

Linear Algebra for Computer Science

Homework 1

Read these first:

- i You may write your solutions on paper, under a word processing software (MS-word, Libre Office, etc.), or under L^AT_EX.
- ii If writing on paper, you must use a scanner device or a Camera Scanner (CamScanner) software to scan the document and submit a *single* PDF file.
- iii Up to 15% extra score will be given to solutions written under L^AT_EX, provided that you follow either of the following conventions:
 - (a) Represent scalars with normal (italic) letters (a, A), vectors with bold lower-case letters (\mathbf{a} , using `\mathbf{a}`), and matrices with bold upper-case letters (\mathbf{A} , using `\mathbf{A}`), or
 - (b) represent scalars with normal (italic) letters (a, A), vectors with bold letters (\mathbf{a}, \mathbf{A}), and matrices with typewriter upper-case letters (\mathbf{A} , using `\mathtt{A}`).
 - (c) Your latex document must contain a *title*, a *date*, and your name as the author.
 - (d) In all cases, you must submit a *single* PDF file.
 - (e) If writing under L^AT_EX, you must submit the *.tex* source (and other necessary source files if there are any) in addition to the PDF file.

Here is a short tutorial on L^AT_EX: https://www.overleaf.com/learn/latex/Learn_LaTeX_in_30_minutes

Questions

1. A vector space defined on the field of the real numbers \mathbb{R} is a set V , equipped with a vector addition operator (vector + vector) and a scalar multiplication (scalar times vector) with the following properties
 - (1) *Commutativity*: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for $\mathbf{u}, \mathbf{v} \in V$.
 - (2) *Associativity*: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{v} + \mathbf{u}) + \mathbf{w}$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$.



- (3) *Identity element:* There exists an element $\mathbf{z} \in V$ such that for every $\mathbf{u} \in V$ we have $\mathbf{u} + \mathbf{z} = \mathbf{u}$. (\mathbf{z} is called the identity element and is often denoted by $\mathbf{0}$).
- (4) *Inverse:* for every $\mathbf{u} \in V$ there exist $\mathbf{u}' \in V$ such that $\mathbf{u} + \mathbf{u}' = \mathbf{z}$ (where the identity element \mathbf{z} was defined above. The vector \mathbf{u}' is called the (additive) inverse of \mathbf{u} and is usually denoted by \mathbf{u}^{-1} or $-\mathbf{u}$.)
- (5) $1\mathbf{u} = \mathbf{u}$. (notice that $1 \in \mathbb{R}$.)
- (6) $a(b\mathbf{u}) = (ab)\mathbf{u}$ for all $a, b \in \mathbb{R}$ and $\mathbf{u} \in V$.
- (7) $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ for all $a \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in V$.
- (8) $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$ for all $a, b \in \mathbb{R}$ and $\mathbf{u} \in V$.

Prove the following. In each step state which of the above axioms (1-8) you are using. You might also use the previous results, i.e. to prove d you may use parts a,b,c. Notice that there is no notion of subtraction $(-)$ here.

- (a) If \mathbf{z} is an identity element, then $\mathbf{z} + \mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in V$. (notice that (3) states $\mathbf{u} + \mathbf{z} = \mathbf{u}$.)
 - (b) The identity element \mathbf{z} is unique.
 - (c) The inverse of the identity element is itself.
 - (d) The inverse of any vector is unique.
 - (e) $0\mathbf{u} = \mathbf{z}$ for the scalar $0 \in \mathbb{R}$.
 - (f) $a\mathbf{z} = \mathbf{z}$ for any scalar $a \in \mathbb{R}$.
 - (g) $(-1)\mathbf{u} = \mathbf{u}'$ where \mathbf{u}' is the inverse element of \mathbf{u} .
2. Consider a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, such that $\mathbf{A}\mathbf{x} = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^n$. Prove that $\mathbf{A} = \mathbf{0}_{m \times n}$, that is all the entries of \mathbf{A} are zero.
 3. Consider a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, such that $\mathbf{A}\mathbf{x}_i = \mathbf{0}$ for $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n \in \mathbb{R}^n$, where $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ form a basis for \mathbb{R}^n . Prove that $\mathbf{A} = \mathbf{0}_{m \times n}$.
 4. Consider a square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ for which $\mathbf{A}\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Prove that $\mathbf{A} = \mathbf{I}_n$, the n by n identity matrix.
 5. Give an example of a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$, such that $\mathbf{A}\mathbf{x} = \mathbf{x}$ for some nonzero vector $\mathbf{x} \in \mathbb{R}^n$, \mathbf{A} is not the identity matrix.
 6. Assume that the vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ are linearly independent. Prove that the set of vectors $\mathbf{a}'_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ are also linearly independent where $\mathbf{a}'_1 = \mathbf{a}_1 + \beta\mathbf{a}_2$ for some scalar β .
 7. The dot product can also be defined on matrices. Consider two matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m \times n}$. Their dot product can be defined as $\langle \mathbf{A}, \mathbf{B} \rangle = \sum_{i=1}^m \sum_{j=1}^n A_{ij} B_{ij}$.

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- (a) Prove that $\langle \mathbf{A}, \mathbf{B} \rangle = \text{trace}(\mathbf{A}^T \mathbf{B}) = \text{trace}(\mathbf{B}^T \mathbf{A}) = \text{trace}(\mathbf{A} \mathbf{B}^T)$, where $\text{trace}(\mathbf{S}) = \sum_i S_{ii}$ gives the sum of the diagonal elements of a square matrix \mathbf{S} .
- (b) Prove that $\langle \mathbf{A} \mathbf{B}, \mathbf{C} \rangle = \langle \mathbf{B}, \mathbf{A}^T \mathbf{C} \rangle = \langle \mathbf{A}, \mathbf{C} \mathbf{B}^T \rangle$ Hint: $(\mathbf{A} \mathbf{B})^T = \mathbf{B}^T \mathbf{A}^T$.
8. Prove that a triangular matrix with at least one zero diagonal element is singular.
9. Prove that the set $\{\mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{w}, \mathbf{v} + \mathbf{w}\}$ is linearly independent if \mathbf{u}, \mathbf{v} and \mathbf{w} are linearly independent.
10. Let \mathcal{M} be a family of subspaces of V . Prove that the intersection $\bigcap \mathcal{M}$ of this family is itself a subspace of V .