Foundations of Electrical Engineering

Semester 2, 2021

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Contents

1 Electrical Circuits

1.1 SI Prefixes

Factor	**Name**	**Symbol**
10^{24}	yotta	Y
10^{21}	zetta	Z
10^{18}	exa	E
10^{15}	peta	P
10^{12}	tera	T
10^{9}	giga	G
10^{6}	mega	M
10^{3}	kilo	k

Factor	**Name**	**Symbol**
10^{-3}	milli	m
10^{-6}	micro	µ
10^{-9}	nano	n
10^{-12}	pico	p
10^{-15}	femto	f
10^{-18}	atto	a
10^{-21}	zepto	z
10^{-24}	yocto	y

Table 1: SI prefixes.

1.2 Fundamental Quantities

Name	**Definition**	**Symbol**	**Unit**
Charge	Electric charge is a fundamental property of matter that governs how particles are affected by an electromagnetic field.	q	Coulomb (C)
Work	Work is the energy transferred by an electric charge moving through an electric field.	w	Joule (J)
Current	Current is the rate of flow of electric charge past a point in an electric circuit: $i = \frac{\mathrm{d}q}{\mathrm{d}t} \iff 1\,\mathrm{A} = 1\,\mathrm{C\,s^{-1}}$	i	Ampere (A)
Voltage	Voltage is the electric potential difference between two points in an electric circuit: $v = \frac{\mathrm{d}w}{\mathrm{d}q} \iff 1\,\mathrm{V} = 1\,\mathrm{J\,C^{-1}}$	v	Volt (V)
Power	Power is the rate at which work is done in an electric circuit: $p = \frac{\mathrm{d}w}{\mathrm{d}t} \iff 1\,\mathrm{W} = 1\,\mathrm{J\,s^{-1}}$	p	Watt (W)

Table 2: Fundamental quantities in electrical circuits.

Charge in an electron. $q = 1.6022 \times 10^{-19}\,\mathrm{C}$.

Electric Power. $p = \frac{\mathrm{d}w}{\mathrm{d}t} = vi$.

1.3 Passive Sign Convention

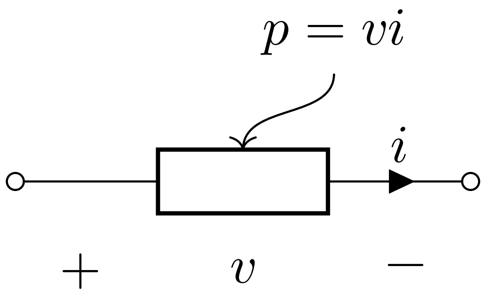

Figure 1: Energy dissipation.

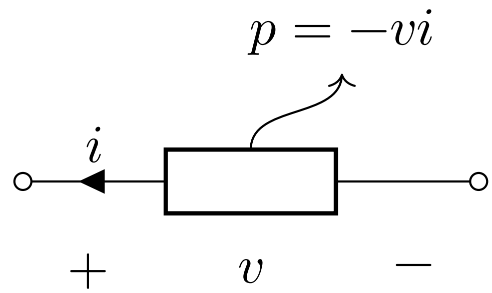

Figure 2: Energy production.

Theorem 1.3.1 (Power Balance)**.**

$$p_{\mathrm{net}} = 0$$

Theorem 1.3.2 (Energy)**.**

$$w\left(\tau\right) = \int_0^\tau p\left(t\right) \mathrm{d}t$$

1.4 Circuits and Sources

Definition 1.1 (Circuits)**.** A circuit is a mathematical model that approximates a real electronic system. It is built from ideal circuit elements connected by ideal wires.

Definition 1.2 (Voltage source)**.** A voltage source produces or dissipates power at a specified voltage with the current required.

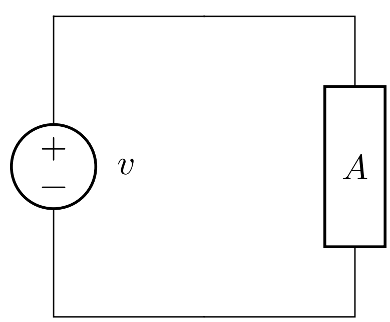

Figure 3: Voltage source — v is specified, i varies depending on circuit element A.

Definition 1.3 (Current source)**.** A current source produces or dissipates power at a specified current with the voltage required.

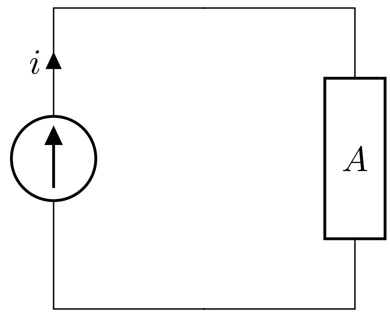

Figure 4: Current source — i is specified, v varies depending on circuit element A.

1.5 Resistors

Definition 1.4 (Resistor)**.** Resistors dissipate power and the voltage across both terminals is proportional to the current through the resistor.

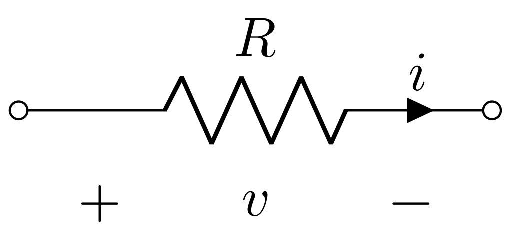

Figure 5: Resistor circuit symbol.

Theorem 1.5.1 (Voltage through a resistor)**.** *The voltage across a resistor is given by Ohm's Law which states*

$$v = iR.$$

Corollary 1.5.1.1 (Power dissipated by a resistor)**.** *The power dissipated by a resistor can therefore be calculated as*

$$p = vi = i^2R = \frac{v^2}{R}.$$

2 Simple Resistive Circuits

2.1 Ignored Physics

1. Electrical effects occur instantaneously, so there is no time delay along the wires.
2. The net charge on every component is zero. Charge is never lost or gained.
3. There is no magnetic coupling between the components.

2.2 Kirchhoff's Laws

Theorem 2.2.1 (Kirchhoff's current law (KCL))**.** *The sum of all currents into a node is zero:*

$$\sum i_{\text{node}} = 0.$$

Theorem 2.2.2 (Kirchhoff's voltage law (KVL))**.** *The sum of all voltages around a loop is zero:*

$$\sum v_{\text{loop}} = 0.$$

2.3 Series and Parallel Circuits

Definition 2.1. Elements connected end-to-end are in series. If both ends of an element are connected directly to another element, the two elements are in parallel.

The following table can be used to simplify quantities for components connected in series and parallel.

Element	**Series**	**Parallel**
Current Source	$i_{\text{eq}} = i_{k\geq 1}$	$i_{\text{eq}} = \sum_{k\geq 1} i_k$
Voltage Source	$v_{\text{eq}} = \sum_{k\geq 1} v_k$	$v_{\text{eq}} = v_{k\geq 1}$
Resistor	$R_{\text{eq}} = \sum_{k\geq 1} R_k$	$\frac{1}{R_{\text{eq}}} = \sum_{k\geq 1} \frac{1}{R_k}$
Inductor	$L_{\text{eq}} = \sum_{k\geq 1} L_k$	$\frac{1}{L_{\text{eq}}} = \sum_{k\geq 1} \frac{1}{L_k}$
Capacitor	$\frac{1}{C_{\text{eq}}} = \sum_{k\geq 1} \frac{1}{C_k}$	$C_{\text{eq}} = \sum_{k\geq 1} C_k$

Table 3: Equivalent values for various components connected in series and parallel.

2.4 Voltage and Current Dividers

Definition 2.2 (Voltage divider)**.** A voltage divider is a circuit that distributes voltage among series resistors in proportion to their resistances.

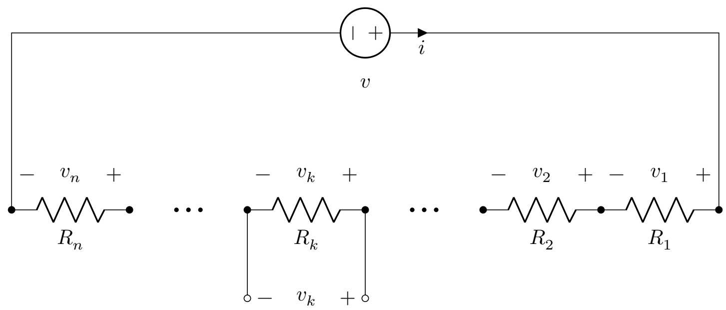

Figure 6: Voltage divider circuit.

Theorem 2.4.1 (Voltage divider)**.**

$$v_k = v\frac{R_k}{R_{\text{eq}}}$$

where R_{eq} is the equivalent resistance of the circuit.

Proof. The current through any resistor is the same,

$$i = \frac{v}{R_{\text{eq}}}.$$

Therefore, the voltage drop across the kth resistor is given by

$$v_k = iR_k$$
$$v_k = \frac{v}{R_{\text{eq}}}R_k.$$

□

Definition 2.3 (Current divider)**.** A current divider is a circuit that distributes current among parallel resistors in proportion to their resistances.

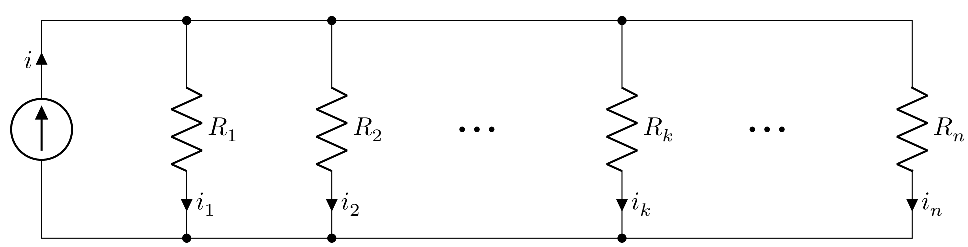

Figure 7: Current divider circuit.

Theorem 2.4.2 (Current divider)**.**

$$i_k = i\frac{R_{\text{eq}}}{R_k}$$

where R_{eq} is the equivalent resistance of the circuit.

Proof. Using KCL, we find

$$\begin{aligned} i &= \sum_{k\geq 1} i_k \\ &= \sum_{k\geq 1} \frac{v}{R_k} \\ &= \frac{v}{R_{\text{eq}}}. \end{aligned}$$

Solving for the voltage v across the current source gives us the relationship:

$$v = iR_{\text{eq}}.$$

Hence, the current through the kth resistor in parallel is given by

$$i_k = \frac{v}{R_k} = \frac{iR_{\text{eq}}}{R_k}.$$

□

3 Diodes

Definition 3.1 (Diode)**.** A diode is a semiconductor component in which current only flows in one direction. A diode requires a voltage to start the flow of current in the forward direction. This forward voltage is commonly 0.7 V.

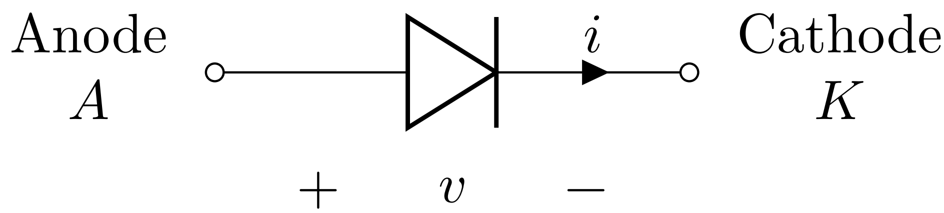

Figure 8: Diode circuit symbol.

3.1 VI Characteristic

The Voltage-Current characteristic of linear circuit elements can be plotted as shown below.

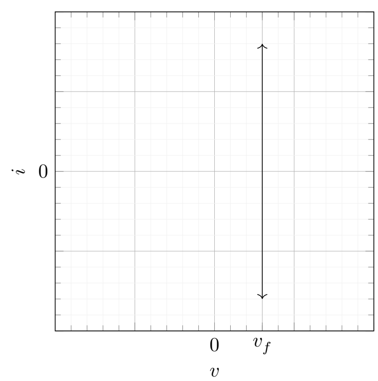

Figure 9: VI characteristic for a voltage source.

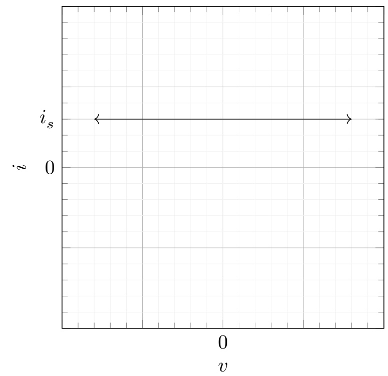

Figure 10: VI characteristic for a current source.

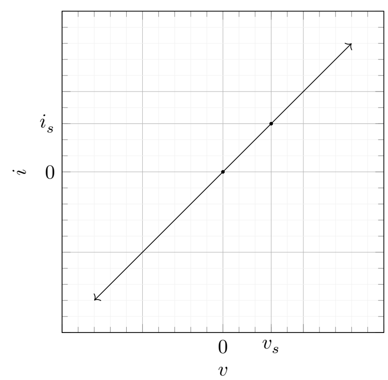

Figure 11: VI characteristic for a resistor.

3.2 VI Characteristic for Diodes

As the diode has a non-linear characteristic curve, it is often simplified using the following model.

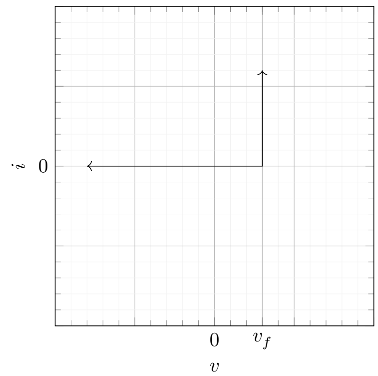

Figure 12: VI characteristic for a diode.

Theorem 3.2.1 (Shockley's diode equation)**.** *A diode can be modelled more precisely using the equation*

$$i = I_S \exp\left(\frac{v}{0.026}\right)$$

where I_S is the saturation current and 0.026 *is the thermal voltage.*

3.3 Operating Points and Load Lines

Definition 3.2 (Operating points)**.** The operating point for two elements can be found by determining the intersection of the two VI characteristic curves.

Definition 3.3 (Load lines)**.** If a circuit contains three or more elements and includes a diode, the VI characteristic curve is found around the diode. This curve is called a load line.

3.4 Operating Point of Non-Linear Component

Given the following circuit with a non-linear component

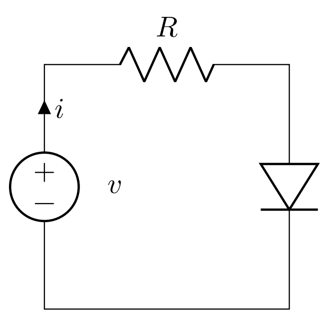

Figure 13: Circuit with non-linear element.

the load line is given by the equation

$$i(v) = -\frac{i_{sc}}{v_{oc}} + i_{sc}$$

where the short circuit current is the current through the non-linear component, and the open circuit voltage is the voltage across the open circuit nodes of the component.

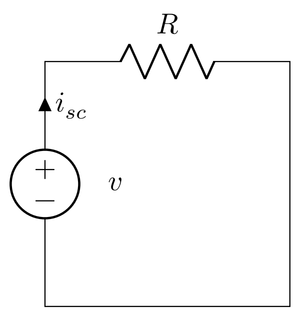

Figure 14: Circuit for short circuit current.

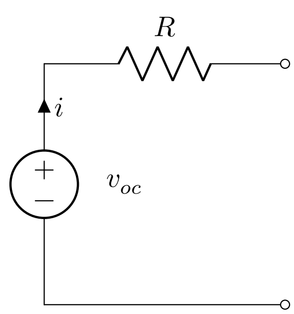

Figure 15: Circuit for open circuit voltage.

4 Mesh Analysis

Definition 4.1 (Node)**.** A point where two or more circuit elements join.

Definition 4.2 (Essential node)**.** A point where three or more circuit elements join.

Definition 4.3 (Loop)**.** A path with the same start and end node.

Definition 4.4 (Mesh)**.** A loop that does not enclose any other loops.

4.1 Mesh Analysis Steps

1. Label the unknown mesh currents.
2. Find the voltage across each of the circuit elements in terms of mesh currents.
3. Use KVL around each mesh to create a system of equations.
4. Solve the system of equations to determine mesh currents.

4.2 Mesh Analysis with Current Sources

If the current source is in a single mesh, we can treat the mesh current as known and solve the circuit as normal. If the current source is between two meshes, then we must use a supermesh.

Definition 4.5 (Supermesh)**.** A supermesh is a special mesh that surrounds a current source.

In this case, we must use the following steps to solve the circuit.

1. Label meshes and identify supermeshes.
2. Use KCL for any current sources.
3. Use KVL around the supermesh.
4. Solve the system of equations to determine mesh currents.

5 Source Transformations

Real sources often have many limitations in terms of voltage and current delivery. Additionally, most circuits in linear circuit theory are modelled as a source and a resistor. Therefore, let us consider the following circuits.

5.1 Thévenin Equivalent Circuit

Definition 5.1. A Thévenin equivalent circuit consists of a single voltage source and a series resistance.

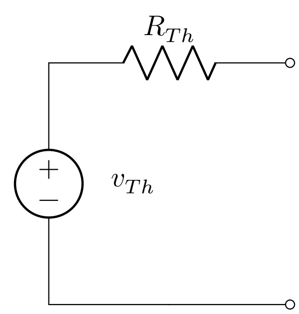

Figure 16: Thévenin equivalent circuit.

This circuit has the following properties:

Open circuit voltage. $v_{oc} = v_{Th}$

Short circuit current. $i_{sc} = \frac{v_{Th}}{R_{Th}}$

5.2 Norton Equivalent Circuit

Definition 5.2. A Norton equivalent circuit consists of a single current source and a parallel resistance. The Norton equivalent circuit is the dual of the Thévenin equivalent circuit.

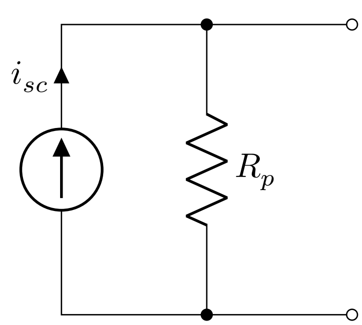

Figure 17: Norton equivalent circuit.

This circuit has the following properties:

Open circuit voltage. $v_{oc} = i_{sc} R_p$

Short circuit current. $i_{sc} = i_{sc}$

5.3 Source Transformations

We can use source transformations to simplify linear circuits by transforming voltage sources into current sources, and vice versa, using Thévenin's Theorem and Norton's Theorem.

Theorem 5.3.1. *Thévenin and Norton equivalent circuits have the following relationship*

$$R_{Th} = R_p.$$

Theorem 5.3.2 (Thévenin's theorem)**.** *Any linear circuit can be replaced by a single voltage source and a resistance in series.*

Theorem 5.3.3 (Norton's theorem)**.** *Any linear circuit can be replaced by a single current source and a resistance in parallel.*

5.4 Superposition

The superposition theorem states that a circuit with multiple sources can be analysed by considering only a single source at a time. Individual component voltages and currents can then be added algebraically to determine the response of the circuit with all sources enabled. In this strategy, setting a voltage source to zero is equivalent to a short circuit, whereas setting a current source to zero is equivalent to an open circuit.

5.4.1 Equivalent Sources

The equivalent voltage or current source across a load can be determined using superposition.

5.4.2 Equivalent Resistance

The equivalent resistance across two nodes can be determined by removing all sources and calculating the equivalent resistance between the two nodes.

5.5 Maximum Power Transfer

In the circuit shown below, the maximum power transfer to the load is given by

$$P_L = \frac{v_{Th}^2}{4R_L}$$

where $R_L = R_{Th}$.

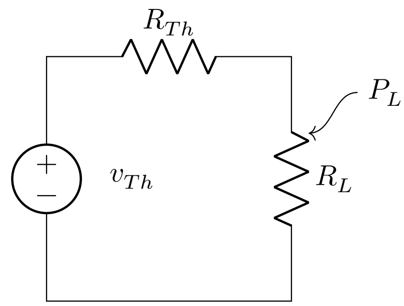

Figure 18: Power transfer in a Thévenin model.

6 Inductors and Capacitors

6.1 Capacitors

Definition 6.1. Capacitors store electrical energy as a voltage. The ratio of voltage to charge across a capacitor is its capacitance, measured in Farads (F).

$$q = Cv$$

Definition 6.2 (VI relationship)**.**

$$i = C\frac{\mathrm{d}v}{\mathrm{d}t}$$

$$v(t) = \frac{1}{C}\int_0^t i(\tau)\,\mathrm{d}\tau + v(0)$$

Definition 6.3 (Energy stored by an capacitor)**.**

$$W = \frac{1}{2}Cv^2$$

Theorem 6.1.1 (Steady state conditions)**.** *When a circuit is in steady state, capacitors can be modelled as open circuits.*

6.2 Inductors

Definition 6.4. Inductors create voltage to oppose a change in current. The ratio of voltage to the rate of change of current through an inductor is its inductance, measured in Henrys (H).

Definition 6.5 (VI relationship)**.**

$$v = L\frac{\mathrm{d}i}{\mathrm{d}t}$$

$$i(t) = \frac{1}{L}\int_0^t v(\tau)\,\mathrm{d}\tau + i(0)$$

Definition 6.6 (Energy stored by an inductor)**.**

$$W = \frac{1}{2}Li^2$$

Theorem 6.2.1 (Steady state conditions)**.** *When a circuit is in steady state, inductors can be modelled as short circuits.*

7 RC and RL Circuits

7.1 Switches

Definition 7.1. A switch engages part of a circuit at a specified point in time, as indicated on a circuit diagram.

Definition 7.2 (Poles)**.** "Pole" refers to the number of circuits a switch can control.

Definition 7.3 (Throw)**.** "Throw" refers to the number of output connections a switch pole can connect its input to.

7.2 Natural Response

Definition 7.4 (RC circuit natural response)**.** For the circuit shown below, using KCL gives the following relationship

$$-C\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{v}{R}.$$

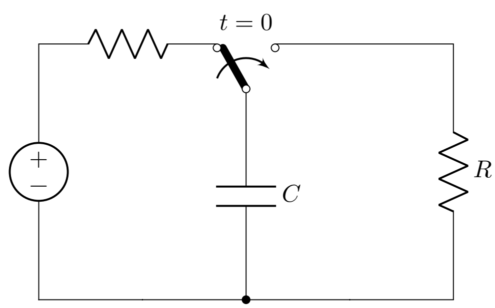

Figure 19: RC circuit.

By solving this differential equation, we can determine the natural response of an RC circuit.

$$v(t) = v(0)\exp\left(-\frac{t}{RC}\right)$$

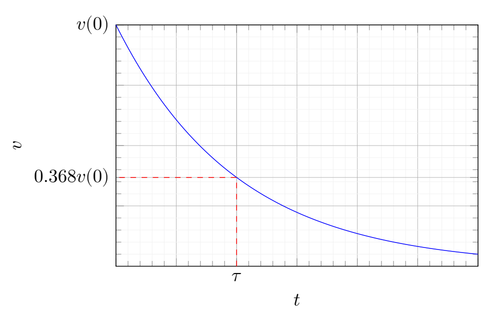

Figure 20: Natural response of an RC circuit.

Definition 7.5 (Time constant)**.** The time constant τ is a parameter for switching circuits. The time constant for an RC circuit is given by

$$\tau = RC$$

The voltage at this time is equal to

$$e^{-1}v(0) \approx 0.368v(0)$$

Definition 7.6 (RL circuit natural response)**.** For the circuit shown below, using mesh analysis gives the following relationship

$$L\frac{\mathrm{d}i}{\mathrm{d}t} = -iR.$$

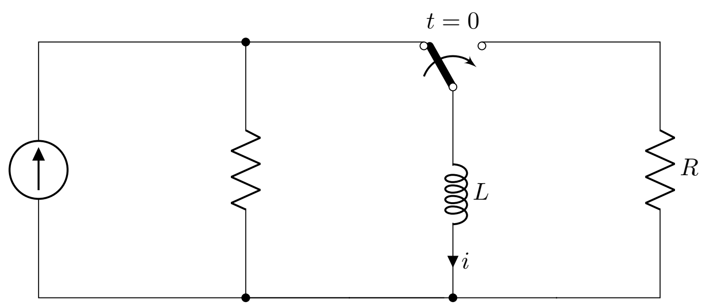

Figure 21: RL circuit.

By solving this differential equation, we can determine the natural response of an RL circuit.

$$i(t) = i(0)\exp\left(-\frac{t}{\frac{1}{R}L}\right)$$

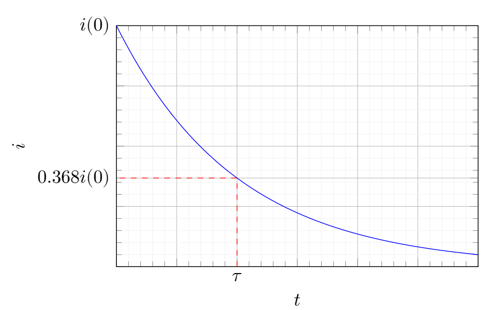

Figure 22: Natural response of an RL circuit.

Definition 7.7 (Time constant)**.** The time constant τ is a parameter for switching circuits. The time constant for an RL circuit is given by

$$\tau = \frac{1}{R}L.$$

The current at this time is equal to

$$e^{-1}i(0) \approx 0.368i(0).$$

7.3 Step Response

Definition 7.8 (RC circuit step response)**.** For the circuit shown below, using KCL at the top node gives the following relationship

$$i_s = C\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{v}{R}.$$

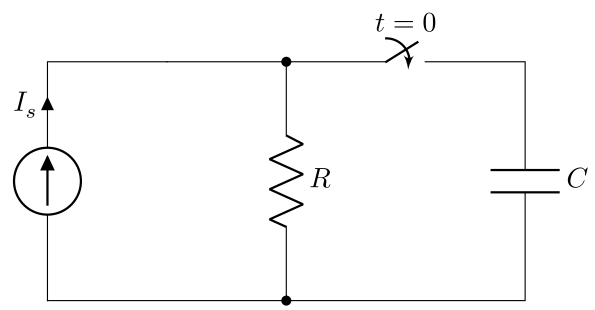

Figure 23: RC circuit.

By solving this differential equation, we can determine the step response of an RC circuit.

$$v(t) = I_sR + (V_0 - I_sR)\exp\left(-\frac{t}{RC}\right)$$

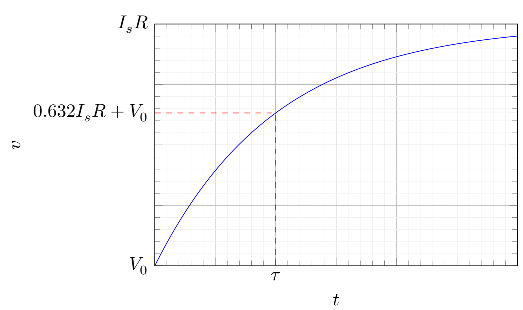

Figure 24: Step response of an RC circuit.

The voltage at $t = \tau$ is equal to

$$v(\tau) = \left(1 - e^{-1}\right) I_s R + V_0$$
$$\approx 0.632 I_s R + V_0.$$

Definition 7.9 (RL circuit step response)**.** For the circuit shown below, using mesh analysis gives the following relationship

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + iR = V_s.$$

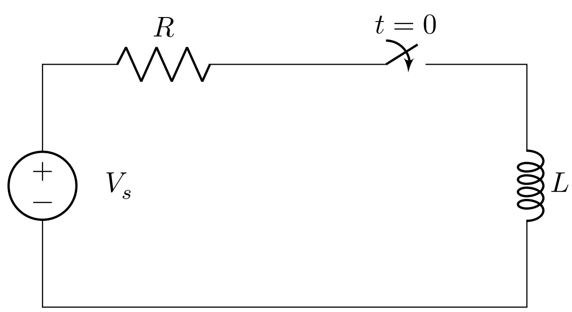

Figure 25: RL circuit.

By solving this differential equation, we can determine the step response of an RL circuit.

$$i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right)\exp\left(-\frac{t}{\frac{1}{R}L}\right)$$

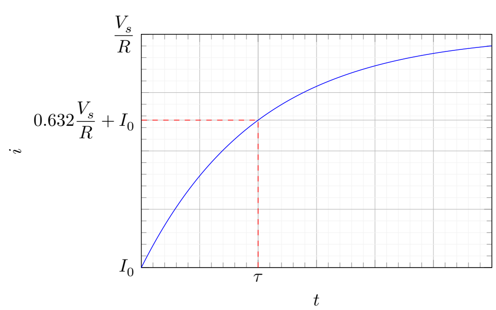

Figure 26: Step response of an RL circuit.

The current at $t = \tau$ is equal to

$$i(\tau) = \left(1 - e^{-1}\right) \frac{V_s}{R} + I_0$$
$$\approx 0.632\frac{V_s}{R} + I_0.$$

8 Operational Amplifiers

Definition 8.1 (Amplifier)**.** An amplifier is a device used to increase the power of a signal using an external energy source. In an electronic amplifier, this signal is usually a voltage or current.

Definition 8.2 (Gain)**.** The gain K of an amplifier is the ratio of the output signal to the input signal.

$$v_{out} = Kv_{in}$$

Definition 8.3 (Operational amplifier)**.** An operational amplifier (op amp) amplifies the voltage difference between its input terminals using a power supply. In an operational amplifier:

1. The output cannot exceed the power supply range.
2. The inputs should remain within the power supply range.

The output voltage has the following possibilities:

1. If $v_p - v_n > 0$, $v_{out} = V_{CC}$.
2. If $v_p - v_n < 0$, $v_{out} = V_{EE}$.
3. If $v_p - v_n = 0$, $v_{out} = 0$.

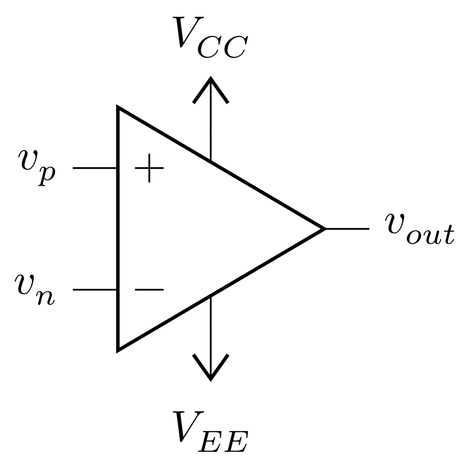

Figure 27: Operational amplifier circuit symbol.

v_p**:** non-inverting input

v_n**:** inverting input

V_{CC}**:** positive power supply

V_{EE}**:** negative power supply

8.1 Golden Rules for Op Amps

When analysing op amps, we can refer to the following golden rules:

1. Assume infinite input impedance, so that input current is zero.
2. In a closed loop, the output drives the voltage difference at the inputs to zero.

Note that the second rule only applies when the op amp has external negative feedback.

8.2 Op Amp Analysis

Using the golden rules above, the following circuits can be constructed.

8.2.1 Inverting Amplifier

$$v_{out} = -\frac{R_2}{R_1}v_{in}$$

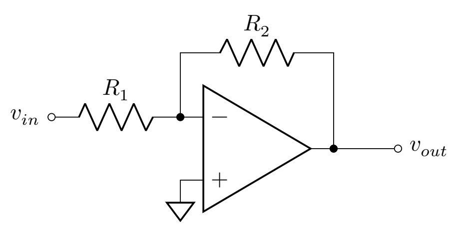

Figure 28: Inverting amplifier.

8.2.2 Non-Inverting Amplifier

$$v_{out} = \frac{R_1 + R_2}{R_1}v_{in}$$

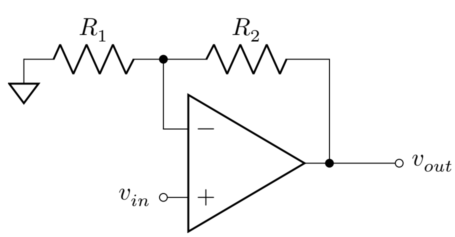

Figure 29: Non-inverting amplifier.

8.2.3 Voltage Follower

$$v_{out} = v_{in}$$

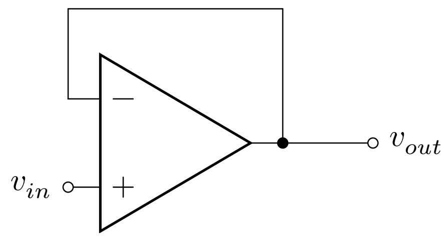

Figure 30: Voltage follower.

8.2.4 Inverting Summing Amplifier

$$v_{out} = -R_f\left(\frac{v_1}{R_1} + \frac{v_2}{R_2} + \cdots + \frac{v_n}{R_n}\right)$$

where R_f is the feedback resistor.

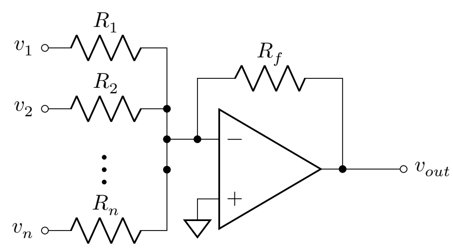

Figure 31: Inverting summing amplifier.

8.2.5 Difference Amplifier

$$v_{out} = \frac{R_2}{R_1}(v_2 - v_1)$$

where the ratio of $\frac{R_2}{R_1}$ must be equal on both legs.

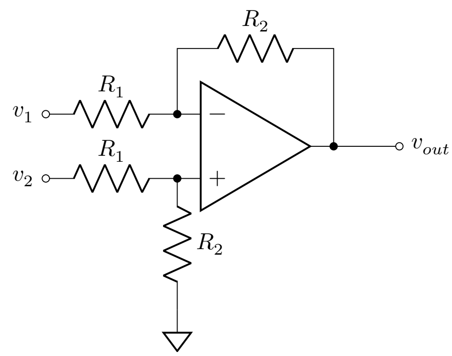

Figure 32: Difference amplifier.

8.3 Designing Op Amps

When designing op amps, it is important to choose resistors in the $1\,\mathrm{k}\Omega$ to $100\,\mathrm{k}\Omega$ range, to prevent damaging the op amp, and to avoid excessive output currents.

9 Sinusoidal Signals

Definition 9.1. A sinusoidal signal is commonly referred to as Alternating Current or AC.

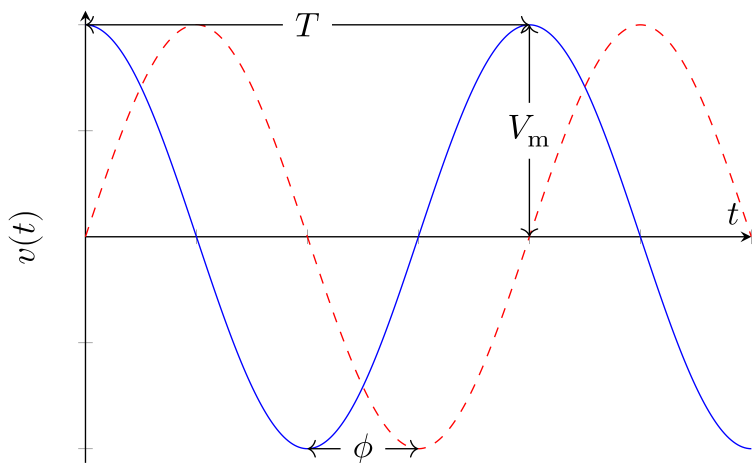

Figure 33: Sinusoidal signals.

9.1 Properties of Sinusoidal Signals

These signals are described using cosine functions of the following form.

$$v(t) = V_{\mathrm{m}} \cos(\omega t + \phi^{\circ})$$

9.1.1 Magnitude

Definition 9.2. The magnitude of a signal is the greatest distance from the line of oscillation, commonly zero, to the peak of the signal.

9.1.2 Angular Frequency

Definition 9.3. The angular frequency ω is measured in radians per second ($\mathrm{rad\,s^{-1}}$), and it allows us to compute the cosine function in radians.

9.1.3 Period and Frequency

Definition 9.4 (Period)**.** The period of a signal T is its cycle time in seconds (s).

Definition 9.5 (Frequency)**.** The frequency of a signal f is the number of cycles per second, or the inverse of the period. The frequency is measured in Hertz (Hz).

The following equations relate the angular frequency, period, and frequency together.

$$f = \frac{1}{T} \qquad T = \frac{2\pi}{\omega} \qquad \omega = 2\pi f$$

9.1.4 Phase

Definition 9.6. The phase ϕ of a signal is the position of a signal relative to a zero phase signal, measured in degrees (°). A **positive** phase corresponds to a "leading" signal, and a **negative** phase corresponds to a "lagging" signal. The phase can be determined by calculating the difference in time τ between two signals.

$$\phi = \frac{\tau}{T}$$

9.2 Root Mean Square

Definition 9.7. Root mean square (rms) is a method of obtaining a useful average of a signal that is symmetric about the horizontal axis. It is defined as the square **root** of the **mean** value of the function **squared**. For a sine wave

$$V_{\text{rms}} = \frac{V_{\text{m}}}{\sqrt{2}}$$

The units in rms quantities are also subscripted with rms, for instance, rms voltage, V_{rms}.

9.3 Power

Definition 9.8. In a resistive load, power can be determined using the rms current and rms voltage.

$$\begin{aligned} p &= vi \\ p &= V_{\text{rms}} I_{\text{rms}} (1 + \cos(2\omega t + 2\phi)) \end{aligned}$$

The average power is given by

$$P_{\text{avg}} = V_{\text{rms}} I_{\text{rms}}$$

9.4 Euler's Formula

Theorem 9.4.1.

$$e^{j\phi} = \cos(\phi) + j\sin(\phi)$$

where $j^2 = -1$ *and* ϕ *is measured in radians.*

9.5 Phasors

Definition 9.9. To assist in AC analysis, we can express the magnitude and phase of a sinusoidal signal using phasors.

9.6 Phasor Transforms

Definition 9.10. A sinusoidal signal can be converted to a phasor using the Phasor Transform

$$\begin{aligned} \mathbf{V} &= \mathscr{P}\{v(t)\} \\ &= \mathscr{P}\{V_{\text{m}} \cos(\omega t + \phi)\} \\ &= V_{\text{m}} e^{j\phi} \end{aligned}$$

Here $\mathbf{V}$ is the phase-domain representation of the time-domain signal $v(t)$. The magnitude and phase of a phasor can be used to represent a phasor in polar form

$$\mathbf{V} = V_\mathrm{m}\angle\underline{\phi}$$

9.7 Inverse Phasor Transforms

Definition 9.11. A phasor can be converted back to a sinusoidal signal using the Inverse Phasor Transform

$$\begin{aligned} v(t) &= \mathscr{P}^{-1}\{\mathbf{V}\} \\ &= \Re\{\mathbf{V}e^{j\omega t}\} \\ &= V_\mathrm{m}\cos(\omega t + \phi) \end{aligned}$$

9.8 Phasor Operations

As phasors are quantities with a magnitude and angle, they can be represented using complex numbers. Using Theorem 9.4.1, we can represent a phasor in rectangular form

$$\mathbf{V} = V_\mathrm{m}\cos(\phi) + jV_\mathrm{m}\sin(\phi)$$

and in polar form, where the magnitude is given by

$$\|\mathbf{V}\| = V_\mathrm{m} = \sqrt{\Re\{\mathbf{V}\}^2 + \Im\{\mathbf{V}\}^2}$$

and the angle is given by

$$\arg(\mathbf{V}) = \phi = \arctan\left(\frac{\Im\{\mathbf{V}\}}{\Re\{\mathbf{V}\}}\right)$$

where $-\pi < \phi \leq \pi$.

9.8.1 Addition with Phasors

Phasors in rectangular form can be added using their real and imaginary parts.

$$\begin{aligned} \mathbf{A} + \mathbf{B} &= \Re\{\mathbf{A} + \mathbf{B}\} + \Im\{\mathbf{A} + \mathbf{B}\} \\ &= (A_\mathrm{m}\cos(\phi_1) + B_\mathrm{m}\cos(\phi_2)) + j(A_\mathrm{m}\sin(\phi_1) + B_\mathrm{m}\sin(\phi_2)) \end{aligned}$$

9.8.2 Multiplication with Phasors

Phasors in polar form can be multiplied using their magnitudes and phases.

$$\begin{aligned} \mathbf{AB} &= \|\mathbf{A}\|\|\mathbf{B}\|\angle\underline{\arg(\mathbf{A}) + \arg(\mathbf{B})} \\ &= A_\mathrm{m}B_\mathrm{m}\angle\underline{\phi_1 + \phi_2} \end{aligned}$$

9.9 Circuit Analysis

Theorem 9.9.1 (Phasor relationship for resistors)**.**

$$\mathbf{V} = \mathbf{I}R$$

Proof. Using a sinusoidal voltage

$$v = V_\mathrm{m}\cos(\omega t + \phi)$$

Ohm's Law gives:

$$\begin{aligned} i &= \frac{v}{R} \\ &= \frac{V_\mathrm{m}}{R}\cos(\omega t + \phi) \end{aligned}$$

Taking the Phasor Transform gives

$$\begin{aligned} \mathscr{P}\{v\} &= \mathscr{P}\{iR\} \\ \mathscr{P}\{V_\mathrm{m}\cos(\omega t + \phi)\} &= \mathscr{P}\left\{\frac{V_\mathrm{m}}{R}\cos(\omega t + \phi)\right\}R \\ \mathbf{V} &= \mathbf{I}R \end{aligned}$$

□

Theorem 9.9.2 (Phasor relationship for inductors)**.**

$$\mathbf{V} = j\omega L\mathbf{I}$$

Proof. Using a sinusoidal current

$$i = I_\mathrm{m}\cos(\omega t + \phi)$$

the voltage drop across an inductor is given by:

$$\begin{aligned} v &= L\frac{\mathrm{d}i}{\mathrm{d}t} \\ &= -\omega L I_\mathrm{m}\sin(\omega t + \phi) \\ &= -\omega L I_\mathrm{m}\cos(\omega t + \phi - 90°) \end{aligned}$$

taking the Phasor Transform gives

$$\begin{aligned} \mathscr{P}\{v\} &= \mathscr{P}\left\{L\frac{\mathrm{d}i}{\mathrm{d}t}\right\} \\ \mathbf{V} &= \mathscr{P}\{-\omega L I_\mathrm{m}\cos(\omega t + \phi - 90°)\} \\ \mathbf{V} &= -\omega L I_\mathrm{m}e^{j(\phi - 90°)} \\ \mathbf{V} &= -\omega L I_\mathrm{m}e^{j\phi}e^{-j90°} \\ \mathbf{V} &= j\omega L I_\mathrm{m}e^{j\phi} \\ \mathbf{V} &= j\omega L\mathbf{I} \end{aligned}$$

□

Theorem 9.9.3 (Phasor relationship for capacitors)**.**

$$\mathbf{V} = \frac{1}{j\omega C}\mathbf{I}$$

Proof. Using a sinusoidal voltage

$$v = V_\mathrm{m}\cos(\omega t + \phi)$$

the voltage-current relationship across a capacitor is given by:

$$\begin{aligned}
i &= C\frac{\mathrm{d}v}{\mathrm{d}t} \\
&= -\omega C V_\mathrm{m}\sin(\omega t + \phi) \\
&= -\omega C V_\mathrm{m}\cos(\omega t + \phi - 90°)
\end{aligned}$$

taking the Phasor Transform gives

$$\begin{aligned}
\mathscr{P}\{i\} &= \mathscr{P}\left\{C\frac{\mathrm{d}v}{\mathrm{d}t}\right\} \\
\mathbf{I} &= \mathscr{P}\{-\omega C V_\mathrm{m}\cos(\omega t + \phi - 90°)\} \\
\mathbf{I} &= -\omega C V_\mathrm{m} e^{j(\phi - 90°)} \\
\mathbf{I} &= -\omega C V_\mathrm{m} e^{j\phi} e^{-j90°} \\
\mathbf{I} &= j\omega C V_\mathrm{m} e^{j\phi} \\
\mathbf{I} &= j\omega C\mathbf{V} \\
\mathbf{V} &= \frac{1}{j\omega C}\mathbf{I}
\end{aligned}$$

□

9.10 Impedance

Definition 9.12. The impedance $\mathbf{Z}$ of an element, measured in Ohms (Ω), is its opposition to alternating current. Impedance is represented as the sum of a real resistance R and an imaginary reactance X.

$$\mathbf{Z} = R + jX$$

Impedance captures the magnitude and phase change associated with a circuit element.

Theorem 9.10.1 (Impedance of resistors)**.**

$$\mathbf{Z} = R$$

Theorem 9.10.2 (Impedance of inductors)**.**

$$\mathbf{Z} = j\omega L$$

Theorem 9.10.3 (Impedance of capacitors)**.**

$$\mathbf{Z} = \frac{1}{j\omega C}$$

9.11 Impedances in Circuits

Theorem 9.11.1. *Impedances behave similarly to resistances in series and parallel.*

10 Frequency Response

Frequency response is a measure of the magnitude and phase of a system as a function of (angular) frequency ω.

10.1 Circuit Analysis with AC Circuits

Theorem 10.1.1. *Any DC analysis techniques can be used with AC circuits as long as resistive elements are represented using complex impedances.*

10.2 Transfer Function

Definition 10.1. The transfer function of a system is given by

$$\mathbf{H}(\omega) = \frac{\mathbf{Y}(\omega)}{\mathbf{X}(\omega)}$$

where $\mathbf{Y}$ is the output and $\mathbf{X}$ is the input of a system.

Definition 10.2 (Linear Gain)**.** The magnitude of the transfer function $\|\mathbf{H}\|$ measures the linear gain of a system.

Definition 10.3 (Gain in decibel)**.** The gain of a system is often represented in dB using the following formula

$$\text{gain} = 20 \log_{10} \|\mathbf{H}\|$$

Definition 10.4 (Phase shift)**.** The phase of the transfer function $\arg(\mathbf{H})$ measures the phase shift of a system.

10.3 Bode Plots

A Bode plot is a graph of the frequency response of a system. In this plot, the frequency (in Hz) is plotted on the horizontal axis using a logarithmic scale.

Definition 10.5 (3 dB point)**.** The frequency f at which the transfer function equals

$$\mathbf{H}(2\pi f) = \frac{1}{\sqrt{2}} \underline{/-45^\circ}$$

is known as the break frequency, corner frequency, 3 dB frequency, or half power point. This is because the gain at this point is equal to $0.707 \approx -3.01$ dB, and the angle $-45°$ is the halfway point between $0°$ and $-90°$.

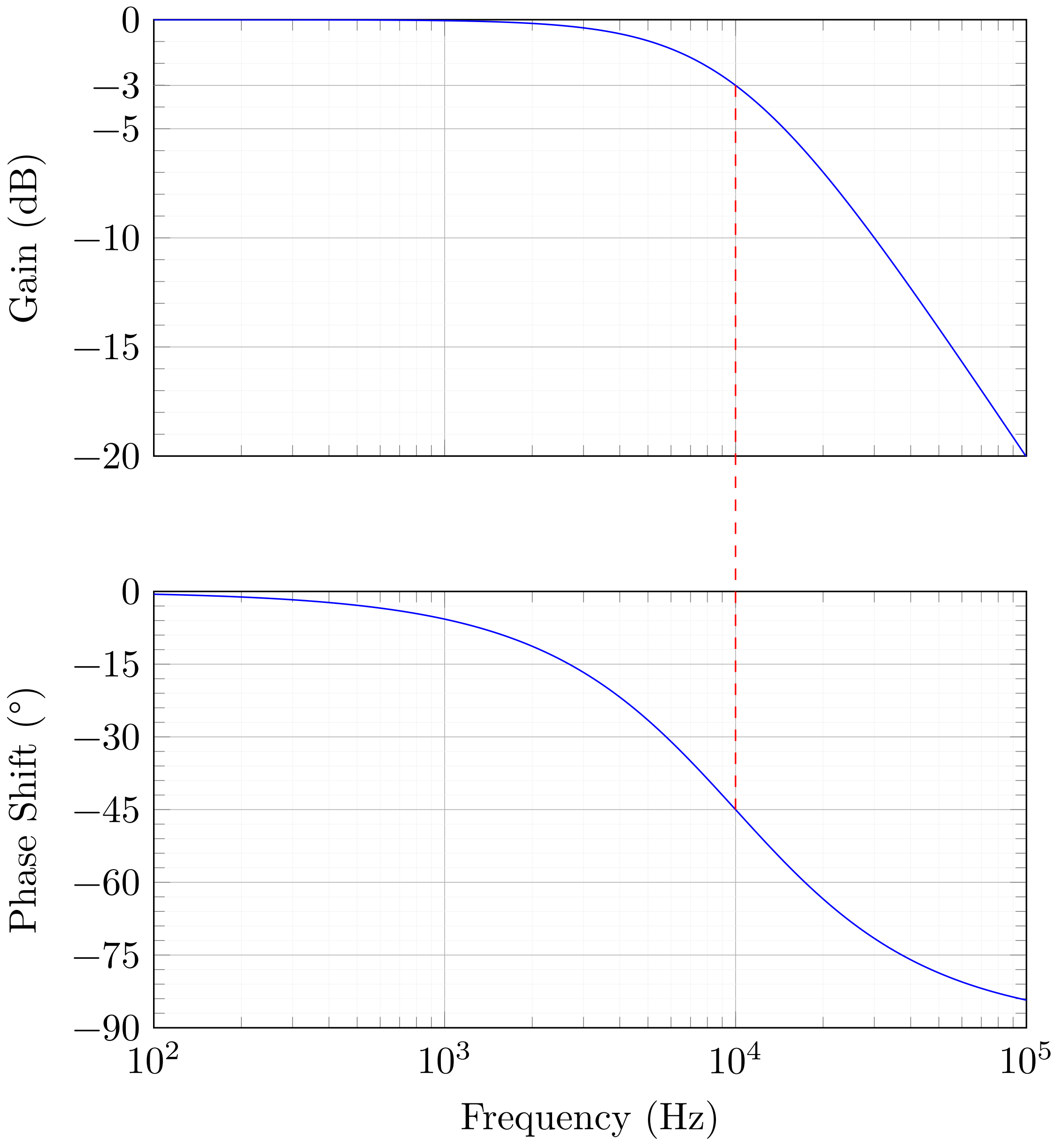

Figure 34: Bode plot of gain and phase shift.

11 Filters and Rectifiers

Definition 11.1 (Filters)**.** A filter is designed to allow certain ranges of frequencies to pass, while other ranges of frequencies are stopped.

Definition 11.2 (Cut-off frequncy)**.** The cut-off (angular) frequency ω_c for a filter is given by

$$\omega_c = \frac{1}{RC}$$

11.1 Common Filters

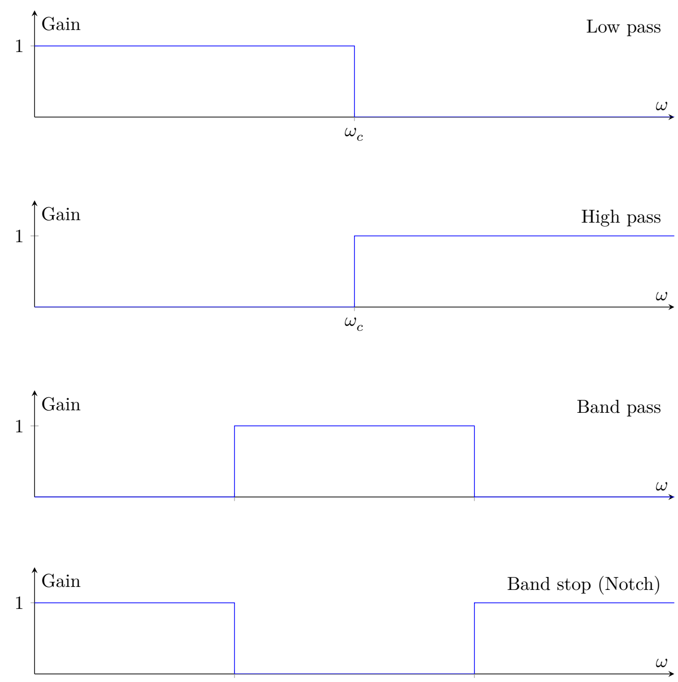

Figure 35: Different types of filters.

11.2 Passive Filters

Passive filters are circuits containing only passive elements (such as resistors, capacitors and inductors), which filter unwanted parts of a signal. Passive filters are affected by loading, such that a varied load will affect the cut-off frequency.

11.2.1 Low Pass Filter

The following low pass filter has the following transfer function.

$$\mathbf{H}(\omega) = \frac{\omega_c}{j\omega + \omega_c}$$

where $\omega_c = \dfrac{1}{RC}$.

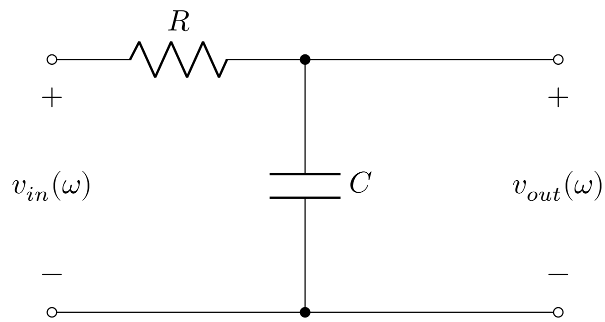

Figure 36: Passive low pass filter.

11.2.2 High Pass Filter

The following high pass filter has the following transfer function.

$$\mathbf{H}(\omega) = \frac{j\omega}{j\omega + \omega_c}$$

where $\omega_c = \dfrac{1}{RC}$.

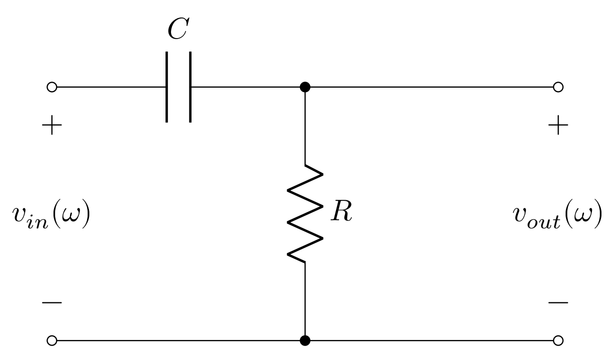

Figure 37: Passive high pass filter.

11.3 Active Filters

Active filters are circuits containing active elements (such as op amps), which filter unwanted parts of a signal. Active filters are not affected by loading, such that a varied load will not affect the cut-off frequency. For the following active filters, the gain component is given by

$$\text{gain} = -\frac{R_2}{R_1}$$

11.3.1 Low Pass Filter

The following low pass filter has the following transfer function.

$$\mathbf{H}(\omega) = -\frac{R_2}{R_1}\frac{\omega_c}{j\omega + \omega_c}$$

where $\omega_c = \dfrac{1}{R_2 C}$.

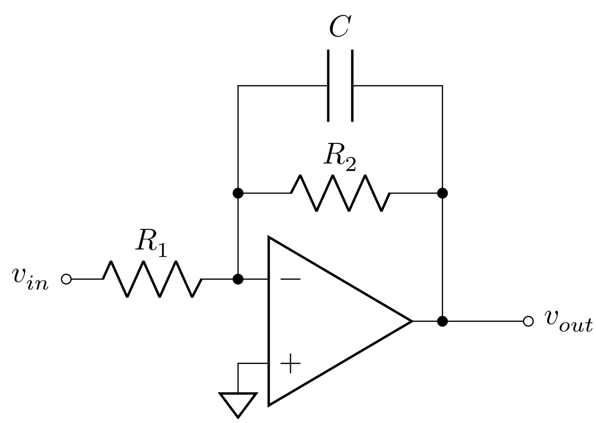

Figure 38: Active low pass filter.

11.3.2 High Pass Filter

The following high pass filter has the following transfer function.

$$\mathbf{H}(\omega) = -\frac{R_2}{R_1}\frac{j\omega}{j\omega + \omega_c}$$

where $\omega_c = \dfrac{1}{R_1 C}$.

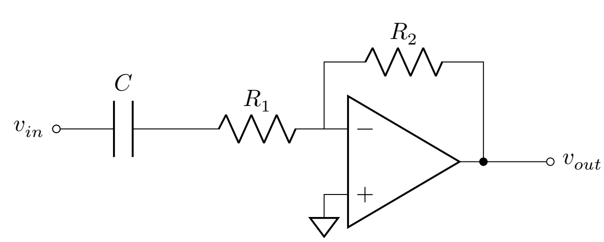

Figure 39: Active high pass filter.

11.4 Rectifiers

Definition 11.3 (Plugpack)**.** An AC plugpack steps the mains AC voltage ($240\,\mathrm{V_{rms}}$) down to a safe AC voltage level.

Definition 11.4 (Rectifier)**.** A rectifier is designed to convert AC signals to DC.

By using a diode, a rectifier only allows current to pass when an AC voltage is positive.

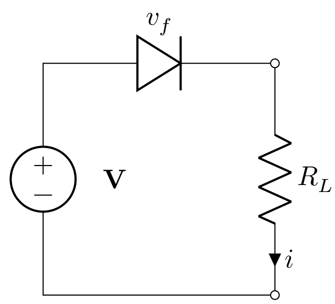

Figure 40: Half-wave rectifier circuit without capacitor.

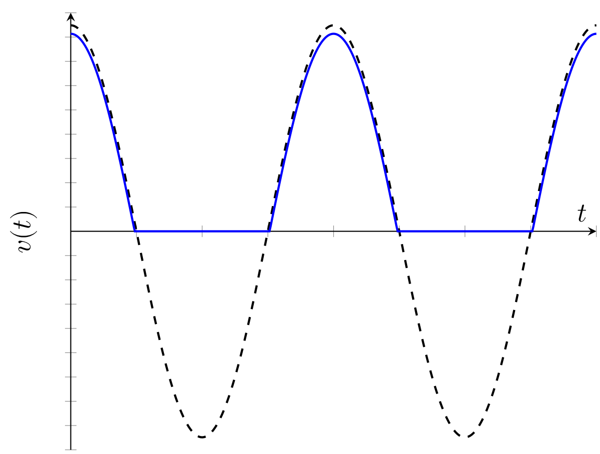

Figure 41: Half-wave rectified signal.

Definition 11.5 (Ripple)**.** Ripple is the residual variation of the DC voltage after rectification. In the circuit shown above, a capacitor can be used to smooth ripples. Ripple can be calculated using the following assumptions:

1. The capacitor instantly charges when the diode is forward biased.
2. The capacitor discharges with a constant current.

The VI relationship for a capacitor gives the following formula for ripple

$$\Delta v = \frac{i}{C}\Delta t$$

where f is the frequency of the voltage source.

11.4.1 Half-wave Rectifier

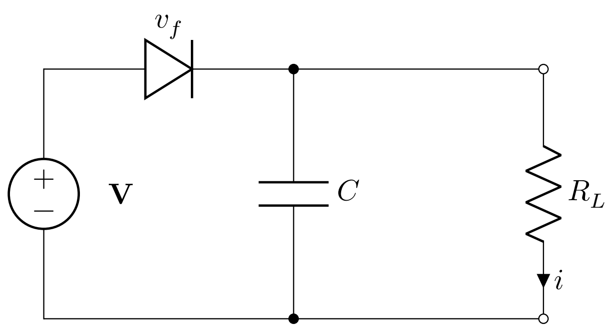

Figure 42: Half-wave rectifier circuit.

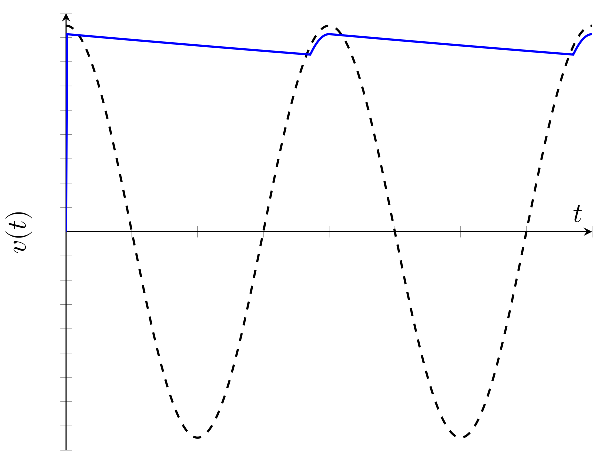

Figure 43: Half-wave rectified signal.

- The diode conducts on the positive half cycle
- The output loses the forward voltage of one diode

The ripple in a half-wave rectifier is calculated using

$$\Delta v = \frac{i}{fC}$$

11.4.2 Full-wave Rectifier

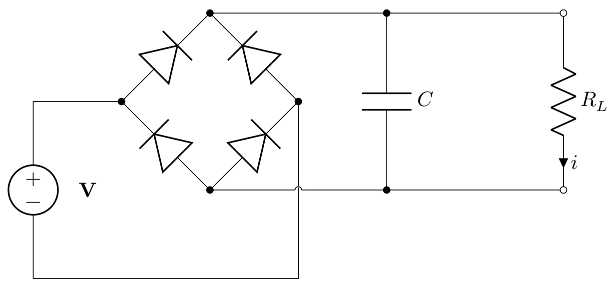

Figure 44: Half-wave rectifier circuit.

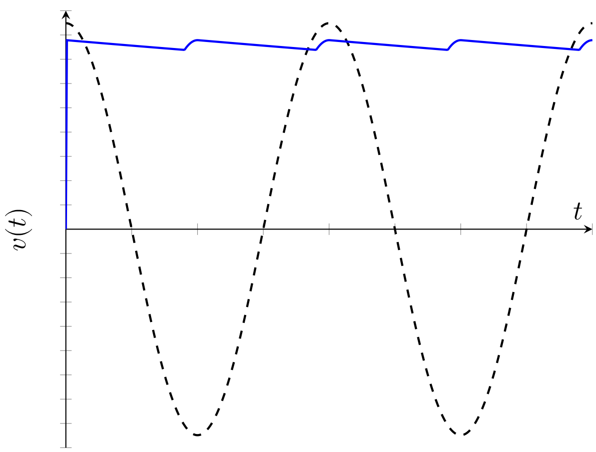

Figure 45: Full-wave rectified signal.

- Two diodes conduct every half cycle
- The output loses the forward voltage of two diodes

The ripple in a full-wave rectifier is calculated using

$$\Delta v = \frac{i}{2fC}$$

11.4.3 Dual half-wave Rectifier

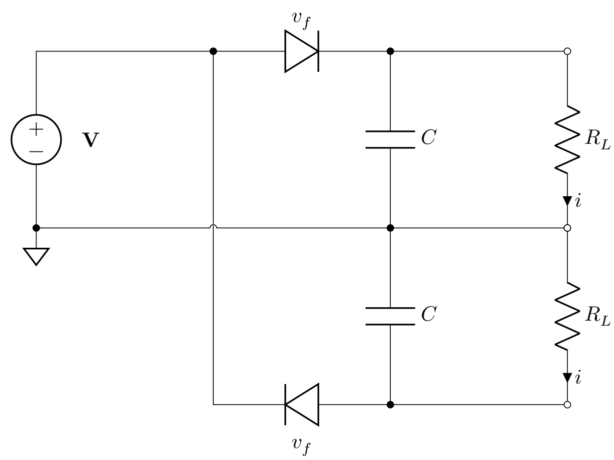

Figure 46: Dual half-wave rectifier circuit.

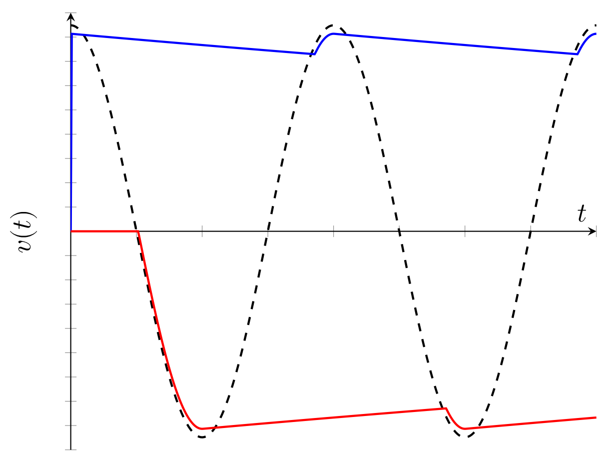

Figure 47: Dual half-wave rectified signal.

- The diodes conduct on both cycles
- The output loses the forward voltage of one diode
- Useful for powering op amps

The ripple in a dual half-wave rectifier is calculated using

$$\Delta v = \frac{i}{fC}$$

12 Zener Diodes and Voltage Regulators

12.1 Zener Diodes

A Zener diode is similar to a regular diode, but it has a specific reverse breakdown voltage. This reverse breakdown voltage is commonly 5 V.

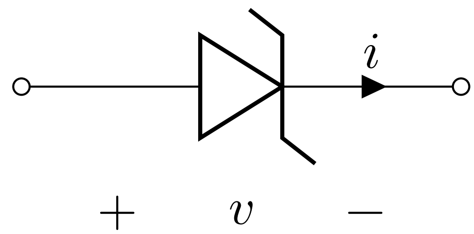

Figure 48: Zener diode circuit diagram.

The ideal VI characteristic for a Zener diode is shown below.

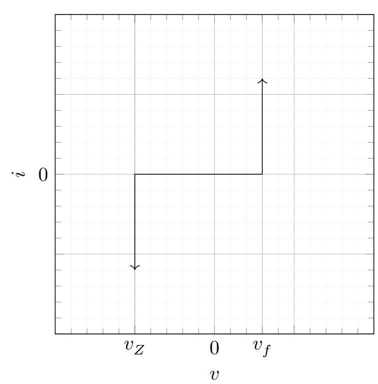

Figure 49: VI characteristic for an ideal Zener diode.

12.2 Voltage Regulators

Most circuits require a clean DC power supply to operate. A voltage regulator ensures that a DC power supply behaves like an ideal voltage source.

12.3 Shunt Regulators

Definition 12.1 (Shunt regulator)**.** A shunt regulator regulates the output voltage by shunting excess current away from the load.

A Zener diode can be used as a shunt regulator to drop voltages to the reverse bias voltage of the diode.

Remark 1. When designing a Zener voltage regulator, the following should be taken into account:

- Power dissipation (particularly when dropping large voltages).
- Large series resistances reduce power dissipation but also reduce the regulated current.
- A Zener voltage regulator always draws power from the source, regardless of the load.
- The load has an effect on the regulator.

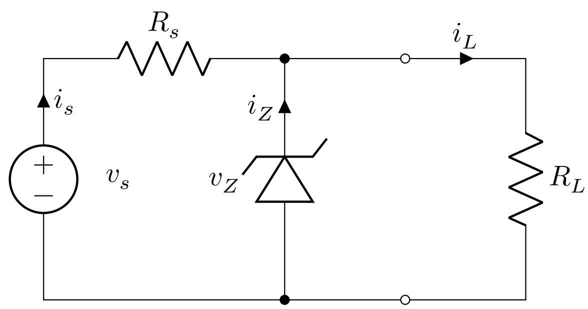

Figure 50: Zener diode regulator.

Theorem 12.3.1 (Maximum load current)**.** *Given a source voltage range v_s and source resistance R_s, the maximum current an ideal Zener diode can regulate is given by*

$$i_L < \frac{v_s - v_Z}{R_s}$$

where the largest value in v_s is chosen.

Theorem 12.3.2 (Maximum source resistance)**.** *Given a source voltage range v_s and load current range i_L, the maximum source resistance required to regulate the output voltage is given by*

$$R_s < \frac{v_s - v_Z}{i_L}$$

where the smallest value in v_s is chosen, and the largest value in i_L is chosen.

Proof. Using KCL gives

$$\begin{aligned} i_s - i_Z &= i_L \\ i_Z &= i_s - i_L. \end{aligned}$$

For a Zener diode to operate at its reverse breakdown voltage, the Zener current must be nonzero. This gives

$$0 < \frac{v_s - v_Z}{R_s} - i_L$$

We must now consider the worst case scenario, namely, when $\frac{v_s - v_Z}{R_s}$ is minimised, and when i_L is maximised. Hence, the maximum load current occurs when v_s is maximal and i_L is minimal; and the maximum source resistance occurs when v_s is minimal and i_L is maximal. □

12.4 Series Regulators

Definition 12.2 (Series regulator)**.** Series regulators regulate the output voltage by limiting the load current by varying the series element.

12.4.1 Series Linear Regulators

LM-series integrated circuits (IC) are a three-terminal linear regulator that regulate fixed voltages of 0.5 V to 24 V. They require an input voltage at least 2 V greater than the regulated output voltage. An LM-series regulator circuit is shown below.

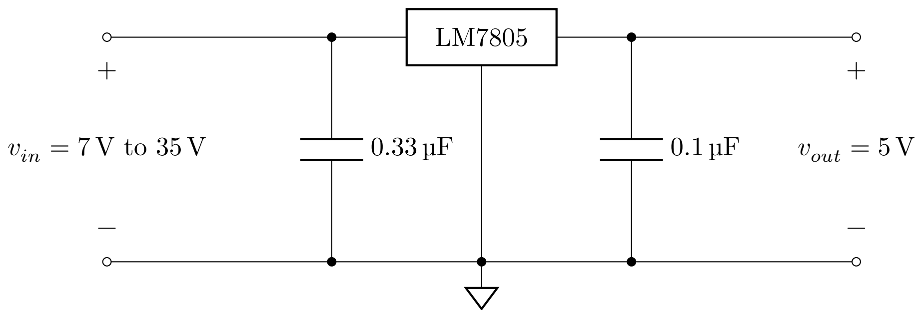

Figure 51: Series Linear Regulator using an LM-series IC.

Note that the capacitors remain the same for any output voltage. The naming convention is as follows,

1. LM78XX regulates a positive voltage XXV.
2. LM79XX regulates a negative voltage -XXV.

The power dissipation can be calculated by assuming 0 ground current.

12.4.2 Power Supply Design

A rectifier and regulator circuit can be combined to create a power supply from an AC voltage plugpack.

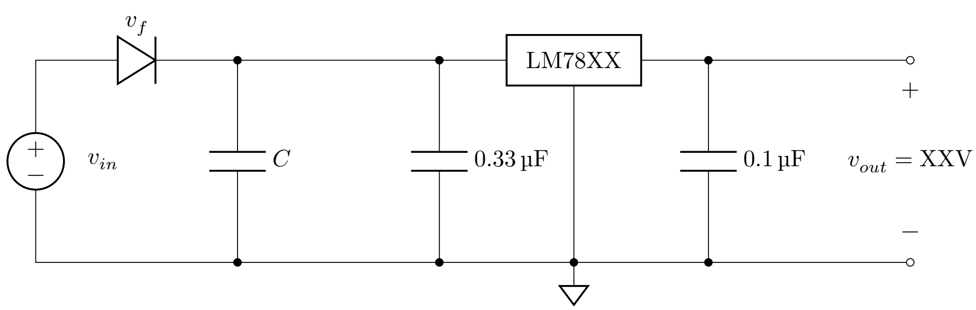

Figure 52: AC to DC power supply circuit.

A dual half-wave rectifier can be used to supply both positive and negative voltage.

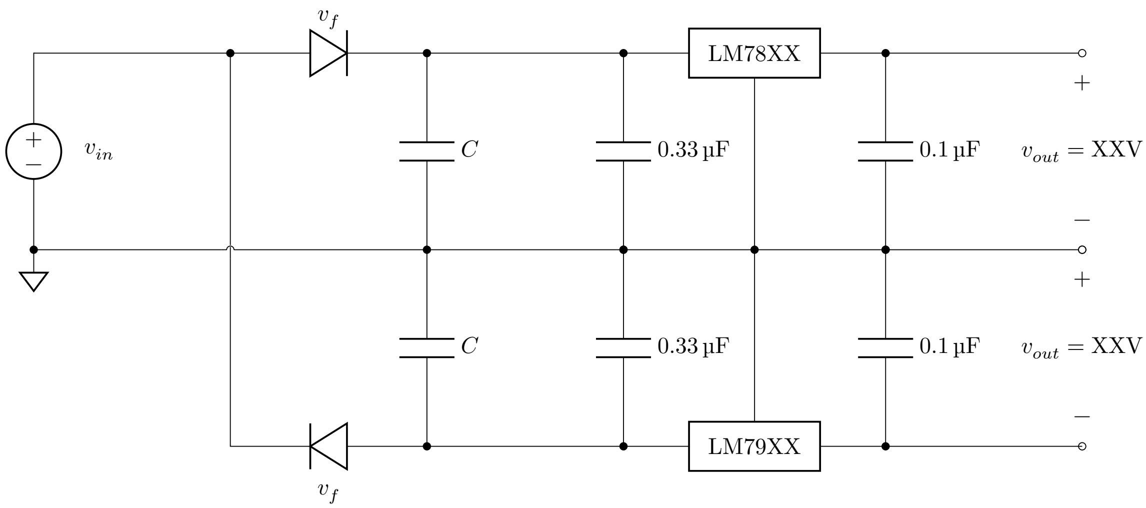

Figure 53: AC to DC dual power supply circuit.

The maximum ripple for these circuits is calculated below.

$$\begin{aligned}
\max \Delta v &= \max v_{in} - v_f - (2 + v_{out}) \\
\therefore \Delta v &> \frac{i}{fC} \\
\max v_{in} - v_f - (2 + v_{out}) &> \frac{i}{fC}.
\end{aligned}$$

12.4.3 Series Op Amp Regulators

Op amps can be utilised to stabilise the output from a regulator. The op amp is powered by the source voltage and the series resistance is chosen to minimise the current through the Zener diode.

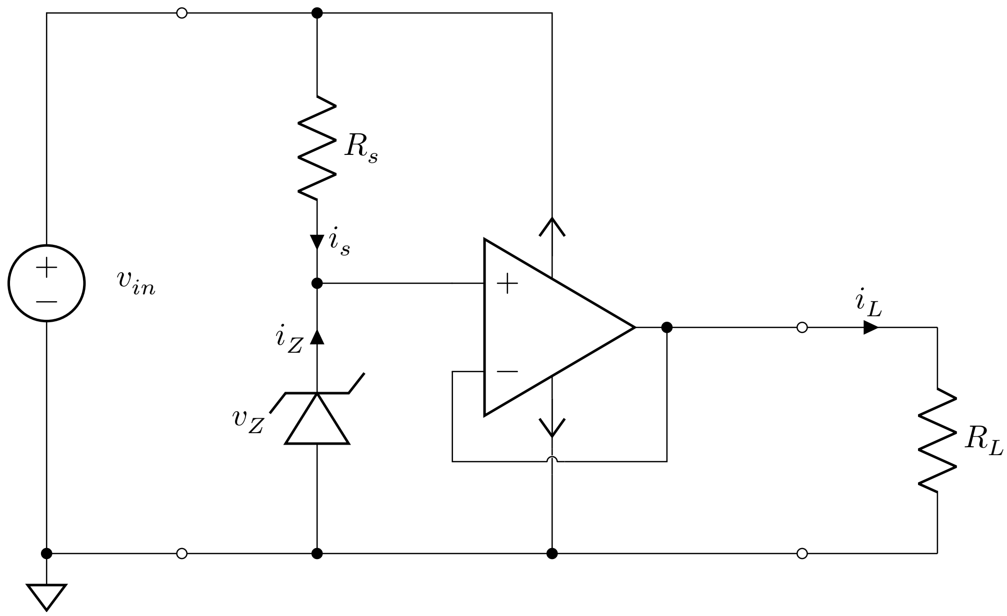

Figure 54: Series op amp regulator circuit.

In the circuit above, $i_Z + i_S = 0$. The power dissipated by the circuit can be determined by finding the power produced at the voltage source which is given by

$$p = v_{in} \left(i_L - i_Z \right).$$

List of Figures