

Lab Instructions - session 3

Row and Column Space, Linear Maps

Column Space and Row Space

The following code creates a figure with two subplots. In the left subplot, we plot a bunch of random 3D points in the column space of matrix A . The right subplot shows a set of 2D points in the row space of A .

`plot1.py`

```
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

# create a 3 x 2 matrix
A = np.array([[1, 2],
              [3, 4],
              [-2, 1]])

fig = plt.figure()

# A 1 by 2 subplot grid, subplot 1 (3D)
ax1 = fig.add_subplot(1, 2, 1, projection='3d')
ax1.set_title('column space')

for i in range(200):
    # create a random column vector
    u = np.random.randn(2, 1)
    # create a point in the column space of A
    v = A @ u
    ax1.scatter(v[0, 0], v[1, 0], v[2, 0], color='b')

# A 1 by 2 subplot grid, subplot 2 (2D)
ax2 = fig.add_subplot(1, 2, 2)
ax2.set_title('row space')

for i in range(200):
    # create a random row vector
    u = np.random.randn(1, 3)
    # create a point in the row space of A
    v = u @ A
    ax2.plot(v[0, 0], v[0, 1], 'ro')

plt.show()
```

- Rotate the 3D plot. Do all the points lie in a lower-dimensional subspace?

- What is the dimension of the column space? What is the dimension of the row space?

Task 1 - Practice vectorized coding

You have to write the above without using the `for` loops. To create an m by n (normally distributed) random matrix use `np.random.randn(m,n)`. Notice that for a 2 by n matrix A containing n points as its columns, you may plot the points by giving the list of the x - and y -coordinates as the first and second argument of the `plot` function respectively:

```
ax.plot(A[0,:], A[1,:], 'o')
```

Similarly, for a 3 by n matrix containing 3D points, you may use

```
ax.scatter(A[0,:], A[1,:], A[2,:])
```

Likewise, you may plot the points represented as rows of a matrix.

Task 2

Repeat task 1 for the matrix

```
1,  2  
3,  6  
-2, -4
```

- What are the dimensions of the row and column spaces?

Task 3

Create a 2 by 3 subplot using `fig.add_subplot(2,3,i, projection='3d')` for plotting the column and row spaces of the following 3 by 3 matrices:

```
A = 1,  2,  1,  
     2, -1, -1,  
    -1,  1, -2
```

```
B = 1,  2, -3  
     3,  1,  1  
     2,  1,  0
```

```
C = 1,  2, -3  
     3,  6, -9  
    -2, -4,  6
```

The row and column spaces must be plotted in the subplot's first and second rows, respectively. The columns of the subplot correspond to the matrices **A**, **B**, and **C**.

- Rotate the plots. For each matrix, what are the dimensions of the row and the column spaces?
- What can you say about the row and column spaces of a matrix?
- Plot (the points in) the row and column spaces of matrix **B** in the same axes using two different colours. Repeat the same for matrix **C**. Are the row and column spaces of matrices equal in general?

Linear Transformations

Remember representing the shape of a face as a set of points from the previous lab. Here, we apply a linear transformation to each point.

face1.py

```
import matplotlib.pyplot as plt
import numpy as np
from face_data import Face1, edges

def plot_face(plt,X,edges,color='b'):
    "plots a face"
    plt.plot(X[:,0], X[:,1], 'o', color=color)

    for i,j in edges:
        xi,yi = X[i]
        xj,yj = X[j]

        plt.plot((xi,xj), (yi,yj), '-', color=color)

    plt.axis('square')
    plt.xlim(-100,100)
    plt.ylim(-100,100)

th = np.pi/6
A = np.array([[np.cos(th), np.sin(th)],
               [-np.sin(th), np.cos(th)]])

X = Face1 @ A
plot_face(plt, X, edges, color='b')
plt.show()
```

- Why does the above rotates the face counterclockwise, while the matrix **A** corresponds to a 30 degrees clockwise rotation (-30°)?

Task 4 - Linear Transformations

- A. Animate the face to rotate around the origin by varying θ from 0 to 2π . Use what you learned from the previous lab.
- B. Apply a scaling transformation:

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$$

- Animate by varying α from $3/4$ to $4/3$.
- What happens when alpha is negative?

- C. Apply a non-uniform scaling transformation:

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

- Animate by varying α from $3/4$ to $4/3$ and taking $\beta = 1/\alpha$.

- D. Shear the face (horizontally) by applying the transformation

$$A = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$$

- Animate by varying s from -0.7 to 0.7 .
- The matrix A above represents a vertical shear. Why does it perform a horizontal shear here?