Linear Algebra for Computer Science
Homework 2

Read this first:

i You may write your solutions on paper, under a word processing software (MS-word, Libre Office, etc.), or under LaTeX.

ii If writing on paper, you must use a scanner device or a Camera Scanner (CamScanner) software to scan the document and submit a *single* PDF file.

iii Up to 15% extra score will be given to solutions written under LaTeX, provided that you follow either of the following conventions:

(a) Represent scalars with normal (italic) letters (a, A), vectors with bold lower-case letters ($\mathbf{a}$, using `\mathbf{a}`), and matrices with bold upper-case letters ($\mathbf{A}$, using `\mathbf{A}`), or

(b) represent scalars with normal (italic) letters (a, A), vectors with bold letters ($\mathbf{a}$, $\mathbf{A}$), and matrices with typewriter upper-case letters ($\mathtt{A}$, using `\mathtt{A}`).

(c) You latex document must contain a *title*, a *date*, and your name as the author.

(d) In all cases, you must submit a *single* PDF file.

(e) If writing under LaTeX, you must submit the *.tex* source (and other necessary source files if there are any) in addition to the PDF file.

Here is a short tutorial on LaTeX: `https://www.overleaf.com/learn/latex/Learn_LaTeX_in_30_minutes`

Questions

Basis

1. Let $\mathcal{S} \subset \mathbb{R}^n$ be a *strict* linear subspace of $\mathbb{R}^n$ (*strict* meaning $\mathcal{S} \neq \mathbb{R}^n$ or $\dim(\mathcal{S}) < n$). I argue that the set of standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \in \mathbb{R}^n$ form a basis for $\mathcal{S}$ because

- $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are linearly independent, and
- every vector in $\mathcal{S}$ can be written as a linear combination of $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$.

How is my argument wrong?

Matrix Multiplication

2. Consider the matrices $\mathtt{A} = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n] \in \mathbb{R}^{m\times n}$, $\mathtt{D} = \mathrm{diag}([d_1, d_2, \ldots, d_n]) \in \mathbb{R}^{n\times n}$, $\mathtt{B} = [\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n] \in \mathbb{R}^{p\times n}$, where $\mathtt{D}$ is a diagonal matrix with diagonal elements d_i. Show that

$$\mathtt{ADB}^T = \sum_{i=1}^{n} d_i\, \mathbf{a}_i \mathbf{b}_i^T$$

.

Row space and Column Space

3. Consider two matrices $\mathtt{A} \in \mathbb{R}^{m\times n}$ and $\mathtt{B} \in \mathbb{R}^{n\times p}$. Prove that $\mathcal{C}(\mathtt{AB}) \subseteq \mathcal{C}(\mathtt{A})$, where $\mathcal{C}(\cdot)$ represents the column space.

4. Consider a matrix $\mathtt{A} \in \mathbb{R}^{m\times n}$ and a square *invertible* matrix $\mathtt{B} \in \mathbb{R}^{n\times n}$. Prove that $\mathcal{C}(\mathtt{A}) = \mathcal{C}(\mathtt{AB})$. (Hint: to prove that two sets S_1 and S_2 are equation you can show $S_1 \subseteq S_2$ and $S_2 \subseteq S_1$).

5. Consider two matrices $\mathtt{A} \in \mathbb{R}^{m\times n}$ and $\mathtt{B} \in \mathbb{R}^{n\times p}$ where $\mathtt{B}$ has *full row rank* (i.e. $\mathrm{rank}(\mathtt{B}) = n$). Prove that $\mathcal{C}(\mathtt{A}) = \mathcal{C}(\mathtt{A\,B})$.