Engineering Mechanics

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1 Stress and Strain

1.1 External Forces

Rigid bodies are subjected to external force and couple moment systems that result from the effects of gravitational, electrical, magnetic, or contact forces. Contact forces can be surface, linear, or concentrated forces.

1.1.1 Types of Forces

- Compressive (pushing)
- Tensile (pulling)
- Shear (sliding)
- Torsional (twisting)
- Biaxial tension
- Hydrostatic compression
- Bending (induces tension, compression and shear)

1.2 Internal Loadings

External forces cause internal loadings that occur in equal and opposite collinear pairs as stresses and strains. Internal loading is associated with **stress** while **strain** is a measure of a body's deformation.
These loadings have no external effects on the body, and are not included on a **Free Body Diagram** (FBD) if the entire body is considered.
To determine the forces in each member, we can use the method of sections to represent the internal loading as external forces.

1.3 Internal Resultant Loadings

Although the exact distribution of the internal loading may be *unknown*, we can determine the resultant force $\mathbf{F}_R$ and resultant moment $(\mathbf{M}_R)_O$ about a point O by applying the equations of equilibrium

$$\sum \mathbf{F} = \mathbf{0}$$
$$\sum \mathbf{M}_O = \mathbf{0}.$$

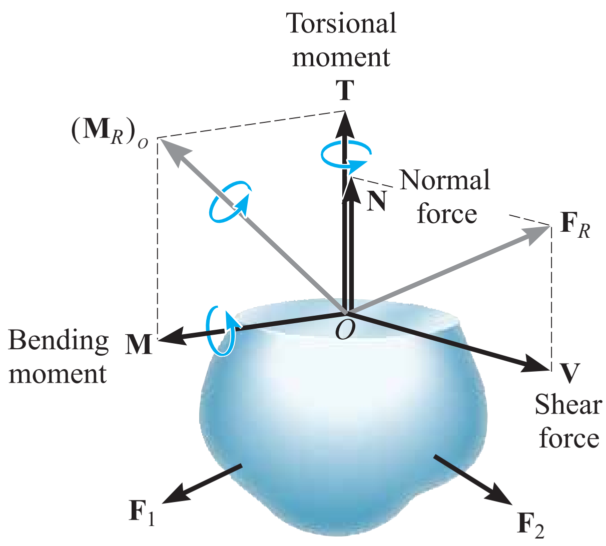

Figure 1: Resultant loadings acting on a body.

1.3.1 In 3D

In 3D, we can represent resultant loadings using four vectors acting over the sectioned area.

Normal force N force acting perpendicular to the area

Shear force V force acting on an axis tangent to the area

Torsional moment T rotation about the perpendicular axis

Bending moment M rotation about an axis tangent to the area

1.3.2 In 2D

In 2D, the body is subjected to a coplanar system of forces, where $\mathbf{T} = \mathbf{0}$.

1.3.3 In 1D

In 1D, the body is only subjected to axial forces, where $\mathbf{V} = \mathbf{T} = \mathbf{M} = \mathbf{0}$.

1.4 Stress

The force and moment acting at a specific point on a sectioned area of a body represent the resultant effects of the distribution of internal loading that acts over the sectioned area.

Definition 1.1 (Stress)**.** Consider the quotient of the force $\Delta\mathbf{F}$ over an area ΔA, then as the $\Delta A \to 0$, so does $\Delta\mathbf{F}$, while the quotient approaches a finite limit. This quotient is called the stress at that point.

$$\boldsymbol{\sigma} = \lim_{\Delta A \to 0} \frac{\Delta\mathbf{F}}{\Delta A}$$

Here the normal and shear stresses can be expressed using σ_z and τ_{zx} and τ_{zy}.

$$\sigma_z = \lim_{\Delta A \to 0} \frac{\Delta F_z}{\Delta A}$$
$$\tau_{zx} = \lim_{\Delta A \to 0} \frac{\Delta F_x}{\Delta A}$$
$$\tau_{zy} = \lim_{\Delta A \to 0} \frac{\Delta F_y}{\Delta A}$$

Stress describes the intensity of the internal force acting on a specific region passing through a point.
The unit for stress is Pascal where $1\,\mathrm{Pa}$ or $1\,\mathrm{N\,m^{-2}}$ and $1\,\mathrm{MPa}$ or $1\,\mathrm{N\,mm^{-2}}$.

1.5 Average Normal Stress

To determine the average stress distribution acting over a cross-sectional area of an axially loaded bar, we assume that the material is both *homogeneous* and *isotropic*. This means the load P applied through the centroid of the cross-sectional area will cause the bar to deform uniformly throughout the central region of its length.
By passing a section through a bar, equilibrium requires the resultant normal force N at the section to be equal to the external force P. And because the material undergoes a uniform deformation, it is necessary that the cross section be subjected to a constant normal stress distribution.
As a result, each small area ΔA on the cross section is subjected to a force $\Delta N = \sigma \Delta A$, where the sum of these forces over the entire cross-sectional area is P. By letting $\Delta A \to \mathrm{d}A$ and therefore also $\Delta N \to \mathrm{d}N$, then as σ is a constant, we have

$$\int \mathrm{d}N = \int_A \sigma \,\mathrm{d}A$$
$$N = \sigma A$$

Therefore

$$\sigma_{\text{avg}} = \frac{N}{A}$$

where in this case $N = P$.

Theorem 1.5.1 (Equilibrium)**.** *For an uniaxially loaded body, the equation of force equilibrium gives*

$$\sigma\,(\Delta A) - \sigma'\,(\Delta A) = 0$$
$$\sigma = \sigma'$$

hence the normal stress components must be equal in magnitude but opposite in direction.
Under this condition, the material is subjected to ***uniaxial stress*** *and this analysis applies to members subjected to tension or compression.*

1.6 Strain

Definition 1.2 (Deformation)**.** Whenever a force is applied to a body, it will tend to change the body's shape and size. These changes are referred to as deformation.

Definition 1.3 (Strain)**.** To describe the deformation of a body through changes in lengths of line segments on the surface, we will develop the concept of strain. If an axial load P is applied to a bar, it will change the bar's length L_0 to L. Then the **average normal strain** of the bar is defined

$$\epsilon_{\text{avg}} = \frac{L - L_0}{L_0}$$

where the numerator is often written as $\delta = L - L_0$ and is known as elongation or extension.
The **normal strain** ϵ at a point in a body with an arbitrary shape is defined similarly. Consider a small line segment Δs which becomes $\Delta s'$ after deformation. Then the limit of the normal strain is

$$\epsilon = \lim_{\Delta s \to 0} \frac{\Delta s' - \Delta s}{\Delta s}$$

In both cases normal strain is positive when the initial length elongates, and negative when the length contracts.
Strain is a dimensionless quantity sometimes expressed mm/mm or m/m, or as a percentage.

2 Tension and Compression Tests

To determine the strength of a material, we must perform a tension or commpression test. This test measures the stress and strain from a load P, and the results can be used to produce a **stress-strain diagram**. There are two ways in which the stress-strain diagram is normally described.

2.1 Stress-Strain Diagram

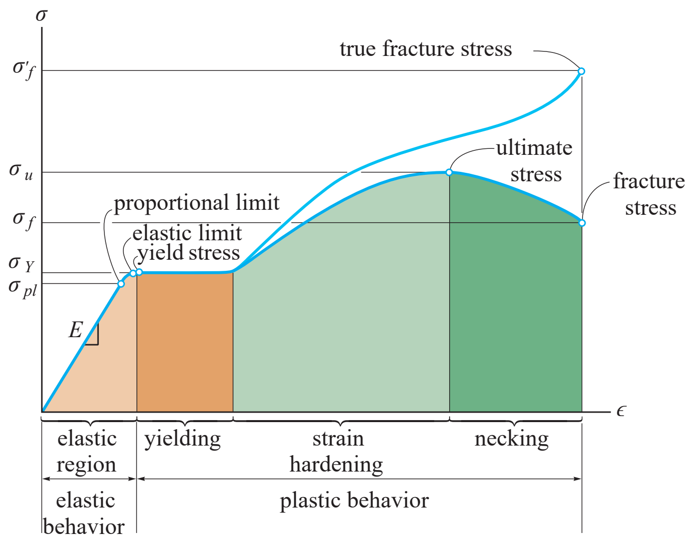

Figure 2: Stress-strain diagram for a typical metal.

2.1.1 Conventional Stress-Strain Diagram

The engineering stress assumes that the area A is constant throughout the gauge length

$$\sigma = \frac{P}{A_0}$$

where A_0 is the *original* cross-sectional area of the specimen.
Likewise, the engineering strain uses the specimen's original length L_0

$$\epsilon = \frac{\delta}{L_0}$$

2.1.2 True Stress-Strain Diagram

The true stress and true strain use the instantaneous area A and length L at each measurement.

2.2 Elastic Behaviour

The initial region of the curve is referred to as the **elastic region** where the deformation is *elastic*, so that unloading causes the specimen to return to its original shape.

2.2.1 Proportional Limit

For the majority of the elastic deformation, the curve is *linear* up to the point where the stress reaches the **proportional limit** at $\left(\sigma_{pl},\, \epsilon_{pl}\right)$.

2.2.2 Modulus of Elasticity

The linear relationship up to this point is characterised by Hooke's law, and is expressed as

$$\sigma = E\epsilon$$

where E is the constant of proportionality, called the **modulus of elasticity** or **Young's modulus**.

2.3 Elastic Limit

When the stress slightly exceeds the proportionality limit, the curve bends until the stress reaches an **elastic limit**.

2.4 Plastic Behaviour

An increase in stress above the elastic limit will result in a breakdown of the material and cause it to deform plastically.

2.4.1 Yielding

This behaviour is called **yielding** and the stress that causes yielding occurs at the **yield point** $(\sigma_Y,\, \epsilon_Y)$. Although not shown in the diagram, the yield point is distinguished as two points.
The **upper yield point** occurs first, followed by a sudden decrease in load-carrying capacity to a **lower yield point**. Once the yield point is reached, *the specimen will continue to elongate* ***without*** *any increase in load*. When the material behaves in this manner, it is often referred to as being **perfectly plastic**.

2.4.2 Yield Strength

Commonly the proportionality limit, the elastic limit, and yield point are indistinguishable, due to this, the **yield strength** is defined at $(\sigma_{YS},\, \epsilon_{YS})$.
To determine this point, a 0.2% strain is chosen, and a line with gradient E is drawn from the ϵ axis. The point where this line intersects the curve defines $(\sigma_{YS},\, \epsilon_{YS})$.

2.4.3 Strain Hardening

Yielding ends when any loading causes the stress to increase, this rise in the curve is referred to as **strain hardening**.
When a plastically deformed ductile material is unloaded, the elastic strain is recovered as the material returns to its equilibrium state.
However the plastic strain is maintained, resulting in a **permanent set**.

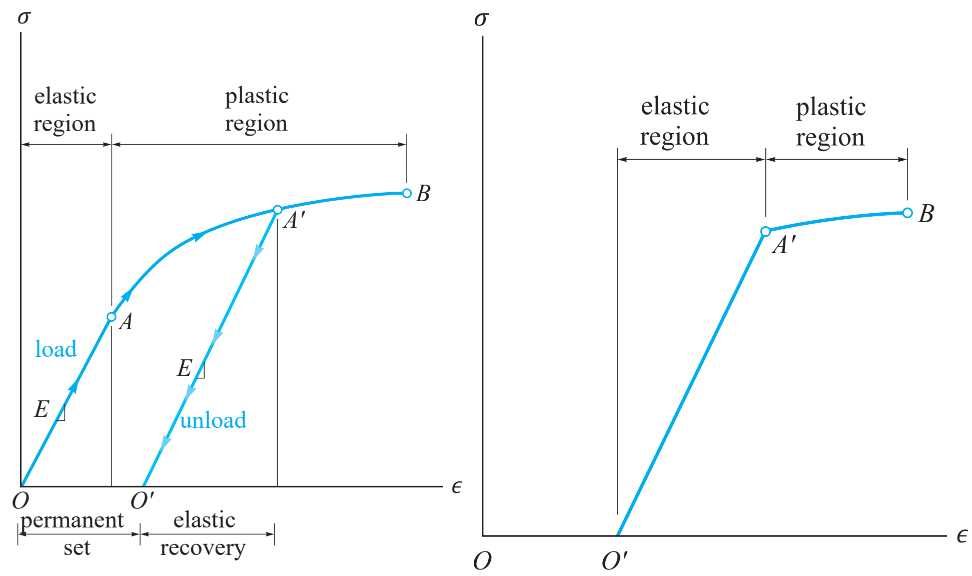

Figure 3: Elastic strain recovery under strain hardening.

2.4.4 Ultimate Tensile Stress

The maximum stress reached on the diagram is referred to as the **ultimate tensile stress** $(\sigma_{UTS},\ \epsilon_{UTS})$.

2.4.5 Necking

While the specimen elongates up to ϵ_{UTS}, its cross-sectional area will decrease *uniformally* over its gauge length. However after reaching ϵ_{UTS}, the cross-sectional area will decrease *locally*, causing an increase in stress. As a result, a "neck" forms at this region, and the specimen experiences **necking**.

2.4.6 Fracture Stress

Finally, the specimen breaks where the curve ends at the **fracture point** at $\left(\sigma_f,\ \epsilon_f\right)$.

2.5 Ductility

Definition 2.1 (Ductility)**.** Ductility is a measure of the amount of plastic deformation a material can sustain under tensile stress before failure.
Ductility can be measured using the **percent elongation** (in length) or **percent reduction** (in

area) of a material.

$$\text{Percent Elongation} = \frac{L_f - L_0}{L_0} 100\,\%$$
$$\text{Percent Reduction} = \frac{A_0 - A_f}{A_0} 100\,\%$$

As the elastic region is very brief in most materials, ductility is often measured using the original length and area, rather than the length and area when the material undergoes plastic deformation.

2.6 Brittleness

Definition 2.2 (Brittleness)**.** Brittleness describes the property of a material that fractures with little to no yielding.

2.7 Poisson's Ratio

When a deformable body is subjected to a force, it can elongate longitudinally and also contract laterally. The strain in the longitudinal (or axial) direction is given by

$$\epsilon_{\text{long}} = \frac{\delta}{L}$$

and the strain in the lateral (or radial) direction is given by

$$\epsilon_{\text{lat}} = \frac{\delta'}{r}$$

where δ' is the change in the radius r.
Consider the ratio of these two quantities ν

$$\nu = -\frac{\epsilon_{\text{lat}}}{\epsilon_{\text{long}}}.$$

Within the elastic region, ν will be constant, and it is referred to as **Poisson's ratio**.
Note the negative value is introduced as the longitudinal and lateral strains have opposite signs.

2.8 Strain Energy

As a material is deformed under external load, the load will do external work. This work is stored in the material as internal energy or **strain energy**.

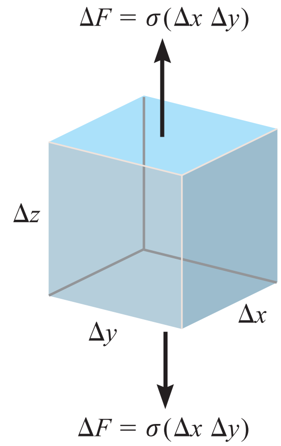

Figure 4: Internal energy in small element.

If we consider a small volume element of the material, then the force is equal to the average force magnitude $\Delta F/2$ and the displacement is given by d. Therefore the strain energy ΔU is given by

$$\begin{aligned}\Delta U &= \frac{1}{2}\Delta F d \\ &= \frac{1}{2}\left(\sigma \Delta x \Delta y\right)\left(\epsilon \Delta z\right) \\ &= \frac{1}{2}\sigma\epsilon\Delta V\end{aligned}$$

where ΔV is the volume of the element. If we consider the strain energy *per unit volume*, then

$$u = \frac{\Delta U}{\Delta V} = \frac{1}{2}\sigma\epsilon$$

where u is the **strain energy density**. u can also be determined by finding the area under the stress-strain diagram, and hence has the units $\mathrm{J\,m^{-3}}$.

2.8.1 Modulus of Resilience

When the stress in a material reaches the proportional limit, the strain energy density is referred to as the modulus of resilience.

$$u_r = \frac{1}{2}\sigma_{pl}\epsilon_{pl} = \frac{1}{2}\frac{\sigma_{pl}^2}{E}$$

The modulus of resilience is also the area under the proportional region of the stress-strain diagram.

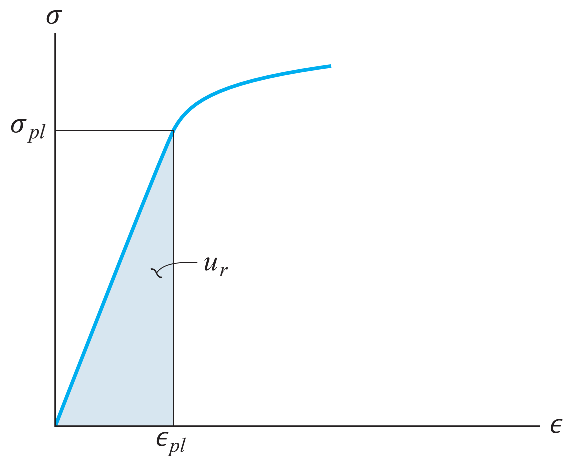

Figure 5: Modulus of resilience u_r.

2.8.2 Modulus of Toughness

Another important property of a material is its modulus of toughness, u_t. This quantity represents the entire area under the stress-strain diagram.

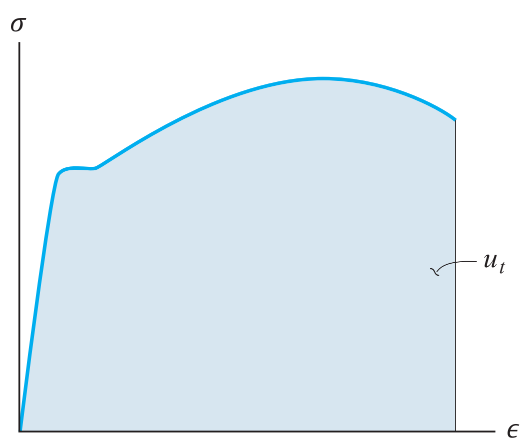

Figure 6: Modulus of toughness u_t.