# MATLAB

```
% Function declaration
function [output1, ...]
    = function_name(input1, ...)
% Iterate over the vector v
for i = v ... end
% Repeat while condition is true
while condition ... end
% Execute first true alternative
if condition_1 ...
elseif condition_2 ...
else ... end
```

### Partial Fraction Decomposition

Given the LHS in the denominator, substitute the RHS.

$$(ax+b)^k \to \frac{A_1}{ax+b} + \cdots + \frac{A_k}{(ax+b)^k}$$

$$(ax^2+bx+c)^k \to \frac{A_1 x + B_1}{ax^2+bx+c} + \cdots + \frac{A_k x + B_k}{(ax^2+bx+c)^k}$$

# Differential Equations

### Mechanical Systems

$$\sum F = \frac{\mathrm{d}p}{\mathrm{d}t}, \quad \sum M = I \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$$

where momentum $p = mv$, moment $M = Fx$ and inertia $I = mr^2$.

$$F_T = (c - v)\, d_f \qquad F_g = -mg$$
$$F_s = -kx \qquad F_f = -bv^2 \text{ (or } -bv\text{)}$$

### Electrical Circuits

$$i = \frac{\mathrm{d}q}{\mathrm{d}t}, \quad \sum v_{loop} = 0, \quad \sum i_{node} = 0$$

Voltage drop across elements:

$$v_R = iR, \quad v_C = \frac{q}{C}, \quad v_L = L\frac{\mathrm{d}i}{\mathrm{d}t}$$

### Separable ODEs

For $\frac{\mathrm{d}y}{\mathrm{d}t} = p(t)q(y)$:

$$\int \frac{1}{q(y)} \frac{\mathrm{d}y}{\mathrm{d}t} \,\mathrm{d}t = \int p(t)\,\mathrm{d}t\,.$$

### Linear ODEs

For $\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)y = q(t)$, use the *integrating factor*: $I(t) = e^{\int p(t)\mathrm{d}t}$, so that

$$y(t) = \frac{1}{I(t)} \int I(t)q(t)\,\mathrm{d}t\,.$$

### Linearisation

$$f(t) \approx f(t_0) + f'(t_0)(t - t_0)$$

$$f(y(t)) \approx f(y(t_0)) + f'(y(t_0))(y(t) - y(t_0))$$

### Euler's Method

For $\boldsymbol{y}' = \boldsymbol{f}(t,\, \boldsymbol{y})$:

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\boldsymbol{f}(t_n,\, \boldsymbol{y}_n)$$

### Modified Euler's Method

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \frac{h}{2}\left(\boldsymbol{f}(t_n,\, \boldsymbol{y}_n) + \boldsymbol{f}(t_{n+1},\, \boldsymbol{y}_{n+1})\right)$$

where $\boldsymbol{f}(t_{n+1},\, \boldsymbol{y}_{n+1})$ is determined using Euler's method.

### Second-Order ODEs

$$ay'' + by' + cy = F(t)$$

### General Solution

$$y(t) = y_H(t) + y_P(t)$$

### Homogeneous Solution

$$y_H(t) = e^{\lambda t}$$

To solve for $\lambda$, substitute the homogeneous form into the homogeneous ODE.

Real distinct roots:

$$y_H(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

Real repeated roots:

$$y_H(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$$

Complex conjugate roots: $\lambda = \alpha \pm \beta i$

$$y_H(t) = e^{\alpha x}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

### Particular Solution

*See table below.* Substitute $y_P$ into the nonhomogeneous ODE, and solve the undetermined coefficients.

### System of ODEs

Given $\boldsymbol{y}' = \mathbf{A}\boldsymbol{y}$,

$$\boldsymbol{y}_H(t) = \boldsymbol{q}e^{\lambda t}.$$

$\lambda_i$ are the eigenvalues of $\boldsymbol{A}$ that satisfy

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

$\boldsymbol{q}_i$ are the associated eigenvectors that satisfy

$$(\mathbf{A} - \lambda_i \mathbf{I})\,\boldsymbol{q}_i = \mathbf{0}.$$

For real distinct roots:

$$\boldsymbol{y}_H(t) = c_1 \boldsymbol{q}_1 e^{\lambda_1 t} + c_2 \boldsymbol{q}_2 e^{\lambda_2 t}$$

### Higher-Order ODEs

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = 0$$

Let $y_1 = y$, $y_2 = y'$, ..., $y_n = y^{(n-1)}$ so that $\boldsymbol{y} = \langle y_1, y_2, \dots, y_n \rangle$. Then $\boldsymbol{y}' = \mathbf{A}\boldsymbol{y}$, where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

| $F(t)$ | $y_P(t)$ |
|---|---|
| a constant | A |
| a polynomial of degree $n$ | $\sum_{i=0}^{n} A_i t^i$ |
| $e^{kt}$ | $Ae^{kt}$ |
| $\cos(\omega t)$ or $\sin(\omega t)$ | $A_0 \cos(\omega t) + A_1 \sin(\omega t)$ |
| a combination of the above | a combination of the above |
| linearly dependent to $y_H(t)$ | multiply $y_P(t)$ by $t$ until linearly independent |

# Probability

### Events

$$\Pr(A) = \sum_{x \in A} p(x)$$

$$\Pr(A^C) = \Pr(\overline{A}) = 1 - \Pr(A)$$

### Unions

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) = 1 - \Pr(A^C \cap B^C)$$

### Bayes' Theorem

$$\Pr(B \mid A) = \frac{\Pr(A \mid B)\Pr(B)}{\Pr(A)}$$

### Disjoint Events

Events don't have outcomes in common.

$$A \cap B = \varnothing$$
$$\Pr(A \cap B) = 0$$

### Independent Events

Outcome of events do not influence each other. Joint probability:

$$\Pr(A \cap B) = \Pr(A)\Pr(B)$$

### Dependent (Conditional) Events

Outcome of event depends on the outcome of the other. Joint probability of $A$ given $B$:

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

### Total Probability

For disjoint events $B_i$:

$$A = \bigcup_{i=1}^{n} (A \cap B_i)$$

$$\Pr(A) = \sum_{i=1}^{n} \Pr(A \cap B_i)$$

### Expectation

Average output or mean:

$$\mu = \mathrm{E}(X).$$

### Variance

Measure of spread from the mean:

$$\sigma^2 = \mathrm{E}(X^2) - \mathrm{E}(X)^2.$$

# Probability Distributions

### Discrete Random Variables

Has countably many outcomes. Distributed with a probability mass function $p(x)$.

### Continuous Random Variables

Has an infinite number of outcomes. Distributed with a probability density function $f(x)$.

$$\Pr(x_1 \le X \le x_2) = \int_{x_1}^{x_2} f(x)\,\mathrm{d}x$$

$$F(x) = \Pr(X \le x)$$

### General Linear Combinations

$$\mathrm{E}(aX \pm b) = a\,\mathrm{E}(X) \pm b$$
$$\mathrm{E}(aX \pm bY) = a\,\mathrm{E}(X) \pm b\,\mathrm{E}(Y)$$
$$\mathrm{Var}(aX \pm b) = a^2\,\mathrm{Var}(X)$$
$$\begin{aligned}\mathrm{Var}(aX \pm bY) &= a^2\,\mathrm{Var}(X)\\ &\quad + b^2\,\mathrm{Var}(Y)\\ &\quad \pm 2ab\,\mathrm{Cov}(X,\, Y)\end{aligned}$$
$$\begin{aligned}\mathrm{Cov}(X,\, Y) &= \mathrm{E}(XY) - \mathrm{E}(X)\,\mathrm{E}(Y)\\ &= \rho_{XY}\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}.\end{aligned}$$

The correlation $\mathrm{Corr}(X,\, Y)$ or $\rho_{XY}$ is a constant that describes the statistical relationship between $X$ and $Y$. $-1 \le \rho_{XY} \le 1$.

### Binomial (Discrete)

Probability of $x$ successes out of $n$ independent trials, each with chance $p$.

### Poisson (Discrete)

Probability of observing $x$ events over an interval $t$, where events occur at an average rate $\lambda$. $\mu = \lambda t$.

### Exponential (Continuous)

The time $t$ between events, where the events are independent and occur at an average rate $\lambda$.

### Uniform (Continuous)

The probability of any value $x \in [a,\, b]$ is constant.

### Normal (Continuous)

Events occur more frequently near $\mu$ and less frequently further away from $\mu$.

### Standardised Normal (Continuous)

$$Z = \frac{X - \mu}{\sigma} \sim \mathrm{N}(0,\, 1)$$

| | Discrete | Continuous |
|---|---|---|
| Valid probabilities | $0 \le p(x) \le 1$ | $f(x) \ge 0$ |
| Cumulative probability | $\sum_{u \le x} p(u)$ | $\int_{-\infty}^{x} f(u)\,\mathrm{d}u$ |
| Expectation | $\sum_{\Omega} xp(x)$ | $\int_{\Omega} xf(x)\,\mathrm{d}x$ |
| Variance | $\sum_{\Omega} (x-\mu)^2\, p(x)$ | $\int_{\Omega} (x-\mu)^2 f(x)\,\mathrm{d}x$ |

| Distribution | Probability | Cumulative Probability | $\mu$ | $\sigma^2$ |
|---|---|---|---|---|
| Binomial: $X \sim \mathrm{bin}(n,\, p)$ | $\binom{n}{x} p^x (1-p)^{n-x}$ | *See table above.* | $np$ | $np(1-p)$ |
| Poisson: $X \sim \mathrm{Pois}(\mu)$ | $e^{-\mu}\mu^x/x!$ | *See table above.* | $\lambda t$ | $\lambda t$ |
| Exponential: $T \sim \exp\{(\lambda)\}$ | $\lambda e^{-\lambda t}$ | $1 - \lambda e^{-\lambda t}$ | $1/\lambda$ | $1/\lambda^2$ |
| Uniform: $X \sim \mathrm{U}(a,\, b)$ | $1/(b-a)$ | $(x-a)/(b-a)$ | $(a+b)/2$ | $(b-a)^2/12$ |
| Normal: $X \sim \mathrm{N}(\mu,\, \sigma^2)$ | $-$ | $-$ | $\mu$ | $\sigma^2$ |

# Sample Statistics

Given $n$ samples $x_i$:

Mean: $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

$$\text{Variance:}\quad s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2 = \frac{1}{n-1}\left(\sum_{i=1}^{n} x_i^2 - n\overline{x}^2\right)$$

### t Distribution

The sample mean is distributed according to the t distribution where

$$T = \frac{\overline{X} - \mu}{s/\sqrt{n}} \sim t_{n-1}$$
$$\Pr\left(T \le t_{n-1,\,1-\alpha/2}\right) = 1 - \frac{\alpha}{2}$$
$$\Pr\left(T > t_{n-1,\,\alpha/2}\right) = \frac{\alpha}{2}$$

### Confidence Intervals

Given the confidence level $c$:

$$c = 1 - \alpha.$$

The confidence interval for $\overline{x}$:

$$CI_c = \overline{x} \pm t_{n-1,\,1-\alpha/2}\frac{s}{\sqrt{n}}.$$

# Hypothesis Testing

Hypothesis testing assesses the likelihood of observing the sample <u>if</u> the null hypothesis was true.

### Hypothesis

$$H_0 : \mu = \mu_0 \quad \text{vs} \quad H_1 : \mu \ne \mu_0$$

Generally there is no evidence supporting $H_0$, but only evidence or lack of evidence for rejecting $H_0$.

### Test Statistic

The measure of the distance that the proposed mean is from the sample mean:

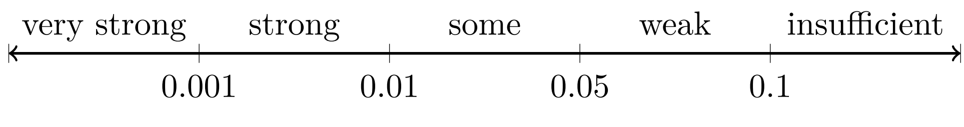
$$T_{\text{test}} = \frac{\overline{x} - \mu_0}{s/\sqrt{n}}$$

### p-Value

Strength of evidence against $H_0$. The $p$-value is the $\alpha$ value that satisfies:

$$|T_{\text{test}}| = t_{n-1,\,1-\alpha/2}.$$
$$\Pr(T \le |T_{\text{test}}|) = 1 - \frac{\alpha}{2}$$
$$\Pr(T > |T_{\text{test}}|) = \frac{\alpha}{2}$$

Strength of evidence against $H_0$




# Linear Regression

Given a set of $n$ points $(x_i,\, y_i)$ that are assumed to have a linear relationship, the model for $y_i$ is

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

where $\varepsilon_i \sim \mathrm{N}(0,\, \sigma^2)$ is the residual. $s$ (RMS error) gives an estimate for $\sigma$ — how close the data is to the model.

### Estimates

$$T_{\text{test}} : \frac{\hat{\beta}_0 - \beta_0}{s_{\hat{\beta}_0}} \sim t_{n-2}, \quad \frac{\hat{\beta}_1 - \beta_1}{s_{\hat{\beta}_1}} \sim t_{n-2}$$

### Inferences about $\beta_0$ and $\beta_1$

$$CI_c = \hat{\beta}_0 \pm t_{n-2,\,1-\alpha/2}\, s_{\hat{\beta}_0}$$
$$CI_c = \hat{\beta}_1 \pm t_{n-2,\,1-\alpha/2}\, s_{\hat{\beta}_1}$$

where $s_{\hat{\beta}_0}$ and $s_{\hat{\beta}_1}$ are the standard errors (SE) for $\hat{\beta}_0$ and $\hat{\beta}_1$. $H_{0,\,\beta_0} : \beta_0 = 0$ tests whether the model crosses the origin. $H_{0,\,\beta_1} : \beta_1 = 0$ tests whether the model is constant.

### R-squared

The percentage of the observed variance in $y$ that is explained by the model.

### Residual Plots

Test the following two assumptions:

1. The relationship between $X$ and $Y$ is best modelled linearly — clear indication of a non-linear trend suggests assumption is not valid.

2. The variance of residuals is the same for all observations, (not affected by $y_i$) — uneven width of residual suggests assumption is not valid. This may lead to inaccurate inferences.