

## MATLAB

```
% Function declaration
function [output1, ...]
    = function_name(input1, ...)
% Iterate over the vector v
for i = v ... end
% Repeat while condition is true
while condition ... end
% Execute first true alternative
if condition_1 ...
elseif condition_2 ...
else ... end
```

## Partial Fraction Decomposition

Given the LHS in the denominator, substitute the RHS.

$$(ax + b)^k \rightarrow \frac{A_1}{ax + b} + \dots + \frac{A_k}{(ax + b)^k}$$

$$(ax^2 + bx + c)^k \rightarrow \frac{A_1x + B_1}{ax^2 + bx + c} + \dots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k}$$

## Differential Equations

### Mechanical Systems

$$\sum F = \frac{dp}{dt}, \quad \sum M = I \frac{d^2\theta}{dt^2}$$

where momentum  $p = mv$ , moment  $M = Fx$  and inertia  $I = mr^2$ .

$$F_T = (c - v) d_f \quad F_g = -mg$$

$$F_s = -kx \quad F_f = -bv^2 \text{ (or } -bv)$$

### Electrical Circuits

$$i = \frac{dq}{dt}, \quad \sum v_{loop} = 0, \quad \sum i_{node} = 0$$

Voltage drop across elements:

$$v_R = iR, \quad v_C = \frac{q}{C}, \quad v_L = L \frac{di}{dt}$$

## Separable ODEs

For  $\frac{dy}{dt} = p(t)q(y)$ :

$$\int \frac{1}{q(y)} \frac{dy}{dt} dt = \int p(t) dt.$$

## Linear ODEs

For  $\frac{dy}{dt} + p(t)y = q(t)$ , use the *integrating factor*:  $I(t) = e^{\int p(t)dt}$ , so that

$$y(t) = \frac{1}{I(t)} \int I(t)q(t) dt.$$

## Linearisation

$$f(t) \approx f(t_0) + f'(t_0)(t - t_0)$$

$$f(y(t)) \approx f(y(t_0)) + f'(y(t_0))(y(t) - y(t_0))$$

## Euler's Method

For  $y' = f(t, y)$ :

$$y_{n+1} = y_n + h f(t_n, y_n)$$

## Modified Euler's Method

$$y_{n+1} =$$

$$y_n + \frac{h}{2} (f(t_n, y_n) + f(t_{n+1}, y_{n+1}))$$

where  $f(t_{n+1}, y_{n+1})$  is determined using Euler's method.

## Second-Order ODEs

$$ay'' + by' + cy = F(t)$$

## General Solution

$$y(t) = y_H(t) + y_P(t)$$

## Homogeneous Solution

$$y_H(t) = e^{\lambda t}$$

To solve for  $\lambda$ , substitute the homogeneous form into the homogeneous

ODE.

Real distinct roots:

$$y_H(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

Real repeated roots:

$$y_H(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$$

Complex conjugate roots:  $\lambda = \alpha \pm \beta i$

$$y_H(t) = e^{\alpha x} (c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

## Particular Solution

See table below. Substitute  $y_P$  into the nonhomogeneous ODE, and solve the undetermined coefficients.

## System of ODEs

Given  $y' = Ay$ ,

$$y_H(t) = q e^{\lambda t}.$$

$\lambda_i$  are the eigenvalues of  $A$  that satisfy

$$\det(A - \lambda I) = 0.$$

$q_i$  are the associated eigenvectors that satisfy

$$(A - \lambda_i I) q_i = 0.$$

For real distinct roots:

$$y_H(t) = c_1 q_1 e^{\lambda_1 t} + c_2 q_2 e^{\lambda_2 t}$$

## Higher-Order ODEs

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = 0$$

Let  $y_1 = y$ ,  $y_2 = y'$ , ...,  $y_n = y^{(n-1)}$  so that  $y = \langle y_1, y_2, \dots, y_n \rangle$ . Then  $y' = Ay$ , where

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix}$$

$F(t)$	$y_P(t)$
a constant	A
a polynomial of degree $n$	$\sum_{i=0}^n A_i t^i$
$e^{kt}$	$A e^{kt}$
$\cos(\omega t)$ or $\sin(\omega t)$	$A_0 \cos(\omega t) + A_1 \sin(\omega t)$
a combination of the above	a combination of the above
linearly dependent to $y_H(t)$	multiply $y_P(t)$ by $t$ until linearly independent

## Probability

### Events

$$\Pr(A) = \sum_{x \in A} p(x)$$

$$\Pr(A^C) = \Pr(\overline{A}) = 1 - \Pr(A)$$

### Unions

$$\begin{aligned} \Pr(A \cup B) &= \Pr(A) + \Pr(B) \\ &\quad - \Pr(A \cap B) \\ &= 1 - \Pr(A^C \cap B^C) \end{aligned}$$

### Bayes' Theorem

$$\Pr(B | A) = \frac{\Pr(A | B) \Pr(B)}{\Pr(A)}$$

### Disjoint Events

Events don't have outcomes in common. For disjoint events  $B_i$ :

$$A \cap B = \emptyset$$

$$\Pr(A \cap B) = 0$$

### Independent Events

Outcome of events do not influence each other. Joint probability:

$$\Pr(A \cap B) = \Pr(A) \Pr(B)$$

### Dependent (Conditional) Events

Outcome of event depends on the outcome of the other. Joint probability of  $A$  given  $B$ :

$$\Pr(A | B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

### Total Probability

$$A = \bigcup_{i=1}^n (A \cap B_i)$$

$$\Pr(A) = \sum_{i=1}^n \Pr(A \cap B_i)$$

### Expectation

Average output or mean:

$$\mu = E(X).$$

### Variance

Measure of spread from the mean:

$$\sigma^2 = E(X^2) - E(X)^2.$$

## Probability Distributions

### Discrete Random Variables

Has countably many outcomes. Distributed with a probability mass function  $p(x)$ .

### Continuous Random Variables

Has an infinite number of outcomes. Distributed with a probability density function  $f(x)$ .

$$\Pr(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} f(x) dx$$

$$F(x) = \Pr(X \leq x)$$

	Discrete	Continuous
Valid probabilities	$0 \leq p(x) \leq 1$	$f(x) \geq 0$
Cumulative probability	$\sum_{u \leq x} p(u)$	$\int_{-\infty}^x f(u) du$
Expectation	$\sum_{\Omega} xp(x)$	$\int_{\Omega} xf(x) dx$
Variance	$\sum_{\Omega} (x - \mu)^2 p(x)$	$\int_{\Omega} (x - \mu)^2 f(x) dx$

## General Linear Combinations

$$E(aX \pm b) = aE(X) \pm b$$

$$E(aX \pm bY) = aE(X) \pm bE(Y)$$

$$\text{Var}(aX \pm b) = a^2 \text{Var}(X)$$

$$\text{Var}(aX \pm bY) = a^2 \text{Var}(X)$$

$$+ b^2 \text{Var}(Y)$$

$$\pm 2ab \text{Cov}(X, Y)$$

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$$

$$= \rho_{XY} \sqrt{\text{Var}(X) \text{Var}(Y)}$$

The correlation  $\text{Corr}(X, Y)$  or  $\rho_{XY}$  is a constant that describes the statistical relationship between  $X$  and  $Y$ .  $-1 \leq \rho_{XY} \leq 1$ .

## Binomial (Discrete)

Probability of  $x$  successes out of  $n$  independent trials, each with chance  $p$ .

### Poisson (Discrete)

Probability of observing  $x$  events over an interval  $t$ , where events occur at an average rate  $\lambda$ .  $\mu = \lambda t$ .

### Exponential (Continuous)

The time  $t$  between events, where the events are independent and occur at an average rate  $\lambda$ .

### Uniform (Continuous)

The probability of any value  $x \in [a, b]$  is constant.

### Normal (Continuous)

Events occur more frequently near  $\mu$  and less frequently further away from  $\mu$ .

### Standardised Normal (Continuous)

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$$

Distribution	Probability	Cumulative Probability	$\mu$	$\sigma^2$
Binomial: $X \sim \text{bin}(n, p)$	$\binom{n}{x} p^x (1-p)^{n-x}$	See table above.	$np$	$np(1-p)$
Poisson: $X \sim \text{Pois}(\mu)$	$e^{-\mu} \mu^x / x!$	See table above.	$\lambda t$	$\lambda t$
Exponential: $T \sim \text{exp}(\lambda)$	$\lambda e^{-\lambda t}$	$1 - \lambda e^{-\lambda t}$	$1/\lambda$	$1/\lambda^2$
Uniform: $X \sim U(a, b)$	$1/(b-a)$	$(x-a)/(b-a)$	$(a+b)/2$	$(b-a)^2/12$
Normal: $X \sim N(\mu, \sigma^2)$	—	—	$\mu$	$\sigma^2$

## Sample Statistics

Given  $n$  samples  $x_i$ :

$$\text{Mean: } \bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

$$\begin{aligned} \text{Variance: } s^2 &= \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2 \\ &= \frac{1}{n-1} \left( \sum_{i=1}^n x_i^2 - n\bar{x}^2 \right) \end{aligned}$$

### t Distribution

The sample mean is distributed according to the  $t$  distribution where

$$T = \frac{\bar{X} - \mu}{s/\sqrt{n}} \sim t_{n-1}$$

$$\Pr(T \leq t_{n-1, 1-\alpha/2}) = 1 - \frac{\alpha}{2}$$

$$\Pr(T > t_{n-1, \alpha/2}) = \frac{\alpha}{2}$$

## Confidence Intervals

Given the confidence level  $c$ :

$$c = 1 - \alpha.$$

The confidence interval for  $\bar{x}$ :

$$CI_c = \bar{x} \pm t_{n-1, 1-\alpha/2} \frac{s}{\sqrt{n}}.$$

## Hypothesis Testing

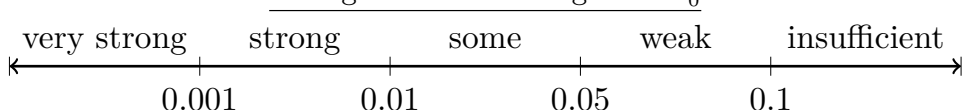
Hypothesis testing assesses the likelihood of observing the sample if the null hypothesis was true.

### Hypothesis

$$H_0 : \mu = \mu_0 \quad \text{vs} \quad H_1 : \mu \neq \mu_0$$

Generally there is no evidence supporting  $H_0$ , but only evidence or lack of evidence

Strength of evidence against  $H_0$



## Linear Regression

Given a set of  $n$  points  $(x_i, y_i)$  that are assumed to have a linear relationship, the model for  $y_i$  is

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

where  $\varepsilon_i \sim N(0, \sigma^2)$  is the residual.  $s$  (RMS error) gives an estimate for  $\sigma$  — how close the data is to the model.

### Estimates

$$T_{\text{test}} : \frac{\hat{\beta}_0 - \beta_0}{s_{\hat{\beta}_0}} \sim t_{n-2}, \quad \frac{\hat{\beta}_1 - \beta_1}{s_{\hat{\beta}_1}} \sim t_{n-2}$$

## Inferences about $\beta_0$ and $\beta_1$

$$CI_c = \hat{\beta}_0 \pm t_{n-2, 1-\alpha/2} s_{\hat{\beta}_0}$$

$$CI_c = \hat{\beta}_1 \pm t_{n-2, 1-\alpha/2} s_{\hat{\beta}_1}$$

where  $s_{\hat{\beta}_0}$  and  $s_{\hat{\beta}_1}$  are the standard errors (SE) for  $\hat{\beta}_0$  and  $\hat{\beta}_1$ .  $H_{0, \beta_0} : \beta_0 = 0$  tests whether the model crosses the origin.  $H_{0, \beta_1} : \beta_1 = 0$  tests whether the model is constant.

### R-squared

The percentage of the observed variance in  $y$  that is explained by the model.

## Residual Plots

Test the following two assumptions:

1. The relationship between  $X$  and  $Y$  is best modelled linearly — clear indication of a non-linear trend suggests assumption is not valid.
2. The variance of residuals is the same for all observations, (not affected by  $y_i$ ) — uneven width of residual suggests assumption is not valid. This may lead to inaccurate inferences.