Engineering Mechanics

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1 Stress and Strain

1.1 External Forces

Rigid bodies are subjected to external force and couple moment systems that result from the effects of gravitational, electrical, magnetic, or contact forces. Contact forces can be surface, linear, or concentrated forces.

1.1.1 Types of Forces

- Compressive (pushing)
- Tensile (pulling)
- Shear (sliding)
- Torsional (twisting)
- Biaxial tension
- Hydrostatic compression
- Bending (induces tension, compression and shear)

1.2 Internal Loadings

External forces cause internal loadings that occur in equal and opposite collinear pairs as stresses and strains. Internal loading is associated with **stress** while **strain** is a measure of a body's deformation.
These loadings have no external effects on the body, and are not included on a **Free Body Diagram** (FBD) if the entire body is considered.
To determine the forces in each member, we can use the method of sections to represent the internal loading as external forces.

1.3 Internal Resultant Loadings

Although the exact distribution of the internal loading may be *unknown*, we can determine the resultant force $\mathbf{F}_R$ and resultant moment $(\mathbf{M}_R)_O$ about a point O by applying the equations of equilibrium

$$\sum \mathbf{F} = \mathbf{0}$$
$$\sum \mathbf{M}_O = \mathbf{0}.$$

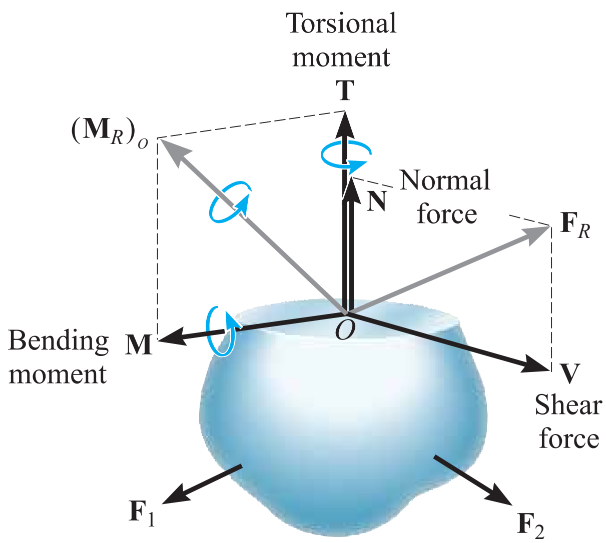


Figure 1: Resultant loadings acting on a body.

1.3.1 In 3D

In 3D, we can represent resultant loadings using four vectors acting over the sectioned area.

Normal force N force acting perpendicular to the area

Shear force V force acting on an axis tangent to the area

Torsional moment T rotation about the perpendicular axis

Bending moment M rotation about an axis tangent to the area

1.3.2 In 2D

In 2D, the body is subjected to a coplanar system of forces, where $\mathbf{T} = \mathbf{0}$.

1.3.3 In 1D

In 1D, the body is only subjected to axial forces, where $\mathbf{V} = \mathbf{T} = \mathbf{M} = \mathbf{0}$.

1.4 Stress

The force and moment acting at a specific point on a sectioned area of a body represent the resultant effects of the distribution of internal loading that acts over the sectioned area.

Definition 1.1 (Stress)**.** Consider the quotient of the force $\mathbf{\Delta F}$ over an area ΔA, then as the $\Delta A \to 0$, so does $\mathbf{\Delta F}$, while the quotient approaches a finite limit. This quotient is called the stress at that point.

$$\boldsymbol{\sigma} = \lim_{\Delta A \to 0} \frac{\mathbf{\Delta F}}{\Delta A}$$

Here the normal and shear stresses can be expressed using σ_z and τ_{zx} and τ_{zy}.

$$\begin{aligned} \sigma_z &= \lim_{\Delta A \to 0} \frac{\Delta F_z}{\Delta A} \\ \tau_{zx} &= \lim_{\Delta A \to 0} \frac{\Delta F_x}{\Delta A} \\ \tau_{zy} &= \lim_{\Delta A \to 0} \frac{\Delta F_y}{\Delta A} \end{aligned}$$

Stress describes the intensity of the internal force acting on a specific region passing through a point.
The unit for stress is Pascal where $1\,\mathrm{Pa}$ or $1\,\mathrm{N\,m^{-2}}$ and $1\,\mathrm{MPa}$ or $1\,\mathrm{N\,mm^{-2}}$.

To determine the average stress distribution acting over a cross-sectional area of an axially loaded bar, we assume that the material is both *homogeneous* and *isotropic*. This means the load P applied through the centroid of the cross-sectional area will cause the bar to deform uniformly throughout the central region of its length.

1.4.1 Average Normal Stress

By passing a section through a bar, equilibrium requires the resultant normal force N at the section to be equal to the external force P. And because the material undergoes a uniform deformation, it is necessary that the cross section be subjected to a constant normal stress distribution.
As a result, each small area ΔA on the cross section is subjected to a force $\Delta N = \sigma \Delta A$, where the sum of these forces over the entire cross-sectional area is P. By letting $\Delta A \to \mathrm{d}A$ and therefore also $\Delta N \to \mathrm{d}N$, then as σ is a constant, we have

$$\begin{gathered} \int \mathrm{d}N = \int_A \sigma \,\mathrm{d}A \\ N = \sigma A \end{gathered}$$

Therefore

$$\sigma_{\text{avg}} = \frac{N}{A}$$

where in this case $N = P$.

Theorem 1.4.1 (Equilibrium)**.** *For an uniaxially loaded body, the equation of force equilibrium gives*

$$\begin{aligned} \sigma\,(\Delta A) - \sigma'\,(\Delta A) &= 0 \\ \sigma &= \sigma' \end{aligned}$$

hence the normal stress components must be equal in magnitude but opposite in direction.
Under this condition, the material is subjected to ***uniaxial stress*** *and this analysis applies to members subjected to tension or compression.*

1.4.2 Average Shear Stress

Shear stress is the stress component that acts in the plane of the sectioned area. Here we must consider the number of planes that are in stress due to the applied force, so that the shear force $V = F/n$ for n planes, is applied to hold the segment in equilibrium.
The average shear stress distributed over each sectioned area that develops this shear force is defined by

$$\tau_{\text{avg}} = \frac{V}{A}$$

The loading case discussed is an example of **simple or direct shear** as the shear is caused by the direct action of the applied load $\mathbf{F}$.

1.5 Strain

Definition 1.2 (Deformation)**.** Whenever a force is applied to a body, it will tend to change the body's shape and size. These changes are referred to as deformation.

Definition 1.3 (Strain)**.** To describe the deformation of a body through changes in lengths of line segments on the surface, we will develop the concept of strain. If an axial load P is applied to a bar, it will change the bar's length L_0 to L. Then the **average normal strain** of the bar is defined

$$\epsilon_{\text{avg}} = \frac{L - L_0}{L_0}$$

where the numerator is often written as $\delta = L - L_0$ and is known as elongation or extension.
The **normal strain** ϵ at a point in a body with an arbitrary shape is defined similarly. Consider a small line segment Δs which becomes $\Delta s'$ after deformation. Then the limit of the normal strain is

$$\epsilon = \lim_{\Delta s \to 0} \frac{\Delta s' - \Delta s}{\Delta s}$$

In both cases normal strain is positive when the initial length elongates, and negative when the length contracts.
Strain is a dimensionless quantity sometimes expressed mm/mm or m/m, or as a percentage.

1.6 Shear Strain

Deformations not only cause line segments to elongate or contract, but they also cause them to change direction. If we consider two line segments that are originally perpendicular to one another, then the change in angle that occurs between them is referred to as **shear strain**. This angle is denoted by γ and is always measured in radians.
The shear strain of a block can be measured using

$$\gamma = \frac{\pi}{2} - \theta$$

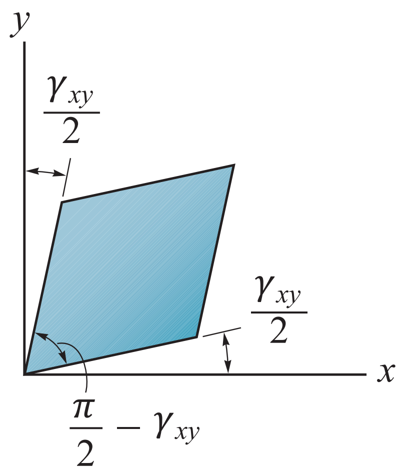


Figure 2: Shear block diagram.

We can also define shear stress as the change in angle

$$\gamma = \gamma_f - \gamma_o$$

1.7 Small Strain Analysis

Most engineering design involve applications for which only small deformations are allowed (i.e., $\epsilon \ll 1$). Hence we can make the following approximations for the small change in angle $\Delta\theta$.

$$\begin{aligned} \sin(\Delta\theta) &\approx \Delta\theta \\ \cos(\Delta\theta) &\approx 1 \\ \tan(\Delta\theta) &\approx \Delta\theta \end{aligned}$$

2 Tension and Compression Tests

To determine the strength of a material, we must perform a tension or commpression test. This test measures the stress and strain from a load P, and the results can be used to produce a **stress-strain diagram**. There are two ways in which the stress-strain diagram is normally described.

2.1 Stress-Strain Diagram

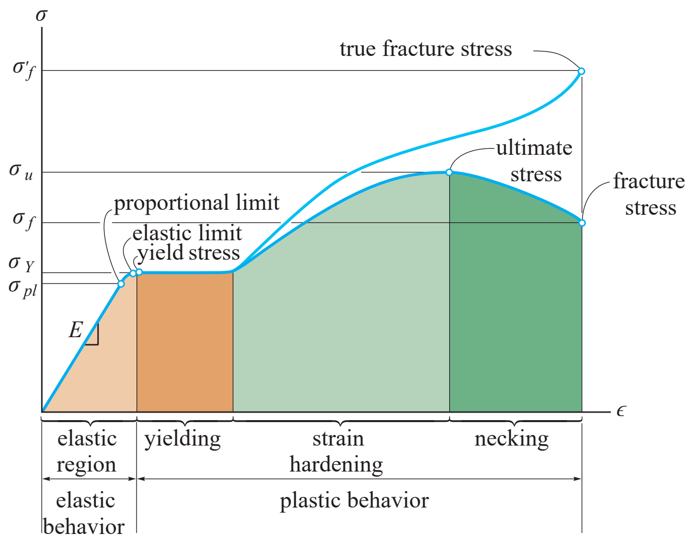


Figure 3: Stress-strain diagram for a typical metal.

2.1.1 Conventional Stress-Strain Diagram

The engineering stress assumes that the area A is constant throughout the gauge length

$$\sigma = \frac{P}{A_0}$$

where A_0 is the *original* cross-sectional area of the specimen.
Likewise, the engineering strain uses the specimen's original length L_0

$$\epsilon = \frac{\delta}{L_0}$$

2.1.2 True Stress-Strain Diagram

The true stress and true strain use the instantaneous area A and length L at each measurement.

2.2 Elastic Behaviour

The initial region of the curve is referred to as the **elastic region** where the deformation is *elastic*, so that unloading causes the specimen to return to its original shape.

2.2.1 Proportional Limit

For the majority of the elastic deformation, the curve is *linear* up to the point where the stress reaches the **proportional limit** at $(\sigma_{pl},\ \epsilon_{pl})$.

2.2.2 Modulus of Elasticity

The linear relationship up to this point is characterised by Hooke's law, and is expressed as

$$\sigma = E\epsilon$$

where E is the constant of proportionality, called the **modulus of elasticity** or **Young's modulus**.

2.2.3 Elastic Limit

When the stress slightly exceeds the proportionality limit, the curve bends until the stress reaches an **elastic limit**.

2.3 Plastic Behaviour

An increase in stress above the elastic limit will result in a breakdown of the material and cause it to deform plastically.

2.3.1 Yielding

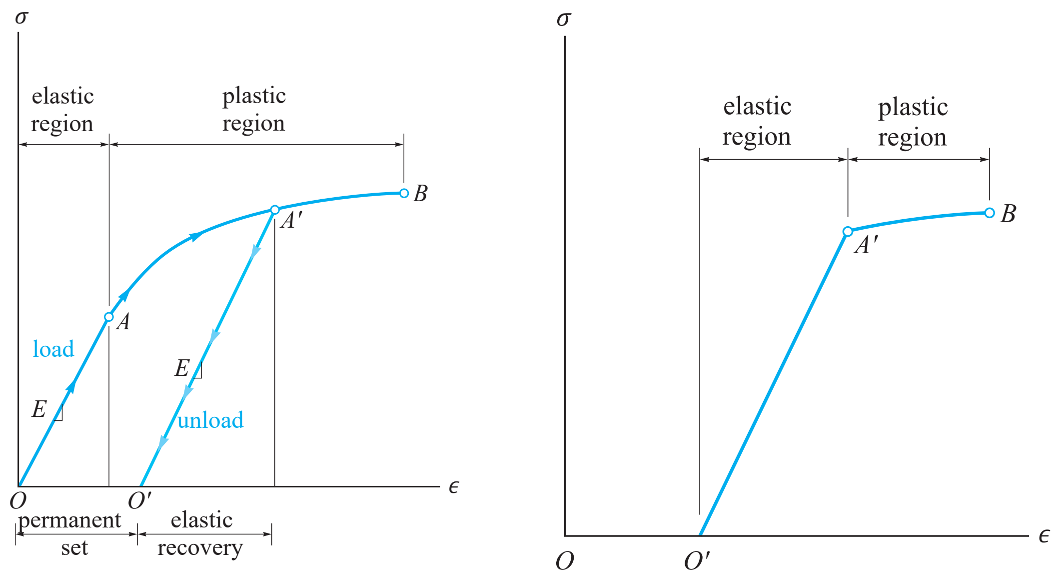
This behaviour is called **yielding** and the stress that causes yielding occurs at the **yield point** $(\sigma_Y,\ \epsilon_Y)$. Although not shown in the diagram, the yield point is distinguished as two points.
The **upper yield point** occurs first, followed by a sudden decrease in load-carrying capacity to a **lower yield point**. Once the yield point is reached, *the specimen will continue to elongate **without** any increase in load.* When the material behaves in this manner, it is often referred to as being **perfectly plastic**.

2.3.2 Yield Strength

Commonly the proportionality limit, the elastic limit, and yield point are indistinguishable, due to this, the **yield strength** is defined at $(\sigma_{YS},\ \epsilon_{YS})$.
To determine this point, a 0.2% strain is chosen, and a line with gradient E is drawn from the ϵ axis. The point where this line intersects the curve defines $(\sigma_{YS},\ \epsilon_{YS})$.

2.3.3 Strain Hardening

Yielding ends when any loading causes the stress to increase, this rise in the curve is referred to as **strain hardening**.
When a plastically deformed ductile material is unloaded, the elastic strain is recovered as the material returns to its equilibrium state.
However the plastic strain is maintained, resulting in a **permanent set**.



(a) During loading. (b) After loading.

Figure 4: Elastic strain recovery under strain hardening.

2.3.4 Ultimate Tensile Stress

The maximum stress reached on the diagram is referred to as the **ultimate tensile stress** $(\sigma_{UTS}, \epsilon_{UTS})$.

2.3.5 Necking

While the specimen elongates up to ϵ_{UTS}, its cross-sectional area will decrease *uniformally* over its gauge length. However after reaching ϵ_{UTS}, the cross-sectional area will decrease *locally*, causing an increase in stress. As a result, a "neck" forms at this region, and the specimen experiences **necking**.

2.3.6 Fracture Stress

Finally, the specimen breaks where the curve ends at the **fracture point** at $\left(\sigma_f, \epsilon_f\right)$.

2.4 Allowable Stress Design

To ensure the safety of a structural member, it is necessary to restrict the applied load to one that is *less than* the load a member can support.
This is done by specifying a **factor of safety** F.S. which determines the allowable load F_{allow} a member should be designed for.

$$\text{F.S.} = \frac{F_{\text{fail}}}{F_{\text{allow}}}$$

Here F_{fail} is found from experimental testing of the material. When the load is linearly related to the stress, we can express the factor of safety using the ratio of failure stress and allowable stress.

$$\text{F.S.} = \frac{\sigma_{\text{fail}}}{\sigma_{\text{allow}}}$$
$$\text{F.S.} = \frac{\tau\text{fail}}{\tau_{\text{allow}}}$$

2.5 Ductility

Definition 2.1 (Ductility)**.** Ductility is a measure of the amount of plastic deformation a material can sustain under tensile stress before failure.
Ductility can be measured using the **percent elongation** (in length) or **percent reduction** (in area) of a material.

$$\text{Percent Elongation} = \frac{L_f - L_0}{L_0} 100\,\%$$
$$\text{Percent Reduction} = \frac{A_0 - A_f}{A_0} 100\,\%$$

As the elastic region is very brief in most materials, ductility is often measured using the original length and area, rather than the length and area when the material undergoes plastic deformation.

2.6 Brittleness

Definition 2.2 (Brittleness)**.** Brittleness describes the property of a material that fractures with little to no yielding.

2.7 Poisson's Ratio

When a deformable body is subjected to a force, it can elongate longitudinally and also contract laterally. The strain in the longitudinal (or axial) direction is given by

$$\epsilon_{\text{long}} = \frac{\delta}{L}$$

and the strain in the lateral (or radial) direction is given by

$$\epsilon_{\text{lat}} = \frac{\delta'}{r}$$

where δ' is the change in the radius r.
Consider the ratio of these two quantities ν

$$\nu = -\frac{\epsilon_{\text{lat}}}{\epsilon_{\text{long}}}.$$

Within the elastic region, ν will be constant, and it is referred to as **Poisson's ratio**.
Note the negative value is introduced as the longitudinal and lateral strains have opposite signs.

2.8 Strain Energy

As a material is deformed under external load, the load will do external work. This work is stored in the material as internal energy or **strain energy**.

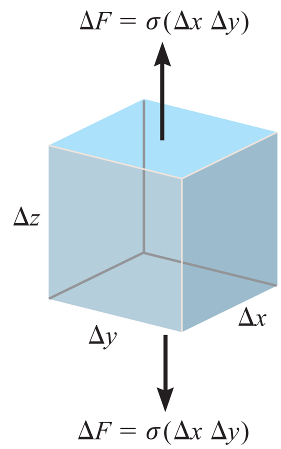


Figure 5: Internal energy in small element.

If we consider a small volume element of the material, then the force is equal to the average force magnitude $\Delta F/2$ and the displacement is given by d. Therefore the strain energy ΔU is given by

$$\begin{aligned}\Delta U &= \frac{1}{2}\Delta F d \\ &= \frac{1}{2}\left(\sigma \Delta x \Delta y\right)\left(\epsilon \Delta z\right) \\ &= \frac{1}{2}\sigma\epsilon\Delta V\end{aligned}$$

where ΔV is the volume of the element. If we consider the strain energy *per unit volume*, then

$$u = \frac{\Delta U}{\Delta V} = \frac{1}{2}\sigma\epsilon$$

where u is the **strain energy density**. u can also be determined by finding the area under the stress-strain diagram, and hence has the units $\mathrm{J\,m^{-3}}$.

2.8.1 Modulus of Resilience

When the stress in a material reaches the proportional limit, the strain energy density is referred to as the modulus of resilience.

$$u_r = \frac{1}{2}\sigma_{pl}\epsilon_{pl} = \frac{1}{2}\frac{\sigma_{pl}^2}{E}$$

The modulus of resilience is also the area under the proportional region of the stress-strain diagram.

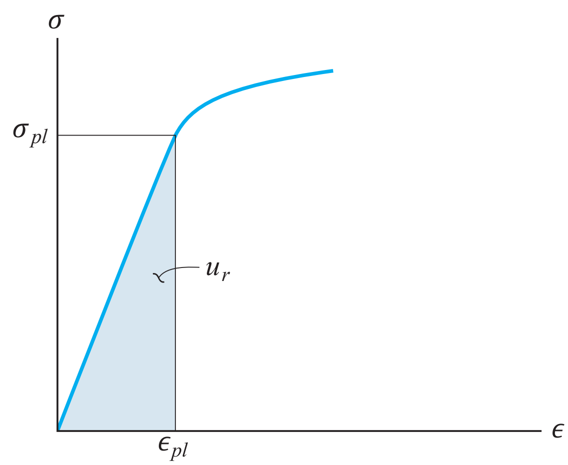


Figure 6: Modulus of resilience u_r.

2.8.2 Modulus of Toughness

Another important property of a material is its modulus of toughness, u_t. This quantity represents the entire area under the stress-strain diagram.

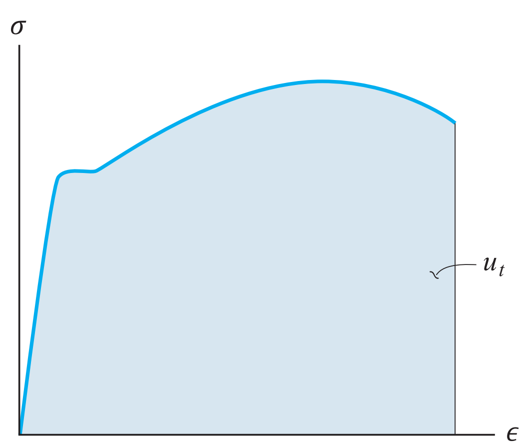


Figure 7: Modulus of toughness u_t.

3 Shear Stress-Strain Diagram

Similar to a tensile test, we can use a thin tube and subject it to torsional loading. Using the data for the applied torque and resulting angle of twist, we can form a shear stress-strain diagram.

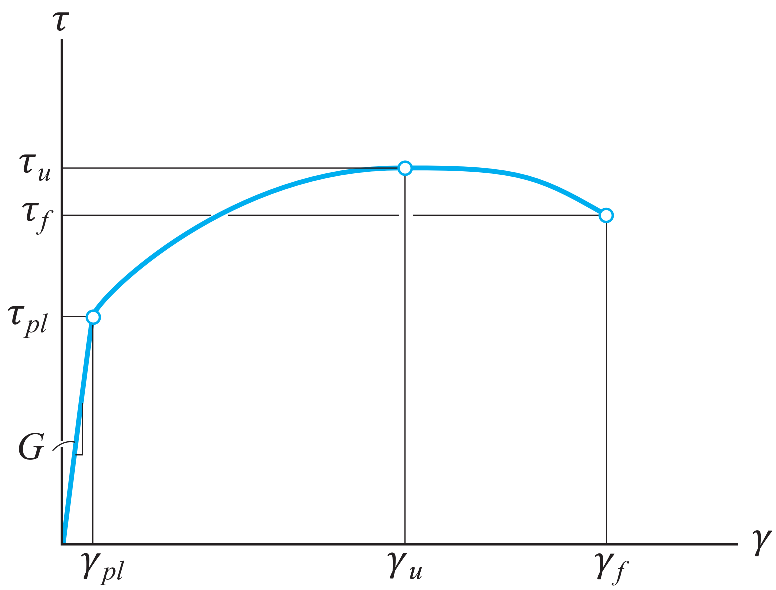


Figure 8: Shear stress-strain diagram.

3.1 Elastic and Plastic Regions

The curve will have similar properties to the stress-strain curve, including the elastic and plastic deformation regions.

3.2 Shear Modulus

Within the proportional region of the curve, the elastic behaviour is linear, so we can apply Hooke's law for shear to obtain

$$\tau = G\gamma$$

where G is called the **shear modulus of elasticity** or the **modulus of rigidity**.

3.3 Relationship between E, G, and ν

The three material constants, E, G, and nu can be related by the equation:

$$G = \frac{E}{2\left(1+\nu\right)}$$

4 Forces

4.1 Rigid Body Forces and Moments

Definition 4.1 (Resultant force)**.** In a system of n concurrent forces, the resultant force vector $\mathbf{F}_R$ is given by

$$\mathbf{F}_R = \sum_{i=1}^{n} \mathbf{F}_i$$

Definition 4.2 (Moment)**.** When a force is applied to a body it will cause the body to rotate about a point that is not on the line of action of the force. This tendency is called the moment of a force. The direction of the rotation is also known as the sense of direction of $\mathbf{M}_O$.
A moment is a vector quantity measured in Newton metres (N m).

Definition 4.3 (Resultant moment)**.** In a system of n concurrent forces, the resultant moment vector $\mathbf{M}_R$ about a point O is given by

$$(\mathbf{M}_R)_O = \sum_{i=1}^{n} (\mathbf{M}_O)_i$$

4.1.1 Moments — Scalar Formulation

The magnitude of a moment about the point O can be determined using the formula

$$\|\mathbf{M}_O\| = \|\mathbf{F}\| d$$

where $\mathbf{F}$ acts perpendicular to a line with distance d.

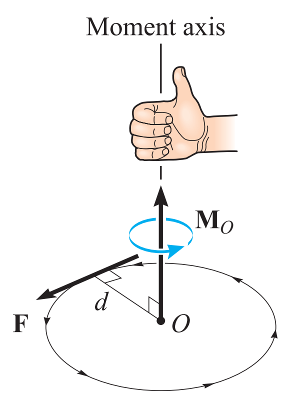


Figure 9: Moment scalar formulation when $\mathbf{F}$ is perpendicular.

When the force is not perpendicular to the distance, we can use the **sliding vector r** along with the angle between the two vectors to determine the magnitude of the moment.

$$\|\mathbf{M}_O\| = \|\mathbf{F}\|\|\mathbf{r}\| \sin(\theta)$$

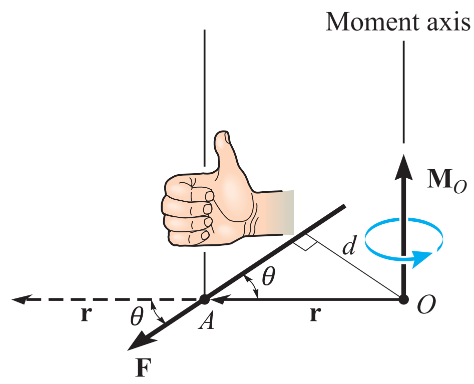


Figure 10: Moment scalar resolution when **F** is not perpendicular.

This results in the relationship

$$d = \|\mathbf{r}\| \sin(\theta)$$

4.1.2 Moments — Vector Formulation

Using the vector $\mathbf{r}$ with the force $\mathbf{F}$ we can define a relationship for the moment $\mathbf{M}_O$

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$$

where the sense of direction is determined through the right-hand rule.

Theorem 4.1.1 (Principle of transmissibility)**.** *Any position vector* $\mathbf{r}$ *from the point* O *to any point on the line of action of the force* $\mathbf{F}$ *will yield the same moment vector* $\mathbf{M}_O$*.*

$$\mathbf{M}_O = \mathbf{r}_1 \times \mathbf{F} = \mathbf{r}_2 \times \mathbf{F} = \mathbf{r}_3 \times \mathbf{F}$$

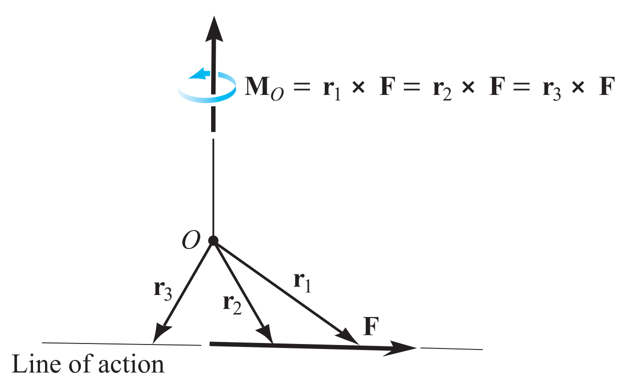


Figure 11: Principle of transmissibility.

Theorem 4.1.2 (Principle of moments)**.** *The moment of a force* **F** *about a point* O *is equal to the sum of the moments of the components of the force about the same point.*

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2) = \mathbf{r} \times \mathbf{F}_1 + \mathbf{r} \times \mathbf{F}_2$$

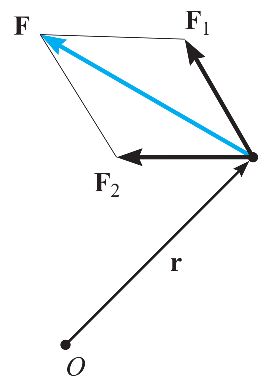
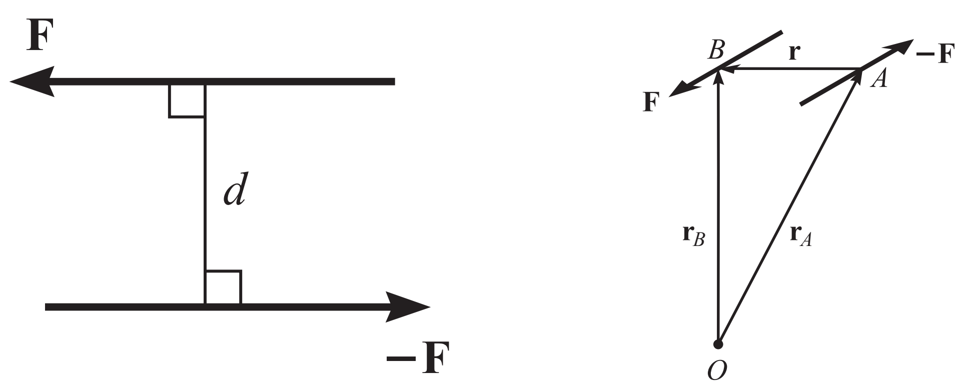


Figure 12: Principle of moments.

4.2 Couples

Definition 4.4 (Couple)**.** A couple is defined as two parallel forces that have the same magnitude, but opposite directions, that are separated by a perpendicular distance d.
Since the resultant force is zero, the only effect of a couple is to produce a rotation.

The moment produced by a couple is called a *couple moment*. We can determine its value by finding the sum of the moments of both couple forces about any arbitrary point.



(a) In 2d. (b) In 3d.

Figure 13: Moment of a couple.

The figures above illustrate how this moment can act at *any point* since **M** depends *only* on the position vector **r**.

The couple moment is determined using

$$\mathbf{M} = \mathbf{r}_B \times \mathbf{F} + \mathbf{r}_A \times (-\mathbf{F}) = (\mathbf{r}_B - \mathbf{r}_A) \times \mathbf{F}$$

However, $\mathbf{r} = \mathbf{r}_B - \mathbf{r}_A$, so that

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

where $\mathbf{M}$ does not require a subscript.

Definition 4.5 (Resultant couple moment)**.** In a system of n concurrent forces, the resultant couple moment vector $\mathbf{M}_R$ is given by

$$\mathbf{M}_R = \sum_{i=1}^{n} \mathbf{M}_i$$