LA-HW1

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Question 1

(a) $\mathbf{z} + \mathbf{u} \overset{1}{=} \mathbf{u} + \mathbf{z} \overset{3}{=} \mathbf{u}$.

(b) Consider there exists a $\mathbf{w}$ such that $\mathbf{u} + \mathbf{w} = \mathbf{u}$ for every $\mathbf{u} \in \mathbf{V}$.
$\mathbf{u} + \mathbf{w} = \mathbf{u} \xrightarrow{\mathbf{u}=\mathbf{z}} \mathbf{z} + \mathbf{w} = \mathbf{z} \overset{1}{=} \mathbf{w} + \mathbf{z} \overset{3}{=} \mathbf{w}$.
Therefore, $\mathbf{z} = \mathbf{w}$.

(c) $\mathbf{z}' = \mathbf{w} \xrightarrow{+\mathbf{z}} \mathbf{z} + \mathbf{z}' = \mathbf{z} + \mathbf{w}$.
$\mathbf{z} + \mathbf{z}' \overset{4}{=} \mathbf{z}$.
$\mathbf{z} + \mathbf{w} \overset{a}{=} \mathbf{w}$.
Thus, $\mathbf{z} = \mathbf{w}$.

(d) $\mathbf{u} + \mathbf{w} = \mathbf{z} \overset{4}{=} \mathbf{u} + \mathbf{u}' \xrightarrow{+\mathbf{u}^{-1}} \mathbf{u}' + \mathbf{u} + \mathbf{w} = \mathbf{u}' + \mathbf{u} + \mathbf{u}'$.
Using (4, 1), $\mathbf{z} + \mathbf{w} = \mathbf{z} + \mathbf{u}' \overset{a}{\to} \mathbf{z} = \mathbf{u}'$.

(e) $\mathbf{u} + 0\mathbf{u} \overset{5}{=} 1\mathbf{u} + 0\mathbf{u} \overset{8}{=} (1+0)\mathbf{u} = 1\mathbf{u} \overset{5}{=} \mathbf{u} \xrightarrow{3\text{b}} 0\mathbf{u} = \mathbf{z}$.

(f) $a\mathbf{z} \overset{e}{=} a(0\mathbf{u}) \overset{6}{=} (a0)\mathbf{u} = 0\mathbf{u} \overset{e}{=} \mathbf{z}$.

(g) $0\mathbf{u} = (1-1)\mathbf{u} = 1\mathbf{u} + (-1)\mathbf{u} = \mathbf{u} + (-1)\mathbf{u}$.
$\mathbf{z} = \mathbf{u} + \mathbf{u}^{-1}$.
$0\mathbf{u} = \mathbf{z} \to \mathbf{u} + \mathbf{u}^{-1} = \mathbf{u} + (-1)\mathbf{u} \overset{d}{\to} \mathbf{u}^{-1} = (-1)\mathbf{u}$.

Question 2

Let

$$\mathbf{A}\mathbf{x} = \mathbf{z}, \quad \text{where } \mathbf{A} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

$$\mathbf{A}\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n.$$

If $\mathbf{a}_k = \mathbf{a}_k + 0$, then

$$\mathbf{a}_k = \mathbf{a}_k + x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n \implies x_1\mathbf{a}_1 + \cdots + (x_k+1)\mathbf{a}_k + \cdots + x_n\mathbf{a}_n = \mathbf{A}\mathbf{x}'.$$

The problem states that for every $\mathbf{x}$, the equation $\mathbf{A}\mathbf{x} = 0$ holds. Therefore, we constructed a new $\mathbf{x}$:

$$\mathbf{A}\mathbf{x}' = 0 \implies \mathbf{a}_k = 0 \implies \mathbf{A} = \mathbf{0}_{m\times n}.$$

Question 3

Let

$$\mathbf{x} = m_1\mathbf{x}_1 + \cdots + m_n\mathbf{x}_n.$$

Then,

$$\mathbf{A}\mathbf{x} = m_1\mathbf{A}\mathbf{x}_1 + \cdots + m_n\mathbf{A}\mathbf{x}_n \implies \mathbf{A}\mathbf{x} = 0.$$

In Question 2, we proved that since $\mathbf{x}$ can be any vector, if $\mathbf{A}\mathbf{x} = 0$, then $\mathbf{A} = 0$. Therefore:

$$\mathbf{A} = \mathbf{0}_{m\times n}.$$

Question 4

$$\mathbf{A}\mathbf{x} = \mathbf{x} \implies \mathbf{A}\mathbf{x} - \mathbf{x} = 0 \implies (\mathbf{A} - 1)\mathbf{x} = 0.$$

$$\mathbf{A}\mathbf{x} - \mathbf{I}_n\mathbf{x} = 0 \implies (\mathbf{A} - \mathbf{I}_n)\mathbf{x} = 0.$$

Based on Question 2, if the above equation holds for every $\mathbf{x}$, then $\mathbf{A} - \mathbf{I}_n = 0$.

$$\mathbf{A} = \mathbf{I}_n.$$

Question 5

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Then,

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{x}.$$

Question 6

Given:

$$x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = 0 \implies \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{0}.$$

Substituting:

$$x_1\mathbf{a}_1' + \cdots + x_n\mathbf{a}_n = 0 \implies x_1(\mathbf{a}_1 + \beta\mathbf{a}_2) + \cdots + x_n\mathbf{a}_n = 0.$$

$$\implies x_1\mathbf{a}_1 + (x_2 + \beta x_1)\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = 0.$$

Since:

$$x_1 = 0 \implies \beta x_1 = 0.$$

And:

$$x_2 = 0.$$

Thus:

$$x_2 + \beta x_1 = 0.$$

Question 7

(a)

$$\langle \mathbf{A}, \mathbf{B} \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} \mathbf{A}_{ij}\mathbf{B}_{ij}.$$

$$\text{trace}(\mathbf{S}) = \sum_{i} \mathbf{S}_{ii}.$$

Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times t}$, then:

$$\mathbf{C}_{m\times t} = \mathbf{A}_{m\times n}\mathbf{B}_{n\times t}, \quad c_{ij} = \sum_{k=1}^{n} \mathbf{a}_{ik}\mathbf{b}_{kj}.$$

$$\text{(I)} \quad (\mathbf{A}^T\mathbf{B})_{ij} = \sum_{k=1}^{m} \mathbf{a}_{ki}\mathbf{b}_{kj} \implies \text{trace}(\mathbf{A}^T\mathbf{B}) =$$

$$\sum_{i=1}^{n} \sum_{k=1}^{m} \mathbf{a}_{ki}\mathbf{b}_{ki} = \sum_{k=1}^{m} \sum_{i=1}^{n} \mathbf{a}_{ki}\mathbf{b}_{ki} = \langle \mathbf{A}, \mathbf{B} \rangle.$$

$$\text{(II)} \quad (\mathbf{B}^T\mathbf{A})_{ij} = \sum_{k=1}^{m} \mathbf{b}_{ki}\mathbf{a}_{kj} \implies \text{trace}(\mathbf{B}^T\mathbf{A}) =$$

$$\sum_{i=1}^{n}\sum_{k=1}^{m}\mathbf{b}_{ki}\mathbf{a}_{ki}=\sum_{k=1}^{m}\sum_{i=1}^{n}\mathbf{a}_{ki}\mathbf{b}_{ki}=\langle\mathbf{A},\mathbf{B}\rangle.$$

$$\text{(III)}\quad (\mathbf{A}\mathbf{B}^T)_{ij}=\sum_{k=1}^{n}\mathbf{a}_{ik}\mathbf{b}_{jk}\implies \text{trace}(\mathbf{A}\mathbf{B}^T)=$$

$$\sum_{i=1}^{m}\sum_{k=1}^{n}\mathbf{a}_{ik}\mathbf{b}_{ik}=\langle\mathbf{A},\mathbf{B}\rangle.$$

(b)

$$\text{(IV)}\quad \langle\mathbf{AB},\mathbf{C}\rangle\overset{(I)}{=}\text{trace}((\mathbf{AB})^T\mathbf{C})=\text{trace}(\mathbf{B}^T\mathbf{A}^T\mathbf{C})$$

$$=\text{trace}(\mathbf{B}^T(\mathbf{A}^T\mathbf{C}))\overset{(I)}{=}\langle\mathbf{B},\mathbf{A}^T\mathbf{C}\rangle.$$

$$\text{(V)}\quad \langle\mathbf{AB},\mathbf{C}\rangle\overset{\text{(III)}}{=}\text{trace}(\mathbf{ABC}^T)=\text{trace}(\mathbf{A}(\mathbf{CB}^T)^T)$$

$$=\langle\mathbf{A},\mathbf{CB}^T\rangle.$$

Question 8

Let

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ 0 & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & a_{nn}\end{bmatrix},$$

where $\mathbf{A}$ is an upper triangular matrix. We assume $a_{kk}=0$ for $1\leq k\leq n$. Our goal is to prove that there exists a non-zero vector $\mathbf{x}$ such that $\mathbf{Ax}=0$.

Define:

$$\mathbf{A}=\begin{bmatrix}\mathbf{A}_1 & \mathbf{A}_2\\ 0 & \mathbf{A}_3\end{bmatrix}.$$

Where $\mathbf{A}_{3_{11}}=0$.

$$\mathbf{Ax}=0\implies\begin{bmatrix}\mathbf{A}_1 & \mathbf{A}_2\\ 0 & \mathbf{A}_3\end{bmatrix}\mathbf{x}=0\implies\mathbf{x}=\begin{bmatrix}-\mathbf{A}_1^{-1}\mathbf{A}_2\mathbf{z}\\ \mathbf{z}\end{bmatrix}.$$

$$\mathbf{Ax}=\begin{bmatrix}\mathbf{A}_1 & \mathbf{A}_2\\ 0 & \mathbf{A}_3\end{bmatrix}\begin{bmatrix}-\mathbf{A}_1^{-1}\mathbf{A}_2\mathbf{z}\\ \mathbf{z}\end{bmatrix}=\begin{bmatrix}-\mathbf{A}_1\mathbf{A}_1^{-1}\mathbf{A}_2\mathbf{z}+\mathbf{A}_2\mathbf{z}\\ \mathbf{A}_3\mathbf{z}\end{bmatrix}=\begin{bmatrix}-\mathbf{A}_2\mathbf{z}+\mathbf{A}_2\mathbf{z}\\ \mathbf{A}_3\mathbf{z}\end{bmatrix}=\begin{bmatrix}0\\ \mathbf{A}_3\mathbf{z}\end{bmatrix}.$$

We can find a $\mathbf{z}$ where $\mathbf{A}_3\mathbf{z}=0$.

Question 9

$$a(\mathbf{u}+\mathbf{v})+b(\mathbf{u}+\mathbf{w})+c(\mathbf{v}+\mathbf{w})=\mathbf{0}.$$

Expanding, we get:

$$(a+b)\mathbf{u}+(a+c)\mathbf{v}+(b+c)\mathbf{w}=\mathbf{0}.$$

From this, we have:

$$a+b=0, \quad a+c=0, \quad b+c=0.$$

Solving the equations:

$$a+b=0 \implies a=-b,$$

$$a+c=0 \implies a=-c,$$

$$b+c=0 \implies c=-b.$$

Substituting $c=-b$ into $a=-c$, we get:

$$a=b.$$

Thus, we have:

$$a=b=c=0.$$

Question 10

1. Non-emptiness: For $\mathbf{W}$, each vector $\mathbf{W}$ is a subspace of $\mathbf{V}$.

$$\mathbf{0}\in\mathbf{W} \implies \forall\mathbf{v}\in\mathbf{W},\ \mathbf{0}\in\mathcal{M}.$$

2. Closed under addition: If $\mathbf{u},\mathbf{v}\in\mathcal{M}$, then $\mathbf{u}+\mathbf{v}\in\mathbf{W}$.

$$\forall\mathbf{u},\mathbf{v}\in\mathbf{W},\ \mathbf{u}+\mathbf{v}\in\mathcal{M}.$$

3. Closed under scalar multiplication: Let $\mathbf{u}\in\mathcal{M}$ and $c\in\mathbb{F}$ (where $\mathbb{F}$ is the field over which $\mathbf{V}$ is defined).

$$\forall\mathbf{u}\in\mathbf{W},\ c\mathbf{u}\in\mathbf{W}.$$

Since $\mathbf{W}$ is closed under scalar multiplication:

$$\forall\mathbf{u}\in\mathbf{W},\ c\mathbf{u}\in\mathcal{M} \implies c\mathbf{u}\in\mathcal{M}.$$