Engineering Computation

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Contents

1 MATLAB Functions

Function Syntax	Function Output
`y = sin(x)`	Sine with x in radians.
`y = sind(x)`	Sine with x in degrees.
`y = asin(x)`	Arcsine with y in radians.
`y = exp(x)`	e^x.
`y = log(x)`	$\ln(x)$.

Table 1: Common Mathematical Functions in MATLAB.

All the above functions are element-wise.

Function Syntax	Function Output(s)
`A = zeros(m, n)`	Creates an $m \times n$ matrix containing zeros.
`A = ones(m, n)`	Creates an $m \times n$ matrix containing ones.
`I = eye(m)`	Creates an $m \times m$ identity matrix.
`a = linspace(a, b, x)`	Creates an evenly spaced vector with bounds $[a, b]$.
`y = length(A)`	The largest dimension of A.
`[m, n] = size(A)`	The dimensions of A.
`y = min(a)`	The minimum value in the vector a.
`y = max(a)`	The maximum value in the vector a.

Table 2: Matrices and Arrays in MATLAB.

When manipulating matrices, `*`, `^`, perform matrix operations, while prepending an operator with a dot (`.`) performs an element-wise operation.

1.1 Plotting

Function Syntax	Function Output(s)
`plot(x, y)`	Plots given x and y coordinate vectors.
`fplot(@f, [a, b])`	Plots the anonymous function over the domain $[a,\ b]$.
`title('string')`	Adds title to current plot.
`xlabel('string')`	Adds x-axis label to current plot.
`ylabel('string')`	Adds y-axis label to current plot.
`legend('string1', ...)`	Adds legend to plot.
`figure`	Creates a new figure.

Table 3: Plotting in MATLAB.

2 Operations in MATLAB

2.1 Conditional Operations

```
if expression
    statements
else if expression
    statements
else
    statements
end
```

Code inside an **`if`** statement only executes if the expression is true. Note that only one branch will execute depending on which expression is true.

2.2 Iterative Operations

```
while expression
    statements
end
```

Statements inside a **`while`** loop execute repeatedly until the expression is false.

```
for index = values
    statements
end
```

Statements inside a **`for`** loop execute a specific number of times, based on the length of `values`.

3 Differential Equations

Definition 3.0.1. A differential equation is an equation that involves the derivatives of a function as well as the function itself. An ordinary differential equation (ODE) is a differential equation of a function with only one independent variable.

3.1 Electrical Systems

Theorem 3.1.1 (VI Relationship between Resistors)**.**

$$v = iR$$

Theorem 3.1.2 (VI Relationship between Inductors)**.**

$$v = L\frac{\mathrm{d}i}{\mathrm{d}t}$$

Theorem 3.1.3 (VI Relationship between Capacitors)**.**

$$i = C\frac{\mathrm{d}v}{\mathrm{d}t}$$

Theorem 3.1.4 (Kirchoff's Voltage Law)**.** *The sum of all voltages around a loop equals zero.*

$$\sum v_{\text{loop}} = 0$$

Theorem 3.1.5 (Kirchoff's Current Law)**.** *The sum of all currents into a node equals zero.*

$$\sum i_{\text{node}} = 0$$

3.2 Mechanical Systems

Theorem 3.2.1 (Newton's Second Law)**.**

$$\sum F = \frac{\mathrm{d}p}{\mathrm{d}t}$$

where $p = mv$*.*

Theorem 3.2.2 (Thrust Force)**.**

$$F_T = (c - v)d_f$$

Theorem 3.2.3 (Force of Gravity)**.**

$$F_g = mg$$

Theorem 3.2.4 (Force of a Spring)**.**

$$F = -kx$$

4 First-Order Ordinary Differential Equations

4.1 Separable ODEs

$$\frac{\mathrm{d}y}{\mathrm{d}t} = F(y,\ t)$$

1. Rewrite the equation in the form: $f(y)\,\mathrm{d}y = g(t)\,\mathrm{d}t$.
2. Integrate both sides: $\int f(y)\,\mathrm{d}y = \int g(t)\,\mathrm{d}t$.
3. Rearrange for the explicit form of $y(t)$.

4.2 Linear ODEs

Let $P = P(t)$, $Q = Q(t)$ and $\mu = \mu(t)$

$$\frac{\mathrm{d}y}{\mathrm{d}t} + Py = Q$$

1. Determine the integrating factor: $\mu = \exp\left(\int P\,\mathrm{d}t\right)$.
2. Solve:

$$y = \frac{1}{\mu}\left(\int Q\mu\,\mathrm{d}t + C\right)$$

Proof. To solve a first-order linear differential equation, determine an integrating factor $\mu = \mu(t)$ such that

$$P\mu = \frac{\mathrm{d}\mu}{\mathrm{d}t} \tag{1}$$

Multiplying the equation by μ gives

$$\begin{aligned}
\mu\frac{\mathrm{d}y}{\mathrm{d}t} + P\mu y &= Q\mu \\
\mu\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\mathrm{d}\mu}{\mathrm{d}t}y &= Q\mu \\
\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= Q\mu \\
\int \frac{\mathrm{d}}{\mathrm{d}t}(\mu y)\,\mathrm{d}t &= \int Q\mu\,\mathrm{d}t \\
\mu y &= \int Q\mu\,\mathrm{d}t \\
y &= \frac{1}{\mu}\left(\int Q\mu\,\mathrm{d}t + C\right)
\end{aligned}$$

To determine μ we can rearrange Equation 1 into

$$P = \frac{1}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}t}$$

By recognition, this is the derivative of the natural logarithm of μ with respect to t.

$$\begin{aligned}
P &= \frac{\mathrm{d}}{\mathrm{d}t}(\ln(\mu)) \\
\int P\,\mathrm{d}t &= \int \frac{\mathrm{d}}{\mathrm{d}t}(\ln(\mu))\,\mathrm{d}t \\
\int P\,\mathrm{d}t &= \ln(\mu) \\
\mu &= \exp\left(\int P\,\mathrm{d}t\right)
\end{aligned}$$

□

4.3 Solution using Linearisation

A function can be linearised by using its 1st degree Taylor polynomial near a.

$$f(x) \approx f(a) + f'(a)(x-a) + \mathcal{O}(x^2)$$

This new polynomial can be substituted to form a linear ODE, which can be solved using an integrating factor.

5 Second-Order Ordinary Differential Equations

5.1 Constant Coefficient Linear ODEs

$$a\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + b\frac{\mathrm{d}y}{\mathrm{d}t} + cy = Q(t)$$

where a, b, c are constants.

5.2 Linearity of Solutions

Theorem 5.2.1 (Principle of Superposition)**.** *As the given ODE is linear, if $y_1(t)$ is a solution to the equation*

$$a\frac{\mathrm{d}^2y_1}{\mathrm{d}t^2} + b\frac{\mathrm{d}y_1}{\mathrm{d}t} + cy_1 = Q_1(t)$$

and $y_2(t)$ is a solution to

$$a\frac{\mathrm{d}^2y_2}{\mathrm{d}t^2} + b\frac{\mathrm{d}y_2}{\mathrm{d}t} + cy_2 = Q_2(t)$$

then for the function $y = c_1y_1 + c_2y_2$

$$a\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + b\frac{\mathrm{d}y}{\mathrm{d}t} + cy = c_1Q_1(t) + c_2Q_2(t)$$

where c_1 and c_2 are constants.

5.3 Homogeneous ODEs

Definition 5.3.1. A homogeneous ODE has $Q(t) = 0$, which gives

$$a\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + b\frac{\mathrm{d}y}{\mathrm{d}t} + cy = 0$$

This differential equation has a solution of the form:

$$y_h = e^{rt}$$

5.4 Characteristic Equation

By making the substitution $y = e^{rt}$, we get

$$\begin{aligned}
a\frac{\mathrm{d}^2y_h}{\mathrm{d}t^2} + b\frac{\mathrm{d}y_h}{\mathrm{d}t} + cy_h &= 0 \\
ar^2e^{rt} + bre^{rt} + ce^{rt} &= 0 \\
(ar^2 + br + c)e^{rt} &= 0 \\
ar^2 + br + c &= 0
\end{aligned}$$

This is known as the characteristic or *auxiliary* equation. The next step is to calculate the roots of the equation.

Real Distinct Roots. If $b^2 > 4ac$.

Real Repeated Roots. If $b^2 = 4ac$.

Complex Conjugate Roots. If $b^2 < 4ac$.

5.4.1 Real Distinct Roots

Given r_1 and r_2 are real and distinct:

$$y_1(t) = e^{r_1 t} \qquad y_2(t) = e^{r_2 t}$$

Hence the solution to the homogeneous equation is given by:

$$y_h(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

5.4.2 Real Repeated Roots

Given r is a repeated root:

$$y_1(t) = e^{rt} \qquad y_2(t) = te^{rt}$$

Hence the solution to the homogeneous equation is given by:

$$y_h(t) = c_1 e^{rt} + c_2 te^{rt}$$

5.4.3 Complex Conjugate Roots

Given $r_1 = \alpha + \beta i$ and $r_2 = \alpha - \beta i$ are complex conjugates:

$$y_1(t) = e^{r_1 t} \qquad y_2(t) = e^{r_2 t}$$

Hence the solution to the homogeneous equation is given by:

$$y_h(t) = c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t)$$

5.5 Nonhomogeneous ODE

A nonhomogeneous differential equation is of the form

$$a\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + b\frac{\mathrm{d}y}{\mathrm{d}t} + cy = Q(t)$$

where $Q(t) \neq 0$.

5.6 General Solution of a Nonhomogeneous ODE

Recall that the solutions to any linear ODE are additive, so that if a solution y_p satisfies the nonhomogeneous ODE, and y_h satisfies the homogeneous ODE,

$$y = y_h + y_p$$

must also satisfy the ODE.

5.7 Undetermined Coefficients

To solve for y_p, we substitute a guess, and determined the coefficients from the ODE itself.
The particular solution will depend on what $Q(t)$ looks like.

$Q(t)$	y_p
a constant	A
nth degree polynomial	$A_0 + A_1 t + \cdots + A_{n-1}t^{n-1} + A_n t^n$
$e^{\alpha t}$	$Ae^{\alpha t}$
$\cos(\alpha t)$	$A\cos(\alpha t) + B\sin(\alpha t)$
$\sin(\alpha t)$	$A\cos(\alpha t) + B\sin(\alpha t)$
$\cos(\alpha t) + \sin(\alpha t)$	$A\cos(\alpha t) + B\sin(\alpha t)$

Table 4: Particular Solutions for Undetermined Coefficients

5.8 Special Forms

5.8.1 Product of Forms

If $Q(t)$ is a product of the functions shown above, then we write the particular solution for both functions separately and multiply the results together.
For example, with $Q(t) = te^{4t}$, we have

$$y_p = (At + B)\left(Ce^{4t}\right)$$

the next step is to expand the function simplify any constants.

$$y_p = (ACt + BC)\, e^{4t} y_p \qquad = (A_1 t + B_1)\, e^{4t}$$

5.8.2 Sum of Forms

If $Q(t)$ is a sum of the functions shown above, then we can use Theorem 5.2.1 and add the particular solutions together.

5.8.3 Linearly Dependent Case

If $Q(t)$ is similar to any homogenous solution, then by *definition* of a homogeneous solution, the solution will be 0. Hence, y_p must be multiplied by t to ensure that the particular solution is linearly independent to the homogeneous solutions, in order to form a *fundamental set of solutions*.

5.9 Solving the Particular Solution

1. Solve y_h
2. Find an appropriate form for y_p
3. Ensure that y_p is linearly independent to the homogeneous solutions
4. Substitute y_p into the nonhomogeneous ODE and solve for the undetermined coefficients

5. Find the general solution $y = y_h + y_p$
6. Apply initial conditions to solve for any constants

6 Systems of Ordinary Differential Equations

A first-order system of differential equations has the form

$$\begin{cases} x_1' = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ x_2' = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \quad\;\; \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\ x_n' = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{cases}$$

where $x_1 = x_1(t),\ x_2 = x_2(t),\ \ldots,\ x_n = x_n(t)$ are the functions to be determined. In matrix form, the system can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\mathbf{x}' = \boldsymbol{A}\mathbf{x}$$

6.1 Higher-Order ODEs

A higher-order linear differential equation can be solved by first converting it to a first-order linear system. Consider the nth-order homogeneous differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = 0$$

Let

$$\begin{aligned} x_1 &= y \\ x_2 &= y' \\ &\vdots \\ x_n &= y^{(n-1)} \end{aligned}$$

so that $\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^\top$. Then the differential equation can be expressed as the following first-order linear system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

6.2 Solution Form

Like the homogeneous case, we will guess a solution of the form

$$\mathbf{x} = \mathbf{q}e^{\lambda t}$$

which allows for the following substitution

$$\begin{aligned} \lambda \mathbf{q}e^{\lambda t} &= \boldsymbol{A}\mathbf{q}e^{\lambda t} \\ \boldsymbol{A}\mathbf{q}e^{\lambda t} - \lambda \mathbf{q}e^{\lambda t} &= \mathbf{0} \\ \left(\boldsymbol{A} - \lambda \boldsymbol{I}\right) \mathbf{q}e^{\lambda t} &= \mathbf{0} \\ \left(\boldsymbol{A} - \lambda \boldsymbol{I}\right) \mathbf{q} &= \mathbf{0} \end{aligned}$$

This equation has the trivial solution $\mathbf{q} = \mathbf{0}$, however for a fundamental set of solutions, we must let $\boldsymbol{A} - \lambda \boldsymbol{I}$ be singular.

6.2.1 Characteristic Equation

To determine the eigenvalues λ of the matrix $\mathbf{A}$, we must solve the characteristic equation associated with the system of ODEs. Namely,

$$\det\left(\boldsymbol{A} - \lambda \boldsymbol{I}\right) = 0$$

These eigenvalues can then be used to solve the eigenvectors of $\boldsymbol{A}$

6.3 Solving a System of ODEs

1. Model the system of ODEs in the form $\mathbf{x}' = \boldsymbol{A}\mathbf{x}$
2. Solve the characteristic equation for the eigenvalues of $\boldsymbol{A}$
3. Solve the corresponding eigenvectors of $\boldsymbol{A}$ by solving $\left(\boldsymbol{A} - \lambda \boldsymbol{I}\right) \mathbf{q} = \mathbf{0}$
4. Write the general solution: $\mathbf{x} = c_1 \mathbf{q}_1 e^{\lambda_1 t} + c_2 \mathbf{q}_2 e^{\lambda_2 t}$
5. Apply initial conditions to solve c_1 and c_2

7 Probability

Definition 7.0.1 (Random Variables)**.** A random variable X is a measurable variable that doesn't hold a definitive value.

Definition 7.0.2 (Discrete Random Variables)**.** A discrete random variable X has a countable number of possible values.

Definition 7.0.3 (Continuous Random Variables)**.** A continuous random variable X can take all values in a given interval.

Definition 7.0.4 (Probability)**.** Probability is used to mean the chance that a particular event (or set of events) will occur, expressed on a linear scale from 0 to 1. The probability of the random variable X taking the value x is denoted

$$\Pr\left(X = x\right)$$

Definition 7.0.5 (Sample Space)**.** Let the set of all possible outcomes of a random variables be called the sample space, denoted Ω, of that random variable.
Let X be a random variable that can take on values $x \in \Omega$. Then for all $x \in \Omega$ there is an associated probability $p(x)$, such that

$$\forall x \in \Omega : 0 < p(x) \leq 1$$
$$\sum p(x) = 1.$$

7.1 Events

Definition 7.1.1 (Events)**.** An event A is a set of individual outcomes within Ω. Then for some event $A \subset \Omega$, the probability is given by

$$\Pr(A) = \sum_{x \in A} p(x).$$

The complementary event, denoted A^C (also $\overline{A}$) is the set of all outcomes within the sample space that are not within A.

$$\Pr(A^C) = 1 - \Pr(A).$$

Definition 7.1.2 (Combination of Events)**.** Events can be combined with the two logical connectors AND and OR, which are equivalent to the intersection ($\cap$) and union ($\cup$) of set.

Definition 7.1.3 (Mutually Exclusive Events)**.** If two events have no possible outcomes in common, they are mutually exclusive or disjoint events.

$$A \cap B = \varnothing.$$

It follows that

$$\Pr(A \cap B) = 0.$$

Theorem 7.1.1 (Probability of Union)**.**

$$\begin{aligned} \Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B) \\ &= 1 - \Pr\left(A^C \cap B^C\right) \end{aligned}$$

Definition 7.1.4 (Independent Events)**.** Two events are independent if the outcome of one event has no influence on the outcome of the other. For these cases, the joint probability is given by

$$\Pr(A \cap B) = \Pr(A) \Pr(B).$$

7.2 Dependent Events

Two events are dependent if the outcome of one event influences the outcome of the other.

Definition 7.2.1 (Conditional Probability)**.** In the case of dependent events, we must use conditional probability concepts in calculating joint probabilities. The probability of A given that the event B has occurred is

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

Theorem 7.2.1 (Total Probability)**.** *If B is a sample space of disjoint events $B_1,\ B_2,\ \ldots,\ B_N$, then $A\cap B_1,\ A\cap B_2,\ \ldots,\ A\cap B_N$ are also disjoint, and*

$$A = (A\cap B_1) + (A\cap B_2) + \cdots + (A\cap B_N)$$

This gives

$$\begin{aligned}\Pr(A) &= \sum_{i=1}^{N} \Pr(A\cap B_i)\\ &= \sum_{i=1}^{N} \Pr(A \mid B_i)\Pr(B_i)\end{aligned}$$

Theorem 7.2.2 (Bayes' Rule)**.** *Using the commutativity of intersections, the rule for conditional probability gives*

$$\Pr(A\cap B) = \Pr(A \mid B)\Pr(B) = \Pr(B \mid A)\Pr(A)$$

Therefore

$$\Pr(A) = \frac{\Pr(A \mid B)\Pr(B)}{\Pr(A)}$$

8 Probability Distributions

Definition 8.0.1 (Discrete Random Variables)**.** A discrete random variable has countably many outcomes.

Definition 8.0.2 (Continuous Random Variables)**.** A continuous random variable can take an infinite number of individual outcomes.

Definition 8.0.3 (Probability Distributions)**.** The probabilities of random variable make up a probability distribution.
For discrete random variables, the distribution is described with a Probability Mass Function (PMF)

$$p(x) = \Pr(X = x)$$

For continuous variables, the distribution is described with a Probability Density Function (PDF) and the associated Cumulative Distribution Function (CDF).
Here, probabilities are represented by areas under the PDF:

$$\Pr(x_1 \le X \le x_2) = \int_{x_1}^{x_2} f(u)\,\mathrm{d}u$$

and the CDF is defined as

$$F(x) = \Pr(X \le x) = \int_{-\infty}^{x} f(u)\,\mathrm{d}u\,.$$

Note that $f(x)$ is a valid PDF provided

$$f(x) \ge 0 : \forall x \quad \text{and} \quad \int_{-\infty}^{\infty} f(u)\,\mathrm{d}u = 1$$

while $F(x)$ is a valid CDF if:

1. F is a non-decreasing right continuous function
2. $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$

8.1 Statistics Summary

Definition 8.1.1 (Expectation)**.** The expected value $\mathrm{E}(X)$, of a random variable is the average outcome that could be expected from an infinite number of observations of that variable. This is also known as the mean of the variable, denoted μ.

$$\mathrm{E}(X) = \begin{cases} \sum_X xp(x) & \text{for discrete variables} \\ \int_X xf(x)\,\mathrm{d}x & \text{for continuous variables} \end{cases}$$

Definition 8.1.2 (Variance)**.** The variance $\mathrm{Var}(X)$, of a random variable is a measure of spread of the distribution (defined as the average squared distance of each value from the mean). $\mathrm{Var}(X)$ is also denoted as σ^2.

$$\begin{aligned} \mathrm{Var}(X) &= \begin{cases} \sum_X (x-\mu)^2 p(x) & \text{for discrete variables} \\ \int_X (x-\mu)^2 p(x)f(x)\,\mathrm{d}x & \text{for continuous variables} \end{cases} \\ &= \mathrm{E}(X^2) - \mathrm{E}(X)^2 \end{aligned}$$

Definition 8.1.3 (Standard Deviation)**.** The standard deviation is defined as

$$\sigma = \sqrt{\mathrm{Var}(X)}$$

8.1.1 General Linear Combinations

For a simple linear function of a random variable

$$\begin{aligned} \mathrm{E}(aX+b) &= a\,\mathrm{E}(X) + b \\ \mathrm{Var}(aX+b) &= a^2\,\mathrm{Var}(X) \end{aligned}$$

For a general linear combination of two random variables,

$$\begin{aligned} \mathrm{E}(aX+bY) &= a\,\mathrm{E}(X) + b\,\mathrm{E}(Y) \\ \mathrm{Var}(aX+bY) &= a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y) + 2ab\,\mathrm{Cov}(X,\,Y) \end{aligned}$$

Definition 8.1.4 (Covariance)**.** Covariance is the joint variability of two random variables.

$$\begin{aligned} \mathrm{Cov}(X,\,Y) &= \mathrm{E}(XY) - \mathrm{E}(X)\,\mathrm{E}(Y) \\ &= \mathrm{Corr}(X,\,Y)\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)} \end{aligned}$$

Definition 8.1.5 (Correlation)**.** The correlation of two random variables is any statistical relationship between those two variables. The correlation $\mathrm{Corr}(X,\,Y)$ is usually denoted ρ_{XY} or ρ, and it always satisfies $-1 \le \rho \le 1$.

9 Common Probability Distributions

9.1 Binomial Distribution

9.2 Bernoulli Distribution

9.3 Poisson Distribution

9.4 Uniform Distribution

9.5 Exponential Distribution

9.6 Normal Distribution