Assignment 3 - Q49, Dec 2018

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Github Link

https://github.com/Tarandeep97/AI5030

1 Problem

(Q49, Dec 2018) Let $X \geq 0$ be a random variable on $(\Omega, \mathcal{F}, P)$ with $\mathbb{E}(X) = 1$. Let $A \in \mathcal{F}$ be an event with $0 < P(A) < 1$. Which of the following defines another probability measure on $(\Omega, \mathcal{F})$?

1) $Q(B) = P(A \cap B)\ \forall B \in \mathcal{F}$
2) $Q(B) = P(A \cup B)\ \forall B \in \mathcal{F}$
3) $Q(B) = \mathbb{E}(XI_B)\ \forall B \in \mathcal{F}$
4) $Q(B) = \begin{cases} P(A/B) & P(B) > 0 \\ 0 & P(B) = 0 \end{cases}$

2 Solution

Properties of Probability measure

A Probability measure on $(\Omega, \mathcal{F})$ is a function $P : \mathcal{F} \to [0, 1]$ satisfying:

1) $P(\Omega) = 1, P(\phi) = 0$
2) If $A_1, A_2, A_3... \in \mathcal{F}$ is a collection of disjoint members in $\mathcal{F}$, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Investigating Options

Option 1, if $B = \Omega$, then

$$Q(\Omega) = P(A \cap \Omega) = P(A) \neq 1 \tag{2.0.1}$$

Option 2, if $B = \phi$, then

$$Q(\phi) = P(A \cup \phi) = P(A) \neq 0 \tag{2.0.2}$$

Option 4, if $B = \Omega$, then

$$Q(\Omega) = P(A/\Omega) = \frac{P(A \cap \Omega)}{P(\Omega)} \tag{2.0.3}$$

$$= P(A) \neq 1 \tag{2.0.4}$$

Option 3, Here the indicator function I_B defines a Bernoulli random variable.

$$I_B = \begin{cases} 1 & \text{if } x \in B, \\ 0 & \text{if } x \notin B, \end{cases} \tag{2.0.5}$$

if $B = \phi$,

$$Q(\phi) = E(XI_\phi) \tag{2.0.6}$$

$$\sum_{x \in \Omega} \sum_{x \in \Omega} x.I_B(x) P((X = x).(I_B(x))) \tag{2.0.7}$$

Here, $\forall x \in \Omega$, $I_B(x) = 0$, as $B = \phi$. Hence,

$$Q(\phi) = 0 \tag{2.0.8}$$

if $B = \Omega$,

$$Q(\Omega) = E(XI_\Omega) \tag{2.0.9}$$

$$= \sum_{x \in \Omega} \sum_{x \in \Omega} x.I_B(x) P((X = x).(I_B(x))) \tag{2.0.10}$$

Here, $\forall x \in \Omega$, $I_B(x) = 1$, as $B = \Omega$. Hence,

$$Q(\Omega) = \mathbb{E}(X) = 1 \tag{2.0.11}$$

Checking for Countable additivity of Q(property 2),

$$Q\left(\bigcup_{i=1}^{\infty} A_i\right) = E(XI_{\bigcup_{i=1}^{\infty} A_i}) \tag{2.0.12}$$

Using Property of Countable additivity on indicator functions, expression can be re-written as below,

$$= E\left(X \sum_{i=1}^{\infty} I_{A_i}\right) = E\left(\sum_{i=1}^{\infty} XI_{A_i}\right) \tag{2.0.13}$$

$$= \sum_{i=1}^{\infty} E\left(XI_{A_i}\right) \tag{2.0.14}$$

$$= \sum_{i=1}^{\infty} Q(A_i) \tag{2.0.15}$$

Option 3 satisfies properties of a probability measure. Hence, it is the correct option.