_2 > 0)] - 1 \tag{2.0.6}$$

$$= 2.[P(Z_1 > 0, Z_2 > 0) + P(Z_1 < 0, Z_2 < 0)] - 1 \tag{2.0.7}$$

By symmetry,

$$= 4.[P(Z_1 < 0, Z_2 < 0)] - 1 \tag{2.0.8}$$

Expectation of Z_1 and Z_2

$$E[Z_1] = E[X - X'] = E[X] - E[X'] = 0 \tag{2.0.9}$$

$$E[Z_2] = E[Y - Y'] = E[Y] - E[Y'] = 0 \tag{2.0.10}$$

Variance of Z_1 and Z_2

$$Var(Z_1) = Var(X) + Var(X') = 2 \tag{2.0.11}$$

$$Var(Z_2) = Var(Y) + Var(Y') = 2 \tag{2.0.12}$$

$$Cov(Z_1, Z_2) = Cov(X - X', Y - Y') \tag{2.0.13}$$

$$= Cov(X, Y) - Cov(X, Y') - Cov(X', Y) + Cov(X', Y') \tag{2.0.14}$$

$$Cov(Z_1, Z_2) = 2\rho \tag{2.0.15}$$

So, distribution of bivariate (Z1,Z2) is

$$\begin{pmatrix}Z_1\\Z_2\end{pmatrix} = \begin{pmatrix}X - X'\\Y - Y'\end{pmatrix} \sim N_2\left[\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}2 & 2\rho\\ 2\rho & 2\end{pmatrix}\right] \tag{2.0.16}$$

This implies,

$$\frac{1}{\sqrt{2}}\begin{pmatrix}Z_1\\Z_2\end{pmatrix} = \begin{pmatrix}X\\Y\end{pmatrix} \tag{2.0.17}$$

So, equation 2.0.8 can be rewritten as,

$$\tau_\rho(X, Y) = 4.[P(X < 0, Y < 0)] - 1 \tag{2.0.18}$$

$$\tau_\rho(X, Y) = 4.\phi_\rho(0, 0) - 1 \tag{2.0.19}$$

Hence, option 1 is correct.