Assignment 4 - Q53, June 2018

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Github Link

https://github.com/Tarandeep97/AI5030

1 Problem

(Q53, June 2018) Suppose that the lifetime of an electric bulb follows an exponential distribution with mean θ hours. In order to estimate θ, n bulbs are switched on at the same time. After t hours, $n-m(>0)$ bulbs were found to be in functioning state. If the lifetimes of the other $m(>0)$ bulbs are noted as $x_1, x_2, x_3, ..., x_m$, respectively, then the maximum likelihood estimate of θ is given by

2 Solution

A continuous random variable X is said to have an exponential distribution with mean θ, $\theta > 0$, if its probability density function is given by

$$f(x|\theta) = \begin{cases} \frac{1}{\theta}e^{-\frac{1}{\theta}x}, & x \geq 0; \\ 0 & x < 0 \end{cases}$$

and its is CDF is given as,

$$F_X(x|\theta) = \begin{cases} 1 - e^{-\frac{1}{\theta}x}, & x \geq 0; \\ 0 & x < 0 \end{cases}$$

Lifetime of each m bulb is given as $x_1, x_2, ..x_m$. Then PDF of each of these m bulbs can be written as,

$$f(x_i|\theta) = \frac{1}{\theta}e^{-\frac{1}{\theta}x_i} \tag{2.0.1}$$

Since, n-m bulbs are still functioning, probability that their lifetime is greater than t, i.e.

$$P(X_i > t) = 1 - P(X_i \leq t) \tag{2.0.2}$$

$$= 1 - (1 - e^{-\frac{1}{\theta}x}) \tag{2.0.3}$$

$$= e^{-\frac{1}{\theta}x} \tag{2.0.4}$$

The Likelihood function for θ given data of each bulb is,

$$L(\theta|x_1, x_2, ..x_m, t, t.., (n-m)\ times) \tag{2.0.5}$$

$$= \left(\prod_{i=1}^{m} f(x_i|\theta)\right) \cdot \left(\prod_{i=1}^{n-m} P(X_i > t|\theta)\right) \tag{2.0.6}$$

$$= \frac{1}{\theta^m}e^{-\frac{1}{\theta}\sum_{i=1}^{m} x_i + (n-m)t} \tag{2.0.7}$$

Using log of the likelihood function,

$$l(\theta|x_1, x_2, ..x_m, t, t.., (n-m)\ times) \tag{2.0.8}$$

$$= \ln(L(\theta|x_1, x_2, ..x_m, t, t.., (n-m)\ times)) \tag{2.0.9}$$

$$= -m. \ln\theta - \frac{1}{\theta}\left(\sum_{i=1}^{m} x_i + (n-m)t\right) \tag{2.0.10}$$

Maxima of the above function gives us MLE of θ, i.e. when

$$\frac{d\left(-m. \ln\theta - \frac{1}{\theta}\left(\sum_{i=1}^{m} x_i + (n-m)t\right)\right)}{d\theta} = 0 \tag{2.0.11}$$

$$-m\frac{1}{\theta} + \frac{1}{\theta^2}\left(\sum_{i=1}^{m} x_i + (n-m)t\right) = 0 \tag{2.0.12}$$

$$\hat{\theta} = \frac{\sum_{i=1}^{m} x_i + (n-m)t}{m} \tag{2.0.13}$$