LA-HW4

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Question 1

$$\text{Proj}_S(\mathbf{Y}) = \mathtt{P}_S\mathbf{Y}$$

$$\mathtt{P}_S = \mathtt{U}_c\mathtt{U}_c^T, \quad \text{where } \mathtt{U}_c = [\mathbf{u}_1, \dots, \mathbf{u}_k] \text{ is the basis of } S.$$

$$\mathbf{y} \in S \quad \Longrightarrow^{\text{(I)}} \quad \mathbf{y} \in \mathcal{C}(\mathtt{U}_c)$$

$$\mathtt{P}_S\mathbf{y} \overset{\text{(I)}}{\to} \mathtt{U}_c\mathtt{U}_c^T\mathbf{y} \quad \overset{\text{(I)}}{=} \mathtt{U}_c\mathbf{U}_c^T\mathbf{U}_c$$

$$\mathtt{U}_c \text{ is orthonormal.} \quad \to \quad \mathtt{P}_S\mathbf{y} = \mathbf{y}$$

Question 2

$$S^\perp = \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{y}^T\mathbf{x} = 0 \text{ for all } \mathbf{x} \in S\}.$$

$$\text{(I)} \quad \forall \mathbf{x} \in S, \exists \mathbf{0} \in \mathbb{R}^n : \mathbf{0}^T\mathbf{x} = 0 \quad \to \quad \mathbf{0} \in S^\perp.$$

$$\text{(II)} \quad \mathbf{y}_1, \mathbf{y}_2 \in S^\perp \quad \to \quad \mathbf{y}_1^T\mathbf{x} = 0, \, \mathbf{y}_2^T\mathbf{x} = 0, \, \forall \mathbf{x} \in S.$$

$$(\mathbf{y}_1 + \mathbf{y}_2)^T\mathbf{x} = \mathbf{y}_1^T\mathbf{x} + \mathbf{y}_2^T\mathbf{x} = 0 + 0 = 0 \quad \to \quad \mathbf{y}_1 + \mathbf{y}_2 \in S^\perp.$$

$$\to \quad \text{closed under addition.}$$

$$\text{(III)} \quad \mathbf{y} \in S^\perp \text{ and } c \in \mathbb{R}.$$

$$\mathbf{y}^T\mathbf{x} = 0, \, \forall \mathbf{x} \in S.$$

$$(c\mathbf{y})^T\mathbf{x} = c(\mathbf{y}^T\mathbf{x}) = c(0) = 0 \quad \to \quad c\mathbf{y} \in S^\perp.$$

$$\rightarrow \quad \text{closed under scalar multiplication.}$$

$$\text{(I), (II), (III)} \quad \rightarrow \quad S^{\perp} \text{ is a linear subspace of } \mathbb{R}^n.$$

Question 3

$$\mathbf{x} \in \text{null}(\mathtt{A}) \quad \rightarrow \quad \mathtt{A}\mathbf{x} = \mathbf{0}.$$

Orthogonal complement: $\mathbf{r}_i \perp \mathbf{x}$, $\forall \mathbf{x} \in R(\mathtt{A})$.

$$\mathtt{A} = \begin{bmatrix} \mathbf{r}_1^T \\ \vdots \\ \mathbf{r}_m^T \end{bmatrix}.$$

$$\mathtt{A}\mathbf{x} = \mathbf{0} \quad \rightarrow \quad \mathbf{r}_1\mathbf{x} = 0, \ldots, \mathbf{r}_m\mathbf{x} = 0$$

$$\rightarrow \quad \text{null}(\mathtt{A}) \subseteq R(\mathtt{A})^{\perp} \quad \text{(I)}.$$

$$\dim(\text{null}(\mathtt{A})) + \dim(R(\mathtt{A})) = n$$

and

$$\dim(R(\mathtt{A})^{\perp}) = n - \dim(R(\mathtt{A}))$$

$$\rightarrow \quad \dim(\text{null}(\mathtt{A})) = \dim(R(\mathtt{A})^{\perp}) \quad \text{(II)}.$$

$$\rightarrow \quad \text{null}(\mathtt{A}) = R(\mathtt{A})^{\perp}.$$

Question 4

We know:

$$\mathtt{P}^2 = \mathtt{P}, \quad \mathtt{P}^T = \mathtt{P}.$$

$$(\mathtt{I} - \mathtt{P})^2 = (\mathtt{I} - \mathtt{P})(\mathtt{I} - \mathtt{P}) = \mathtt{I} - \mathtt{P} - \mathtt{P} + \mathtt{P}^2.$$

Since $\mathtt{P}^2 = \mathtt{P}$:

$$(\mathtt{I} - \mathtt{P})^2 = \mathtt{I} - \mathtt{P}.$$

$$(\mathtt{I} - \mathtt{P})^T = \mathtt{I}^T - \mathtt{P}^T = \mathtt{I} - \mathtt{P}.$$

$$\therefore \mathtt{P} \text{ is a projection matrix.}$$

Decompose $\mathbf{v}$:

$$\mathbf{v} = \mathbf{v}_S + \mathbf{v}_S^{\perp}, \quad \mathtt{P}\mathbf{v} = \mathbf{v}_S.$$

$$(\mathtt{I} - \mathtt{P})\mathbf{v} = \mathbf{v} - \mathtt{P}\mathbf{v} = \mathbf{v}_S^{\perp} \quad \to \quad (\mathtt{I} - \mathtt{P})\mathbf{v} \in V_S^{\perp}.$$

Question 5

$$\mathtt{Q}^T\mathtt{Q} = \mathtt{I}, \quad \mathtt{Q}^T = \mathtt{Q}^{-1}.$$

$$\det(\mathtt{Q}^T\mathtt{Q}) = \det(\mathtt{I}) = 1.$$

$$\det(\mathtt{AB}) = \det(\mathtt{A}) \cdot \det(\mathtt{B}) \quad \to$$

$$\det(\mathtt{Q}^T\mathtt{Q}) = \det(\mathtt{Q}^T) \cdot \det(\mathtt{Q}) = 1.$$

$$\det(\mathtt{Q}^T) = \det(\mathtt{Q}), \quad \det(\mathtt{Q})^2 = 1.$$

$$\therefore \det(\mathtt{Q}) = \pm 1.$$

Question 6

$$\mathtt{P}^2 = \mathtt{P} \quad \to \quad \det(\mathtt{P}^2) = \det(\mathtt{P}).$$

$$\det(\mathtt{P}^2) = \det(\mathtt{P})^2 \quad \to \quad \det(\mathtt{P})^2 = \det(\mathtt{P}).$$

$$\det(\mathtt{P})^2 - \det(\mathtt{P}) = 0.$$

$$\det(\mathtt{P})(\det(\mathtt{P}) - 1) = 0 \quad \to \quad \det(\mathtt{P}) = 0 \text{ or } 1.$$

Geometrical interpretation: If the matrix is already in the aimed space, then $\mathtt{P} = \mathtt{I}$ and $\det(\mathtt{P}) = 1$.

Otherwise, the column space of the matrix will be dependent, and $\det(\mathtt{P}) = 0$.