LA-HW5

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Question 1

If we represent the eigenvalues of A with λ, then the eigenvalues of $A - \alpha I$ are $\lambda - \alpha$.

The eigenvectors of A are also eigenvectors of $A - \alpha I$:

$$A\mathbf{v} = \lambda\mathbf{v} \implies (A - \alpha I)\mathbf{v} = A\mathbf{v} - \alpha I\mathbf{v}$$

$$= \lambda\mathbf{v} - \alpha\mathbf{v} = (\lambda - \alpha)\mathbf{v}.$$

Question 2

The characteristic polynomial of A is defined as:

$$p_A(\lambda) = \det(A - \lambda I).$$

Similarly, the characteristic polynomial of A^T is:

$$p_{A^T}(\lambda) = \det(A^T - \lambda I).$$

Now, consider the determinant of $A^T - \lambda I$:

$$\det(A^T - \lambda I) = \det((A - \lambda I)^T) = \det(A - \lambda I).$$

This implies:

$$p_{A^T}(\lambda) = p_A(\lambda).$$

Therefore, the eigenvalues of A are the same as the eigenvalues of A^T.

Question 3

Let A be a stochastic matrix. This implies:

$$A_{ij} \geq 0 \quad \text{and} \quad \sum_i A_{ij} = 1 \quad \text{for all } j.$$

Similarly, for A^T, we have:

$$\sum_j A^T_{ij} = \sum_j A_{ji} = 1 \quad \text{for all } i,$$

which implies that A^T is also stochastic.

If we consider the vector $\mathbf{1} = [1, 1, \ldots, 1]^T$, then:

$$A^T\mathbf{1} = \mathbf{1}.$$

This shows that $\mathbf{1}$ is an eigenvector of A^T corresponding to the eigenvalue $\lambda = 1$.

From Question 2, we conclude that the eigenvalues of A and A^T are the same.

Question 4

(a)

Given:

$$A\mathbf{v} = \lambda\mathbf{v}.$$

We know that $A\mathbf{v}$ is in the column space of A, and $\lambda\mathbf{v}$ is also in the column space of A. Since λ is a nonzero scalar, $\mathbf{v}$ is in the column space of A.

(b)

Let $\mathbf{v} = \mathbf{v}_r + \mathbf{v}_n$, where $\mathbf{v}_r$ is in the row space of A and $\mathbf{v}_n$ is in the null space of A. Then:

$$A\mathbf{v} = A(\mathbf{v}_r + \mathbf{v}_n) = A\mathbf{v}_r + A\mathbf{v}_n.$$

Since $\mathbf{v}_n$ is in the null space of A, we have $A\mathbf{v}_n = \mathbf{0}$. Therefore:

$$A\mathbf{v} = A\mathbf{v}_r + \mathbf{0} = A\mathbf{v}_r.$$

From part (a), we concluded that $\mathbf{v}$ is in the column space of A. Hence:

$$A\mathbf{v} = \lambda\mathbf{v} \neq \mathbf{0}.$$

This implies:

$$A\mathbf{v}_r \neq \mathbf{0}, \quad \text{so } \mathbf{v}_r \neq \mathbf{0}.$$

Question 5

Given:

$$A\mathbf{v}_i = \lambda_i\mathbf{v}_i \quad \text{and} \quad A\mathbf{v}_j = \lambda_j\mathbf{v}_j.$$

Consider the inner product $\mathbf{v}_j^T(A\mathbf{v}_i)$:

$$\mathbf{v}_j^T(A\mathbf{v}_i) = \mathbf{v}_j^T(\lambda_i\mathbf{v}_i) = \lambda_i(\mathbf{v}_j^T\mathbf{v}_i).$$

Since A is symmetric, $A = A^T$. Therefore:

$$\mathbf{v}_j^T(A\mathbf{v}_i) = (A\mathbf{v}_j)^T\mathbf{v}_i = (\lambda_j\mathbf{v}_j)^T\mathbf{v}_i = \lambda_j(\mathbf{v}_j^T\mathbf{v}_i).$$

Equating the two expressions:

$$\lambda_i(\mathbf{v}_j^T\mathbf{v}_i) = \lambda_j(\mathbf{v}_j^T\mathbf{v}_i).$$

This implies:

$$(\lambda_i - \lambda_j)(\mathbf{v}_j^T\mathbf{v}_i) = 0.$$

If $\lambda_i \neq \lambda_j$, then:

$$\mathbf{v}_j^T\mathbf{v}_i = 0.$$

Thus, $\mathbf{v}_i$ and $\mathbf{v}_j$ are orthogonal when $\lambda_i \neq \lambda_j$.

Question 6

A matrix A is positive definite if:

$$\mathbf{x}^T A\mathbf{x} > 0 \quad \text{for all nonzero vectors } \mathbf{x}.$$

If $A = V\Lambda V^T$, then:

$$\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T(V\Lambda V^T)\mathbf{x}.$$

Let $\mathbf{y} = V^T\mathbf{x}$, then:

$$\mathbf{x}^T A\mathbf{x} = \mathbf{y}^T\Lambda\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2.$$

Since $\mathbf{x}^T A\mathbf{x} > 0$ for all nonzero $\mathbf{x}$:

$$\sum_{i=1}^{n} \lambda_i y_i^2 > 0 \quad \text{for all nonzero } \mathbf{y}.$$

This implies that all $\lambda_i > 0$. (If any λ_i were negative, the equation would not hold as the sum could become negative.)

Thus:

If A is positive definite, then all eigenvalues of A are positive.

Conversely, if all $\lambda_i > 0$, then:

$$\mathbf{x}^T A\mathbf{x} = \mathbf{y}^T\Lambda\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 > 0.$$

Hence:

If all eigenvalues of A are positive, then A is positive definite.

Question 7

If A is positive definite, then:

$$\mathbf{x}^T A\mathbf{x} > 0 \quad \text{for all nonzero vectors } \mathbf{x}.$$

Let $\mathbf{x} = \mathbf{e}_i$, where $\mathbf{e}_i$ is the i-th standard basis vector. Then:

$$\mathbf{e}_i^T A\mathbf{e}_i = A_{ii}.$$

Since $\mathbf{x}^T A\mathbf{x} > 0$ for all nonzero $\mathbf{x}$:

$$\mathbf{e}_i^T A\mathbf{e}_i > 0 \implies A_{ii} > 0.$$

Thus, the diagonal entries of a positive definite matrix A are positive.

Question 8

(a)

For all $\mathbf{u} \in \mathbb{R}^n$, we have:

$$\langle \mathbf{u}, \mathbf{u} \rangle_A > 0.$$

Since A is positive definite:

$$\mathbf{u}^T A\mathbf{u} > 0 \quad \text{for all nonzero } \mathbf{u},$$

and:

$$\mathbf{u}^T A\mathbf{u} = 0 \quad \text{if } \mathbf{u} = 0.$$

(b)

$\langle \mathbf{u}, \mathbf{u} \rangle_A = 0$ if and only if $\mathbf{u} = 0$.

From part (a), we conclude that the statement is true.

(c)

The bilinear form $\langle \mathbf{u}, \mathbf{v} \rangle_A$ satisfies:

$$\langle \mathbf{u}, \mathbf{v} \rangle_A = \langle \mathbf{v}, \mathbf{u} \rangle_A.$$

Since A is symmetric ($A = A^T$):

$$\mathbf{u}^T A\mathbf{v} = (\mathbf{v}^T A\mathbf{u})^T = \mathbf{v}^T A\mathbf{u}.$$

(d)

The bilinear form is linear in its arguments:

$$\langle \alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w}\rangle_A = (\alpha\mathbf{u} + \beta\mathbf{v})^T A\mathbf{w}.$$

Expanding this, we get:

$$\langle \alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w}\rangle_A = \alpha(\mathbf{u}^T A\mathbf{w}) + \beta(\mathbf{v}^T A\mathbf{w}),$$

which simplifies to:

$$\langle \alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w}\rangle_A = \alpha\langle \mathbf{u}, \mathbf{w}\rangle_A + \beta\langle \mathbf{v}, \mathbf{w}\rangle_A.$$