LA-HW3

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Question 1

$$\mathtt{A}\mathbf{x} = \mathbf{b} \quad \rightarrow \quad \mathbf{x} = \{\mathbf{x}_p + \mathbf{x}_n \mid \mathbf{x}_n \in \mathcal{N}(\mathtt{A})\}$$

$$\text{rank}(\mathtt{A}) = m$$

$$\dim(\mathcal{N}(\mathtt{A})) = n - \text{rank}(\mathtt{A}) = n - m$$

$$\mathtt{A} \in \mathbb{R}^{m \times n},\, m < n \quad \rightarrow \quad n - m > 0$$

$$\text{there are infinite vectors in } \mathcal{N}(\mathtt{A})$$

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n \quad \text{forms an infinite set.}$$

Question 2

$$\text{rank}(\mathtt{A}) = n$$

$$m > n \quad \rightarrow \quad \text{more rows than columns} \quad \rightarrow \quad \mathtt{A} \text{ is overdetermined.}$$

$$\text{If } \mathbf{b} \notin \mathcal{C}(\mathtt{A}) \quad \rightarrow \quad \text{no solution for } \mathbf{b} \quad \text{(I)}$$

$$\text{If } \mathbf{b} \in \mathcal{C}(\mathtt{A}) \quad \rightarrow \quad \exists\, \mathbf{x} : \mathtt{A}\mathbf{x} = \mathbf{b}$$

$$\mathcal{N}(\mathtt{A}) = \{\mathbf{0}\} \quad \rightarrow \quad \mathbf{x} = \mathbf{x}_p \quad \rightarrow \quad \text{only one solution for } \mathbf{b} \quad \text{(II)}$$

Question 3

$$\text{rank}(\mathtt{A}) < \min(m, n)$$

$$\mathcal{N}(\mathtt{A}) = n - \text{rank}(\mathtt{A}) > 0$$

$$\mathbf{b} \in \mathbb{R}^m, \quad \text{and rank}(\mathtt{A}) \leq m, \quad \text{if:}$$

$$\mathbf{b} \notin \mathcal{C}(\mathtt{A}) \quad \rightarrow \quad \mathbf{b} \text{ has no solution} \quad \text{(I)}$$

$$\mathbf{b} \in \mathcal{C}(\mathtt{A}) \quad \rightarrow \quad \exists\, \mathbf{x}_p : \mathtt{A}\mathbf{x}_p = \mathbf{b} \quad \text{and}$$

$$\mathcal{N}(\mathtt{A}) > 0 \quad \rightarrow \quad \mathbf{x} = \mathbf{x}_p + \mathbf{x}_n \quad \rightarrow \quad \mathbf{b} \text{ has infinite solutions} \quad \text{(II)}$$

Question 4

(a)

We know that S is a linear subspace if:
1. Contains the zero vector.
2. Closed under addition.
3. Closed under scalar multiplication.

$$\mathbf{b} = \mathbf{0} \quad \rightarrow \quad \mathtt{A}\mathbf{x} = \mathbf{0} \quad \rightarrow \quad S = \mathcal{N}(\mathtt{A}) \quad \rightarrow$$
$$\text{Null space is a linear subspace.} \quad \rightarrow \quad S \text{ is a linear subspace (I)}$$

$$\mathbf{b} \neq \mathbf{0} \quad \rightarrow \quad \mathtt{A}\mathbf{x} = \mathbf{b} \quad \rightarrow \quad S = \{\mathbf{x}_p + \mathbf{x}_n \mid \mathbf{x}_n \in \mathcal{N}(\mathtt{A})\}$$

If $\mathbf{0} \notin S$, then $\mathtt{A}(\mathbf{0}) = \mathbf{b} \quad \not\Leftrightarrow \quad \mathbf{0} = \mathbf{b} \quad$ (II)
S is nonempty $\rightarrow \mathtt{A}\mathbf{x}_p = \mathbf{b}$
Consider $\mathbf{y} = \mathbf{x}_p$:

$$\{\mathbf{z} - \mathbf{y} \mid \mathbf{z} \in S\} = \{\mathbf{z} - \mathbf{x}_p \mid \mathbf{z} \in S\}$$

We know that $\mathbf{z} \in S \rightarrow \mathbf{z} = \mathbf{x}_p + \mathbf{x}_n$:

$$\{\mathbf{z} - \mathbf{x}_p \mid \mathbf{z} \in S\} = \{(\mathbf{x}_p + \mathbf{x}_n) - \mathbf{x}_p \mid \mathbf{x}_n \in \mathcal{N}(\mathtt{A})\}$$

$$= \{\mathbf{x}_n \mid \mathbf{x}_n \in \mathcal{N}(\mathtt{A})\} \rightarrow \mathbf{x}_n \in \mathcal{N}(\mathtt{A})$$

$\mathbf{0} \notin S$, then $\mathtt{A}(\mathbf{0}) = \mathbf{b} \quad \not\Leftrightarrow \quad \mathbf{0} = \mathbf{b} \quad$ (II)

(b)

S is nonempty $\rightarrow \mathtt{A}\mathbf{x}_p = \mathbf{b}$

Consider $\mathbf{y} = \mathbf{x}_p$:

$$\{\mathbf{z} - \mathbf{y} \mid \mathbf{z} \in S\} = \{\mathbf{z} - \mathbf{x}_p \mid \mathbf{z} \in S\}$$

We know that $\mathbf{z} \in S \rightarrow \mathbf{z} = \mathbf{x}_p + \mathbf{x}_n$:

$$\{\mathbf{z} - \mathbf{x}_p \mid \mathbf{z} \in S\} = \{(\mathbf{x}_p + \mathbf{x}_n) - \mathbf{x}_p \mid \mathbf{x}_n \in \mathcal{N}(\mathtt{A})\}$$

$$= \{\mathbf{x}_n \mid \mathbf{x}_n \in \mathcal{N}(\mathtt{A})\} \rightarrow \mathbf{x}_n \in \mathcal{N}(\mathtt{A})$$

Question 5

$$\mathtt{A}\mathtt{A}^T = (\mathcal{C}(\mathtt{R})(\mathcal{C}\mathtt{A}))^T = \mathcal{C}(\mathtt{R}\mathtt{R}^T)\mathcal{C}^T$$

$$\mathcal{C}(\mathtt{A}\mathtt{A}^T) = \mathcal{C}(\mathcal{C}(\mathtt{R}\mathtt{R}^T)\mathcal{C}^T)$$

$$\mathtt{R}\mathtt{R}^T \in \mathbb{R}^{r\times r} \quad \rightarrow \quad \mathtt{R} \text{ is square and full row rank.}$$

$$\rightarrow \mathcal{C}(\mathcal{C}(\mathtt{R}\mathtt{R}^T)\mathcal{C}^T) \subseteq \mathcal{C}(\mathcal{C}) \quad \text{(I)}$$

$$\mathcal{C} \text{ has full column rank.} \quad \rightarrow \quad \mathcal{C}(\mathtt{A}) = \mathcal{C}(\mathcal{C}) \quad \text{(II)}$$

$$\mathcal{C}(\mathtt{A}) = \mathcal{C}(\mathtt{A}\mathtt{A}^T)$$

Question 6

$$\mathcal{C}(\mathtt{A}) \subseteq \mathcal{C}(\mathtt{A}\mathtt{A}^T) \quad \rightarrow \quad \text{rank}(\mathtt{A}) = \text{rank}(\mathtt{A}\mathtt{A}^T) \quad \text{(I)}$$

$$\mathcal{C}(\mathtt{A}^T) \subseteq \mathcal{C}(\mathtt{A}^T\mathtt{A}) \quad \rightarrow \quad \text{rank}(\mathtt{A}^T) = \text{rank}(\mathtt{A}^T\mathtt{A})$$

$$\text{rank}(\mathtt{A}) = \text{rank}(\mathtt{A}^T) \quad \rightarrow \quad \text{rank}(\mathtt{A}^T) = \text{rank}(\mathtt{A}^T\mathtt{A}) \quad \text{(II)}$$

Combining (I) and (II):

$$\text{rank}(\mathtt{A}) = \text{rank}(\mathtt{A}\mathtt{A}^T) = \text{rank}(\mathtt{A}^T\mathtt{A})$$

Question 7

$$\mathtt{A} = \mathtt{CR} \quad \rightarrow \quad \mathtt{A}\mathbf{x} = \mathtt{CR}\mathbf{x}$$

$$\mathtt{A}\mathbf{x} = \mathbf{0} \quad \rightarrow \quad \mathcal{C}(\mathtt{R}\mathbf{x}) = \mathbf{0}$$

$$\mathtt{C} \text{ has full column rank } \rightarrow \text{ if } \mathcal{C}(\mathtt{R}\mathbf{x}) = \mathbf{0},$$

then $\mathtt{R}\mathbf{x} = \mathbf{0}$, $\text{null}(\mathtt{A}) \subseteq \text{null}(\mathtt{R}) \quad \text{(I)}$

$$\mathtt{R}\mathbf{x} = \mathbf{0} \quad \rightarrow \quad \mathtt{CR}\mathbf{x} = \mathbf{C}(\mathbf{0}) = \mathbf{0} \quad \rightarrow \quad \mathtt{A}\mathbf{x} = \mathbf{0}$$

$$\text{null}(\mathtt{CR}) \subseteq \text{null}(\mathtt{A}) \quad \text{(II)}$$

Combining (I) and (II):

$$\text{null}(\mathtt{A}) = \text{null}(\mathtt{R})$$