When solving an initial value problem, always solve the general solution first.

First-Order ODEs

Separable ODEs. $\frac{\mathrm{d}y}{\mathrm{d}t} = f(y)g(t)$

1. Rewrite as: $f\,\mathrm{d}y = g\,\mathrm{d}t$.
2. Integrate both sides: $\int f\,\mathrm{d}y = \int g\,\mathrm{d}t + C$.
3. Rearrange for the explicit form of $y(t)$.

Linear ODEs. $\frac{\mathrm{d}y}{\mathrm{d}t} + P(t)y = Q(t)$

1. Determine the integrating factor: $\mu(t) = \exp\left(\int P\,\mathrm{d}t\right)$.
2. Solve: $y(t) = \frac{1}{\mu}\left(\int Q\mu\,\mathrm{d}t + C\right)$.

Linearisation.

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$
$$f(y(x)) \approx f(y(x_0)) + f'(y(x))(y(x) - y(x_0))$$

Constant Coefficient Linear ODEs

Homogeneous ODEs. $a\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + b\frac{\mathrm{d}y}{\mathrm{d}t} + cy = 0$

1. Substitute $y_h = \mathrm{e}^{rt}$ and solve characteristic equation: $ar^2 + br + c = 0$.
2. Find homogeneous solution:

   Real Distinct Roots (r_1, r_2).
   $y_h(t) = c_1\mathrm{e}^{r_1t} + c_2\mathrm{e}^{r_2t}$.

   Real Repeated Roots (r).
   $y_h(t) = c_1\mathrm{e}^{rt} + c_2t\mathrm{e}^{rt}$

   Complex Conjugate Roots ($r_{1,2} = \alpha \pm \beta i$).
   $y_h(t) = c_1\mathrm{e}^{\alpha t}\cos(\beta t) + c_2\mathrm{e}^{\alpha t}\sin(\beta t)$.

Nonhomogeneous ODEs. $a\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + b\frac{\mathrm{d}y}{\mathrm{d}t} + cy = Q(t)$

1. Determine y_h.
2. If y_p is linearly independent to y_h, multiply y_p by t.
3. Substitute y_p and solve for undetermined coefficients, using the table below.

$Q(t)$	y_p
a constant	A
nth degree polynomial	$\sum_{i=0}^{n} A_it^i$
$\mathrm{e}^{\alpha t}$	$A\mathrm{e}^{\alpha t}$
$\cos(\alpha t)$ or $\sin(\alpha t)$	$A\cos(\alpha t) + B\sin(\alpha t)$
sum/product of above	sum/product of above

If $Q(t)$ contains multiple forms, simplify the coefficients before substituting y_p.

4. Find general solution: $y = y_h + y_p$.

1 Systems of Ordinary Differential Equations

A first-order system of differential equations has the form

$$\begin{cases} x_1' = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ x_2' = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ x_n' = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{cases}$$

where $x_1 = x_1(t)$, $x_2 = x_2(t)$, ..., $x_n = x_n(t)$ are the functions to be determined. In matrix form, the system can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$
$$\mathbf{x}' = \boldsymbol{A}\mathbf{x}$$

Higher-Order ODEs A higher-order linear differential equation can be solved by first converting it to a first-order linear system. Consider the nth-order homogeneous differential equation

$$y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = 0$$

Let

$$\begin{aligned} x_1 &= y \\ x_2 &= y' \\ &\vdots \\ x_n &= y^{(n-1)} \end{aligned}$$

so that $\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^\top$. Then the differential equation can be expressed as the following first-order linear system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Solution Form Like the homogeneous case, we will guess a solution of the form

$$\mathbf{x} = \mathbf{q}e^{\lambda t}$$

which allows for the following substitution

$$\begin{aligned}
\lambda \mathbf{q}e^{\lambda t} &= \boldsymbol{A}\mathbf{q}e^{\lambda t} \\
\boldsymbol{A}\mathbf{q}e^{\lambda t} - \lambda \mathbf{q}e^{\lambda t} &= \mathbf{0} \\
\left(\boldsymbol{A} - \lambda \boldsymbol{I}\right) \mathbf{q}e^{\lambda t} &= \mathbf{0} \\
\left(\boldsymbol{A} - \lambda \boldsymbol{I}\right) \mathbf{q} &= \mathbf{0}
\end{aligned}$$

This equation has the trivial solution $\mathbf{q} = \mathbf{0}$, however for a fundamental set of solutions, we must let $\boldsymbol{A} - \lambda \boldsymbol{I}$ be singular.

1.0.1 Characteristic Equation

To determine the eigenvalues λ of the matrix $\mathbf{A}$, we must solve the characteristic equation associated with the system of ODEs. Namely,

$$\det\left(\boldsymbol{A} - \lambda \boldsymbol{I}\right) = 0$$

These eigenvalues can then be used to solve the eigenvectors of $\boldsymbol{A}$ **Solving a System of ODEs**

1. Model the system of ODEs in the form $\mathbf{x}' = \boldsymbol{A}\mathbf{x}$
2. Solve the characteristic equation for the eigenvalues of $\boldsymbol{A}$
3. Solve the corresponding eigenvectors of $\boldsymbol{A}$ by solving $\left(\boldsymbol{A} - \lambda \boldsymbol{I}\right) \mathbf{q} = \mathbf{0}$
4. Write the general solution: $\mathbf{x} = c_1 \mathbf{q}_1 e^{\lambda_1 t} + c_2 \mathbf{q}_2 e^{\lambda_2 t}$
5. Apply initial conditions to solve c_1 and c_2