LA-HW1

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Question 1

$\mathbf{S}$ is a strict subspace of $\mathbb{R}^n$, $\rightarrow$ its columns are linearly dependent, $\rightarrow$ the number of independent columns is less than n, $\rightarrow$ the number of its bases is less than n.

Question 2

$\mathbf{D}$ scales each column $\mathbf{a}_i$ by d_i:

$$\mathbf{AD} = \begin{bmatrix} d_1\mathbf{a}_1 & \cdots & d_n\mathbf{a}_n \end{bmatrix}.$$

$$\mathbf{B}^T = \begin{bmatrix} \mathbf{b}_1^T \\ \vdots \\ \mathbf{b}_n^T \end{bmatrix}.$$

$(\mathbf{ADB}^T)_{ij} = d_i\mathbf{a}_i \cdot (\text{j-th column of } \mathbf{B}^T).$

$$\mathbf{ADB}^T = \sum_{i=1}^{n} d_i\mathbf{a}_i\mathbf{b}_i^T.$$

Question 3

$$\mathbf{x} \in \mathcal{C}(\mathbf{AB}) \implies \exists\mathbf{y} : \mathbf{ABy} = \mathbf{x} \implies \mathbf{Az} = \mathbf{x}, \text{ where } \mathbf{z} = \mathbf{By}.$$

$$\mathbf{Az} = \mathbf{x} \implies \mathbf{x} \in \mathcal{C}(\mathbf{A}).$$

Question 4

$$\mathbf{x} \in \mathcal{C}(\mathbf{AB}) \implies \exists\mathbf{y} : \mathbf{ABy} = \mathbf{x} \implies \mathbf{Az} = \mathbf{x}, \text{ where } \mathbf{z} = \mathbf{By}.$$

$$\mathbf{x} \in \mathcal{C}(\mathbf{A}) \implies \mathcal{C}(\mathbf{AB}) \subseteq \mathcal{C}(\mathbf{A}) \quad \text{(I)}.$$

$$\mathbf{x} \in \mathcal{C}(\mathbf{A}) \implies \exists \mathbf{y} : \mathbf{A}\mathbf{y} = \mathbf{x}, \quad \mathbf{B} \text{ is invertible.}$$

$$\mathbf{A}(\mathbf{B}\mathbf{B}^{-1})\mathbf{y} = \mathbf{x} \implies \mathbf{A}\mathbf{B}(\mathbf{B}^{-1}\mathbf{y}) = \mathbf{x} \implies \mathbf{A}\mathbf{B}\mathbf{z} = \mathbf{x}, \text{ where } \mathbf{z} = \mathbf{B}^{-1}\mathbf{y}.$$

$$\mathbf{x} \in \mathcal{C}(\mathbf{A}\mathbf{B}) \implies \mathcal{C}(\mathbf{A}) \subseteq \mathcal{C}(\mathbf{A}\mathbf{B}) \quad \text{(II)}.$$

Combining (I) and (II), we conclude:

$$\mathcal{C}(\mathbf{A}) = \mathcal{C}(\mathbf{A}\mathbf{B}).$$

Question 5

$$\mathbf{x} \in \mathcal{C}(\mathbf{A}) \implies \exists \mathbf{y} : \mathbf{A}\mathbf{y} = \mathbf{x}.$$

Since $\text{rank}(\mathbf{B}) = n$ and $\mathbf{y} \in \mathbb{R}^n$, we can write:

$$\mathbf{y} = \mathbf{B}\mathbf{z} \implies \mathbf{A}\mathbf{y} = \mathbf{A}\mathbf{B}\mathbf{z} = \mathbf{x}.$$

$$\mathbf{x} \in \mathcal{C}(\mathbf{A}\mathbf{B}) \implies \mathcal{C}(\mathbf{A}) \subseteq \mathcal{C}(\mathbf{A}\mathbf{B}) \quad \text{(I)}.$$

$$\mathbf{x} \in \mathcal{C}(\mathbf{A}\mathbf{B}) \implies \exists \mathbf{y} : \mathbf{A}\mathbf{B}\mathbf{y} = \mathbf{x}, \text{ where } \mathbf{z} = \mathbf{B}\mathbf{y}.$$

$$\mathbf{A}\mathbf{z} = \mathbf{x} \implies \mathbf{x} \in \mathcal{C}(\mathbf{A}) \implies \mathcal{C}(\mathbf{A}\mathbf{B}) \subseteq \mathcal{C}(\mathbf{A}) \quad \text{(II)}.$$

From (I) and (II), we conclude:

$$\mathcal{C}(\mathbf{A}) = \mathcal{C}(\mathbf{A}\mathbf{B}).$$