LA-HW6

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January 21, 2025

Question 1

Using the singular value decomposition (SVD) of a matrix A, we have:

$$\det(A) = \det(U)\det(\Sigma)\det(V^T),$$

where U and V are orthogonal matrices, and Σ is the diagonal matrix of singular values.

Since $\det(V^T) = \det(V)$:

$$\det(A) = \det(U)\det(\Sigma)\det(V).$$

Now, consider $\det(\Sigma) > 0$ (as the singular values are positive). The sign of $\det(A)$ is determined by the product:

$$\text{sign}(\det(A)) = \text{sign}(\det(U)\det(V)).$$

Since $\det(U)$ and $\det(V)$ are either $+1$ or -1 (as U and V are orthogonal):

$$\det(U)\det(V) = \pm 1.$$

Thus:

$$\text{sign}(\det(U)\det(V)) = \text{sign}(\det(A)).$$

Question 2

The eigenvalue decomposition of a symmetric matrix A is given by:

$$A = V\Lambda V^T,$$

where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ and all $\lambda_i > 0$ if A is positive definite.

The singular value decomposition (SVD) of A is:

$$A = U\Sigma V^T.$$

If A is symmetric, then $U = V$.

The matrix Σ represents the singular values of A. For symmetric positive definite matrices, the singular values are the square roots of the eigenvalues of $A^T A$ or A^2.

If A is positive definite, the eigenvalues of A are positive. Thus:

$$\text{Singular values} = \text{Eigenvalues}.$$

In conclusion:

$$A = U\Sigma V^T = V\Lambda V^T, \quad \Sigma = \Lambda, \quad U = V.$$

Question 3

The eigenvalue decomposition of A is given by:

$$A = V\Lambda V^T,$$

where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$.

The singular value decomposition (SVD) adjusts Λ to include absolute values of eigenvalues:

$$\Sigma = \text{diag}(|\lambda_1|, \ldots, |\lambda_n|).$$

If $\lambda_i < 0$, then the corresponding eigenvector $\mathbf{v}_i$ is adjusted to $-\mathbf{v}_i$ in V to ensure positivity in Σ.

The adjusted matrices U and V are used in the SVD representation:

$$A = U\Sigma V^T.$$

Question 4

Let $B = PAQ$, where $P^T P = I$ and $Q^T Q = I$. Then:

$$B^T B = (PAQ)^T (PAQ) = Q^T A^T P^T PAQ.$$

Since $P^T P = I$, we have:

$$B^T B = Q^T A^T AQ.$$

Here, Q is orthogonal, so the eigenvalues of $B^T B$ are equal to the eigenvalues of $A^T A$.

The singular values of B are the square roots of the eigenvalues of $B^T B$. Therefore:

$$\text{Singular values of } B = \text{Singular values of } A.$$