MATLAB

```
% Function declaration
function [output1, ...]
    = function_name(input1, ...)
% Iterate over the vector v
for i = v ... end
% Repeat while condition is true
while condition ... end
% Execute first true alternative
if condition_1 ...
elseif condition_2 ...
else ... end
```

Partial Fraction Decomposition

Given the LHS in the denominator, substitute the RHS.

$$(ax+b)^k \to \frac{A_1}{ax+b} + \cdots + \frac{A_k}{(ax+b)^k}$$

$$(ax^2+bx+c)^k \to \frac{A_1 x + B_1}{ax^2+bx+c} + \cdots + \frac{A_k x + B_k}{(ax^2+bx+c)^k}$$

Differential Equations

Mechanical Systems

$$\sum F = \frac{\mathrm{d}p}{\mathrm{d}t}, \quad \sum M = I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$$

where momentum $p = mv$, moment $M = Fx$ and inertia $I = mr^2$.

$$F_T = (c-v)\,d_f \quad F_g = -mg$$
$$F_s = -kx \quad F_f = -bv^2 \text{ (or } -bv)$$

Electrical Circuits

$$i = \frac{\mathrm{d}q}{\mathrm{d}t}, \quad \sum v_{loop} = 0, \quad \sum i_{node} = 0$$

Voltage drop across elements:

$$v_R = iR, \quad v_C = \frac{q}{C}, \quad v_L = L\frac{\mathrm{d}i}{\mathrm{d}t}$$

Separable ODEs

For $\frac{\mathrm{d}y}{\mathrm{d}t} = p(t)q(y)$:

$$\int \frac{1}{q(y)}\frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t = \int p(t)\,\mathrm{d}t\,.$$

Linear ODEs

For $\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)y = q(t)$, use the *integrating factor*: $I(t) = e^{\int p(t)\mathrm{d}t}$, so that

$$y(t) = \frac{1}{I(t)}\int I(t)q(t)\,\mathrm{d}t\,.$$

Linearisation

$$f(t) \approx f(t_0) + f'(t_0)(t-t_0)$$

$$f(y(t)) \approx f(y(t_0)) + f'(y(t_0))(y(t)-y(t_0))$$

Euler's Method

For $\boldsymbol{y}' = \boldsymbol{f}(t,\,\boldsymbol{y})$:

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\boldsymbol{f}(t_n,\,\boldsymbol{y}_n)$$

Modified Euler's Method

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \frac{h}{2}\left(\boldsymbol{f}(t_n,\,\boldsymbol{y}_n) + \boldsymbol{f}(t_{n+1},\,\boldsymbol{y}_{n+1})\right)$$

where $\boldsymbol{f}(t_{n+1},\,\boldsymbol{y}_{n+1})$ is determined using Euler's method.

Second-Order ODEs

$$ay'' + by' + cy = F(t)$$

General Solution

$$y(t) = y_H(t) + y_P(t)$$

Homogeneous Solution

$$y_H(t) = e^{\lambda t}$$

To solve for λ, substitute the homogeneous form into the homogeneous ODE.

Real distinct roots:

$$y_H(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

Real repeated roots:

$$y_H(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$$

Complex conjugate roots: $\lambda = \alpha \pm \beta i$

$$y_H(t) = e^{\alpha x}(c_1\cos(\beta t) + c_2\sin(\beta t))$$

Particular Solution

See table below. Substitute y_P into the nonhomogeneous ODE, and solve the undetermined coefficients.

System of ODEs

Given $\boldsymbol{y}' = \mathbf{A}\boldsymbol{y}$,

$$\boldsymbol{y}_H(t) = \boldsymbol{q}e^{\lambda t}.$$

λ_i are the eigenvalues of $\boldsymbol{A}$ that satisfy

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

$\boldsymbol{q}_i$ are the associated eigenvectors that satisfy

$$(\mathbf{A} - \lambda_i\mathbf{I})\,\boldsymbol{q}_i = \mathbf{0}.$$

For real distinct roots:

$$\boldsymbol{y}_H(t) = c_1\boldsymbol{q}_1 e^{\lambda_1 t} + c_2\boldsymbol{q}_2 e^{\lambda_2 t}$$

Higher-Order ODEs

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = 0$$

Let $y_1 = y$, $y_2 = y'$, …, $y_n = y^{(n-1)}$ so that $\boldsymbol{y} = \langle y_1,\,y_2,\,\ldots,\,y_n\rangle$. Then $\boldsymbol{y}' = \mathbf{A}\boldsymbol{y}$, where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

$F(t)$	$y_P(t)$
a constant	A
a polynomial of degree n	$\sum_{i=0}^{n} A_i t^i$
e^{kt}	Ae^{kt}
$\cos(\omega t)$ or $\sin(\omega t)$	$A_0\cos(\omega t) + A_1\sin(\omega t)$
a combination of the above	a combination of the above
linearly dependent to $y_H(t)$	multiply $y_P(t)$ by t until linearly independent

Probability

Events

$$\Pr(A) = \sum_{x\in A} p(x)$$

$$\Pr(A^C) = \Pr\left(\overline{A}\right) = 1 - \Pr(A)$$

Unions

$$\Pr(A\cup B) = \Pr(A) + \Pr(B) - \Pr(A\cap B) = 1 - \Pr(A^C \cap B^C)$$

Bayes' Theorem

$$\Pr(B \mid A) = \frac{\Pr(A\mid B)\Pr(B)}{\Pr(A)}$$

Disjoint Events

Events don't have outcomes in common.

$$A \cap B = \varnothing$$
$$\Pr(A\cap B) = 0$$

Independent Events

Outcome of events do not influence each other. Joint probability:

$$\Pr(A\cap B) = \Pr(A)\Pr(B)$$

Dependent (Conditional) Events

Outcome of event depends on the outcome of the other. Joint probability of A given B:

$$\Pr(A\mid B) = \frac{\Pr(A\cap B)}{\Pr(B)}$$

Total Probability

For disjoint events B_i:

$$A = \bigcup_{i=1}^{n}(A\cap B_i)$$

$$\Pr(A) = \sum_{i=1}^{n}\Pr(A\cap B_i)$$

Expectation

Average output or mean:

$$\mu = \mathrm{E}(X).$$

Variance

Measure of spread from the mean:

$$\sigma^2 = \mathrm{E}(X^2) - \mathrm{E}(X)^2.$$

Probability Distributions

Discrete Random Variables

Has countably many outcomes. Distributed with a probability mass function $p(x)$.

Continuous Random Variables

Has an infinite number of outcomes. Distributed with a probability density function $f(x)$.

$$\Pr(x_1 \le X \le x_2) = \int_{x_1}^{x_2} f(x)\,\mathrm{d}x$$

$$F(x) = \Pr(X \le x)$$

General Linear Combinations

$$\mathrm{E}(aX \pm b) = a\,\mathrm{E}(X) \pm b$$
$$\mathrm{E}(aX \pm bY) = a\,\mathrm{E}(X) \pm b\,\mathrm{E}(Y)$$
$$\mathrm{Var}(aX \pm b) = a^2\,\mathrm{Var}(X)$$
$$\begin{aligned}\mathrm{Var}(aX \pm bY) &= a^2\,\mathrm{Var}(X)\\ &\quad + b^2\,\mathrm{Var}(Y)\\ &\quad \pm 2ab\,\mathrm{Cov}(X,\,Y)\end{aligned}$$
$$\begin{aligned}\mathrm{Cov}(X,\,Y) &= \mathrm{E}(XY) - \mathrm{E}(X)\,\mathrm{E}(Y)\\ &= \rho_{XY}\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}.\end{aligned}$$

The correlation $\mathrm{Corr}(X,\,Y)$ or ρ_{XY} is a constant that describes the statistical relationship between X and Y. $-1 \le \rho_{XY} \le 1$.

Binomial (Discrete)

Probability of x successes out of n independent trials, each with chance p.

Poisson (Discrete)

Probability of observing x events over an interval t, where events occur at an average rate λ. $\mu = \lambda t$.

Exponential (Continuous)

The time t between events, where the events are independent and occur at an average rate λ.

Uniform (Continuous)

The probability of any value $x \in [a,\,b]$ is constant.

Normal (Continuous)

Events occur more frequently near μ and less frequently further away from μ.

Standardised Normal (Continuous)

$$Z = \frac{X - \mu}{\sigma} \sim \mathrm{N}(0,\,1)$$

	Discrete	Continuous
Valid probabilities	$0 \le p(x) \le 1$	$f(x) \ge 0$
Cumulative probability	$\sum_{u \le x} p(u)$	$\int_{-\infty}^{x} f(u)\,\mathrm{d}u$
Expectation	$\sum_{\Omega} xp(x)$	$\int_{\Omega} xf(x)\,\mathrm{d}x$
Variance	$\sum_{\Omega}(x-\mu)^2\,p(x)$	$\int_{\Omega}(x-\mu)^2\,f(x)\,\mathrm{d}x$

Distribution	Probability	Cumulative Probability	μ	σ^2
Binomial: $X \sim \mathrm{bin}(n,\,p)$	$\binom{n}{x}p^x(1-p)^{n-x}$	*See table above.*	np	$np(1-p)$
Poisson: $X \sim \mathrm{Pois}(\mu)$	$e^{-\mu}\mu^x/x!$	*See table above.*	λt	λt
Exponential: $T \sim \exp\{(\lambda)\}$	$\lambda e^{-\lambda t}$	$1 - \lambda e^{-\lambda t}$	$1/\lambda$	$1/\lambda^2$
Uniform: $X \sim \mathrm{U}(a,\,b)$	$1/(b-a)$	$(x-a)/(b-a)$	$(a+b)/2$	$(b-a)^2/12$
Normal: $X \sim \mathrm{N}(\mu,\,\sigma^2)$	$-$	$-$	μ	σ^2

Sample Statistics

Given n samples x_i:

Mean: $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

Variance: $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2 = \frac{1}{n-1}\left(\sum_{i=1}^{n} x_i^2 - n\overline{x}^2\right)$

t Distribution

The sample mean is distributed according to the t distribution where

$$T = \frac{\overline{X} - \mu}{s/\sqrt{n}} \sim t_{n-1}$$
$$\Pr\left(T \le t_{n-1,\,1-\alpha/2}\right) = 1 - \frac{\alpha}{2}$$
$$\Pr\left(T > t_{n-1,\,\alpha/2}\right) = \frac{\alpha}{2}$$

Confidence Intervals

Given the confidence level c:

$$c = 1 - \alpha.$$

The confidence interval for $\overline{x}$:

$$CI_c = \overline{x} \pm t_{n-1,\,1-\alpha/2}\frac{s}{\sqrt{n}}.$$

Hypothesis Testing

Hypothesis testing assesses the likelihood of observing the sample <u>if</u> the null hypothesis was true.

Hypothesis

$$H_0 : \mu = \mu_0 \quad \text{vs} \quad H_1 : \mu \ne \mu_0$$

Generally there is no evidence supporting H_0, but only evidence or lack of evidence for rejecting H_0.

Test Statistic

The measure of the distance that the proposed mean is from the sample mean:

$$T_{\text{test}} = \frac{\overline{x} - \mu_0}{s/\sqrt{n}}$$

p-Value

Strength of evidence against H_0. The p-value is the α value that satisfies:

$$|T_{\text{test}}| = t_{n-1,\,1-\alpha/2}.$$
$$\Pr(T \le |T_{\text{test}}|) = 1 - \frac{\alpha}{2}$$
$$\Pr(T > |T_{\text{test}}|) = \frac{\alpha}{2}$$

Strength of evidence against H_0

very strong | strong | some | weak | insufficient

0.001 | 0.01 | 0.05 | 0.1

Linear Regression

Given a set of n points $(x_i,\,y_i)$ that are assumed to have a linear relationship, the model for y_i is

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

where $\varepsilon_i \sim \mathrm{N}(0,\,\sigma^2)$ is the residual. s (RMS error) gives an estimate for σ — how close the data is to the model.

Estimates

$$T_{\text{test}} : \frac{\hat{\beta}_0 - \beta_0}{s_{\hat{\beta}_0}} \sim t_{n-2}, \quad \frac{\hat{\beta}_1 - \beta_1}{s_{\hat{\beta}_1}} \sim t_{n-2}$$

Inferences about β_0 and β_1

$$CI_c = \hat{\beta}_0 \pm t_{n-2,\,1-\alpha/2}s_{\hat{\beta}_0}$$
$$CI_c = \hat{\beta}_1 \pm t_{n-2,\,1-\alpha/2}s_{\hat{\beta}_1}$$

where $s_{\hat{\beta}_0}$ and $s_{\hat{\beta}_1}$ are the standard errors (SE) for $\hat{\beta}_0$ and $\hat{\beta}_1$. $H_{0,\,\beta_0} : \beta_0 = 0$ tests whether the model crosses the origin. $H_{0,\,\beta_1} : \beta_1 = 0$ tests whether the model is constant.

R-squared

The percentage of the observed variance in y that is explained by the model.

Residual Plots

Test the following two assumptions:

1. The relationship between X and Y is best modelled linearly — clear indication of a non-linear trend suggests assumption is not valid.

2. The variance of residuals is the same for all observations, (not affected by y_i) — uneven width of residual suggests assumption is not valid. This may lead to inaccurate inferences.