Linear Algebra for Computer Science
Homework 3

Read these first:

i You may write your solutions on paper, under a word processing software (MS-word, Libre Office, etc.), or under LaTeX.

ii If writing on paper, you must use a scanner device or a Camera Scanner (CamScanner) software to scan the document and submit a *single* PDF file.

iii Up to 15% extra score will be given to solutions written under LaTeX, provided that you follow either of the following conventions:

(a) Represent scalars with normal (italic) letters (a, A), vectors with bold lower-case letters ($\mathbf{a}$, using `\mathbf{a}`), and matrices with bold upper-case letters ($\mathbf{A}$, using `\mathbf{A}`), or

(b) represent scalars with normal (italic) letters (a, A), vectors with bold letters ($\mathbf{a}$, $\mathbf{A}$), and matrices with typewriter upper-case letters ($\mathtt{A}$, using `\mathtt{A}`).

(c) You latex document must contain a *title*, a *date*, and your name as the author.

(d) In all cases, you must submit a *single* PDF file.

(e) If writing under LaTeX, you must submit the *.tex* source (and other necessary source files if there are any) in addition to the PDF file.

Here is a short tutorial on LaTeX: `https://www.overleaf.com/learn/latex/Learn_LaTeX_in_30_minutes`

Questions

Linear Equations

To answer the following questions you need to use the fact that the set of solutions to a system of linear equations $\mathtt{A}\,\mathbf{x} = \mathbf{b}$ is in the form of $\{\mathbf{x}_p + \mathbf{x}_n \mid \mathbf{x}_n \in \mathcal{N}(\mathtt{A})\}$, where $\mathbf{x}_p$ is a particular solution.

1. Let $\mathtt{A} \in \mathbb{R}^{m\times n}$ be a fat matrix (i.e. $m < n$) with *full row rank* and $\mathbf{b} \in \mathbb{R}^m$. Show that $\mathtt{A}\,\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

2. Let $\mathtt{A} \in \mathbb{R}^{m\times n}$ be a tall matrix (i.e. $m > n$) with *full column rank* and $\mathbf{b} \in \mathbb{R}^m$. Show that $\mathtt{A}\,\mathbf{x} = \mathbf{b}$ has either no solution or exactly one solution.

3. Let $\mathtt{A} \in \mathbb{R}^{m\times n}$ be rank-deficient ($\operatorname{rank}(\mathtt{A}) < \min(m, n)$) and $\mathbf{b} \in \mathbb{R}^m$. Show $\mathtt{A}\,\mathbf{x} = \mathbf{b}$ has either no solution or infinitely many solutions.

4. Consider the system of linear equations $\mathtt{A}\,\mathbf{x} = \mathbf{b}$ with $\mathtt{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$, and let $\mathcal{S}$ be the set of solutions to it. Show that

   (a) $\mathcal{S}$ is a linear subspace if and only if $\mathbf{b} = \mathbf{0}$.

   (b) If $\mathcal{S}$ is nonempty, then there exists a vector $\mathbf{y} \in \mathbb{R}^n$ such that the set $\{\mathbf{z} - \mathbf{y} \mid \mathbf{z} \in \mathcal{S}\}$ is a linear subspace.

Matrix Rank

5. Consider a matrix $\mathtt{A} \in \mathbb{R}^{m\times n}$. Show that $\mathcal{C}(\mathtt{A}) = \mathcal{C}(\mathtt{A}\,\mathtt{A}^T)$. **Hint:** You may use the fact that any matrix $\mathtt{A} \in \mathbb{R}^{m\times n}$ of rank r can be decomposed as $\mathtt{A} = \mathtt{C}\,\mathtt{R}$, where $\mathtt{C} \in \mathbb{R}^{m\times r}$ and $\mathtt{R} \in \mathbb{R}^{r\times n}$ are of full column rank and full row rank, respectively.

6. From the previous question conclude that $\operatorname{rank}(\mathtt{A}) = \operatorname{rank}(\mathtt{A}\mathtt{A}^T) = \operatorname{rank}(\mathtt{A}^T\mathtt{A})$.

Null space

7. Consider a matrix $\mathtt{A} \in \mathbb{R}^{m\times n}$ of rank r. We perform the low-rank decomposition $\mathtt{A} = \mathtt{C}\,\mathtt{R}$, where $\mathtt{C} \in \mathbb{R}^{m\times r}$ is of full column rank, and $\mathtt{R} \in \mathbb{R}^{r\times n}$ are of full row rank. Show that the null space of $\mathtt{A}$ is the same as the null space of $\mathtt{R}$.