Engineering Mechanics

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Contents

1 Stress and Strain

1.1 External Forces

Rigid bodies are subjected to external force and couple moment systems that result from the effects of gravitational, electrical, magnetic, or contact forces. Contact forces can be surface, linear, or concentrated forces.

1.1.1 Types of Forces

- Compressive (pushing)
- Tensile (pulling)
- Shear (sliding)
- Torsional (twisting)
- Biaxial tension
- Hydrostatic compression
- Bending (induces tension, compression and shear)

1.2 Internal Loadings

External forces cause internal loadings that occur in equal and opposite collinear pairs as stresses and strains. Internal loading is associated with **stress** while **strain** is a measure of a body's deformation.
These loadings have no external effects on the body and are not included on a **Free Body Diagram** (FBD) if the entire body is considered.
To determine the forces in each member, we can use the method of sections to represent the internal loading as external forces.

1.3 Internal Resultant Loadings

Although the exact distribution of the internal loading may be *unknown*, we can determine the resultant force $\mathbf{F}_R$ and resultant moment $(\mathbf{M}_R)_O$ about a point O by applying the equations of equilibrium

$$\sum \mathbf{F} = \mathbf{0}$$
$$\sum \mathbf{M}_O = \mathbf{0}.$$

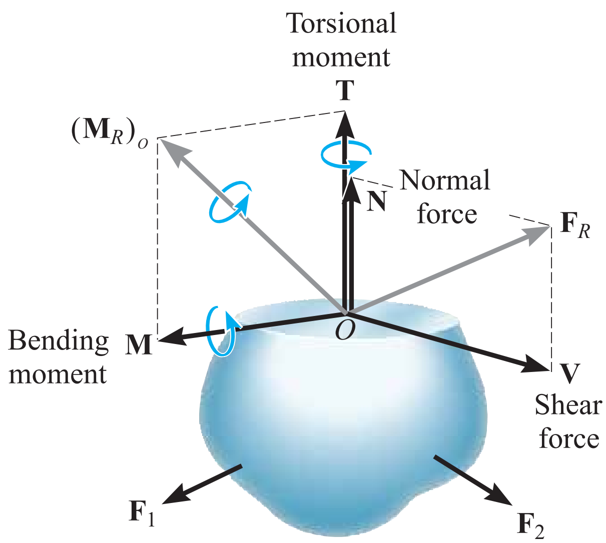


Figure 1: Resultant loadings acting on a body.

1.3.1 In 3D

In 3D, we can represent resultant loadings using four vectors acting over the sectioned area.

Normal force N force acting perpendicular to the area

Shear force V force acting on an axis tangent to the area

Torsional moment T rotation about the perpendicular axis

Bending moment M rotation about an axis tangent to the area

1.3.2 In 2D

In 2D, the body is subjected to a coplanar system of forces, where $\mathbf{T} = \mathbf{0}$.

1.3.3 In 1D

In 1D, the body is only subjected to axial forces, where $\mathbf{V} = \mathbf{T} = \mathbf{M} = \mathbf{0}$.

1.4 Stress

The force and moment acting at a specific point on a sectioned area of a body represent the resultant effects of the distribution of internal loading that acts over the sectioned area.

Definition 1.1 (Stress)**.** Consider the quotient of the force $\Delta\mathbf{F}$ over an area ΔA, then as the $\Delta A \to 0$, so does $\Delta\mathbf{F}$, while the quotient approaches a finite limit. This quotient is called the stress at that point.

$$\boldsymbol{\sigma} = \lim_{\Delta A \to 0} \frac{\Delta\mathbf{F}}{\Delta A}$$

Here the normal and shear stresses can be expressed using σ_z and τ_{zx} and τ_{zy}.

$$\sigma_z = \lim_{\Delta A \to 0} \frac{\Delta F_z}{\Delta A}$$
$$\tau_{zx} = \lim_{\Delta A \to 0} \frac{\Delta F_x}{\Delta A}$$
$$\tau_{zy} = \lim_{\Delta A \to 0} \frac{\Delta F_y}{\Delta A}$$

Stress describes the intensity of the internal force acting on a specific region passing through a point.
The unit for stress is Pascal where $1\,\mathrm{Pa}$ or $1\,\mathrm{N\,m^{-2}}$ and $1\,\mathrm{MPa}$ or $1\,\mathrm{N\,mm^{-2}}$.

To determine the average stress distribution acting over a cross-sectional area of an axially loaded bar, we assume that the material is both *homogeneous* and *isotropic*. This means the load P applied through the centroid of the cross-sectional area will cause the bar to deform uniformly throughout the central region of its length.

1.4.1 Average Normal Stress

By passing a section through a bar, equilibrium requires the resultant normal force N at the section to be equal to the external force P. And because the material undergoes a uniform deformation, it is necessary that the cross-section is subjected to a constant normal stress distribution.
As a result, each small area ΔA on the cross section is subjected to a force $\Delta N = \sigma \Delta A$, where the sum of these forces over the entire cross-sectional area is P. By letting $\Delta A \to \mathrm{d}A$ and therefore also $\Delta N \to \mathrm{d}N$, then as σ is a constant, we have

$$\int \mathrm{d}N = \int_A \sigma\,\mathrm{d}A$$
$$N = \sigma A$$

Therefore

$$\sigma_{\text{avg}} = \frac{N}{A}$$

where in this case $N = P$.

Theorem 1.4.1 (Equilibrium)**.** *For an uniaxially loaded body, the equation of force equilibrium gives*

$$\sigma\,(\Delta A) - \sigma'\,(\Delta A) = 0$$
$$\sigma = \sigma'$$

hence the normal stress components must be equal in magnitude but opposite in direction.
Under this condition, the material is subjected to ***uniaxial stress*** *and this analysis applies to members subjected to tension or compression.*

1.4.2 Average Shear Stress

Shear stress is the stress component that acts in the plane of the sectioned area. Here we must consider the number of planes that are in stress due to the applied force, so that the shear force $V = F/n$ for n planes, is applied to hold the segment in equilibrium.
The average shear stress distributed over each sectioned area that develops this shear force is defined by

$$\tau_{\text{avg}} = \frac{V}{A}$$

The loading case discussed is an example of **simple or direct shear** as the shear is caused by the direct action of the applied load **F**.

1.5 Strain

Definition 1.2 (Deformation)**.** Whenever a force is applied to a body, it will tend to change the body's shape and size. These changes are referred to as deformation.

Definition 1.3 (Strain)**.** To describe the deformation of a body through changes in lengths of line segments on the surface, we will develop the concept of strain. If an axial load P is applied to a bar, it will change the bar's length L_0 to L. Then the **average normal strain** of the bar is defined

$$\epsilon_{\text{avg}} = \frac{L - L_0}{L_0}$$

where the numerator is often written as $\delta = L - L_0$ and is known as elongation or extension.
The **normal strain** ϵ at a point in a body with an arbitrary shape is defined similarly. Consider a small line segment Δs which becomes $\Delta s'$ after deformation. Then the limit of the normal strain is

$$\epsilon = \lim_{\Delta s \to 0} \frac{\Delta s' - \Delta s}{\Delta s}$$

In both cases, normal strain is positive when the initial length elongates, and negative when the length contracts.
Strain is a dimensionless quantity sometimes expressed mm/mm or m/m, or as a percentage.

1.6 Shear Strain

Deformations not only cause line segments to elongate or contract, but they also cause them to change direction. If we consider two line segments that are originally perpendicular to one another, then the change in angle that occurs between them is referred to as **shear strain**. This angle is denoted by γ and is always measured in radians.
The shear strain of a block can be measured using

$$\gamma = \frac{\pi}{2} - \theta$$

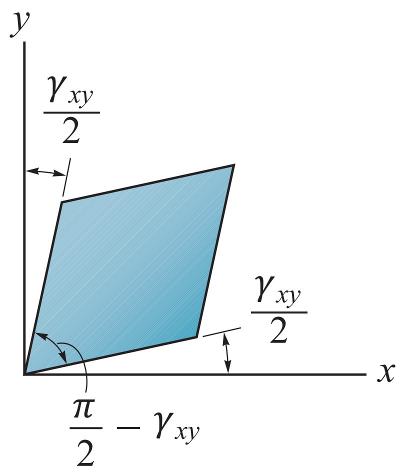


Figure 2: Shear block diagram.

We can also define shear stress as the change in angle

$$\gamma = \gamma_f - \gamma_o$$

1.7 Small Strain Analysis

Most engineering designs involve applications for which only small deformations are allowed (i.e., $\epsilon \ll 1$). Hence we can make the following approximations for the small change in angle $\Delta\theta$.

$$\sin(\Delta\theta) \approx \Delta\theta$$
$$\cos(\Delta\theta) \approx 1$$
$$\tan(\Delta\theta) \approx \Delta\theta$$

2 Tension and Compression Tests

To determine the strength of a material, we must perform a tension or compression test. This test measures the stress and strain from a load P, and the results can be used to produce a **stress-strain diagram**. There are two ways in which the stress-strain diagram is normally described.

2.1 Stress-Strain Diagram

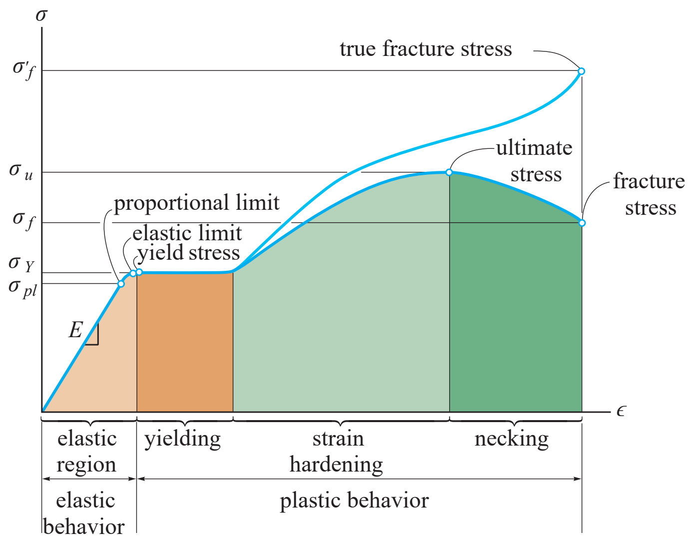


Figure 3: Stress-strain diagram for a typical metal.

2.1.1 Conventional Stress-Strain Diagram

The engineering stress assumes that the area A is constant throughout the gauge length

$$\sigma = \frac{P}{A_0}$$

where A_0 is the *original* cross-sectional area of the specimen.
Likewise, the engineering strain uses the specimen's original length L_0

$$\epsilon = \frac{\delta}{L_0}$$

2.1.2 True Stress-Strain Diagram

The true stress and true strain use the instantaneous area A and length L at each measurement.

2.2 Elastic Behaviour

The initial region of the curve is referred to as the **elastic region** where the deformation is *elastic* so that unloading causes the specimen to return to its original shape.

2.2.1 Proportional Limit

For the majority of the elastic deformation, the curve is *linear* up to the point where the stress reaches the **proportional limit** at $(\sigma_{pl}, \epsilon_{pl})$.

2.2.2 Modulus of Elasticity

The linear relationship up to this point is characterised by Hooke's law and is expressed as

$$\sigma = E\epsilon$$

where E is the constant of proportionality, called the **modulus of elasticity** or **Young's modulus**.

2.2.3 Elastic Limit

When the stress slightly exceeds the proportionality limit, the curve bends until the stress reaches an **elastic limit**.

2.3 Plastic Behaviour

An increase in stress above the elastic limit will result in a breakdown of the material and cause it to deform plastically.

2.3.1 Yielding

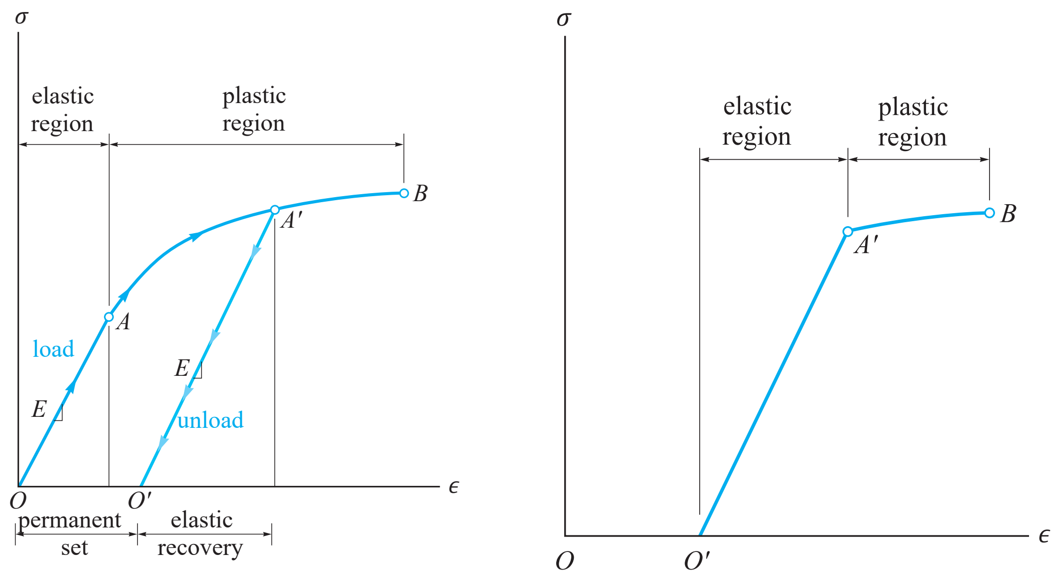
This behaviour is called **yielding** and the stress that causes yielding occurs at the **yield point** (σ_Y, ϵ_Y). Although not shown in the diagram, the yield point is distinguished as two points.
The **upper yield point** occurs first, followed by a sudden decrease in load-carrying capacity to a **lower yield point**. Once the yield point is reached, *the specimen will continue to elongate **without** any increase in load.* When the material behaves in this manner, it is often referred to as being **perfectly plastic**.

2.3.2 Yield Strength

Commonly the proportionality limit, the elastic limit, and yield point are indistinguishable, due to this, the **yield strength** is defined at $(\sigma_{YS}, \epsilon_{YS})$.
To determine this point, a 0.2% strain is chosen, and a line with gradient E is drawn from the ϵ axis. The point where this line intersects the curve defines $(\sigma_{YS}, \epsilon_{YS})$.

2.3.3 Strain Hardening

Yielding ends when any loading causes the stress to increase, this rise in the curve is referred to as **strain hardening**.
When a plastically deformed ductile material is unloaded, the elastic strain is recovered as the material returns to its equilibrium state.
However the plastic strain is maintained, resulting in a **permanent set**.



(a) During loading.

(b) After loading.

Figure 4: Elastic strain recovery under strain hardening.

2.3.4 Ultimate Tensile Stress

The maximum stress reached on the diagram is referred to as the **ultimate tensile stress** $(\sigma_{UTS},\ \epsilon_{UTS})$.

2.3.5 Necking

While the specimen elongates up to ϵ_{UTS}, its cross-sectional area will decrease *uniformally* over its gauge length. However after reaching ϵ_{UTS}, the cross-sectional area will decrease *locally*, causing an increase in stress. As a result, a "neck" forms in this region, and the specimen experiences **necking**.

2.3.6 Fracture Stress

Finally, the specimen breaks where the curve ends at the **fracture point** at $\left(\sigma_f,\ \epsilon_f\right)$.

2.4 Allowable Stress Design

To ensure the safety of a structural member, it is necessary to restrict the applied load to one that is *less than* what a member can support.
This is done by specifying a **factor of safety** F.S. which determines the allowable load F_{allow} a member should be designed for.

$$\text{F.S.} = \frac{F_{\text{fail}}}{F_{\text{allow}}}$$

Here F_{fail} is found from experimental testing of the material. When the load is linearly related to the stress, we can express the factor of safety using the ratio of failure stress and allowable stress.

$$\text{F.S.} = \frac{\sigma_{\text{fail}}}{\sigma_{\text{allow}}}$$
$$\text{F.S.} = \frac{\tau \text{fail}}{\tau_{\text{allow}}}$$

2.5 Ductility

Definition 2.1 (Ductility)**.** Ductility is a measure of the amount of plastic deformation a material can sustain under tensile stress before failure.
Ductility can be measured using the **percent elongation** (in length) or **percent reduction** (in area) of a material.

$$\text{Percent Elongation} = \frac{L_f - L_0}{L_0} 100\,\%$$
$$\text{Percent Reduction} = \frac{A_0 - A_f}{A_0} 100\,\%$$

As the elastic region is very brief in most materials, ductility is often measured using the original length and area, rather than the length and area when the material undergoes plastic deformation.

2.6 Brittleness

Definition 2.2 (Brittleness)**.** Brittleness describes the property of a material that fractures with little to no yielding.

2.7 Poisson's Ratio

When a deformable body is subjected to a force, it can elongate longitudinally and also contract laterally. The strain in the longitudinal (or axial) direction is given by

$$\epsilon_{\text{long}} = \frac{\delta}{L}$$

and the strain in the lateral (or radial) direction is given by

$$\epsilon_{\text{lat}} = \frac{\delta'}{r}$$

where δ' is the change in the radius r.
Consider the ratio of these two quantities ν

$$\nu = -\frac{\epsilon_{\text{lat}}}{\epsilon_{\text{long}}}.$$

Within the elastic region, ν will be constant, and it is referred to as **Poisson's ratio**.
Note the negative value is introduced as the longitudinal and lateral strains have opposite signs.

2.8 Strain Energy

As a material is deformed under external load, the load will do external work. This work is stored in the material as internal energy or **strain energy**.

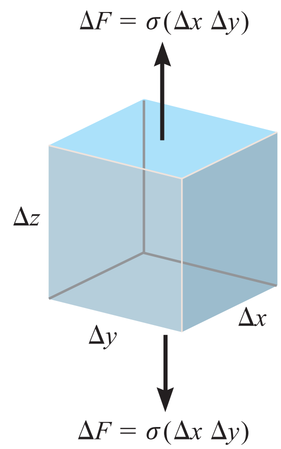


Figure 5: Internal energy in small element.

If we consider a small volume element of the material, then the force is equal to the average force magnitude $\Delta F/2$ and the displacement is given by d. Therefore the strain energy ΔU is given by

$$\begin{aligned}\Delta U &= \frac{1}{2}\Delta F d \\ &= \frac{1}{2}\left(\sigma\Delta x\Delta y\right)\left(\epsilon\Delta z\right) \\ &= \frac{1}{2}\sigma\epsilon\Delta V\end{aligned}$$

where ΔV is the volume of the element. If we consider the strain energy *per unit volume*, then

$$u = \frac{\Delta U}{\Delta V} = \frac{1}{2}\sigma\epsilon$$

where u is the **strain energy density**. u can also be determined by finding the area under the stress-strain diagram, and hence has the units $\mathrm{J\,m^{-3}}$.

2.8.1 Modulus of Resilience

When the stress in a material reaches the proportional limit, the strain energy density is referred to as the modulus of resilience.

$$u_r = \frac{1}{2}\sigma_{pl}\epsilon_{pl} = \frac{1}{2}\frac{\sigma_{pl}^2}{E}$$

The modulus of resilience is also the area under the proportional region of the stress-strain diagram.

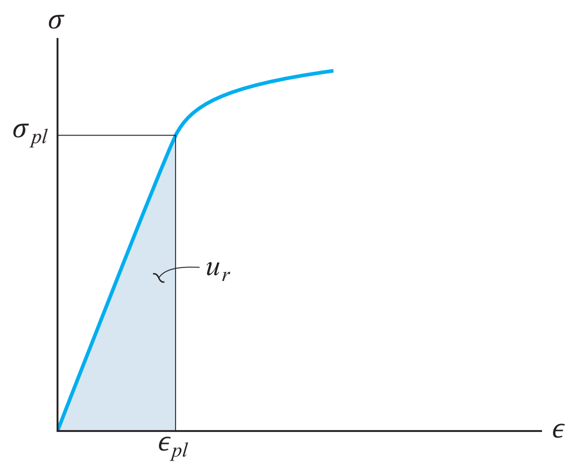


Figure 6: Modulus of resilience u_r.

2.8.2 Modulus of Toughness

Another important property of a material is its modulus of toughness, u_t. This quantity represents the entire area under the stress-strain diagram.

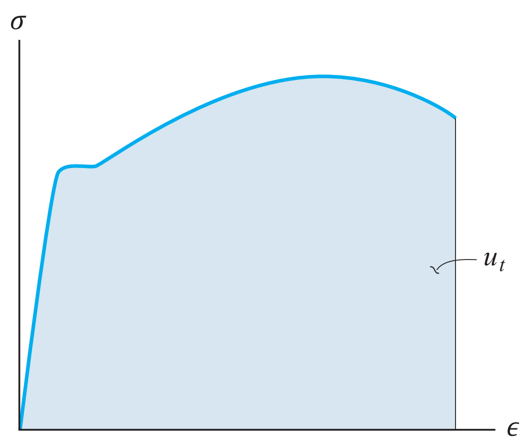


Figure 7: Modulus of toughness u_t.

3 Shear Stress-Strain Diagram

Similar to a tensile test, we can use a thin tube and subject it to torsional loading. Using the data for the applied torque and resulting angle of twist, we can form a shear stress-strain diagram.

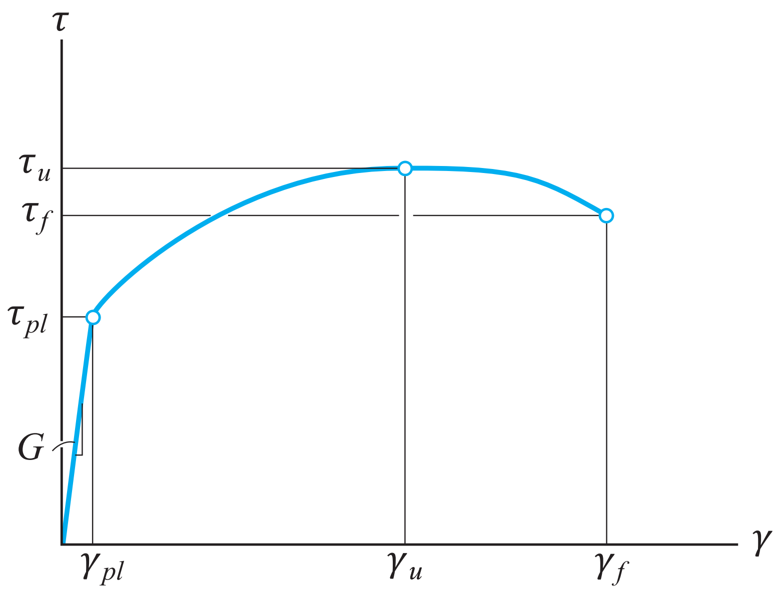


Figure 8: Shear stress-strain diagram.

3.1 Elastic and Plastic Regions

The curve will have similar properties to the stress-strain curve, including the elastic and plastic deformation regions.

3.2 Shear Modulus

Within the proportional region of the curve, the elastic behaviour is linear, so we can apply Hooke's law for shear to obtain

$$\tau = G\gamma$$

where G is called the **shear modulus of elasticity** or the **modulus of rigidity**.

3.3 Relationship between E, G, and ν

The three material constants, E, G, and nu can be related by the equation:

$$G = \frac{E}{2\,(1+\nu)}$$

4 Forces

4.1 Rigid Body Forces and Moments

Definition 4.1 (Resultant force)**.** In a system of n concurrent forces, the resultant force vector $\mathbf{F}_R$ is given by

$$\mathbf{F}_R = \sum_{i=1}^{n} \mathbf{F}_i$$

Definition 4.2 (Moment)**.** When a force is applied to a body it will cause the body to rotate about a point that is not on the line of action of the force. This tendency is called the moment of a force. The direction of the rotation is also known as the sense of direction of $\mathbf{M}_O$.
A moment is a vector quantity measured in Newton metres (N m).

Definition 4.3 (Resultant moment)**.** In a system of n concurrent forces, the resultant moment vector $\mathbf{M}_R$ about a point O is given by

$$(\mathbf{M}_R)_O = \sum_{i=1}^{n} (\mathbf{M}_O)_i$$

4.1.1 Moments — Scalar Formulation

The magnitude of a moment about the point O can be determined using the formula

$$\|\mathbf{M}_O\| = \|\mathbf{F}\| d$$

where $\mathbf{F}$ acts perpendicular to a line with distance d.

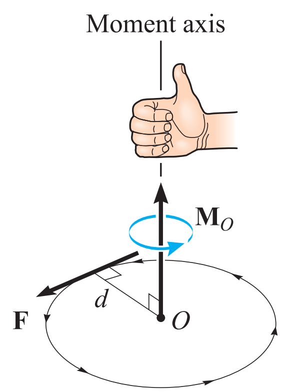


Figure 9: Moment scalar formulation when $\mathbf{F}$ is perpendicular.

When the force is not perpendicular to the distance, we can use the **sliding vector r** along with the angle between the two vectors to determine the magnitude of the moment.

$$\|\mathbf{M}_O\| = \|\mathbf{F}\|\|\mathbf{r}\| \sin(\theta)$$

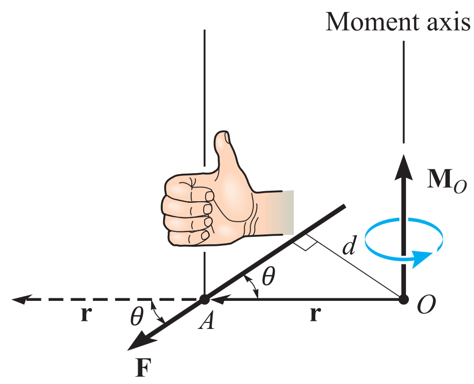


Figure 10: Moment scalar resolution when $\mathbf{F}$ is not perpendicular.

This results in the relationship

$$d = \|\mathbf{r}\| \sin(\theta)$$

4.1.2 Moments — Vector Formulation

Using the vector $\mathbf{r}$ with the force $\mathbf{F}$ we can define a relationship for the moment $\mathbf{M}_O$

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$$

where the sense of direction is determined through the right-hand rule.

Theorem 4.1.1 (Principle of transmissibility)**.** *Any position vector* $\mathbf{r}$ *from the point* O *to any point on the line of action of the force* $\mathbf{F}$ *will yield the same moment vector* $\mathbf{M}_O$*.*

$$\mathbf{M}_O = \mathbf{r}_1 \times \mathbf{F} = \mathbf{r}_2 \times \mathbf{F} = \mathbf{r}_3 \times \mathbf{F}$$

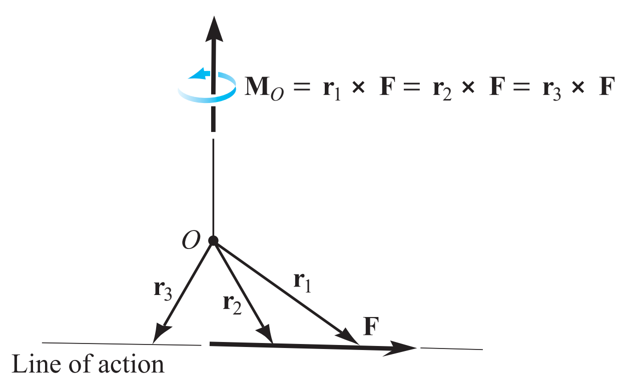


Figure 11: Principle of transmissibility.

Theorem 4.1.2 (Principle of moments)**.** *The moment of a force* **F** *about a point* O *is equal to the sum of the moments of the components of the force about the same point.*

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2) = \mathbf{r} \times \mathbf{F}_1 + \mathbf{r} \times \mathbf{F}_2$$

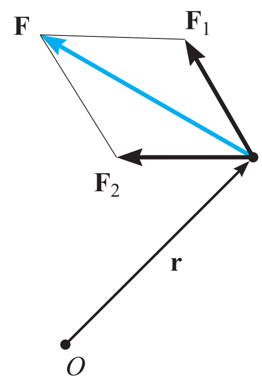
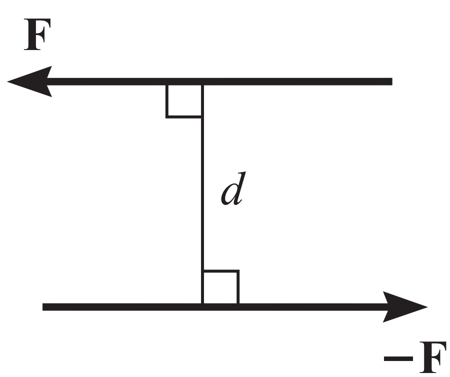


Figure 12: Principle of moments.

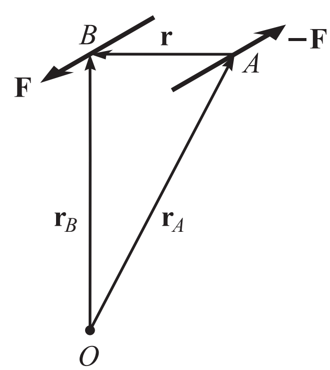
4.2 Couples

Definition 4.4 (Couple)**.** A couple is defined as two parallel forces that have the same magnitude, but opposite directions, that are separated by a perpendicular distance d.
Since the resultant force is zero, the only effect of a couple is to produce a rotation.

The moment produced by a couple is called a *couple moment*. We can determine its value by finding the sum of the moments of both couple forces about any arbitrary point.



(a) In 2d.



(b) In 3d.

Figure 13: Moment of a couple.

The figures above illustrate how this moment can act at *any point* since **M** depends *only* on the position vector **r**.

The couple moment is determined using

$$\mathbf{M} = \mathbf{r}_B \times \mathbf{F} + \mathbf{r}_A \times (-\mathbf{F}) = (\mathbf{r}_B - \mathbf{r}_A) \times \mathbf{F}$$

However, $\mathbf{r} = \mathbf{r}_B - \mathbf{r}_A$, so that

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

where $\mathbf{M}$ does not require a subscript.

Definition 4.5 (Resultant couple moment)**.** In a system of n concurrent forces, the resultant couple moment vector $\mathbf{M}_R$ is given by

$$\mathbf{M}_R = \sum_{i=1}^{n} \mathbf{M}_i$$

In summary, we can reduce a force and couple system to a resultant force $\mathbf{F}_R$, and resultant moment $(\mathbf{M}_R)_O$ about a point O, so that

$$\mathbf{F}_R = \sum \mathbf{F}$$
$$(\mathbf{M}_R)_O = \sum \mathbf{M}_O + \sum \mathbf{M}$$

4.3 Rigid-Body Equilibrium

When applying the equations of equilibrium we assume that the body remains rigid and does not deform under the applied load.

4.4 Weight

In a gravitational field, the gravitational force acting on an object is called its weight. Using Newton's Law of Gravitational Attraction, with $g = G\frac{M}{r^2}$ where M and r are the mass and radius of the earth, we can calculate the force of gravity on an object of mass m with

$$W = mg.$$

4.5 Centre of Gravity

In a gravitational field, each particle in a body will have a weight $\mathrm{d}W$. These weights form a parallel force system, so that the resultant of the system is the total weight of the body, which passes through a single point called the center of gravity G.

4.6 Friction

Friction is a force F that resists the movement P between two contacting surfaces that slide relative to one another. This force always acts tangent to the surface at the points of contact, and is directed so that it opposes the possible or existing motion between the surfaces.

4.6.1 Static Friction

When the frictional force F is not great enough to balance the force P, F is called the limiting static frictional force F_s as any increase in P will cause the object to move. This force is directly proportional to the resultant normal force N:

$$F_s = \mu_s N$$

where the constant of proportionality μ_s is called the coefficient of static friction.

4.6.2 Kinetic Friction

If the magnitude of P is increased so that it is greater than F_s, the frictional force at the contacting surface drops to a smaller value F_k, called the kinetic frictional force. The object will then begin to slide with increasing speed. The kinetic friction force is calculated similar to the static friction force,

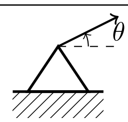
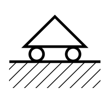
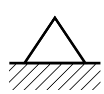
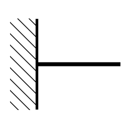
$$F_k = \mu_k N$$

where the constant of proportionality μ_k is called the coefficient of kinetic friction.

5 Structural Analysis

5.1 Supports

A support prevents the translation or rotation of a body by exerting a force or couple moment on the body.
The following table summarises basic supports in 2d, along with their symbols and reactions.

Name	**Symbol**	**Reaction**
Cable		$\mathbf{F}$ at fixed angle θ
Roller		$\mathbf{F}_y$
Pin		$\mathbf{F}$
Fixed		$\mathbf{F}$ and $\mathbf{M}$

5.2 Free Body Diagrams

To construct a free body diagram, we can use the following steps:

1. Draw outlined shape
   (a) Isolate body
   (b) Define a coordinate system
2. Show all external forces and couple moments
   (a) Label all known forces and moments with magnitudes and directions
   (b) Label all unknown forces and moments with appropriate letters
   (c) Indicate any required dimensions on body

5.3 Trusses

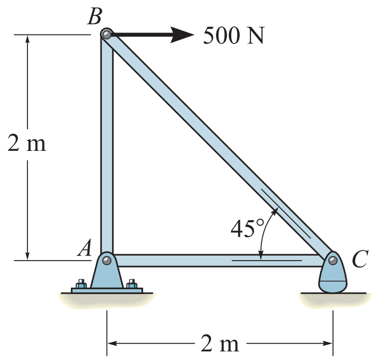
A truss is a structure composed of members joined at their end points. Using the dimensions and loads on a truss, we can apply analysis techniques to determine the forces developed in the truss members.

5.4 Method of Joints

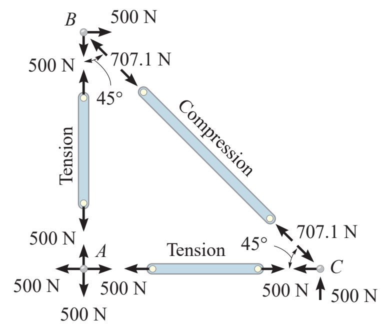
By drawing a free body diagram at each joint, we can use the force equilibrium equations to obtain the forces acting on each joint.
Generally, this requires starting at the external force and traversing each joint individually.
When applying the equilibrium equations assume that unknown forces are in tension so that positive magnitudes correspond to members in tension, and negative magnitudes correspond to members in compression.

Theorem 5.4.1 (Determinate trusses)**.** *Given the number of members m, joints j and reactions r, the following relationship must hold for a statically determinate system.*

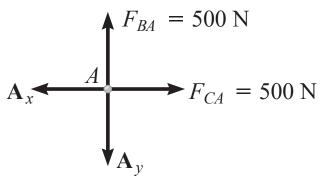
$$m = 2j - r$$



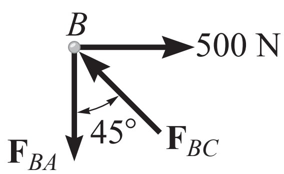
(a) Example truss.



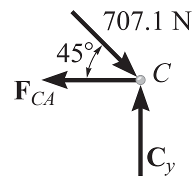
(b) Internal forces with FBDs.



(c) Joint A FBD.



(d) Joint B FBD.

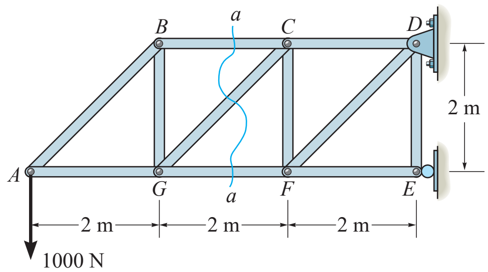


(e) Joint C FBD.

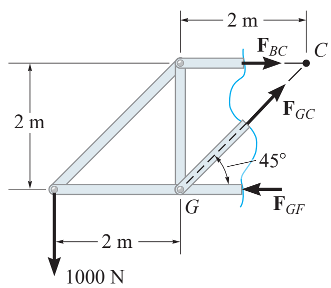
Figure 14: Method of joints example.

5.5 Method of Sections

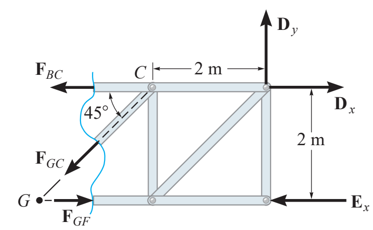
To calculate the force of an arbitrary member in a truss we can utilise the method of sections. In this method, we can create a section that cuts through a maximum of 3 members, corresponding to the three equilibrium equations. Then the sectioned members can be treated as tension forces and their internal forces can be determined using the equilibrium equations.



(a) Example truss.



(b) Left section.



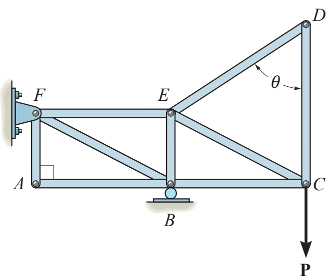
(c) Right section.

Figure 15: Method of sections example.

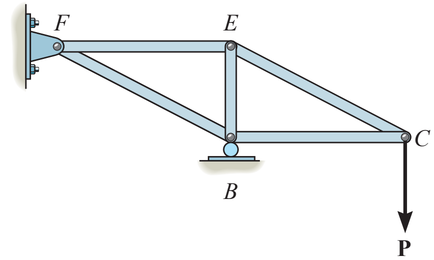
5.6 Zero Force Members

A zero force member does not carry any load and thus has an internal force of 0. These members are used to increase the stability of structures in the event that loading changes.
By identifying these members, we can reduce the number of members when applying the method of joints.
Zero force members can be identified by two properties:

1. At a **two** member joint, if both members are **not** parallel and no external loads are applied at the joint, then **both** members are zero force members.



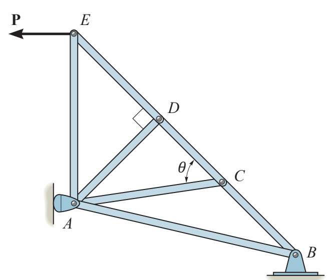
(a) Example truss.



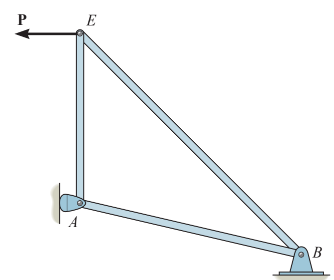
(b) Simplified truss.

Figure 16: Identifying two member joint zero force members.

2. At a **three** member joint, if **two** members are parallel and no external loads are applied at the joint, then the **third** member is a zero force member.



(a) Example truss.



(b) Simplified truss.

Figure 17: Identifying three member joint zero force members.

5.7 Frames and Machines

Frames and machines are two types of structures which are often composed of pin-connected multi-force members, i.e., members that are subjected to more than two forces. Frames are used to support loads, whereas machines contain moving parts and are designed to transmit and alter the effect of forces.
Provided a frame or machine contains no more supports or members than are necessary to prevent its collapse, then the forces acting at the joints and supports can be determined by applying the equations of equilibrium to each of its members.

5.7.1 Free Body Diagrams

To determine the forces acting at the joints and supports of a frame or machine, the structure must be disassembled and the free-body diagrams of its parts must be drawn.
The following important points must be observed:

- Isolate each part by drawing its outlined shape and show all the forces and couple moments that act on the part.
- Identify all the two-force members in the structure and represent their free-body diagrams as having two equal but opposite collinear forces acting at their points of application (i.e., the pins). By doing this, we can avoid solving an unnecessary number of equilibrium equations.
- Forces common to any two contacting members act with equal magnitudes but opposite sense on the free-body diagrams of the respective members.

6 Centroids

Given the 1d region $L : [x_1, x_2]$, 2d region $R : [x_1, x_2] \times [y_1, y_2]$ and 3d region $V : [x_1, x_2] \times [y_1, y_2] \times [z_1, z_2]$, the following definitions represent various geometric properties of a body.

6.1 Mass

The mass of a body with density ρ is a measure of the quantity of matter.
For a length:

$$M = \int_L \rho \, \mathrm{d}L$$

For an area:

$$M = \iint_A \rho \, \mathrm{d}A$$

For a volume:

$$M = \iiint_V \rho \, \mathrm{d}V$$

6.2 Weight

The weight of a body is its attraction to another body (e.g., Earth), developed by a gravitational field with acceleration g.
For a length:

$$W = \int_L \rho g \, \mathrm{d}L$$

For an area:

$$W = \iint_A \rho g \, \mathrm{d}A$$

For a volume:

$$W = \iiint_V \rho g \, \mathrm{d}V$$

6.3 Centre of Mass

The resultant mass of the particles in a body pass through a single point called the center of mass.
For a length:

$$\overline{x} = \frac{1}{M} \int_L \rho x \, \mathrm{d}L$$

For an area:

$$\overline{x} = \frac{1}{M} \iint_A \rho x \, \mathrm{d}A \qquad \overline{y} = \frac{1}{M} \iint_A \rho y \, \mathrm{d}A$$

For a volume:

$$\overline{x} = \frac{1}{M} \iiint_V \rho x \, \mathrm{d}A \qquad \overline{y} = \frac{1}{M} \iiint_V \rho y \, \mathrm{d}V \qquad \overline{z} = \frac{1}{M} \iiint_V \rho z \, \mathrm{d}V$$

6.4 Center of Gravity

The resultant weight of the particles in a body pass through a single point called the center of gravity.
For a length:

$$\overline{x} = \frac{1}{W} \int_L \rho g x \, \mathrm{d}L$$

For an area:

$$\overline{x} = \frac{1}{W} \iint_A \rho g x \, \mathrm{d}A \qquad \overline{y} = \frac{1}{W} \iint_A \rho g y \, \mathrm{d}A$$

For a volume:

$$\overline{x} = \frac{1}{W} \iiint_V \rho g x \, \mathrm{d}V \qquad \overline{y} = \frac{1}{W} \iiint_V \rho g y \, \mathrm{d}V \qquad \overline{z} = \frac{1}{W} \iiint_V \rho g z \, \mathrm{d}V$$

6.5 Uniform Gravitational Field

In a uniform gravitational field where $g \in \mathbb{R}$ is a constant. The weight simplifies to

$$W = Mg$$

and the center of mass and center gravity are equal.

6.6 Geometric Measures

The measure of a body represents its length in 1d, area in 2d, and volume in 3d.
For a length:

$$L = \int_L \mathrm{d}L$$

For an area:

$$A = \iint_A \mathrm{d}A$$

For a volume:

$$V = \iiint_V \mathrm{d}V$$

6.7 Centroids

The geometric center of a body is known as its centroid.
For a length:

$$\overline{x} = \frac{1}{L} \int_L \mathbf{r} \, \mathrm{d}L$$

For an area:

$$\overline{x} = \frac{1}{A} \iint_A x \, \mathrm{d}A \qquad \overline{y} = \frac{1}{A} \iint_A y \, \mathrm{d}A$$

For a volume:

$$\overline{x} = \frac{1}{V} \iiint_V x \, \mathrm{d}V \qquad \overline{y} = \frac{1}{V} \iiint_V y \, \mathrm{d}V \qquad \overline{z} = \frac{1}{V} \iiint_V z \, \mathrm{d}V$$

6.8 Composite Bodies

A composite body consists of n connected "simpler" shaped bodies. To determine the center of gravity of such a body, we can use the sum of individual components:

$$\overline{x} = \frac{\sum_{i=1}^{n} \overline{x}_i A_i}{\sum_{i=1}^{n} A_i} \qquad \overline{y} = \frac{\sum_{i=1}^{n} \overline{y}_i A_i}{\sum_{i=1}^{n} A_i}$$

where A_i is the area of each shape, and $\overline{x}_i$ and $\overline{y}_i$ are the centroids of each shape.

6.9 Distributed Loads

A distributed load on a 1d rod of length L can be represented as a point load $\mathbf{F}_R$ applied at the position $0 \leq \overline{x} \leq L$ on the rod.

$$\mathbf{F}_R = \int_0^L w(x) \, \mathrm{d}x$$

$$\overline{x} = \frac{1}{\mathbf{F}_R} \int_0^L w(x) \, x \, \mathrm{d}x$$

here each section can be represented as

$$\mathbf{F}_R = \int_a^b w(x) \, \mathrm{d}x$$

$$\overline{x} = \frac{1}{\mathbf{F}_R} \int_a^b w(x) \, x \, \mathrm{d}x$$

over the interval $a < x < b$.

6.10 Moment of Inertia

The moment of inertia describes the resistance of a moment when a force is applied perpendicular to the axis of rotation.
The moment of inertia about the x and y axes are given by

$$I_x = \iint_A y^2 \, \mathrm{d}A$$
$$I_y = \iint_A x^2 \, \mathrm{d}A$$

and the polar moment of inertia is given by

$$J_O = I_x + I_y.$$

Theorem 6.10.1 (Parallel axis theorem)**.** *If the moment of inertia is known about an axis passing through the centroid of a region, then the parallel-axis theorem can be used to find the moment of inertia about any axis that is parallel to the centroidal axis.*

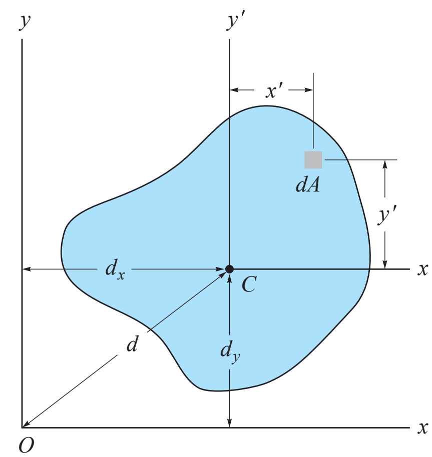


Figure 18: Region with known moment of inertia about its centroidal axis.

In the figure above, if $I_{x'}$ represents the moment of inertia about the centroidal axis x', then the moment of inertia about the x-axis is given by

$$\begin{aligned} I_x &= \iint_R \left(y' + d_y\right)^2 \mathrm{d}A \\ &= \iint_R y'^2 \, \mathrm{d}A + 2d_y \iint y' \, \mathrm{d}A + d_y^2 \iint_R \mathrm{d}A \\ &= \iint_R y'^2 \, \mathrm{d}A + A d_y^2 \\ &= I_{x'} + A d_y^2 \end{aligned}$$

Similarly for $I_{y'}$, we have

$$I_y = I_{y'} + Ad_x^2$$

so that the polar moment of inertia is given by

$$J_O = J_C + Ad^2$$

where $J_C = I_{x'} + I_{y'}$ and $d^2 = d_x^2 + d_y^2$.

7 Axial Load

This section covers the methods used to determine the deformation of an axially loaded member.

7.1 Saint-Venant's Principle

The difference between the effects of two different but statically equivalent loads becomes very small at sufficiently large distances from the load.

7.2 Axial Deformation

Using Hooke's Law, we can show that the axial displacement of a member is given by

$$\delta = \frac{PL}{AE}$$

If the member varies in cross-sectional area, elasticity, or load, then we can use the following derivation instead:

$$\begin{aligned} \sigma(x) &= E(x)\,\epsilon(x) \\ \frac{P(x)}{A(x)} &= E(x)\frac{\mathrm{d}\delta}{\mathrm{d}x} \\ \delta &= \int_0^L \frac{PL}{AE}\,\mathrm{d}x \end{aligned}$$

similarly for n piecewise members,

$$\delta = \sum_{i=1}^{n} \frac{P_i L_i}{A_i E_i}$$

where we can use the method of sections and determine displacements between changes in internal force, cross-sectional area, or elasticity.

7.3 Principle of Superposition

The principle of superposition is used to determine the stress or displacement at a point in a member which is subjected to complicated loading. By subdiving the loading into components, the resultant stress or displacement at a point can be determined by algebraically summing the stress of displacement caused by each load component applied separately to the member.
The following two conditions must be satisfied:

1. The loading must be linearly related to the stress or displacement. For example, $\sigma = P/A$ or $\delta = PL/(AE)$, but not $\sigma = P^2/A$ or $\delta = \sin(P)L/(AE)$.

2. The loading must not significantly change the original geometry or configuration of the member. This is because the moment caused by each load will not equal the resultant moment.

7.4 Statically Indeterminate Axially Loaded Members

If a system is statically indeterminate, we can establish an additional **kinematic condition** that considers the displacement of a particular point on the bar. For example, if both ends of the bar are fixed, then the following equation may be used:

$$\delta_{AB} = 0.$$

8 Shear Force & Bending Moment Diagrams

Members that are slender and support loadings that are applied perpendicular to their longitudinal axis are called beams. Beams can be supported in a number of ways, the most common are:

- Simply supported (pin and roller)
- Cantilever (fixed and hanging)
- Overhanging (pin and roller with roller side hanging)

The shear force and bending moment along the length of a beam can be described as functions of distance x along the beam. We can use these functions to develop shear force and bending moment diagrams.

8.1 Beam Positive Sign Convention

When applying the method of sections along a beam, we start from the left side and move toward the right side of the beam. The following sign conventions apply:

- Positive loads point upwards
- Positive shear forces point downwards (reverse if starting from the right)
- Positive moments rotate anticlockwise (reverse if starting from the right)

This is illustrated on the following diagram.

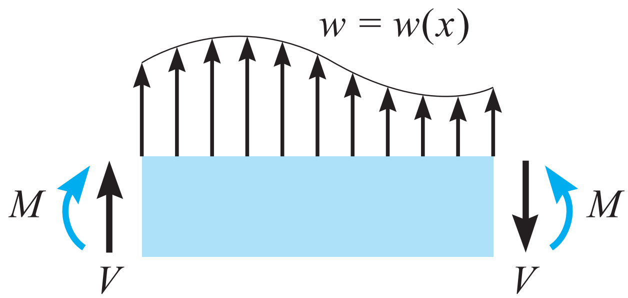


Figure 19: Positive sign convention on a section of a beam.

8.2 Determining the Shear Force and Bending Moment

In each section, the length of the beam is treated as a variable x, allowing us to determine V and M for the ith section on the interval $s_i < x \leq s_{i+1}$. By constructing the piecewise function comprising of the entire length of the beam, we can determine the shear force and bending moment diagrams.

8.3 Determining the Shear Force and Bending Moment Analytically

If we consider the following segment Δx on a beam of length L:

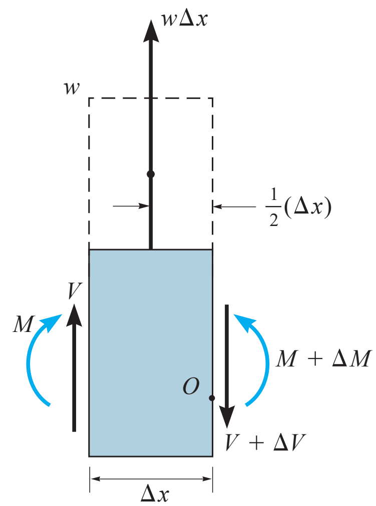


Figure 20: Small section of a beam.

then by the equilibrium of forces in the vertical direction:

$$V(x) = \int w(x)\,\mathrm{d}x$$

similarly the equilibrium of moments about O give:

$$M(x) = \int V(x)\,\mathrm{d}x$$

8.4 Point Loads

The result of a point load $\mathbf{F}$ on a beam is a step on the shear force diagram in the direction of $\mathbf{F}$,

8.5 Point Moments

The result of a clockwise moment $\mathbf{M}_O$ applied at a point O on a beam is a step on the bending moment diagram, where clockwise moments are positive.

8.6 Bending Deformation

Consider the following beam under bending

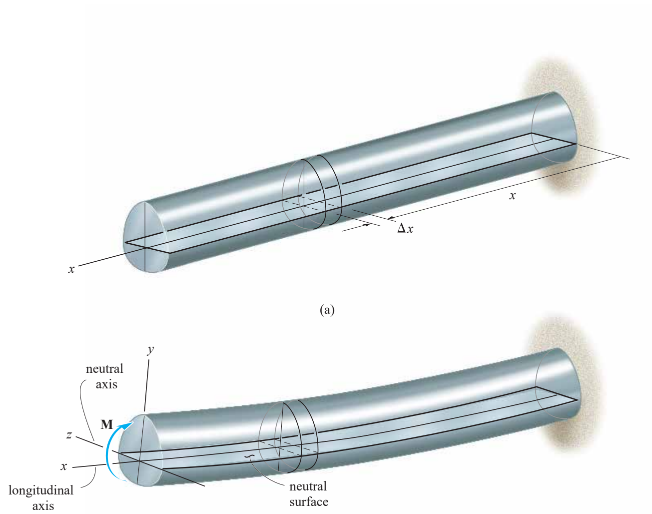
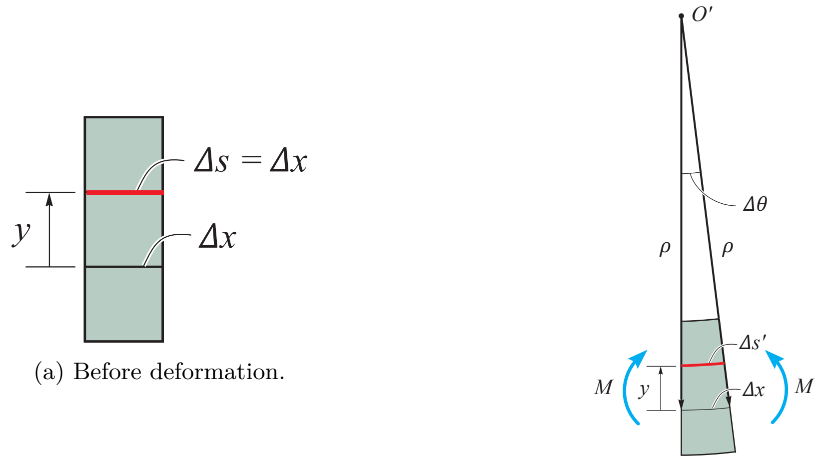


Figure 21: Beam under bending deformation.

with the following section



(a) Before deformation.

(b) After deformation.

Figure 22: Beam section.

where ρ is the radius of curvature of the beam and Δx, located on the neutral surface, does not change in length whereas Δs located at an arbitrary distance y above the neutral axis, contracts to $\Delta s'$ through deformation.
By definition, the normal strain along Δs is given by

$$\epsilon = \lim_{\Delta s \to 0} \frac{\Delta s' - \Delta s}{\Delta s}$$

Before deformation, we have

$$\Delta x = \Delta s = \rho \Delta \theta$$

through trigonometry, we can also show that for small angles

$$\Delta s' = (\rho - y) \sin(\Delta\theta) \approx (\rho - y)\, \Delta\theta$$

therefore the above equation becomes

$$\begin{aligned} \epsilon &= \lim_{\Delta\theta \to 0} \frac{(\rho - y)\, \Delta\theta - \rho\Delta\theta}{\rho\Delta\theta} \\ &= -\frac{y}{\rho}. \end{aligned}$$

As $\frac{1}{\rho}$ is constant at x, the longitudinal normal strain will vary linearly with y measured from the neutral axis. Therefore at distance $y = c$ from the neutral axis the strain will be maximised.

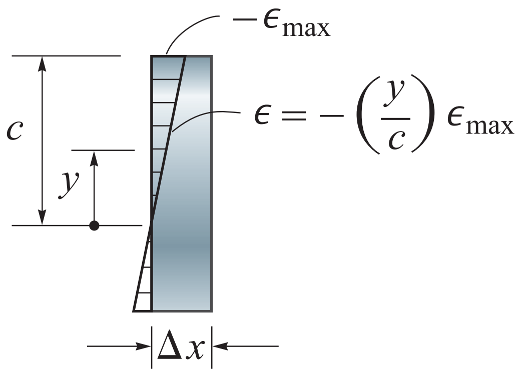


Figure 23: Linear strain in beam.

Here

$$\epsilon_{\max} = \frac{c}{\rho}$$

so that

$$\begin{aligned} \frac{\epsilon}{\epsilon_{\max}} &= \frac{-y/\rho}{c/\rho} \\ &= -\frac{y}{c}. \end{aligned}$$

Given elastic deformation, we also have

$$\frac{\sigma}{\sigma_{\max}} = -\frac{y}{c}$$

8.7 Flexure Formula

8.7.1 Location of the Neutral Axis

To locate the neutral axis in Figure (x), we require the resultant force produced by the stress distribution acting over the cross-sectional area to be equal to zero.

$$\begin{aligned} \mathbf{F}_R &= \sum F_x \\ 0 &= \iint_A \mathrm{d}F \\ &= \iint_A \sigma \, \mathrm{d}A \\ &= \iint_A -\frac{y}{c}\sigma_{\max} \, \mathrm{d}A \\ &= -\frac{\sigma_{\max}}{c} \iint_A y \, \mathrm{d}A \end{aligned}$$

as $\frac{\sigma_{\max}}{c} \neq 0$, $\iint_A y \, \mathrm{d}A = 0$.

8.7.2 Bending Moment

We can determine the stress in the beam if we require the moment M to be equal to the moment produced by the stress distribution about the neutral axis.

$$\begin{aligned}
(M_R)_z &= \sum M_z \\
M &= \iint_A y\,\mathrm{d}F \\
&= \iint_A y\sigma\,\mathrm{d}A \\
&= \iint_A y\frac{y}{c}\sigma_{\max}\,\mathrm{d}A \\
&= \frac{\sigma_{\max}}{c}\iint_A y^2\,\mathrm{d}A \\
&= \frac{\sigma_{\max}}{c} I
\end{aligned}$$

therefore,

$$\sigma_{\max} = \frac{Mc}{I}$$

and the normal stress at any distance y is given by

$$\sigma = -\frac{My}{I}$$

8.8 Beam Deflection

Applying Hooke'a Law, we obtain the following equation that relates the radius of curvature with the properties of the beam

$$\begin{aligned}
\sigma &= E\epsilon \\
-\frac{My}{I} &= E\left(-\frac{y}{\rho}\right) \\
\frac{1}{\rho} &= \frac{M}{EI}
\end{aligned}$$

Using Calculus, we can show that the radius of curvature of a curve $\nu(x)$ is given by

$$\frac{1}{\rho} = \frac{\frac{\mathrm{d}^2\nu}{\mathrm{d}x^2}}{\left(1+\left(\frac{\mathrm{d}\nu}{\mathrm{d}x}\right)^2\right)^{\frac{3}{2}}}$$

but for small deformations we suppose $\frac{\mathrm{d}\nu}{\mathrm{d}x} = 0$, so that

$$\frac{1}{\rho} = \frac{\mathrm{d}^2\nu}{\mathrm{d}x^2}.$$

Substituting this into the equation above gives

$$\frac{\mathrm{d}^2\nu}{\mathrm{d}x^2} = \frac{M}{EI}.$$

This yields

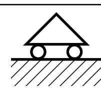
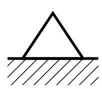
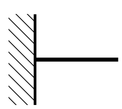
$$M(x) = EI\frac{\mathrm{d}^2\nu}{\mathrm{d}x^2}$$
$$V(x) = EI\frac{\mathrm{d}^3\nu}{\mathrm{d}x^3}$$
$$w(x) = EI\frac{\mathrm{d}^4\nu}{\mathrm{d}x^4}$$

Finally, the slope θ (in radians) of this deflection is simply the rate of change

$$\theta(x) = \frac{\mathrm{d}\nu}{\mathrm{d}x}$$

8.9 Statically Indeterminate Beams

Beams where reaction forces cannot be solved using force and moment resolution are statically indeterminate. Here we must use the slope and deflection functions, in combination with certain beam conditions, to solve for reaction forces.

Name	**Symbol**	**Condition**
Roller		$\nu = 0$
Pin		$\nu = 0$
Fixed		$\nu = 0$ and $\theta = 0$

8.10 Principle of Superposition

Similar to axial loading, we can utilise the principle of superposition on complex beams.

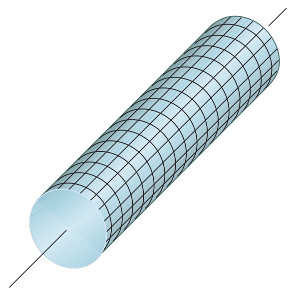
9 Torsion

Definition 9.1 (Torque)**.** Torque is a moment that tends to twist a member about its longitudinal axis.

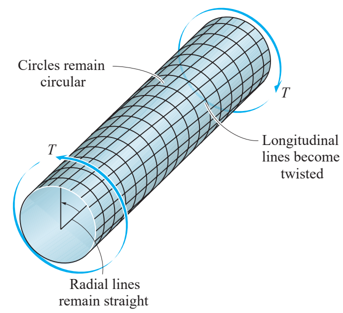
Torsion induces shear strain γ and shear stress τ.

9.1 Torsional Deformation

Given a shaft that is fixed at one end with a torque at its other end. Provided the angle of twist is small, the length and radius will remain practically unchanged.



(a) Before deformation.



(b) After deformation.

Due to the deformation, the front and read faces will undergo rotation and the difference in these rotations dϕ, causes the element to be subjected to a shear strain, γ.

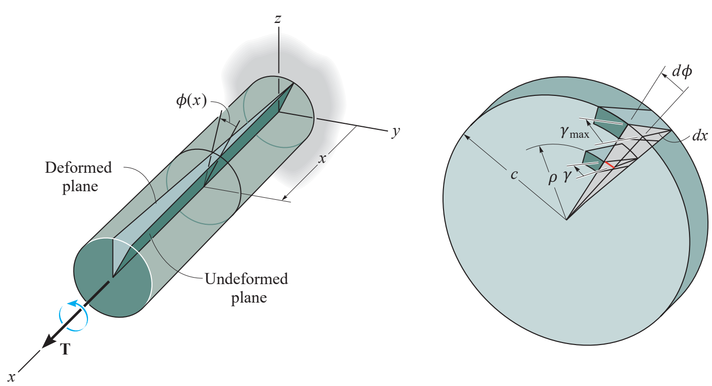


Figure 25: Cross section of a shaft.

Consider a small section along the longitudinal axis of the shaft. The angle dϕ can be related to the width of the section as follows

$$\begin{aligned}\rho\,\mathrm{d}\phi &= \gamma\,\mathrm{d}x \\ \gamma &= \rho\frac{\mathrm{d}\phi}{\mathrm{d}x}.\end{aligned}$$

As $\frac{\mathrm{d}\phi}{\mathrm{d}x}$ is constant over the cross section, we can rewrite the slope as γ_{max}/c.

$$\gamma = \frac{\rho}{c}\gamma_{\mathrm{max}}$$

so that the shear strain within the shaft varies linearly along any radial line.

9.2 Torsion Formula

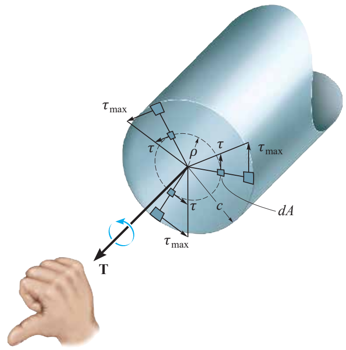


Figure 26: Shear stress on a shaft.

Using Hooke's law for shear stress and strain, we also have the following equation for shear strain acting on a corss section of the shaft

$$\tau = \frac{\rho}{c}\tau_{\max}.$$

As each element of area $\mathrm{d}A$ is subjected to a force $\mathrm{d}F = \tau\,\mathrm{d}A$, the torque produced by this force is given by

$$\begin{aligned}
\mathrm{d}T &= \rho\,\mathrm{d}F \\
\mathrm{d}T &= \rho\tau\,\mathrm{d}A \\
T &= \iint_A \rho\tau\,\mathrm{d}A \\
T &= \iint_A \rho\frac{\rho}{c}\tau_{\max}\,\mathrm{d}A \\
T &= \frac{\tau_{\max}}{c}\iint_A \rho^2\,\mathrm{d}A \\
T &= \frac{\tau_{\max}}{c}J
\end{aligned}$$

so that the maximum shear stress in the shaft is given by

$$\tau_{\max} = \frac{Tc}{J}.$$

Similarly, the shear stress at any distance ρ is expressed as

$$\tau = \frac{T\rho}{J}.$$

Either of these equations are known as the torsion formula.

9.3 Power Transmission

Shafts with circular cross sections are often used to transmit power developed by a machine. They are subjected to a torque that depends both on the power generated by the machine and the angular speed of the shaft. This is expressed mathematically as

$$P = T\omega = 2\pi f T.$$

9.4 Angle of Twist

The angle of twist ϕ of one end of a shaft with respect to its other end is determined through a similar formulation. By rearranging Hooke's law, and substituting the torsion formula we have,

$$\begin{aligned} \tau &= G\gamma \\ \frac{T\rho}{J} &= G\gamma \\ \gamma &= \frac{T\rho}{GJ} \end{aligned}$$

so that

$$\begin{aligned} \mathrm{d}\phi &= \frac{T}{GJ}\,\mathrm{d}x \\ \phi &= \int_0^L \frac{T}{GJ}\,\mathrm{d}x. \end{aligned}$$

If T, G, and J are constant, then

$$\phi = \frac{TL}{GJ}.$$

If a shaft has multiple torques, or the cross-sectional area or shear modulus changes abruptly from one section of the shaft to the next, we can find the angle of twist from the algebraic sum of the angles of twist of each section.

$$\phi = \sum \frac{TL}{GJ}.$$

9.5 Sign Convention

To apply this equation, use the right-hand rule where the thumb is directed outward from the shaft and the fingers indicate the direction of the torque.

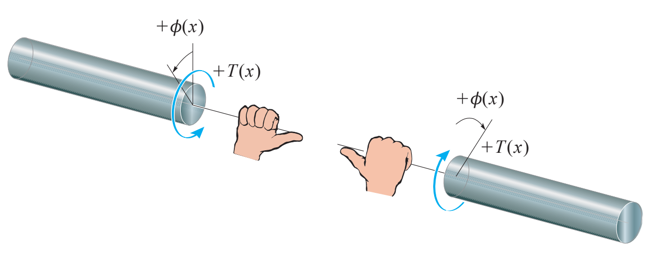


Figure 27: Positive sign convention for T and ϕ.