Engineering Computation

Semester 2, 2021

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Contents

1 MATLAB Functions

Function Syntax	Function Output
`y = sin(x)`	Sine with x in radians.
`y = sind(x)`	Sine with x in degrees.
`y = asin(x)`	Arcsine with y in radians.
`y = exp(x)`	e^x.
`y = log(x)`	$\ln(x)$.

Table 1: Common Mathematical Functions in MATLAB.

All the above functions are element-wise.

Function Syntax	Function Output(s)
`A = zeros(m, n)`	Creates an $m \times n$ matrix containing zeros.
`A = ones(m, n)`	Creates an $m \times n$ matrix containing ones.
`I = eye(m)`	Creates an $m \times m$ identity matrix.
`a = linspace(a, b, x)`	Creates an evenly spaced vector with bounds $[a, b]$.
`y = length(A)`	The largest dimension of A.
`[m, n] = size(A)`	The dimensions of A.
`y = min(a)`	The minimum value in the vector a.
`y = max(a)`	The maximum value in the vector a.

Table 2: Matrices and Arrays in MATLAB.

When manipulating matrices, `*`, `^`, perform matrix operations, while prepending an operator with a dot (`.`) performs an element-wise operation.

1.1 Plotting

Function Syntax	Function Output(s)
`plot(x, y)`	Plots given x and y coordinate vectors.
`fplot(@f, [a, b])`	Plots the anonymous function over the domain $[a,\ b]$.
`title('string')`	Adds title to current plot.
`xlabel('string')`	Adds x-axis label to current plot.
`ylabel('string')`	Adds y-axis label to current plot.
`legend('string1', ...)`	Adds legend to plot.
`figure`	Creates a new figure.

Table 3: Plotting in MATLAB.

2 Operations in MATLAB

2.1 Conditional Operations

```
if expression
    statements
else if expression
    statements
else
    statements
end
```

Code inside an `if` statement only executes if the expression is true. Note that only one branch will execute depending on which expression is true.

2.2 Iterative Operations

```
while expression
    statements
end
```

Statements inside a `while` loop execute repeatedly until the expression is false.

```
for index = values
    statements
end
```

Statements inside a `for` loop execute a specific number of times, based on the length of `values`.

3 Differential Equations

Definition 3.0.1. A differential equation is an equation that involves the derivatives of a function as well as the function itself. An ordinary differential equation (ODE) is a differential equation of a function with only one independent variable.

3.1 Electrical Systems

4 First-Order Ordinary Differential Equations

4.1 Separable ODEs

$$\frac{\mathrm{d}y}{\mathrm{d}t} = F(y,\, t)$$

1. Rewrite the equation in the form: $f(y)\,\mathrm{d}y = g(t)\,\mathrm{d}t$.
2. Integrate both sides: $\int f(y)\,\mathrm{d}y = \int g(t)\,\mathrm{d}t$.
3. Rearrange for the explicit form of $y(t)$.

4.2 Linear ODEs

Let $P = P(t)$, $Q = Q(t)$ and $\mu = \mu(t)$

$$\frac{\mathrm{d}y}{\mathrm{d}t} + Py = Q$$

1. Determine the integrating factor: $\mu = \exp\left(\int P \,\mathrm{d}t\right)$.

2. Solve:

$$y = \frac{1}{\mu}\left(\int Q\mu \,\mathrm{d}t + C\right)$$

Proof. To solve a first-order linear differential equation, determine an integrating factor $\mu = \mu(t)$ such that

$$P\mu = \frac{\mathrm{d}\mu}{\mathrm{d}t} \tag{1}$$

Multiplying the equation by μ gives

$$\begin{aligned}
\mu\frac{\mathrm{d}y}{\mathrm{d}t} + P\mu y &= Q\mu \\
\mu\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\mathrm{d}\mu}{\mathrm{d}t}y &= Q\mu \\
\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= Q\mu \\
\int \frac{\mathrm{d}}{\mathrm{d}t}(\mu y)\,\mathrm{d}t &= \int Q\mu \,\mathrm{d}t \\
\mu y &= \int Q\mu \,\mathrm{d}t \\
y &= \frac{1}{\mu}\left(\int Q\mu \,\mathrm{d}t + C\right)
\end{aligned}$$

To determine μ we can rearrange Equation 1 into

$$P = \frac{1}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}t}$$

By recognition, this is the derivative of the natural logarithm of μ with respect to t.

$$\begin{aligned}
P &= \frac{\mathrm{d}}{\mathrm{d}t}(\ln(\mu)) \\
\int P \,\mathrm{d}t &= \int \frac{\mathrm{d}}{\mathrm{d}t}(\ln(\mu))\,\mathrm{d}t \\
\int P \,\mathrm{d}t &= \ln(\mu) \\
\mu &= \exp\left(\int P \,\mathrm{d}t\right)
\end{aligned}$$

□

4.3 Solution using Linearisation

A function can be linearised by using its 1st degree Taylor polynomial near a.

$$f(x) \approx f(a) + f'(a)(x - a) + \mathcal{O}(x^2)$$

This new polynomial can be substituted to form a linear ODE, which can be solved using an integrating factor.

5 Second-Order Ordinary Differential Equations

5.1 Constant Coefficient Linear ODEs

$$a\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + b\frac{\mathrm{d}y}{\mathrm{d}t} + cy = Q(t)$$

where a, b, c are constants.

5.2 Linearity of Solutions

Theorem 5.2.1 (Principle of Superposition)**.** *As the given ODE is linear, if $y_1(t)$ is a solution to the equation*

$$a\frac{\mathrm{d}^2y_1}{\mathrm{d}t^2} + b\frac{\mathrm{d}y_1}{\mathrm{d}t} + cy_1 = Q_1(t)$$

and $y_2(t)$ is a solution to

$$a\frac{\mathrm{d}^2y_2}{\mathrm{d}t^2} + b\frac{\mathrm{d}y_2}{\mathrm{d}t} + cy_2 = Q_2(t)$$

then for the function $y = c_1y_1 + c_2y_2$

$$a\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + b\frac{\mathrm{d}y}{\mathrm{d}t} + cy = c_1Q_1(t) + c_2Q_2(t)$$

where c_1 and c_2 are constants.

5.3 Homogeneous ODEs

Definition 5.3.1. A homogeneous ODE has $Q(t) = 0$, which gives

$$a\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + b\frac{\mathrm{d}y}{\mathrm{d}t} + cy = 0$$

This differential equation has a solution of the form:

$$y_h = \mathrm{e}^{rt}$$

5.4 Characteristic Equation

By making the substitution $y = \mathrm{e}^{rt}$, we get

$$\begin{aligned} a\frac{\mathrm{d}^2y_h}{\mathrm{d}t^2} + b\frac{\mathrm{d}y_h}{\mathrm{d}t} + cy_h &= 0 \\ ar^2\mathrm{e}^{rt} + br\mathrm{e}^{rt} + c\mathrm{e}^{rt} &= 0 \\ (ar^2 + br + c)\mathrm{e}^{rt} &= 0 \\ ar^2 + br + c &= 0 \end{aligned}$$

This is known as the characteristic or *auxiliary* equation. The next step is to calculate the roots of the equation.

Real Distinct Roots. If $b^2 > 4ac$.

Real Repeated Roots. If $b^2 = 4ac$.

Complex Conjugate Roots. If $b^2 < 4ac$.

5.4.1 Real Distinct Roots

Given r_1 and r_2 are real and distinct:

$$y_1(t) = \mathrm{e}^{r_1 t} \qquad\qquad y_2(t) = \mathrm{e}^{r_2 t}$$

Hence the solution to the homogeneous equation is given by:

$$y_h(t) = c_1 \mathrm{e}^{r_1 t} + c_2 \mathrm{e}^{r_2 t}$$

5.4.2 Real Repeated Roots

Given r is a repeated root:

$$y_1(t) = \mathrm{e}^{rt} \qquad\qquad y_2(t) = t\mathrm{e}^{rt}$$

Hence the solution to the homogeneous equation is given by:

$$y_h(t) = c_1 \mathrm{e}^{rt} + c_2 t\mathrm{e}^{rt}$$

5.4.3 Complex Conjugate Roots

Given $r_1 = \alpha + \beta i$ and $r_2 = \alpha - \beta i$ are complex conjugates:

$$y_1(t) = \mathrm{e}^{r_1 t} \qquad\qquad y_2(t) = \mathrm{e}^{r_2 t}$$

Hence the solution to the homogeneous equation is given by:

$$y_h(t) = c_1 \mathrm{e}^{\alpha t} \cos(\beta t) + c_2 \mathrm{e}^{\alpha t} \sin(\beta t)$$

5.5 Nonhomogeneous ODE

A nonhomogeneous differential equation is of the form

$$a\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + b\frac{\mathrm{d}y}{\mathrm{d}t} + cy = Q(t)$$

where $Q(t) \neq 0$.

5.6 General Solution of a Nonhomogeneous ODE

Recall that the solutions to any linear ODE are additive, so that if a solution y_p satisfies the nonhomogeneous ODE, and y_h satisfies the homogeneous ODE,

$$y = y_h + y_p$$

must also satisfy the ODE.

5.7 Undetermined Coefficients

To solve for y_p, we substitute a guess like the homogeneous case, and the coefficients in this guess will be determined from the ODE itself.
The particular solution will depend on what $Q(t)$ looks like.

$Q(t)$	y_p
a constant	A
nth degree polynomial	$A_0 + A_1 t + \cdots + A_{n-1} t^{n-1} + A_n t^n$
$\mathrm{e}^{\alpha t}$	$A\mathrm{e}^{\alpha t}$
$\cos(\alpha t)$	$A\cos(\alpha t) + B\sin(\alpha t)$
$\sin(\alpha t)$	$A\cos(\alpha t) + B\sin(\alpha t)$
$\cos(\alpha t) + \sin(\alpha t)$	$A\cos(\alpha t) + B\sin(\alpha t)$

Table 4: Particular Solutions for Undetermined Coefficients

5.8 Special Forms

5.8.1 Product of Forms

If $Q(t)$ is a product of the functions shown above, then we write the particular solution for both functions separately and multiply the results together.
For example, with $Q(t) = t\mathrm{e}^{4t}$, we have

$$y_p = (At + B)\left(C\mathrm{e}^{4t}\right)$$

the next step is to expand the function simplify any constants.

$$y_p = (ACt + BC)\,\mathrm{e}^{4t} y_p \qquad = (A_1 t + B_1)\,\mathrm{e}^{4t}$$

5.8.2 Sum of Forms

If $Q(t)$ is a sum of the functions shown above, then we can use Theorem 5.2.1 and add the particular solutions together.

5.8.3 Linearly Dependent Case

If $Q(t)$ is similar to any homogenous solution, then by *definition* of a homogeneous solution, the solution will be 0. Hence, y_p must be multiplied by t to ensure that the particular solution is linearly independent to the homogeneous solutions, in order to form a *fundamental set of solutions*.

5.9 Solving the Particular Solution

1. Solve y_h
2. Find an appropriate form for y_p
3. Ensure that y_p is linearly independent to the homogeneous solutions

4. Substitute y_p into the nonhomogeneous ODE and solve for the undetermined coefficients
5. Find the general solution $y = y_h + y_p$
6. Apply initial conditions to solve for any constants

6 Systems of Ordinary Differential Equations

A first-order system of differential equations has the form

$$\begin{cases} x_1' = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ x_2' = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \quad\;\; \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\ x_n' = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{cases}$$

where $x_1 = x_1(t)$, $x_2 = x_2(t)$, ..., $x_n = x_n(t)$ are the functions to be determined. In matrix form, the system can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$
$$\mathbf{x}' = \boldsymbol{A}\mathbf{x}$$

6.1 Higher-Order ODEs

A higher-order linear differential equation can be solved by first converting it to a first-order linear system. Consider the nth-order homogeneous differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = 0$$

Let

$$\begin{aligned} x_1 &= y \\ x_2 &= y' \\ &\vdots \\ x_n &= y^{(n-1)} \end{aligned}$$

so that $\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^\top$. Then the differential equation can be expressed as the following first-order linear system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

6.2 Solution Form

Like the homogeneous case, we will guess a solution of the form

$$\mathbf{x} = \mathbf{q}e^{\lambda t}$$

which allows for the following substitution

$$\begin{aligned}
\lambda\mathbf{q}e^{\lambda t} &= \boldsymbol{A}\mathbf{q}e^{\lambda t} \\
\boldsymbol{A}\mathbf{q}e^{\lambda t} - \lambda\mathbf{q}e^{\lambda t} &= \mathbf{0} \\
(\boldsymbol{A} - \lambda\boldsymbol{I})\,\mathbf{q}e^{\lambda t} &= \mathbf{0} \\
(\boldsymbol{A} - \lambda\boldsymbol{I})\,\mathbf{q} &= \mathbf{0}
\end{aligned}$$

This equation has the trivial solution $\mathbf{q} = \mathbf{0}$, however for a fundamental set of solutions, we must let $\boldsymbol{A} - \lambda\boldsymbol{I}$ be singular.

6.2.1 Characteristic Equation

To determine the eigenvalues λ of the matrix $\mathbf{A}$, we must solve the characteristic equation associated with the system of ODEs. Namely,

$$\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0$$

These eigenvalues can then be used to solve the eigenvectors of $\boldsymbol{A}$

6.3 Solving a System of ODEs

1. Model the system of ODEs in the form $\mathbf{x}' = \boldsymbol{A}\mathbf{x}$
2. Solve the characteristic equation for the eigenvalues of $\boldsymbol{A}$
3. Solve the corresponding eigenvectors of $\boldsymbol{A}$ by solving $(\boldsymbol{A} - \lambda\boldsymbol{I})\,\mathbf{q} = \mathbf{0}$
4. Write the general solution: $\mathbf{x} = c_1\mathbf{q}_1e^{\lambda_1 t} + c_2\mathbf{q}_2e^{\lambda_2 t}$
5. Apply initial conditions to solve c_1 and c_2

7 Probability

Definition 7.0.1 (Random Variables)**.** A random variable X is a measurable variable that doesn't hold a definitive value.

Definition 7.0.2 (Discrete Random Variables)**.** A discrete random variable X has a countable number of possible values.

Definition 7.0.3 (Continuous Random Variables)**.** A continuous random variable X can take all values in a given interval.

Definition 7.0.4 (Probability)**.** Probability is used to mean the chance that a particular event (or set of events) will occur, expressed on a linear scale from 0 to 1. The probability of the random variable X taking the value x is denoted

$$\Pr(X = x)$$

Definition 7.0.5 (Sample Space)**.** Let the set of all possible outcomes of a random variables be called the sample space, denoted Ω, of that random variable.
Let X be a random variable that can take on values $x \in \Omega$. Then for all $x \in \Omega$ there is an associated probability $p(x)$, such that

$$\forall x \in \Omega : 0 < p(x) \leq 1$$
$$\sum p(x) = 1.$$

7.1 Events

Definition 7.1.1 (Events)**.** An event A is a set of individual outcomes within Ω. Then for some event $A \subset \Omega$, the probability is given by

$$\Pr(A) = \sum_{x \in A} p(x).$$

The complementary event, denoted A^C (also $\overline{A}$) is the set of all outcomes within the sample space that are not within A.

$$\Pr(A^C) = 1 - \Pr(A).$$

Theorem 7.1.1 (Combination of Events)**.** *Events can be combined with the two logical connectors AND and OR, which are equivalent to the intersection ($\cap$) and union ($\cup$) of set.*

Theorem 7.1.2 (Mutually Exclusive Events)**.** *If two events have no possible outcomes in common, they are mutually exclusive or disjoint events.*

$$A \cap B = \emptyset.$$

It follows that

$$\Pr(A \cap B) = 0.$$

Theorem 7.1.3 (Probability of Union)**.**

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

Theorem 7.1.4 (AND Statements)**.**

7.2 Dependent Events

Definition 7.2.1 (Conditional Probability)**.**

Theorem 7.2.1 (Total Probability)**.**

Theorem 7.2.2 (Bayes' Rule)**.**

8 Probability Distributions

Definition 8.0.1 (Expectation)**.**

Definition 8.0.2 (Variance)**.**

8.1 Binomial Distribution

8.2 Bernouilli Distribution

8.3 Poisson Distribution

8.4 Uniform Distribution

8.5 Exponential Distribution

8.6 Normal Distribution