Foundations of Electrical Engineering

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Contents

1 Electrical Circuits

1.1 Fundamental Quantities

Name	Definition	Symbol	Unit
Charge	Electric charge is a fundamental property of matter that governs how particles are affected by an electromagnetic field.	q	Coulomb (C)
Current	$i = \frac{\mathrm{d}q}{\mathrm{d}t} \iff 1\,\mathrm{A} = 1\,\mathrm{C\,s^{-1}}$	i	Ampere (A)
Voltage	$v = \frac{\mathrm{d}w}{\mathrm{d}q} \iff 1\,\mathrm{V} = 1\,\mathrm{J\,C^{-1}}$	v	Volt (V)
Power	$p = \frac{\mathrm{d}w}{\mathrm{d}t} \iff 1\,\mathrm{W} = 1\,\mathrm{J\,s^{-1}}$	p	Watt (W)

Charge in an electron. $q = 1.6022 \times 10^{-19}\,\mathrm{C}$.

Electric Power. $p = \frac{\mathrm{d}w}{\mathrm{d}t} = vi$.

1.2 Passive Sign Convention

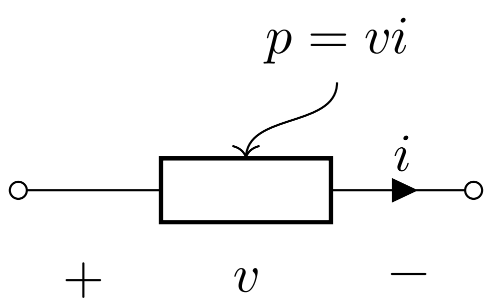

Figure 1: Energy dissipated.

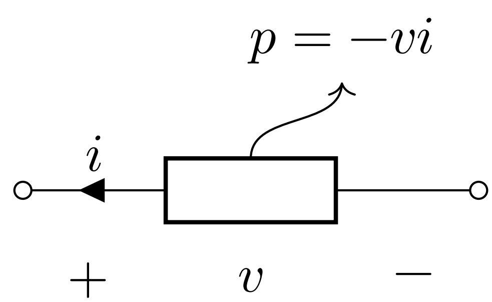

Figure 2: Energy produced.

Theorem 1.2.1 (Power Balance)**.**

$$p_{\text{net}} = 0$$

Theorem 1.2.2 (Energy)**.**

$$w(\tau) = \int_0^\tau p(t)\,\mathrm{d}t$$

1.3 Circuits and Sources

Definition 1.3.1 (Circuits)**.** A circuit is a mathematical model that approximates a real system. It is built from ideal circuit elements connected by ideal wires.

Definition 1.3.2 (Voltage Source)**.** Produces or dissipates power at a specified voltage with whatever current is required.

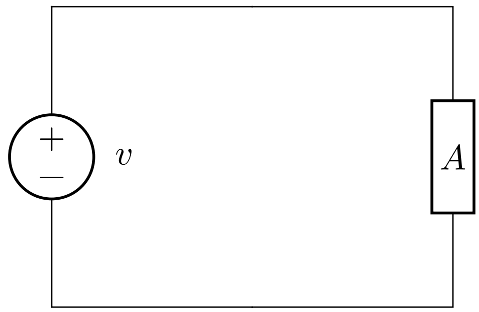

Figure 3: Voltage source – v is specified, i varies depending on circuit element A.

Definition 1.3.3 (Current Source)**.** Produces or dissipates power at a specified current with whatever voltage is required.

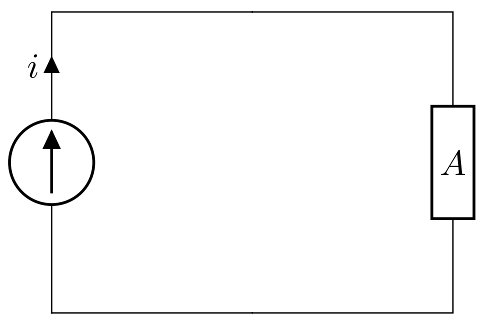

Figure 4: Current source – i is specified, v varies depending on circuit element A.

1.4 Resistors

Definition 1.4.1 (Resistor)**.** Resistors dissipate power, and the voltage across both terminals is proportional to the current.

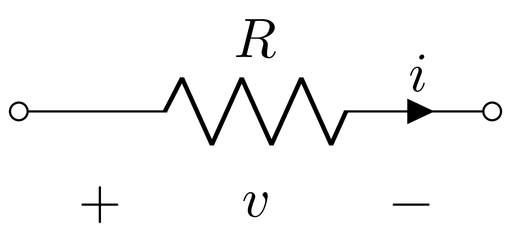

Figure 5: Resistor circuit symbol.

Theorem 1.4.1 (Voltage through a resistor)**.**

$$v = iR$$

Corollary 1.4.1.1 (Power dissipated by a resistor)**.**

$$p = vi = i^2R = \frac{v^2}{R}$$

2 Simple Resistive Circuits

2.1 Ignored Physics

1. Electrical effects occur instantaneously, so there is no time delay along the wires.
2. The net charge on every component is zero. Charge is never lost or gained.
3. There is no magnetic coupling between the components.

2.2 Kirchhoff's Laws

Theorem 2.2.1 (Kirchhoff's Current Law (KCL))**.** *The sum of all currents into a node equals zero.*

$$\sum i_{\text{node}} = 0$$

Theorem 2.2.2 (Kirchhoff's Voltage Law (KVL))**.** *The sum of all voltages around a loop equals zero.*

$$\sum v_{\text{loop}} = 0$$

2.3 Series and Parallel Circuits

Definition 2.3.1. Elements connected end-to-end are in series. If both ends of an element are connected directly to another element, the two elements are in parallel.

Element	Series	Parallel
Current Source	$i_{\text{eq}} = i_{k\geq 1}$	$i_{\text{eq}} = \sum_{k\geq 1} i_k$
Voltage Source	$v_{\text{eq}} = \sum_{k\geq 1} v_k$	$v_{\text{eq}} = v_{k\geq 1}$
Resistor	$R_{\text{eq}} = \sum_{k\geq 1} R_k$	$\frac{1}{R_{\text{eq}}} = \sum_{k\geq 1} \frac{1}{R_k}$
Inductor	$L_{\text{eq}} = \sum_{k\geq 1} L_k$	$\frac{1}{L_{\text{eq}}} = \sum_{k\geq 1} \frac{1}{L_k}$
Capacitor	$\frac{1}{C_{\text{eq}}} = \sum_{k\geq 1} \frac{1}{C_k}$	$C_{\text{eq}} = \sum_{k\geq 1} C_k$

Table 1: Equivalent values for various components connected in series and parallel.

These equations can be used to simplify a complex circuit.

2.4 Voltage and Current Dividers

Definition 2.4.1 (Voltage Divider)**.** A voltage divider is a circuit that divides a voltage in the proportion of the series resistances.

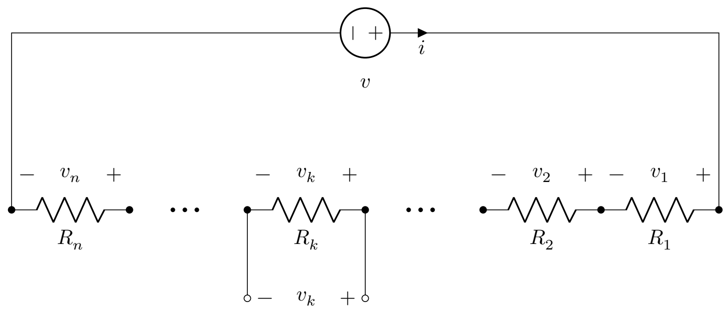

Figure 6: Voltage divider circuit.

Theorem 2.4.1 (Voltage Divider)**.**

$$v_k = v\frac{R_k}{R_{\text{eq}}}$$

Proof. The current through any resistor is

$$i = \frac{v}{R_{\text{eq}}}.$$

Therefore the voltage drop in any resistor is

$$v_k = iR_k$$
$$v_k = \frac{v}{R_{\text{eq}}}R_k.$$

□

Definition 2.4.2 (Current Divider)**.** A voltage divider is a circuit that divides a voltage in the proportion of the series resistances.

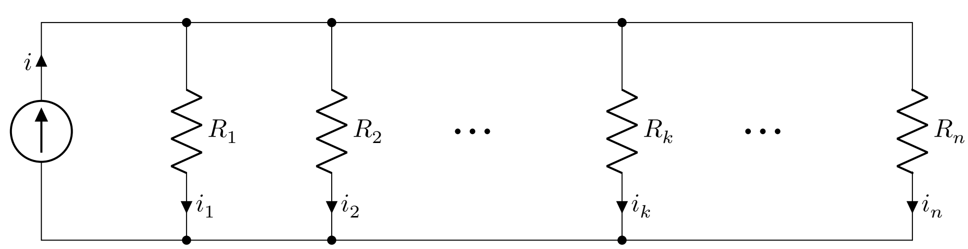

Figure 7: Current divider circuit.

Theorem 2.4.2 (Current Divider)**.**

$$i_k = i\frac{R_{\text{eq}}}{R_k}$$

Proof. Using KCL we have

$$
\begin{aligned}
i &= \sum_{k\geq 1} i_k \\
i &= \sum_{k\geq 1} \frac{v}{R_k} \\
i &= \frac{v}{R_{\text{eq}}}.
\end{aligned}
$$

Solving for v gives

$$v = iR_{\text{eq}}.$$

Hence the current through any resistor in parallel is given by

$$i_k = \frac{v}{R_k} = \frac{iR_{\text{eq}}}{R_k}.$$

□

3 Diodes

Definition 3.0.1 (Diode)**.** A diode is a semiconductor component in which current flows only in one direction. A diodes requires a voltage to start the flow of current in the forward direction.

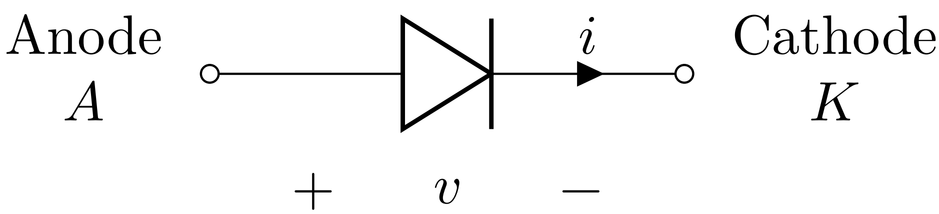

Figure 8: Diode circuit symbol.

3.1 VI Characteristic

The Voltage-Current characteristic of linear circuit elements can be plotted as shown below.

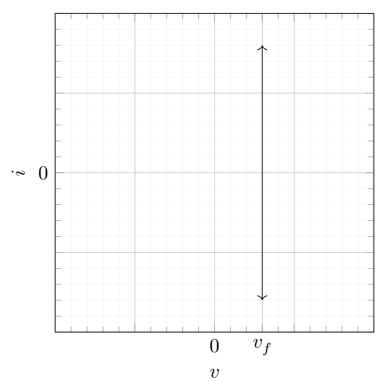

Figure 9: VI characteristic for a voltage source.

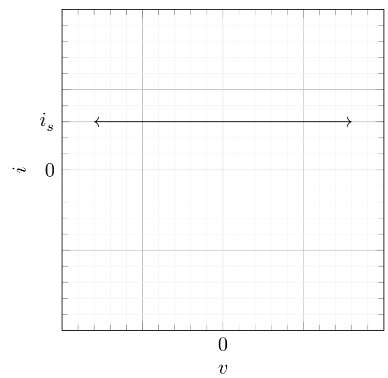

Figure 10: VI characteristic for a current source.

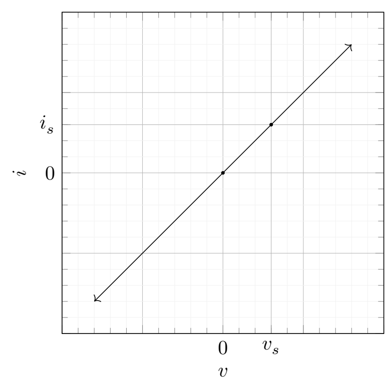

Figure 11: VI characteristic for a resistor.

3.2 VI Characteristic for Diodes

A diode has a non-linear characteristic curve, hence it is often simplified.

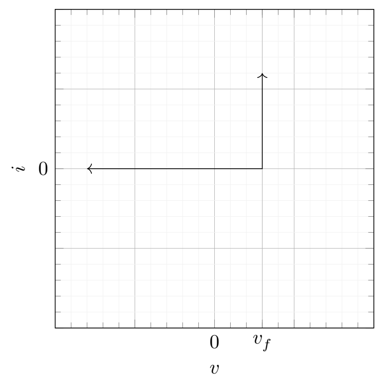

Figure 12: VI characteristic for a diode.

Theorem 3.2.1 (Shockley's Diode Equation)**.** *A diode can be better modelled using Shockley's equation.*

$$i = I_S \exp\left(\frac{v}{0.026}\right)$$

where I_S is the saturation current and 0.026 *is the thermal voltage.*

3.3 Operating Points and Load Lines

Definition 3.3.1 (Operating Point)**.** The operating point for two elements can be found by determining the intersection of the two VI characteristic curves.

Definition 3.3.2 (Load Lines)**.** If a circuit contains three or more elements including a diode, the VI characteristic curve is found around the diode. This curve is called a load line.

3.4 Operating Point of Non-Linear Component

Given the following circuit with a non-linear component

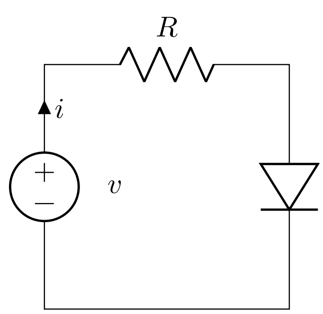

Figure 13: Circuit with non-linear element.

the load line is given by the equation

$$i(v) = -\frac{i_{sc}}{v_{oc}} + i_{sc}$$

where the short circuit current is the current through the non-linear component, and the open circuit voltage is the voltage across the open circuit nodes of the component.

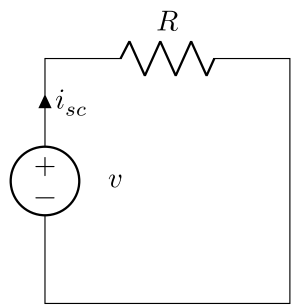

Figure 14: Circuit for short circuit current.

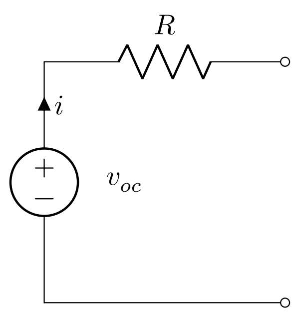

Figure 15: Circuit for open circuit voltage.

4 Mesh Analysis

Definition 4.0.1 (Node)**.** A point where two or more circuit elements join.

Definition 4.0.2 (Essential Node)**.** A point where three or more circuit elements join.

Definition 4.0.3 (Loop)**.** A path with the same start and end node.

Definition 4.0.4 (Mesh)**.** A loop that does not enclose any other loops.

4.1 Mesh Analysis Steps

1. Label the unknown mesh currents.
2. Find the voltage across each of the circuit elements in terms of mesh currents.
3. Use KVL around each mesh to create simultaneous equations.
4. Solve simultaneous equations for mesh currents.

4.2 Mesh Analysis with Current Sources

If the current source is in a single mesh, then treat the mesh current as known and solve as before. If the current source is between two meshes, then we must use a supermesh.

Definition 4.2.1 (Supermesh)**.** A supermesh is a special mesh that surrounds the current source.

1. Label meshes and identify the supermesh.
2. Write KCL equation for the current source.
3. Write the supermesh equation.
4. Use KVL around the supermesh.

5 Source Transformations

Real sources often have many limitations in terms of voltage and current delivery. The most commonly modelled, and most useful for linear circuit theory, is some form of resistance associated with the source.

5.1 Thévenin Equivalent Circuit

Definition 5.1.1. The Thévenin equivalent circuit is a voltage source with a series resistance.

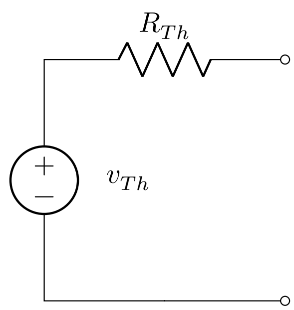

Figure 16: Thévenin equivalent circuit.

This circuit has the following properties:

Open circuit voltage. $v_{oc} = v_{Th}$

Short circuit current. $i_{sc} = \frac{v_{Th}}{R_{Th}}$

5.2 Norton Equivalent Circuit

Definition 5.2.1. Similar to the Thévenin equivalent model, we can use a current source with a resistor in parallel to model the same circuit.

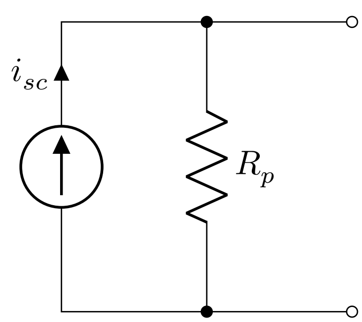

Figure 17: Norton equivalent circuit.

This circuit has the following properties:

Open circuit voltage. $v_{oc} = i_{sc} R_p$

Short circuit current. $i_{sc} = i_{sc}$

5.3 Source Transformations

A source transformation is the process of simplifying a circuit by transforming voltage sources into current sources, and vice versa, using Thévenin's Theorem and Norton's Theorem.

Theorem 5.3.1. *Thévenin and Norton equivalent circuits have the following relationship*

$$R_{Th} = R_p.$$

5.4 Superposition

Theorem 5.4.1 (Thévenin's Theorem)**.** *Any linear circuit can be replaced by a voltage source and a resistance in series.*

Theorem 5.4.2 (Norton's Theorem)**.** *Any linear circuit can be replaced by a current source and a resistance in parallel.*

Definition 5.4.1 (Turning Sources "off")**.** Turning a source "off" means to set its value to 0.

For a voltage source: $v = 0\,\mathrm{V}$, hence we can treat it as a short circuit.

For a current source: $i = 0\,\mathrm{A}$, hence we can treat it as an open circuit.

5.4.1 Equivalent Sources

To determine the equivalent voltage or current source between two nodes, we must turn on each source in the circuit one by one, and calculate the voltage or current between those nodes. The equivalent voltage or current is equal to the sum of contributions from each source.

5.4.2 Equivalent Resistance

The equivalent resistance is determined by turning off all sources and calculating the equivalent resistance between the two nodes.

5.5 Maximum Power Transfer

In the circuit shown below, the maximum power transfer to the load is given by

$$P_L = \frac{v_{Th}^2}{4R_L}$$

where $R_L = R_{Th}$.

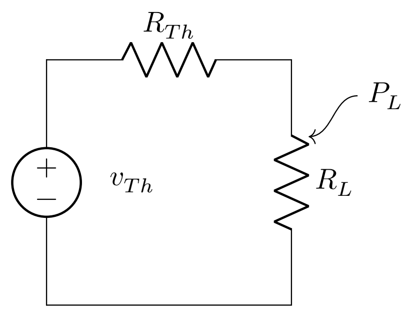

Figure 18: Power transfer in a Thévenin model.

6 Inductors and Capacitors

6.1 Capacitors

Definition 6.1.1. Capacitors store electrical energy as a voltage. The ratio of voltage to charge across a capacitor is its capacitance, measured in Farads (F).

$$q = Cv$$

Definition 6.1.2 (VI Relationship)**.**

$$i = C\frac{\mathrm{d}v}{\mathrm{d}t}$$

$$v(t) = \frac{1}{C}\int_0^t i(\tau)\,\mathrm{d}\tau + v(0)$$

Definition 6.1.3 (Energy Stored by an Capacitor)**.**

$$W = \frac{1}{2}Cv^2$$

Theorem 6.1.1 (Steady State Conditions)**.** *When a circuit is in steady state, capacitors can be modelled as open circuits.*

6.2 Inductors

Definition 6.2.1. Inductors create voltage to oppose a change in current. The ratio of voltage to the rate of change of current through an inductor is its inductance, measured in Henrys (H).

Definition 6.2.2 (VI Relationship)**.**

$$v = L\frac{\mathrm{d}i}{\mathrm{d}t}$$

$$i(t) = \frac{1}{L}\int_0^t v(\tau)\,\mathrm{d}\tau + i(0)$$

Definition 6.2.3 (Energy Stored by an Inductor)**.**

$$W = \frac{1}{2}Li^2$$

Theorem 6.2.1 (Steady State Conditions)**.** *When a circuit is in steady state, inductors can be modelled as short circuits.*

7 RC and RL Circuits

7.1 Switches

Definition 7.1.1. A switch will engage part of a circuit at a specified point in time, as indicated on the circuit diagram.

Definition 7.1.2 (Poles)**.** "Pole" refers to the number of circuits one switch can control.

Definition 7.1.3 (Throw)**.** "Throw" refers to the number of output connections each switch pole can connect its input to.

7.2 Natural Response

Definition 7.2.1 (RC Circuit Natural Response)**.** For the circuit shown below, using KCL gives the following relationship

$$-C\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{v}{R}.$$

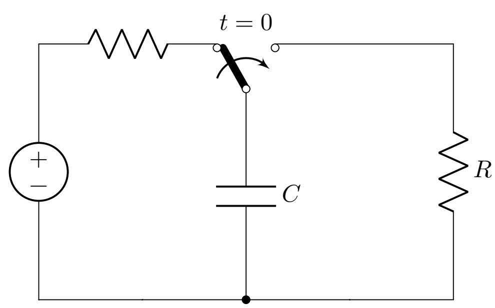

Figure 19: RC circuit.

By solving this differential equation, we can determine the natural response of a RC circuit.

$$v(t) = v(0)\exp\left(-\frac{t}{RC}\right)$$

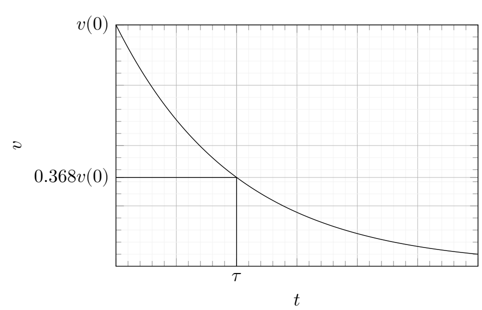

Figure 20: Natural response of a RC circuit.

Definition 7.2.2 (Time Constant)**.** The time constant τ is a parameter for switching circuits. The time constant for a RC circuit is given by

$$\tau = RC$$

The voltage at this time is equal to

$$e^{-1}v(0) \approx 0.368v(0)$$

Definition 7.2.3 (RL Circuit Natural Response)**.** For the circuit shown below, using mesh analysis gives the following relationship

$$L\frac{\mathrm{d}i}{\mathrm{d}t} = -iR.$$

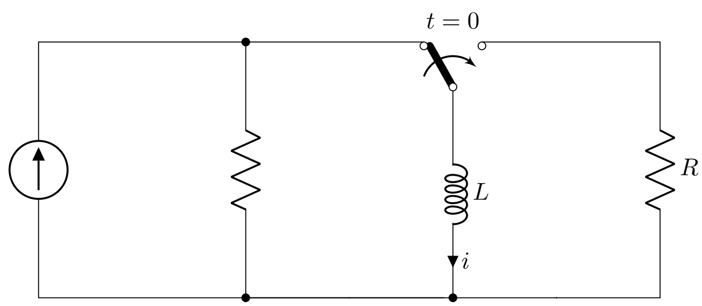

Figure 21: RL circuit.

By solving this differential equation, we can determine the natural response of a RL circuit.

$$i(t) = i(0)\exp\left(-\frac{t}{\frac{1}{R}L}\right)$$

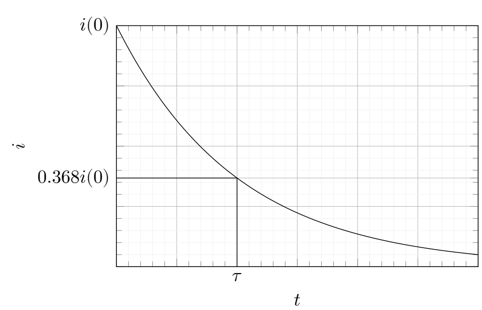

Figure 22: Natural response of a RL circuit.

Definition 7.2.4 (Time Constant)**.** The time constant τ is a parameter for switching circuits. The time constant for a RL circuit is given by

$$\tau = \frac{1}{R}L.$$

The current at this time is equal to

$$e^{-1}i(0) \approx 0.368i(0).$$

7.3 Step Response

Definition 7.3.1 (RC Circuit Step Response)**.** For the circuit shown below, using KCL at the top node gives the following relationship

$$i_s = C\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{v}{R}.$$

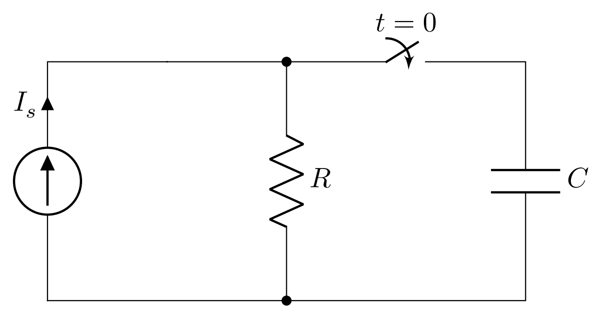

Figure 23: RC circuit.

By solving this differential equation, we can determine the step response of a RC circuit.

$$v(t) = I_sR + (V_0 - I_sR)\exp\left(-\frac{t}{RC}\right)$$

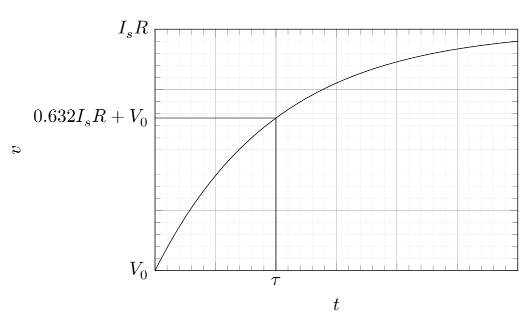

Figure 24: Step response of a RC circuit.

The voltage at $t = \tau$ is equal to

$$\begin{aligned} v(\tau) &= \left(1 - e^{-1}\right) I_s R + V_0 \\ &\approx 0.632 I_s R + V_0. \end{aligned}$$

Definition 7.3.2 (RL Circuit Step Response)**.** For the circuit shown below, using mesh analysis gives the following relationship

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + iR = V_s.$$

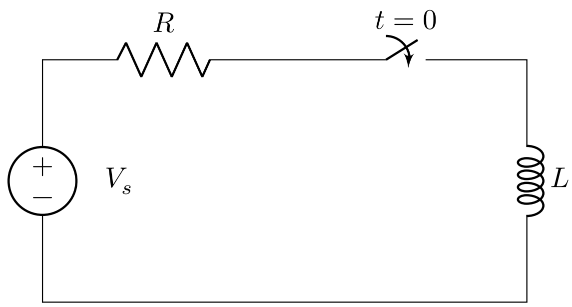

Figure 25: RL circuit.

By solving this differential equation, we can determine the step response of a RL circuit.

$$i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right) \exp\left(-\frac{t}{\frac{1}{R}L}\right)$$

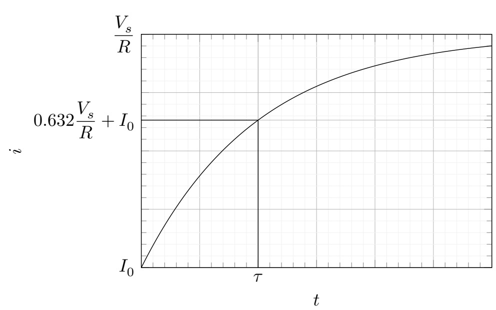

Figure 26: Step response of a RL circuit.

The current at $t = \tau$ is equal to

$$\begin{aligned} i(\tau) &= \left(1 - e^{-1}\right) \frac{V_s}{R} + I_0 \\ &\approx 0.632\frac{V_s}{R} + I_0. \end{aligned}$$

8 Operational Amplifiers

Definition 8.0.1 (Amplifier)**.** An amplifier is a device for increasing the power of a signal through an external energy source. In an electronic amplifier, the "signal" is usually a voltage or current.

Definition 8.0.2 (Gain)**.** The gain K of an amplifier, is the ratio of the output signal to the input signal.

$$v_{out} = Kv_{in}$$

Definition 8.0.3 (Operational Amplifier)**.** An operational amplifier (op amp) amplifies the voltage difference between its input terminals. Operational amplifiers require a power supply to amplify voltage. In an operational amplifier:

1. The output cannot exceed the power supply range.
2. The inputs should remain within the power supply range.

The output voltage has the following possibilities:

1. If $v_p - v_n > 0$, $v_{out} = V_{CC}$.
2. If $v_p - v_n < 0$, $v_{out} = V_{EE}$.
3. If $v_p - v_n = 0$, $v_{out} = 0$.

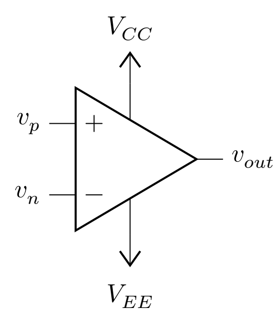

Figure 27: Operational amplifier circuit symbol.

v_p: non-inverting input

v_n: inverting input

V_{CC}: positive power supply

V_{EE}: negative power supply

8.1 Golden Rules for Op Amps

1. Infinite input impedance, therefore, zero input current.
2. In a closed loop, the output drives the voltage difference at the inputs to zero.

Note that the second rule only applies when the op amp has external negative feedback.

8.2 Op Amp Analysis

Using the golden rules above, the following circuits can be constructed.

8.2.1 Inverting Amplifier

$$v_{out} = -\frac{R_2}{R_1} v_{in}$$

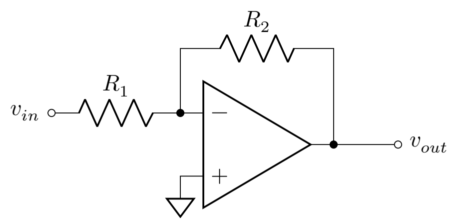

Figure 28: Inverting amplifier.

8.2.2 Non-Inverting Amplifier

$$v_{out} = \frac{R_1 + R_2}{R_1} v_{in}$$

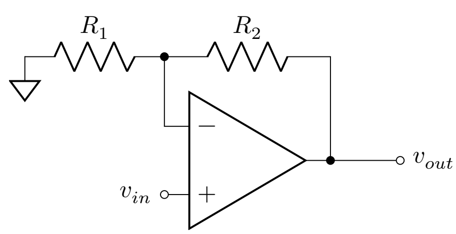

Figure 29: Non-inverting amplifier.

8.2.3 Voltage Follower

$$v_{out} = v_{in}$$

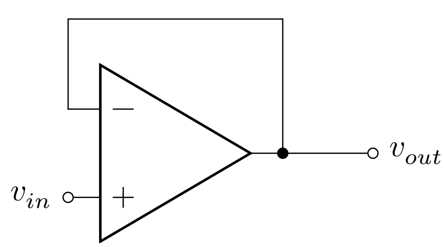

Figure 30: Voltage follower.

8.2.4 Inverting Summing Amplifier

$$v_{out} = -R_f \left(\frac{v_1}{R_1} + \frac{v_2}{R_2} + \cdots + \frac{v_n}{R_n} \right)$$

where R_f is the feedback resistor.

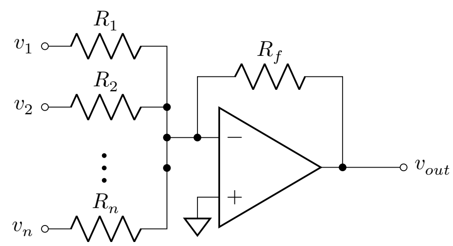

Figure 31: Inverting summing amplifier.

8.2.5 Difference Amplifier

$$v_{out} = \frac{R_2}{R_1} (v_2 - v_1)$$

where the ratio of $\frac{R_2}{R_1}$ must be equal on both legs.

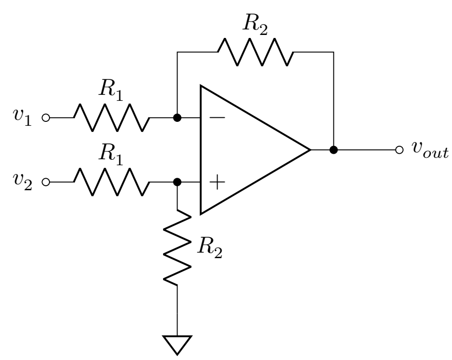

Figure 32: Difference amplifier.

8.3 Designing Op Amps

When designed op amps, it is important to choose resistors in the $1\,\mathrm{k}\Omega$ to $100\,\mathrm{k}\Omega$ range, to prevent damaging the op amp, and to avoid excessive output currents.

9 Sinusoidal Signals

Definition 9.0.1. A sinusoidal signal is commonly referred to as Alternating Current or AC.

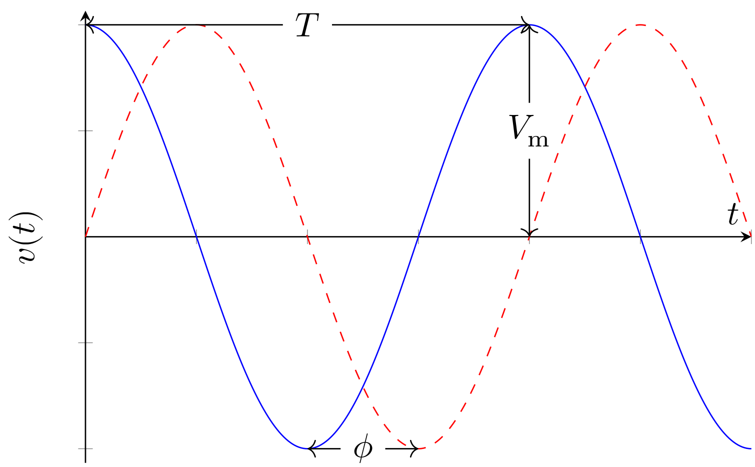

Figure 33: Sinusoidal signals.

9.1 Properties of Sinusoidal Signals

These signals are described using cosine functions in the following form.

$$v(t) = V_{\mathrm{m}} \cos(\omega t + \phi^\circ)$$

9.1.1 Magnitude

Definition 9.1.1. The magnitude of a signal is the greatest distance from the line of oscillation, commonly zero, to the peak of the signal.

9.1.2 Angular Frequency

Definition 9.1.2. The angular frequency ω is measured in radians per second ($\mathrm{rad\,s^{-1}}$), and it allows us to compute the cosine function in radians.

9.1.3 Period and Frequency

Definition 9.1.3 (Period)**.** The period of a signal T is its cycle time in seconds (s).

Definition 9.1.4 (Frequency)**.** The frequency of a signal f is the number of cycles per second, or the inverse of the period. The frequency is measured in Hertz (Hz).

The following equations relate the angular frequency, period and frequency together.

$$f = \frac{1}{T} \qquad T = \frac{2\pi}{\omega} \qquad \omega = 2\pi f$$

9.1.4 Phase

Definition 9.1.5. The phase ϕ of a signal is the position of a signal relative to a zero phase signal, measured in degrees (°). A **positive** phase corresponds to a "leading" signal, and a **negative** phase corresponds to a "lagging" signal. The phase can be determined by calculating the difference in time τ between two signals.

$$\phi = \frac{\tau}{T}$$

9.2 Root Mean Square

Definition 9.2.1. Root mean square (rms) is a method of obtaining a useful average of a signal that is symmetric about the horizontal axis. It is defined as the square **root** of the **mean** value of the function **squared**. For a sine wave

$$V_{\mathrm{rms}} = \frac{V_{\mathrm{m}}}{\sqrt{2}}$$

The units in rms quantities are also subscripted with rms, for example, rms voltage, $\mathrm{V}_{\mathrm{rms}}$.

9.3 Power

Definition 9.3.1. In a resistive load, the power can be determined using the rms current and rms voltage.

$$\begin{aligned} p &= vi \\ p &= V_{\mathrm{rms}} I_{\mathrm{rms}} (1 + \cos(2\omega t + 2\phi)) \end{aligned}$$

The average power is given by

$$P_{\mathrm{avg}} = V_{\mathrm{rms}} I_{\mathrm{rms}}$$

9.4 Euler's Formula

Theorem 9.4.1.

$$e^{j\phi} = \cos(\phi) + j\sin(\phi)$$

where $j^2 = -1$ *and* ϕ *is measured in radians.*

9.5 Phasors

Definition 9.5.1. To assist in AC analysis, we can express the magnitude and phase of a sinusoidal signal as a phasor.

9.6 Phasor Transforms

Definition 9.6.1. A sinusoidal signal can be converted to a phasor using the Phasor Transform

$$\begin{aligned} \mathbf{V} &= \mathcal{P}\{v(t)\} \\ &= \mathcal{P}\{V_{\mathrm{m}} \cos(\omega t + \phi)\} \\ &= V_{\mathrm{m}} e^{j\phi} \end{aligned}$$

Here $\mathbf{V}$ is the phase-domain representation of the time-domain signal $v(t)$. The magnitude and phase of a phasor can be used to represent a phasor in polar form

$$\mathbf{V} = V_{\mathrm{m}}\angle\phi$$

9.7 Inverse Phasor Transforms

Definition 9.7.1. A phasor can be converted back to a sinusoidal signal using the Inverse Phasor Transform

$$\begin{aligned} v(t) &= \mathcal{P}^{-1}\left\{\mathbf{V}\right\} \\ &= \Re\left\{\mathbf{V}e^{j\omega t}\right\} \\ &= V_{\mathrm{m}}\cos\left(\omega t + \phi\right) \end{aligned}$$

9.8 Phasor Operations

As phasors are quantities with a magnitude and angle, they can be represented as complex numbers. Using Theorem 9.4.1, we can represent a phasor in rectangular form

$$\mathbf{V} = V_{\mathrm{m}}\cos\left(\phi\right) + jV_{\mathrm{m}}\sin\left(\phi\right)$$

in polar form, the magnitude is given by

$$\|\mathbf{V}\| = V_{\mathrm{m}} = \sqrt{\Re\left\{\mathbf{V}\right\}^2 + \Im\left\{\mathbf{V}\right\}^2}$$

and the angle

$$\arg\left(\mathbf{V}\right) = \phi = \arctan\left(\frac{\Im\left\{\mathbf{V}\right\}}{\Re\left\{\mathbf{V}\right\}}\right)$$

where $-\pi < \phi \leq \pi$.

9.8.1 Addition with Phasors

Phasors in rectangular form can be added using their real and imaginary parts.

$$\begin{aligned} \mathbf{A} + \mathbf{B} &= \Re\left\{\mathbf{A} + \mathbf{B}\right\} + \Im\left\{\mathbf{A} + \mathbf{B}\right\} \\ &= \left(A_{\mathrm{m}}\cos\left(\phi_1\right) + B_{\mathrm{m}}\cos\left(\phi_2\right)\right) + j\left(A_{\mathrm{m}}\sin\left(\phi_1\right) + B_{\mathrm{m}}\sin\left(\phi_2\right)\right) \end{aligned}$$

9.8.2 Multiplication with Phasors

Phasors in polar form can be multiplied using their magnitudes and phases.

$$\begin{aligned} \mathbf{AB} &= \|\mathbf{A}\|\|\mathbf{B}\|\angle\arg\left(\mathbf{A}\right) + \arg\left(\mathbf{B}\right) \\ &= A_{\mathrm{m}}B_{\mathrm{m}}\angle\phi_1 + \phi_2 \end{aligned}$$

9.9 Circuit Analysis

Theorem 9.9.1 (Phasor Relationship for Resistors)**.**

$$\mathbf{V} = \mathbf{I}R$$

Proof. Using a sinusoidal voltage

$$v = V_{\mathrm{m}} \cos(\omega t + \phi)$$

Ohm's Law gives:

$$\begin{aligned} i &= \frac{v}{R} \\ &= \frac{V_{\mathrm{m}}}{R} \cos(\omega t + \phi) \end{aligned}$$

Taking the Phasor Transform gives

$$\begin{aligned} \mathcal{P}\{v\} &= \mathcal{P}\{iR\} \\ \mathcal{P}\{V_{\mathrm{m}} \cos(\omega t + \phi)\} &= \mathcal{P}\left\{\frac{V_{\mathrm{m}}}{R} \cos(\omega t + \phi)\right\} R \\ \mathbf{V} &= \mathbf{I}R \end{aligned}$$

□

Theorem 9.9.2 (Phasor Relationship for Inductors)**.**

$$\mathbf{V} = j\omega L\mathbf{I}$$

Proof. Using a sinusoidal current

$$i = I_{\mathrm{m}} \cos(\omega t + \phi)$$

the voltage drop across an inductor is given by:

$$\begin{aligned} v &= L\frac{\mathrm{d}i}{\mathrm{d}t} \\ &= -\omega L I_{\mathrm{m}} \sin(\omega t + \phi) \\ &= -\omega L I_{\mathrm{m}} \cos(\omega t + \phi - 90°) \end{aligned}$$

taking the Phasor Transform gives

$$\begin{aligned} \mathcal{P}\{v\} &= \mathcal{P}\left\{L\frac{\mathrm{d}i}{\mathrm{d}t}\right\} \\ \mathbf{V} &= \mathcal{P}\{-\omega L I_{\mathrm{m}} \cos(\omega t + \phi - 90°)\} \\ \mathbf{V} &= -\omega L I_{\mathrm{m}} e^{j(\phi - 90°)} \\ \mathbf{V} &= -\omega L I_{\mathrm{m}} e^{j\phi} e^{-j90°} \\ \mathbf{V} &= j\omega L I_{\mathrm{m}} e^{j\phi} \\ \mathbf{V} &= j\omega L\mathbf{I} \end{aligned}$$

□

Theorem 9.9.3 (Phasor Relationship for Capacitors)**.**

$$\mathbf{V} = \frac{1}{j\omega C}\mathbf{I}$$

Proof. Using a sinusoidal voltage

$$v = V_\mathrm{m}\cos(\omega t + \phi)$$

the voltage-current relationship across a capacitor is given by:

$$\begin{aligned} i &= C\frac{\mathrm{d}v}{\mathrm{d}t} \\ &= -\omega C V_\mathrm{m}\sin(\omega t + \phi) \\ &= -\omega C V_\mathrm{m}\cos(\omega t + \phi - 90°) \end{aligned}$$

taking the Phasor Transform gives

$$\begin{aligned} \mathcal{P}\{i\} &= \mathcal{P}\left\{C\frac{\mathrm{d}v}{\mathrm{d}t}\right\} \\ \mathbf{I} &= \mathcal{P}\{-\omega C V_\mathrm{m}\cos(\omega t + \phi - 90°)\} \\ \mathbf{I} &= -\omega C V_\mathrm{m} e^{j(\phi - 90°)} \\ \mathbf{I} &= -\omega C V_\mathrm{m} e^{j\phi} e^{-j90°} \\ \mathbf{I} &= j\omega C V_\mathrm{m} e^{j\phi} \\ \mathbf{I} &= j\omega C\mathbf{V} \\ \mathbf{V} &= \frac{1}{j\omega C}\mathbf{I} \end{aligned}$$

□

9.10 Impedance

Definition 9.10.1. The impedance $\mathbf{Z}$ of an element, measured in Ohms (Ω), is its opposition to alternating current. Impedance is represented as the sum of a real resistance R and imaginary reactance X.

$$\mathbf{Z} = R + jX$$

Impedance captures the magnitude and phase change associated with a circuit element.

Theorem 9.10.1 (Impedance of Resistors)**.**

$$\mathbf{Z} = R$$

Theorem 9.10.2 (Impedance of Inductors)**.**

$$\mathbf{Z} = j\omega L$$

Theorem 9.10.3 (Impedance of Capacitors)**.**

$$\mathbf{Z} = \frac{1}{j\omega C}$$

9.11 Impedances in Circuits

Theorem 9.11.1. *Impedances behave similarly to resistances in series and parallel.*

10 Frequency Response

Frequency response is a measure of the magnitude and phase of a system as a function of (angular) frequency ω.

10.1 Circuit Analysis with AC Circuits

Theorem 10.1.1. *Any DC analysis technique can be used with AC circuits, as long as resistive elements are represented using complex impedances.*

10.2 Transfer Function

Definition 10.2.1. The transfer function of a system is given by

$$\mathbf{H}(\omega) = \frac{\mathbf{Y}(\omega)}{\mathbf{X}(\omega)}$$

where $\mathbf{Y}$ is the output and $\mathbf{X}$ is the input of a system.

Definition 10.2.2 (Linear Gain)**.** The magnitude of the transfer function $\|\mathbf{H}\|$ measures the linear gain of a system.

Definition 10.2.3 (Gain in dB)**.** The gain of a system is often represented in dB using the following formula

$$\text{gain} = 20 \log_{10} \|\mathbf{H}\|$$

Definition 10.2.4 (Phase Shift)**.** The phase of the transfer function $\arg(\mathbf{H})$ measures the phase shift of a system.

10.3 Bode Plots

A Bode plot is a graph of the frequency response of a system. In this plot, the frequency (in Hz) is plotted on the horizontal axis using a logarithmic scale.

Definition 10.3.1 (3 dB Point)**.** The frequency f at which the transfer function equals

$$\mathbf{H}(2\pi f) = \frac{1}{\sqrt{2}}\underline{/-45^\circ}$$

is known as the break frequency, corner frequency, 3 dB frequency, or half power point.
This is because the gain at this point is equal to $0.707 \approx -3.01\,\text{dB}$, and the angle $-45°$ is the halfway point between $0°$ and $-90°$.

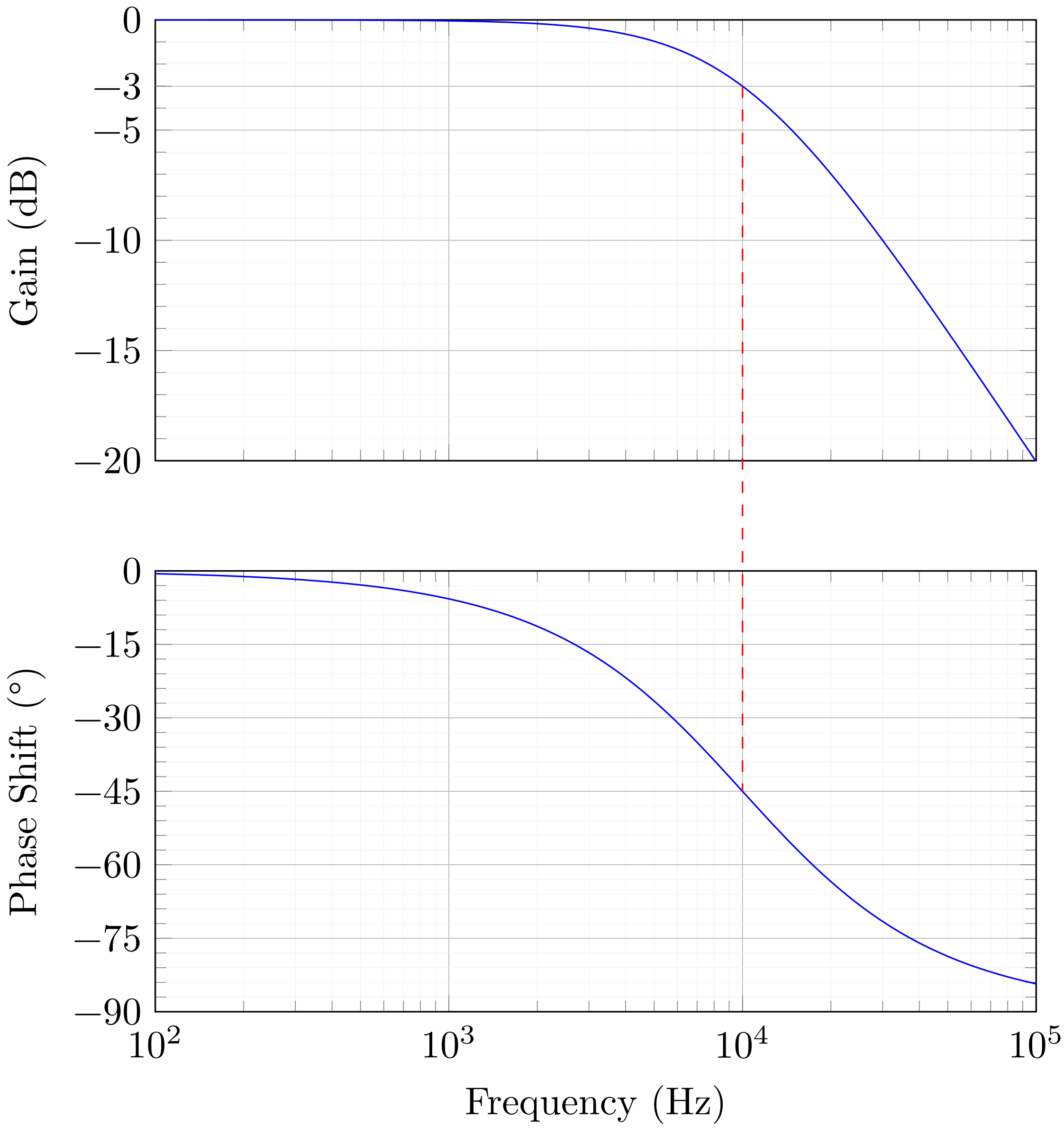

Figure 34: Bode plot of gain and phase shift.

11 Filters and Rectifiers

Definition 11.0.1 (Filters)**.** A filter is designed to allow certain ranges of frequencies to pass, while other ranges of frequencies are stopped.

Definition 11.0.2 (Cut-off Frequncy)**.** The cut-off (angular) frequency ω_c for a filter is given by

$$\omega_c = \frac{1}{RC}$$

11.1 Common Filters

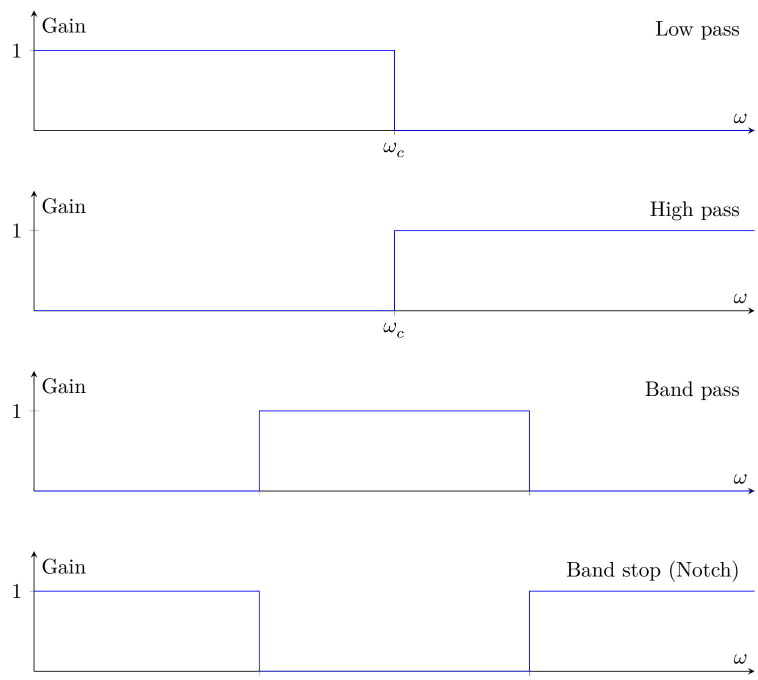

Figure 35: Different types of filters.

11.2 Passive Filters

Passive filters are circuits containing only passive elements (such as resistors, capacitors and inductors), which filter unwanted parts of a signal. Passive filters are affected by loading, such that a varied load will affect the cut-off frequency.

11.2.1 Low Pass Filter

The following low pass filter has the following transfer function.

$$\mathbf{H}(\omega) = \frac{\omega_c}{j\omega + \omega_c}$$

where $\omega_c = \dfrac{1}{RC}$.

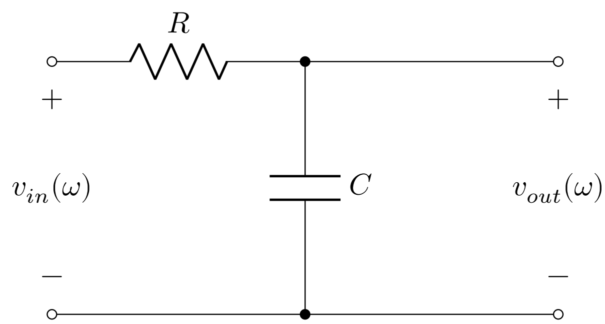

Figure 36: Passive low pass filter.

11.2.2 High Pass Filter

The following high pass filter has the following transfer function.

$$\mathbf{H}(\omega) = \frac{j\omega}{j\omega + \omega_c}$$

where $\omega_c = \dfrac{1}{RC}$.

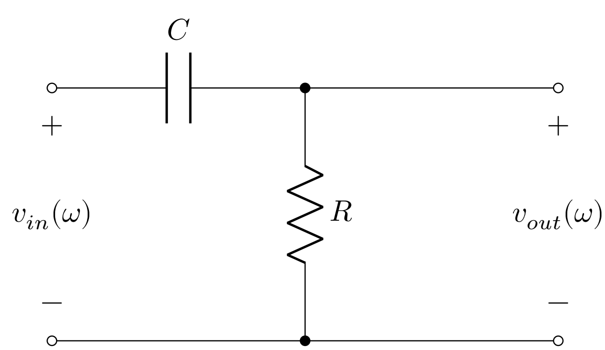

Figure 37: Passive high pass filter.

11.3 Active Filters

Active filters are circuits containing active elements (such as op amps), which filter unwanted parts of a signal. Active filters are not affected by loading, such that a varied load will not affect the cut-off frequency.
For the following active filters, the gain component is given by

$$\text{gain} = -\frac{R_2}{R_1}$$

11.3.1 Low Pass Filter

The following low pass filter has the following transfer function.

$$\mathbf{H}(\omega) = -\frac{R_2}{R_1}\frac{\omega_c}{j\omega + \omega_c}$$

where $\omega_c = \dfrac{1}{R_2 C}$.

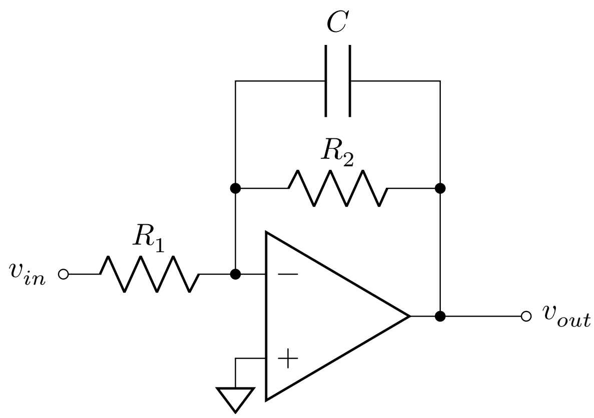

Figure 38: Active low pass filter.

11.3.2 High Pass Filter

The following high pass filter has the following transfer function.

$$\mathbf{H}(\omega) = -\frac{R_2}{R_1}\frac{j\omega}{j\omega + \omega_c}$$

where $\omega_c = \dfrac{1}{R_1 C}$.

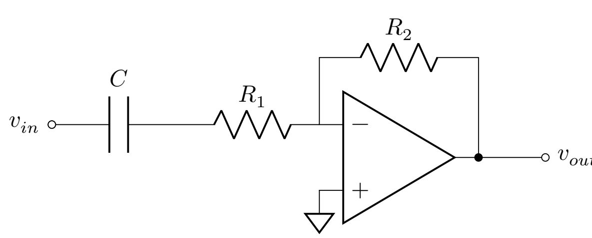

Figure 39: Active high pass filter.

11.4 Rectifiers

12 Zener Diodes and Voltage Regulators

12.1 Zener Diodes

A Zener diode is similar to a regular diode, but it has a specific reverse breakdown voltage.

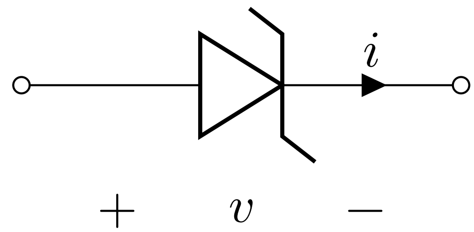

Figure 40: Zener diode circuit diagram.

The ideal VI characteristic for a Zener diode is shown below.

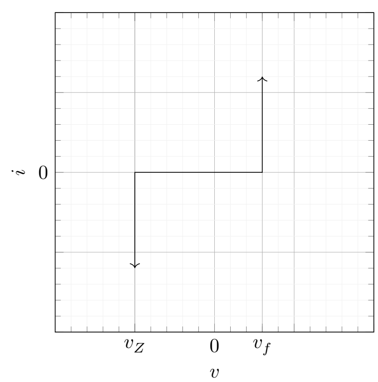

Figure 41: VI characteristic for an ideal Zener diode.

12.2 Shunt Regulators

A Zener diode can be used as a voltage regulator, to maintain a constant output voltage, given a varying voltage source and varying output current.

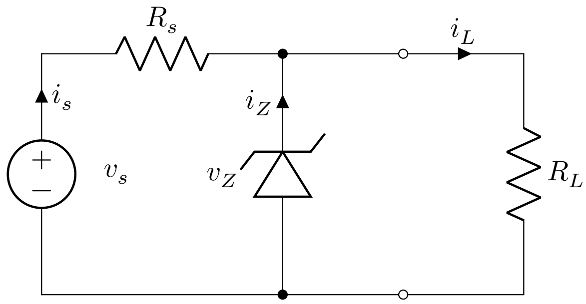

Figure 42: Zener diode regulator.

Theorem 12.2.1 (Maximum Load Current.)**.** *Given a source voltage range v_s and source resistance R_s, the maximum current an ideal Zener diode can regulate is given by*

$$i_L < \frac{v_s - v_Z}{R_s}$$

where the largest value in v_s is chosen.

Theorem 12.2.2 (Maximum Source Resistance.)**.** *Given a source voltage range v_s and load current range i_L, the maximum source resistance required to regulate the output voltage is given by*

$$R_s < \frac{v_s - v_Z}{i_L}$$

where the smallest value in v_s is chosen, and the largest value in i_L is chosen.

Proof. Using KCL gives

$$\begin{aligned} i_s - i_Z &= i_L \\ i_Z &= i_s - i_L. \end{aligned}$$

For a Zener diode to operate at its reverse breakdown voltage, the Zener current must be nonzero. This gives

$$0 < \frac{v_s - v_Z}{R_s} - i_L$$

We must now consider the worst case scenario, namely, when $\frac{v_s - v_Z}{R_s}$ is minimised, and when i_L is maximised.
Hence the maximum load current is when v_s is maximal.
And the maximum source resistance is when v_s is minimal, and i_L is maximal. $\square$

In a Zener regulator, the power dissipation

12.3 Series Regulators