Linear Algebra

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Contents

1 Euclidean Vector Spaces

1.1 Vectors

Definition 1.1. An n-dimensional **vector** is an ordered list of n numbers.

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$$

Theorem 1.1.1. $\mathbb{R}^n$ *is the set of all ordered n-tuples of real numbers.*

$$\mathbb{R}^n = \{(v_1,\ v_2,\ \dots,\ v_n) : v_1,\ v_2,\ \dots,\ v_n \in \mathbb{R} : n \in \mathbb{N}\}$$

Notation:

1. Component form: $\mathbf{v} = \langle v_1,\ v_2 \rangle = (v_1,\ v_2) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$
2. Unit vector form: $\mathbf{v} = v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}}$, where $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are basis vectors along the x and y axes respectively.
3. Denotation: $\mathbf{v} = \underset{\sim}{v} = \vec{v}$

1.2 Position and Displacement Vectors

Definition 1.2. The **displacement vector** $\overrightarrow{AB}$ from $\mathbf{a}$ to $\mathbf{b}$ can be defined as $\mathbf{b} - \mathbf{a}$.

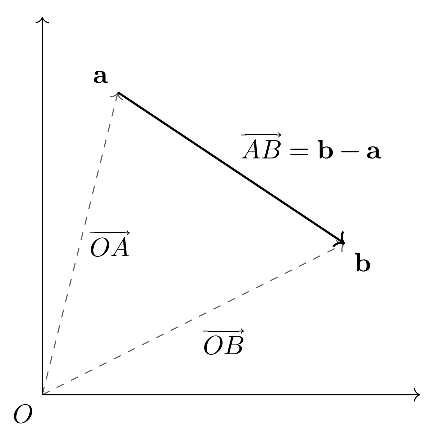

Figure 1: Displacement vector between two points.

1.3 Vector Addition

Definition 1.3. **Vector addition** is performed by adding the corresponding components of two vectors of the same dimension.

$$\mathbf{a} + \mathbf{b} = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

1.4 Scalar Multiplication

Definition 1.4. **Scalar multiplication** is performed by multiplying each element of the vector by the scalar.

$$a\mathbf{v} = \begin{bmatrix} av_1 \\ \vdots \\ av_n \end{bmatrix}$$

1.5 Norm of a Vector

Definition 1.5. The **norm** of a vector $\mathbf{v}$, denoted by $\|\mathbf{v}\|$, is the *length* or *magnitude* of $\mathbf{v}$.

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

1.6 The Unit Vector

Definition 1.6. A **unit vector** is a vector, denoted $\hat{\mathbf{v}}$, that has a length of 1 in the direction of $\mathbf{v}$.

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

1.7 The Dot Product

Definition 1.7. The **dot product** is a function that associates each pair of vectors $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^n$ a real number $\mathbf{v} \cdot \mathbf{w}$.

$$\begin{aligned} \mathbf{v} \cdot \mathbf{w} &= v_1 w_1 + v_2 w_2 + \cdots + v_n w_n \\ &= \|\mathbf{v}\| \|\mathbf{w}\| \cos(\theta) \end{aligned}$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

Theorem 1.7.1. *If* $\mathbf{v} \cdot \mathbf{w} = 0$ *then* $\mathbf{v}$ *and* $\mathbf{w}$ *are orthogonal.*

1.8 The Cross Product

Definition 1.8. The **cross product** is a function that associates each ordered pair of vectors $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^3$ a vector $\mathbf{v} \times \mathbf{w} \in \mathbb{R}^3$.

$$\begin{aligned} \mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\ &= \|\mathbf{v}\| \|\mathbf{w}\| \sin(\theta) \hat{\mathbf{n}} \end{aligned}$$

where $\hat{\mathbf{n}}$ is the normal vector given by the right-hand rule.

2 Vector Identities

Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V$ with $r \in \mathbb{R}$.

Theorem 2.0.1. *Commutativity of vector addition.*

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

Theorem 2.0.2.

$$\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$$

Theorem 2.0.3. *Commutativity of dot products.*

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

Theorem 2.0.4. *Distributivity of dot products over vector addition.*

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

Theorem 2.0.5. *Associativity of dot products over scalar multiplication.*

$$(r\mathbf{a}) \cdot \mathbf{b} = r\left(\mathbf{a} \cdot \mathbf{b}\right) = \mathbf{a} \cdot (r\mathbf{b})$$

Theorem 2.0.6. *Bilinearity of dot products.*

$$\mathbf{a} \cdot (r\mathbf{b} + \mathbf{c}) = r\left(\mathbf{a} \cdot \mathbf{b}\right) + \left(\mathbf{a} \cdot \mathbf{c}\right)$$

Theorem 2.0.7.

$$\mathbf{a} \times \mathbf{a} = \mathbf{0}$$

Theorem 2.0.8. *Anticommutativity of cross products.*

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

Theorem 2.0.9. *Distributivity of cross products over vector addition.*

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

Theorem 2.0.10. *Associativity of cross products over scalar multiplication.*

$$(r\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (r\mathbf{b}) = r\left(\mathbf{a} \times \mathbf{b}\right)$$

Theorem 2.0.11.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

Theorem 2.0.12.

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}\left(\mathbf{a} \cdot \mathbf{c}\right) - \mathbf{c}\left(\mathbf{a} \cdot \mathbf{b}\right)$$

3 Linear System of Equations

3.1 Linear Equations

Definition 3.1. A **linear equation** in n variables $x_1, x_2, \dots, x_n$ can be expressed in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where the *coefficients* $a_1, a_2, \dots, a_n$ and the *constant term* b are constants.

3.2 Homogeneous Linear Equations

Definition 3.2. In the special case where $b = 0$, the linear equation is called a **homogeneous linear equation**.

3.3 Linear Systems

Definition 3.3. A **linear system of equations** is a set of linear equations, where the variables x_i are called *unknowns*. The general linear system of m equations with n unknowns can be written as

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

A **solution** to the system is an n-tuple $\langle x_1, x_2, \dots, x_n \rangle$ that satisfies each equation.

3.4 Coefficient Matrices

Definition 3.4. The coefficients of the variables in each equation can be placed inside the systems **coefficient matrix**.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

3.5 Augmented Matrices

Definition 3.5. The information of a system can be contained in its **augmented matrix**.

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Definition 3.6. An array having m rows and n columns, is an $m \times n$ **matrix**. This matrix may be denoted as a_{ij}, where a_{ij} is the entry in ith row and jth column of the matrix $\mathbf{A}$.

$$m \text{ rows} \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \right. \atop \underbrace{\hphantom{a_{11} \quad a_{12} \quad \cdots \quad a_{1n}}}_{n \text{ columns}}$$

3.6 Elementary Row Operations

Definition 3.7. A linear system can be solved using the following **elementary row operations**:

1. **scalar multiplication**: multiplying any row by a constant
2. **row addition**: adding a multiple of one row to another
3. **row exchange**: exchanging any two rows

3.7 Pivots

Definition 3.8. The first non-zero entry of the row in a matrix is called the **pivot** of the row.

Theorem 3.7.1. *If a row apart from the first has a pivot, then this pivot must be to the right of the pivot in the preceding row.*

3.8 Gaussian Elimination

Definition 3.9. **Gaussian elimination** is a method for solving linear systems. These systems can be solved by composing the augmented matrix of a system, and performing elementary row operations, to put the matrix in the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{mn} \end{bmatrix}$$

3.9 Row-Echelon Form

Definition 3.10. A matrix that has undergone Gaussian elimination is in **row-echelon form** if the pivots of the augmented matrix are all 1.

$$\begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ & 1 & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix}$$

3.10 Gauss-Jordan Elimination

Definition 3.11. **Gauss-Jordan elimination** extends Gaussian elimination so that the entries in a column containing a pivot are zeros, and the pivots are all 1. This new augmented matrix is then in **reduced row-echelon form**.

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

3.11 Solutions to Linear Systems

Definition 3.12. A **consistent system** of equations has at least one solution, and an **inconsistent system** has no solution.

4 Matrices

Definition 4.1. A **matrix** is an array of numbers arranged into *rows* and *columns*, and can be used to represent a linear transformation.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n}$$

4.1 Matrix Addition

Definition 4.2. **Matrix addition** is performed by adding the corresponding components of two matrices of the same dimension.

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

4.2 Scalar Multiplication

Definition 4.3. **Scalar multiplication** is performed by multiplying each element of a matrix by a scalar.

$$c\mathbf{A} = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix}$$

4.3 Matrix Multiplication

Definition 4.4. **Matrix multiplication** is performed by multiplying each row in the first matrix by the columns of the second matrix.

$$\mathbf{AB} = \mathbf{C}$$

$$\begin{bmatrix} — & \mathbf{a}_1 & — \\ — & \mathbf{a}_2 & — \\ & \vdots & \\ — & \mathbf{a}_m & — \end{bmatrix} \begin{bmatrix} | & | & & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1\mathbf{b}_1 & \mathbf{a}_1\mathbf{b}_2 & \cdots & \mathbf{a}_1\mathbf{b}_n \\ \mathbf{a}_2\mathbf{b}_1 & \mathbf{a}_2\mathbf{b}_2 & \cdots & \mathbf{a}_2\mathbf{b}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{a}_m\mathbf{b}_1 & \mathbf{a}_m\mathbf{b}_2 & \cdots & \mathbf{a}_m\mathbf{b}_n \end{bmatrix}$$

Theorem 4.3.1. *A matrix product is defined if and only if the number of columns in the first matrix is equal to the number of rows in the second matrix.*

4.4 The Identity Matrix

Definition 4.5. The **identity matrix** is the simplest nontrivial **diagonal matrix**, denoted $\mathbf{I}$, such that

$$\mathbf{IA} = \mathbf{A}$$

written explicitly as

$$\mathbf{I} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

4.5 The Inverse Matrix

Definition 4.6. The **inverse** of a **square matrix** is a matrix $\mathbf{A}^{-1}$, such that

$$\mathbf{AA}^{-1} = \mathbf{I}$$

Theorem 4.5.1. *The inverse of a 2×2 matrix is given by*

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Theorem 4.5.2. *The inverse of an $n \times n$ matrix can be determined by solving $[\mathbf{A} \mid \mathbf{I}]$.*

4.6 The Diagonal Matrix

Definition 4.7. A **diagonal** matrix, denoted $\operatorname{diag}(d_{11}, d_{22}, \dots, d_{nn})$, is an $n \times n$ matrix $\mathbf{D}$ in which entries outside the main diagonal are all zero.

$$\mathbf{D} = \operatorname{diag}(d_{11}, d_{22}, \dots, d_{nn}) = \begin{bmatrix} d_{11} & & & \\ & d_{22} & & \\ & & \ddots & \\ & & & d_{nn} \end{bmatrix}$$

4.7 Matrix Transpose

Definition 4.8. The **transpose** of a matrix, denoted by $\mathbf{A}^\top$, is obtained by replacing all a_{ij} elements with a_{ji}, so that the matrix $\mathbf{A}$ is flipped over its main diagonal.

4.8 Matrix Trace

Definition 4.9. The **trace** of an $n \times n$ matrix $\mathbf{A}$, denoted $\operatorname{Tr}(\mathbf{A})$, is defined as

$$\operatorname{Tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}$$

5 General Vector Spaces

5.1 Real Vector Spaces

Definition 5.1. A **vector space** is a set that is closed under vector addition and scalar multiplication.

Theorem 5.1.1. *If the following axioms are satisfied by all objects* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in V$*, and all scalars* k *and* m*, then* V *is a* ***vector space****, and the objects in* V *are vectors.*

Axiom 1 (Closure under addition)**.**

$$\mathbf{u} + \mathbf{v} \in V$$

Axiom 2 (Commutativity of vector addition)**.**

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

Axiom 3 (Associativity of vector addition)**.**

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$$

Axiom 4 (Additive identity)**.**

$$\mathbf{u} + \mathbf{0} = \mathbf{u}$$

Axiom 5 (Additive inverse)**.**

$$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$$

Axiom 6 (Closure under scalar multiplication)**.**

$$k\mathbf{u} \in V$$

Axiom 7 (Distributivity of vector addition)**.**

$$k\left(\mathbf{u} + \mathbf{v}\right) = k\mathbf{u} + k\mathbf{v}$$

Axiom 8 (Distributivity of scalar addition)**.**

$$\left(k + m\right)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$$

Axiom 9 (Associativity of scalar multiplication)**.**

$$k\left(m\mathbf{u}\right) = \left(km\right)\mathbf{u}$$

Axiom 10 (Scalar multiplication identity)**.**

$$1\mathbf{u} = \mathbf{u}$$

To identify that a set with two operations is a vector space:

1. Identify the set V of objects that will become vectors.
2. Identify the addition and scalar multiplication operations on V.

3. Verify Axioms 1 and 6.
4. Confirm that Axioms 2, 3, 4, 5, 7, 8, 9, and 10 hold.

Theorem 5.1.2. *Let V be a vector space. If $\mathbf{v} \in V$, and k is a scalar.*

1. $0\mathbf{v} = \mathbf{0}$
2. $k\mathbf{0} = \mathbf{0}$
3. $(-1)\,\mathbf{v} = -\mathbf{v}$
4. *If $k\mathbf{v} = \mathbf{0}$, then $k = 0$ or $\mathbf{v} = \mathbf{0}$*

5.2 Subspaces

Definition 5.2. A *subset* W of a vector space V is called a **subspace** of V if W is itself a vector space under the addition and scalar multiplication operations defined on V.

Theorem 5.2.1. *Let W be a subspace of the vector space V, then the following axioms must be satisfied.*

1. ***Axiom 1**: Closure under addition*
2. ***Axiom 6**: Closure under scalar multiplication*

Theorem 5.2.2. *Every vector space has at least two subspaces, itself and its zero subspace.*

Theorem 5.2.3. *Subspaces of $\mathbb{R}^2$.*

1. $\{\mathbf{0}\}$
2. *Lines through the origin*
3. $\mathbb{R}^2$

Theorem 5.2.4. *Subspaces of $\mathbb{R}^3$.*

1. $\{\mathbf{0}\}$
2. *Lines through the origin*
3. *Planes through the origin*
4. $\mathbb{R}^3$

Theorem 5.2.5. *Subspaces of $\mathbf{M}_{nn}$.*

1. *Upper triangular matrices*
2. *Lower triangular matrices*
3. *Diagonal matrices*
4. $\mathbf{M}_{nn}$

5.3 Spanning Sets

Definition 5.3. If the vector $\mathbf{w}$ is in a vector space V, then $\mathbf{w}$ is a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$, if $\mathbf{w}$ can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n$$

Theorem 5.3.1. *If $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ is a nonempty set of vectors in a vector space V, then the set W of all possible linear combinations of the vectors in S is a subspace of V. The subspace W is called the subspace of V* ***spanned*** *by S and the vectors in S* ***span*** *W. If a vector in S can be expressed as the linear combination of any vectors in S then the set is* ***linearly dependent****.*

5.4 Linear Independence

Definition 5.4. If S is a set of two or more vectors in a vector space V, then S is **linearly independent** if no vector in S can be expressed as a linear combination of the others.

Theorem 5.4.1. *A set S is linearly independent if and only if there is one solution to the equation*

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n = \mathbf{0}$$

where the coefficients satisfying this equation are $k_1 = 0, k_2 = 0, \ldots, k_n = 0$.

5.5 Basis Vectors

Definition 5.5. If S is a set of vectors in a vector space V, then S is called a **basis** for V if

1. S spans V.
2. S is linearly independent.

5.6 Dimension

Definition 5.6. The **dimension** of a finite-dimensional vector space V, denoted $\dim(V)$, is the number of vectors in a basis for V.

Theorem 5.6.1. *The zero vector space is defined to have dimension zero.*

6 Fundamental Subspaces

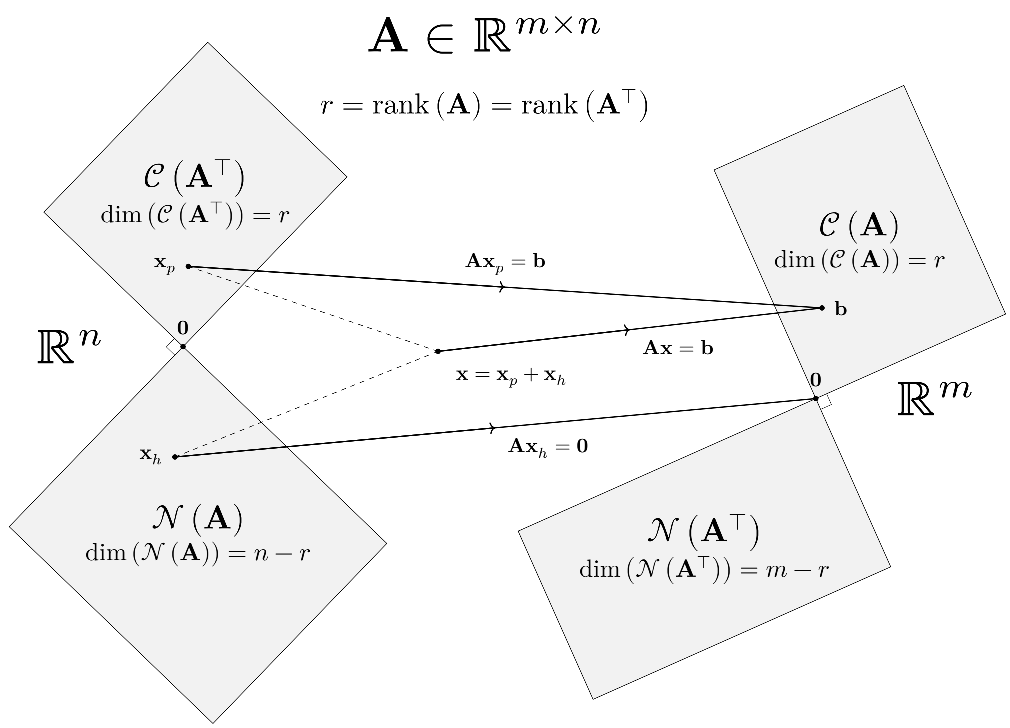

Figure 2: The Four Fundamental Subspaces of a Matrix.

6.1 The Four Fundamental Subspaces of a Matrix

Definition 6.1. If $\mathbf{A} \in \mathbb{R}^{m\times n}$ is an $m \times n$ matrix, then:

1. The subspace spanned by the *column vectors* of $\mathbf{A}$, is the **column space** of $\mathbf{A}$, denoted $\mathcal{C}(\mathbf{A})$.
2. The subspace spanned by the *row vectors* of $\mathbf{A}$, is the **row space** of $\mathbf{A}$, denoted $\mathcal{C}(\mathbf{A}^\top)$.
3. The subspace spanned by the *solution space* of the equation $\mathbf{A}\mathbf{x} = \mathbf{0}$, is the **null space** of $\mathbf{A}$, denoted $\mathcal{N}(\mathbf{A})$.
4. The subspace spanned by the *solution space* of the equation $\mathbf{A}^\top\mathbf{y} = \mathbf{0}$ (or $\mathbf{y}^\top\mathbf{A} = \mathbf{0}$), is the **left null space** of $\mathbf{A}$, denoted $\mathcal{N}(\mathbf{A}^\top)$.

6.2 The General Solution of a System of Equations

Theorem 6.2.1. *The **general solution** to a matrix equation $\mathbf{A}\mathbf{x} = \mathbf{b}$, can be given by adding the particular and homogeneous solutions, where the particular solution is the solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$, or*

$\mathcal{C}(\mathbf{A}^\top)$, *and the homogeneous solution is the solution to* $\mathbf{A}\mathbf{x} = \mathbf{0}$, *or* $\mathcal{N}(\mathbf{A})$.

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$$

6.3 Row Equivalence

Definition 6.2. Two matrices are **row equivalent** if each can be obtained from the other by elementary row operations. These matrices have the same row space and null space.

6.4 Rank

Definition 6.3. The **rank** of a matrix, denoted by $\operatorname{rank}(\mathbf{A})$, is given by $\dim(\mathcal{C}(\mathbf{A}))$.

Theorem 6.4.1. *The column space and row space have the same dimension so that*

$$\operatorname{rank}(\mathbf{A}) = \dim(\mathcal{C}(\mathbf{A})) = \dim(\mathcal{C}(\mathbf{A}^\top))$$

6.5 Nullity

Definition 6.4. The **nullity** of a matrix, denoted by $\operatorname{null}(\mathbf{A})$, is given by $\dim(\mathcal{N}(\mathbf{A}))$.

7 Orthogonality

Definition 7.1. Two vectors are **orthogonal** if the following holds.

$$\mathbf{v} \cdot \mathbf{w} = 0 \iff \mathbf{v}^\top \mathbf{w} = 0$$

Theorem 7.0.1. $\mathbf{0}$ *is orthogonal to every vector in* V.

Theorem 7.0.2. $\mathbf{0}$ *is the only vector in* V, *that is orthogonal to itself.*

Theorem 7.0.3.

$$\|\mathbf{v}\|^2 = \mathbf{v}^\top \mathbf{v}$$

Theorem 7.0.4.

$$\|\mathbf{v}\| = \sqrt{\mathbf{v}^\top \mathbf{v}}$$

7.1 Orthogonal Subspaces

Definition 7.2. Two subspaces U and W of a vector space V, are **orthogonal subspaces** iff every vector in U is orthogonal to every vector in W.

$$\forall \mathbf{u} \in U : \forall \mathbf{w} \in W : \mathbf{u}^\top \mathbf{w} = 0$$

7.2 Orthogonal Complements

Definition 7.3. If U is a subspace of V, then the **orthogonal complement** of U, denoted $U^\perp$, is the set of all vectors in V that are orthogonal to every vector in U.

$$U^\perp = \left\{\forall \mathbf{u} \in U : \mathbf{v} \in V : \mathbf{v}^\top \mathbf{u} = 0\right\}$$

Theorem 7.2.1.

$$\left(U^\perp\right)^\perp = U$$

Theorem 7.2.2.

$$\dim U + \dim U^\perp = \dim V$$

7.3 Vector Projections

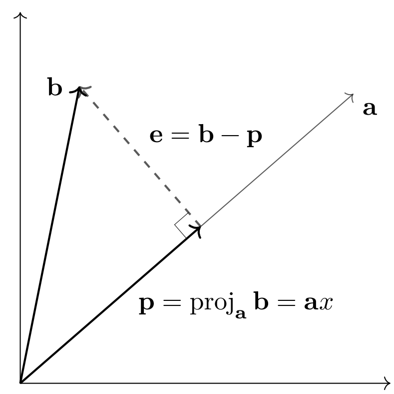

Figure 3: Vector Projection of **b** onto **a**.

Definition 7.4. Let the **vector projection** of $\mathbf{b}$ onto $\mathbf{a}$, denoted as $\operatorname{proj}_{\mathbf{a}} \mathbf{b}$, be the *orthogonal projection* of $\mathbf{b}$ in the direction of $\mathbf{a}$, that minimises the error vector: $\mathbf{e} = \mathbf{b} - \mathbf{p}$.

Theorem 7.3.1. *The projection of* $\mathbf{b}$ *onto* $\mathbf{a}$ *is given by*

$$\operatorname{proj}_{\mathbf{a}} \mathbf{b} = \mathbf{a}x = \mathbf{a}\left(\mathbf{a}^\top \mathbf{a}\right)^{-1} \mathbf{a}^\top \mathbf{b}$$

alternatively

$$\operatorname{proj}_{\mathbf{a}} \mathbf{b} = \mathbf{a}x = \mathbf{a}\frac{\mathbf{a}^\top \mathbf{b}}{\mathbf{a}^\top \mathbf{a}} = \mathbf{a}\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}$$

Proof. As $\mathbf{p}$ lies on line through $\mathbf{a}$, $\mathbf{p} = \mathbf{a}x$, so that $\mathbf{e} = \mathbf{b} - \mathbf{a}x$. As $\mathbf{e}$ is orthogonal to $\mathbf{a}$, we can construct the following relationship.

$$\begin{aligned}
\mathbf{a}^\top \mathbf{e} &= 0 \\
\mathbf{a}^\top \left(\mathbf{b} - \mathbf{a}x\right) &= 0 \\
\mathbf{a}^\top \mathbf{b} - \mathbf{a}^\top \mathbf{a}x &= 0 \\
\mathbf{a}^\top \mathbf{a}x &= \mathbf{a}^\top \mathbf{b} \\
x &= \left(\mathbf{a}^\top \mathbf{a}\right)^{-1} \mathbf{a}^\top \mathbf{b} = \frac{\mathbf{a}^\top \mathbf{b}}{\mathbf{a}^\top \mathbf{a}}
\end{aligned}$$

□

7.4 Projection onto a Subspace

Theorem 7.4.1. *Let W be a subspace of the vector space V such that if $\mathbf{b} \in V$, then $\mathbf{p} = \operatorname{proj}_W \mathbf{b}$ is the **best approximation** of $\mathbf{b}$ on W, so that*

$$\|\mathbf{b} - \mathbf{p}\| < \|\mathbf{b} - \mathbf{w}\|$$

for all $\mathbf{w} \in W$, where $\mathbf{w} \neq \mathbf{p}$.

Theorem 7.4.2. *The projection of $\mathbf{b}$ onto the vector space W is given by*

$$\operatorname{proj}_W \mathbf{b} = \mathbf{A}\hat{\mathbf{x}} = \mathbf{A}\left(\mathbf{A}^\top \mathbf{A}\right)^{-1} \mathbf{A}^\top \mathbf{b}$$

Proof. As $\mathbf{p} \in W$, $\mathbf{p}$ can be represented as the linear combination of the basis vectors $\mathbf{a}_i$ that span W.

$$\begin{aligned}
\mathbf{p} &= \hat{x}_1 \mathbf{a}_1 + \hat{x}_2 \mathbf{a}_2 + \cdots + \hat{x}_n \mathbf{a}_n \\
&= \begin{bmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \cdots \\ \hat{x}_n \end{bmatrix} \\
&= \mathbf{A}\hat{\mathbf{x}}
\end{aligned}$$

Consider the error vector $\mathbf{e} = \mathbf{b} - \mathbf{p}$. As $\mathbf{e}$ is orthogonal to W, it will also be orthogonal to the vectors that span W. Therefore

$$\begin{cases} \mathbf{a}_1^\top \left(\mathbf{b} - \mathbf{A}\hat{\mathbf{x}}\right) = 0 \\ \mathbf{a}_2^\top \left(\mathbf{b} - \mathbf{A}\hat{\mathbf{x}}\right) = 0 \\ \quad\vdots \\ \mathbf{a}_n^\top \left(\mathbf{b} - \mathbf{A}\hat{\mathbf{x}}\right) = 0 \end{cases}$$

which gives the following equation

$$\mathbf{A}^\top \left(\mathbf{b} - \mathbf{A}\hat{\mathbf{x}}\right) = \mathbf{0}$$

where we solve for $\hat{\mathbf{x}}$

$$\begin{aligned}
\mathbf{A}^\top \mathbf{b} - \mathbf{A}^\top \mathbf{A}\hat{\mathbf{x}} &= \mathbf{0} \\
\mathbf{A}^\top \mathbf{A}\hat{\mathbf{x}} &= \mathbf{A}^\top \mathbf{b} \\
\hat{\mathbf{x}} &= \left(\mathbf{A}^\top \mathbf{A}\right)^{-1} \mathbf{A}^\top \mathbf{b}
\end{aligned}$$

□

7.5 Least Squares

Theorem 7.5.1. *Suppose $\mathbf{A}\mathbf{x} = \mathbf{b}$ is an <u>inconsistent</u> linear system. The **least squares** solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$ is given by the orthogonal projection $\operatorname{proj}_{C(\mathbf{A})} \mathbf{b}$.*

8 Linear Maps

8.1 Matrix Transformations

Definition 8.1. A **matrix transformation** $T_{\mathbf{A}} : \mathbb{R}^n \to \mathbb{R}^m$ is a mapping of the form

$$T_{\mathbf{A}}(\mathbf{x}) = \mathbf{A}\mathbf{x}$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$. As this transformation is linear, the following linearity properties hold.

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
2. $T(k\mathbf{u}) = kT(\mathbf{u})$

8.2 General Linear Transformations

Theorem 8.2.1. *If $T : V \to W$ is a mapping between two vector spaces V and W, then T is the **linear transformation** from V to W, and the following properties hold.*

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
2. $T(k\mathbf{u}) = kT(\mathbf{u})$

Theorem 8.2.2. *When $V = W$, the linear map is called a **linear operator**.*

8.3 Subspaces of Linear Transformations

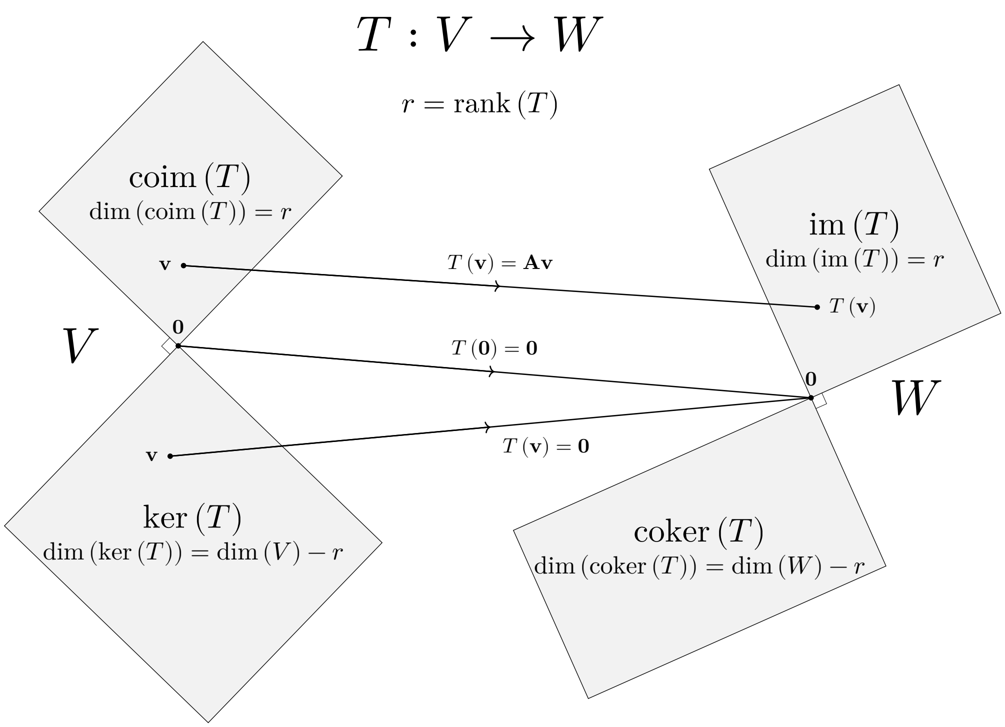

Figure 4: Subspaces of a Linear Transformation.

Definition 8.2. If $T : V \to W$ is a linear transformation between two vector spaces V and W, then:

1. The vector space V is the **domain** of T.
2. The vector space W is the **codomain** of T.
3. The **image** (or **range**) of T is the set of vectors the linear transformation maps to.
$$\text{im}(T) = T(V) = \{T(\mathbf{v}) : \mathbf{v} \in V\} \subset W$$
4. The **kernel** of T is the set of vectors that map to the zero vector.
$$\ker(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}$$

8.4 Constructing a Transformation Matrix

Theorem 8.4.1. *The standard matrix for a linear transformation is given by the formula:*

$$\mathbf{A} = \begin{bmatrix} | & | & & | \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \\ | & | & & | \end{bmatrix}$$

where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

*are the **standard basis vectors** for $\mathbb{R}^n$.*

9 Determinants

9.1 Properties of Determinants

1. $\det(\mathbf{I}) = 1$.
2. Exchanging two rows of a matrix reverses the sign of its determinant.
3. Determinants are multilinear, so that
   (a) $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$
   (b) $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$
4. If $\mathbf{A}$ has two equal rows, then $\det(\mathbf{A}) = 0$.
5. Adding a scalar multiple of one row to another does not change the determinant of a matrix.
6. If $\mathbf{A}$ has a row of zeros, then $\det(\mathbf{A}) = 0$.
7. If $\mathbf{A}$ is triangular, then $\det(\mathbf{A}) = \prod_{i=1}^{n} a_{ii}$.
8. If $\mathbf{A}$ is singular, then $\det(\mathbf{A}) = 0$.
9. $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.
10. $\det(\mathbf{A}^\top) = \det(\mathbf{A})$.

9.2 Matrix Minors

Definition 9.1. The **minor** of a_{ij} in $\mathbf{A}$, denoted $\mathbf{M}_{ij}$, is the determinant of the submatrix formed by deleting the ith row and jth column of $\mathbf{A}$.

9.3 Matrix Cofactors

Definition 9.2. The **cofactor** of a_{ij} in $\mathbf{A}$ is defined as

$$\mathbf{C}_{ij} = (-1)^{i+j}\,\mathbf{M}_{ij}$$

9.4 The Determinant of a Matrix

Theorem 9.4.1. *The determinant of an $n \times n$ matrix $\mathbf{A}$ is given by*

$$\det(\mathbf{A}) = \sum_{j=1}^{n} a_{ij}\mathbf{C}_{ij} = \sum_{i=1}^{n} a_{ij}\mathbf{C}_{ij}$$

where a_{ij} is the entry in the ith row and jth column of $\mathbf{A}$.

9.5 The Cofactor Matrix

Definition 9.3. The **cofactor matrix** of an $n \times n$ matrix $\mathbf{A}$, denoted $\mathbf{C}$, is defined as the matrix of the cofactors of $\mathbf{A}$.

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}$$

9.6 The Adjugate of a Matrix

Definition 9.4. The **adjugate** (or *classical adjoint*) of a square matrix $\mathbf{A}$, denoted $\operatorname{adj}(\mathbf{A})$, is the transpose of its cofactor matrix.

$$\operatorname{adj}(\mathbf{A}) = \mathbf{C}^{\top}$$

9.7 The Inverse of a Matrix

Theorem 9.7.1. *The **inverse** of a nonsingular matrix $\mathbf{A}$ is given by*

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\operatorname{adj}(\mathbf{A})$$

10 Invariant Subspaces

Definition 10.1. Consider the subspace $\mathcal{V}$ of the linear mapping $T : V \to V$ from a vector space V to itself, then $\mathcal{V}$ is an **invariant subspace** of T if

$$T(\mathcal{V}) \subseteq \mathcal{V}$$

Theorem 10.0.1. *If $\mathcal{V}$ is an invariant subspace of a linear mapping $T : V \to V$ from a vector space V to itself, then*

$$\forall \mathbf{v} \in \mathcal{V} \implies T(\mathbf{v}) \in \mathcal{V}$$

10.1 Trivial Invariant Subspaces

1. V.
2. $\{\mathbf{0}\}$.
3. $\ker(T)$.
4. $\operatorname{im}(T)$.
5. Any linear combination of invariant subspaces.

10.2 Eigenspaces

Definition 10.2. If an invariant subspace is one-dimensional, then the subspace is called an **eigenspace** of the linear transformation.

Theorem 10.2.1. *If $\mathcal{V}$ is an eigenspace of the linear mapping $T : V \to V$, then*

$$\mathcal{V} = \{\forall \mathbf{v} \in \mathcal{V} : \exists \lambda \in \mathbb{C} : T(\mathbf{v}) = \lambda \mathbf{v}\}$$

where λ is the ***eigenvalue*** *associated with the* ***eigenvector*** $\mathbf{v}$.

10.3 The Eigenvalue Problem

Theorem 10.3.1. *The eigenvalues λ of an invertible square matrix $\mathbf{A}$, are the solutions to*

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \mathbf{0}$$

Theorem 10.3.2. *The eigenvectors associated with each eigenvalue, of an invertible square matrix $\mathbf{A}$, are the solutions to*

$$(\mathbf{A} - \lambda \mathbf{I})\,\mathbf{v} = \mathbf{0}$$

Proof. The eigenvalues and associated eigenvectors of a square matrix $\mathbf{A}$, are the solutions to $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$.

$$\begin{aligned} \mathbf{A}\mathbf{v} &= \lambda\mathbf{v} \\ \mathbf{A}\mathbf{v} - \lambda\mathbf{v} &= \mathbf{0} \\ (\mathbf{A} - \lambda\mathbf{I})\,\mathbf{v} &= \mathbf{0} \end{aligned}$$

The linear system $(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{v} = \mathbf{0}$ has a nontrivial solution iff $\mathbf{A} - \lambda\mathbf{I}$ is singular. □

10.4 Properties of Eigenvalues

Theorem 10.4.1.

$$\operatorname{Tr}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i$$

Theorem 10.4.2.

$$\det(\mathbf{A}) = \prod_{i=1}^{n} \lambda_i$$

11 Eigen Decomposition

11.1 Similarity Transformations

Definition 11.1. A **similarity transformation** is a linear mapping of the form

$$\mathbf{A} \to \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$$

in which the matrices $\mathbf{A}$ and $\mathbf{V}$ are $n \times n$ invertible matrices. Here we say, "$\mathbf{A}$ is similar to $\mathbf{V}^{-1}\mathbf{A}\mathbf{V}$".

11.2 Matrix Diagonalisation

Definition 11.2. The matrix $\mathbf{A}$ is a **diagonalisable** matrix if it is similar to a diagonal matrix. That is, there exists an invertible matrix $\mathbf{V}$, and diagonal matrix $\mathbf{D}$, such that

$$\mathbf{D} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$$

Theorem 11.2.1. *Let $\mathbf{A}$ be an $n \times n$ matrix with n linearly independent eigenvectors, then $\mathbf{A}$ is diagonalisable if $\mathbf{D} = \operatorname{diag}(\lambda_1,\ \lambda_2,\ \dots,\ \lambda_n)$ and $\mathbf{V}$ is a matrix composed of the eigenvectors of $\mathbf{A}$. Explicitly,*

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \quad \textit{and} \quad \mathbf{V} = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{bmatrix}$$

where $\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n$ are the eigenvectors of $\mathbf{A}$.

Proof. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be linearly independent eigenvectors of $\mathbf{A}$, and $\lambda_1, \lambda_2, \dots, \lambda_n$, the associated eigenvalues. By definition of an eigenspace, we have

$$\begin{cases} \mathbf{A}\mathbf{v}_1 = \lambda_1\mathbf{v}_1 \\ \mathbf{A}\mathbf{v}_2 = \lambda_2\mathbf{v}_2 \\ \quad\vdots \qquad \vdots\ \vdots \\ \mathbf{A}\mathbf{v}_n = \lambda_n\mathbf{v}_n \end{cases}$$

which we can rewrite as

$$\begin{aligned} \mathbf{A}\mathbf{V} &= \mathbf{V}\mathbf{D} \qquad (1) \\ \mathbf{D} &= \mathbf{V}^{-1}\mathbf{A}\mathbf{V} \end{aligned}$$

by rearranging Equation 1, we have $\mathbf{A}$ in terms of its eigenvalues and eigenvectors.

$$\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$$

$\square$

11.3 Powers of a Matrix

Theorem 11.3.1. *Let* $\mathbf{A}$ *be a diagonalisable matrix, then for all* $k \in \mathbb{N}_0$

$$\mathbf{A}^k = \mathbf{V}\mathbf{D}^k\mathbf{V}^{-1}$$

Proof.

$$\begin{aligned}
\mathbf{A}^k &= \left(\mathbf{V}\mathbf{D}\mathbf{V}^{-1}\right)^k \\
&= \underbrace{\left(\mathbf{V}\mathbf{D}\mathbf{V}^{-1}\right)\left(\mathbf{V}\mathbf{D}\mathbf{V}^{-1}\right)\cdots\left(\mathbf{V}\mathbf{D}\mathbf{V}^{-1}\right)}_{k \text{ times}} \\
&= \underbrace{\mathbf{V}\mathbf{D}\mathbf{V}^{-1}\mathbf{V}\mathbf{D}\mathbf{V}^{-1}\cdots\mathbf{V}\mathbf{D}\mathbf{V}^{-1}}_{k \text{ times}} \\
&= \underbrace{\mathbf{V}\mathbf{D}\cancel{\mathbf{V}^{-1}\mathbf{V}}\mathbf{D}\cancel{\mathbf{V}^{-1}\cdots\mathbf{V}}\mathbf{D}\mathbf{V}^{-1}}_{k \text{ times}} \\
&= \mathbf{V}\underbrace{\mathbf{D}\mathbf{D}\cdots\mathbf{D}}_{k \text{ times}}\mathbf{V}^{-1} \\
&= \mathbf{V}\mathbf{D}^k\mathbf{V}^{-1}
\end{aligned}$$

□

Theorem 11.3.2. *The eigenvalues of* $\mathbf{A}^k$, $\forall k \in \mathbb{N}$ *are* $\lambda_1^k,\ \lambda_2^k,\ \ldots,\ \lambda_n^k$.

Theorem 11.3.3. *The eigenvectors of* $\mathbf{A}$ *are equal to the eigenvectors of* $\mathbf{A}^k$.

12 System of Differential Equations

12.1 First-Order Differential Equations

Definition 12.1. A **first-order differential equation** is a differential equation where the highest derivative is of order one.

$$x' = ax$$

Theorem 12.1.1. *The general solution to a first-order linear differential equation is of the form*

$$x(t) = c_1 e^{at}$$

where c_1 *is an arbitrary constant.*

12.2 First-Order System of Differential Equations

Definition 12.2. A **first-order system of differential equations** is of the form

$$\begin{cases}
x_1' &= & a_{11}x_1 &+& a_{12}x_2 &+& \cdots &+& a_{1n}x_n \\
x_2' &= & a_{21}x_1 &+& a_{22}x_2 &+& \cdots &+& a_{2n}x_n \\
\vdots & & \vdots & & \vdots & & & & \vdots \\
x_n' &= & a_{n1}x_1 &+& a_{n2}x_2 &+& \cdots &+& a_{nn}x_n
\end{cases}$$

where $x_1 = x_1(t)$, $x_2 = x_2(t)$, ..., $x_n = x_n(t)$ are the functions to be determined. In matrix form, the system can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$
$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

12.3 Solution using Diagonalisation

Theorem 12.3.1. *The first-order system of differential equations* $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *can be solved using the following substitution*

$$\mathbf{x} = \mathbf{V}\mathbf{u}$$

where $\mathbf{u}$ *is a vector to be determined, and* $\mathbf{V}$ *is the matrix that diagonalises* $\mathbf{A}$*.* $\mathbf{u}$ *is determined by solving*

$$\mathbf{u}' = \mathbf{D}\mathbf{u}$$

where $\mathbf{D}$ *is the diagonal similarity transformation of* $\mathbf{A}$*. This substitution uncouples the system of differential equations so that each equation can be solved as a first-order differential equation.*

Proof.

$$\begin{aligned} \mathbf{x}' &= \mathbf{A}\mathbf{x} \\ (\mathbf{V}\mathbf{u})' &= \left(\mathbf{V}\mathbf{D}\mathbf{V}^{-1}\right)(\mathbf{V}\mathbf{u}) \\ \mathbf{V}\mathbf{u}' &= \mathbf{V}\mathbf{D}\mathbf{V}^{-1}\mathbf{V}\mathbf{u} \\ \cancel{\mathbf{V}}\mathbf{u}' &= \cancel{\mathbf{V}}\mathbf{D}\cancel{\mathbf{V}^{-1}\mathbf{V}}\mathbf{u} \\ \mathbf{u}' &= \mathbf{D}\mathbf{u} \end{aligned}$$

□

Theorem 12.3.2. *If* $\mathbf{A}$ *is a diagonalisable matrix, then the general solution of* $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *can be expressed as*

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + \cdots + c_n e^{\lambda_n t}\mathbf{v}_n$$

12.4 Principle of Superposition

Theorem 12.4.1. *If* x_1 *and* x_2 *are two solutions to a linear differential equation, then*

$$x = c_1 x_1 + c_2 x_2$$

is also a solution to the differential equation.

12.5 Higher-Order Differential Equations

Theorem 12.5.1. *A **higher-order linear differential equation** can be solved by first converting it to a first-order linear system. Consider the nth-order differential equation*

$$x^{(n)} + a_1 x^{(n-1)} + \cdots + a_{n-1}x' + a_n x = 0$$

We then define

$$\begin{aligned} x_1 &= x \\ x_2 &= x' \\ &\vdots \\ x_n &= x^{(n-1)} \end{aligned}$$

Let $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^\top$. *Then the first-order linear system of differential equations can be written as*

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$