Advanced Calculus

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Contents

1 Euclidean Space

The Euclidean space $\mathbb{R}^n$ is an n-dimensional vector space of real numbers. This space is closed under addition and scalar multiplication.

1.1 Operations

1.1.1 Addition

The sum of two vectors $\mathbf{x}$ and $\mathbf{y}$ is defined element-wise

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

In a coordinate system, the vectors $\mathbf{x}$ and $\mathbf{y}$ are added tip-to-tail.

1.1.2 Scalar Multiplication

The scalar multiplication of a vector $\mathbf{x}$ by a scalar $\lambda \in \mathbb{R}$ is defined element-wise

$$\lambda\mathbf{x} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \end{bmatrix}$$

In a coordinate system, λ scales the vector $\mathbf{x}$ along the same line.

1.1.3 Norm

The norm (length) of a vector $\mathbf{x}$ is defined as

$$\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

The norm of a vector $\mathbf{x}$ is the distance from the origin to the tip of the vector. This allows us to define the unit vector $\hat{\mathbf{x}}$ as

$$\hat{\mathbf{x}} = \frac{\mathbf{x}}{\|\mathbf{x}\|}$$

which is a vector of length 1 in the same direction as $\mathbf{x}$.

1.1.4 Scalar Product

The scalar product (dot product) of two vectors $\mathbf{x}$ and $\mathbf{y}$ is defined as

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i$$

The scalar product allows us to define the angle θ between two vectors $\mathbf{x}$ and $\mathbf{y}$ as

$$\cos(\theta) = \hat{\mathbf{x}} \cdot \hat{\mathbf{y}}$$

where we use the unit vectors of $\mathbf{x}$ and $\mathbf{y}$, as the angle between two vectors is invariant under scaling. Additionally, we can determine the projection of the vector $\mathbf{x}$ onto the vector $\mathbf{y}$ using trigonometry

$$\mathrm{proj}_{\mathbf{y}}(\mathbf{x}) = (\|\mathbf{x}\| \cos(\theta))\, \hat{\mathbf{y}} = (\|\mathbf{x}\| (\hat{\mathbf{x}} \cdot \hat{\mathbf{y}}))\, \hat{\mathbf{y}} = (\mathbf{x} \cdot \hat{\mathbf{y}})\, \hat{\mathbf{y}}$$

where $\mathbf{x} \cdot \hat{\mathbf{y}}$ is the norm of the projection vector.

1.2 Additional Properties

1.2.1 Triangle Inequality

$$\|\mathbf{x} + \mathbf{y}\| \leqslant \|\mathbf{x}\| + \|\mathbf{y}\|$$

1.2.2 Inverse Triangle Inequality

$$\|\mathbf{x} - \mathbf{y}\| \geqslant |\|\mathbf{x}\| - \|\mathbf{y}\||$$

1.2.3 Cauchy-Schwarz Inequality

$$|\mathbf{x} \cdot \mathbf{y}| \leqslant \|\mathbf{x}\|\|\mathbf{y}\|$$

1.3 Multivariable Functions

A multivariable function f maps a vector $\mathbf{x} \in \mathbb{R}^n$ to a real number $f(\mathbf{x}) \in \mathbb{R}$. This function can be expressed in **explicit form** as

$$z = f(x,\, y)$$

or in **implicit form** as

$$F(x,\, y,\, z) = z - f(x,\, y)$$

These equations define a surface in $\mathbb{R}^3$.

1.3.1 Level Curves

The level curves of a function $f(x,\, y)$ are the curves in $\mathbb{R}^2$ where

$$f(x,\, y) = c$$

where c is the height of the curve. Implicitly, this is equivalent to

$$F(x,\, y,\, z) = 0.$$

Level curves represent paths of equal height on the surface defined by $z = f(x,\, y)$.

1.4 Special Regions

1.4.1 Balls

In an Euclidean space, an open ball of radius $r > 0$ centred at a point $\mathbf{p} \in \mathbb{R}^n$ is denoted $B_r(\mathbf{p})$, and is defined as

$$B_r(\mathbf{p}) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{p}\| < r\}.$$

This region includes all points less than a distance r from the vector $\mathbf{p}$, where the distance is typically defined by the L_2-norm:

$$\|\mathbf{x} - \mathbf{p}\|_2 = \left(\sum_{i=1}^{n} (x_i - p_i)^2\right)^{1/2}.$$

1.5 Mathematical Representation of Curves

1.5.1 Explicit Form

A curve in $\mathbb{R}^2$ can be represented in explicit form as

$$y = f(x)$$

but this is not possible in $\mathbb{R}^3$ as a 3D curve requires two equations. For a 2D explicit curve:

- x is an independent variable such that we have 1 degree of freedom.

1.5.2 Implicit Form

A curve in $\mathbb{R}^2$ can be represented in implicit form as

$$F(x,\ y) = 0.$$

In 3D, we must impose an additional equation that intersects a surface.

$$\begin{cases} F(x,\ y,\ z) = 0 \\ G(x,\ y,\ z) = 0 \end{cases}$$

In both cases, we have 1 degree of freedom as the degrees of freedom is the difference between the number of variables and the number of equations.

1.5.3 Parametric Form

In parametric form, curves are parametrised in terms of a parameter t. In 2D, this is represented as

$$\mathbf{r}(t) = \langle x(t),\ y(t) \rangle$$

and similarly in 3D,

$$\mathbf{r}(t) = \langle x(t),\ y(t),\ z(t) \rangle$$

1.6 Converting Between Representations

1.6.1 Explicit to Implicit

The equation $y = f(x)$ can always be converted to implicit form by rewriting it as

$$F(x,\ y) = y - f(x) = 0.$$

1.6.2 Implicit to Explicit

The equation $F(x,\ y) = 0$ can be converted to explicit form if we can solve for y (or x):

1.6.3 Parametric to Explicit/Implicit

The equation $\mathbf{r}(t) = \langle x(t),\ y(t)\rangle$ can be written in explicit or implicit form, if the parameter t can be eliminated from the simultaneous equations.

1.6.4 Explicit to Parametric

The equation $y = f(x)$ can always be converted to parametric form by choosing the parameter $t = x$, so that

$$\mathbf{r}(t) = \langle t,\ f(t)\rangle.$$

1.6.5 Implicit to Parametric

The equation $F(x,\ y) = 0$ can be converted to parametric form if we can find $x = p(t)$ and $y = q(t)$, such that $F(p(t),\ q(t)) = 0$, and

$$\mathbf{r}(t) = \langle p(t),\ q(t)\rangle$$

for all t.

1.7 Paramaterisation

To parametrise a curve, consider the following strategies:

- For a closed curve, consider the polar parametrisation in terms of the angle θ:

$$\mathbf{r}(\theta) = \langle R(\theta)\cos(\theta),\ R(\theta)\sin(\theta)\rangle.$$

- For a curve that is the intersection of two surfaces, consider one of the following mappings:

$$x \mapsto \begin{bmatrix} x \\ y(x) \\ z(x) \end{bmatrix} \qquad y \mapsto \begin{bmatrix} x(y) \\ y \\ z(y) \end{bmatrix} \qquad z \mapsto \begin{bmatrix} x(z) \\ y(z) \\ z \end{bmatrix}$$

- Otherwise, consider a vector construction.

1.7.1 Line Segments

To parametrise a line segment from point A to B, define the parameter $t \in [0, 1]$. Then, consider the vectors $\mathbf{a} = \overline{OA}$ and $\mathbf{b} = \overline{OB}$. By scaling the vector from A to B by t, we can parametrise the line segment as

$$\mathbf{r}(t) = \mathbf{a} + t(\mathbf{b} - \mathbf{a}) = \mathbf{a}(1 - t) + \mathbf{b}t.$$

1.7.2 Circles

To parametrise a circle of radius R centred at the $\langle x_0, y_0 \rangle$, first parametrise the curve in terms of the angle θ, then shift the curve by $\langle x_0, y_0 \rangle$.

$$\mathbf{r}(\theta) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} R\cos(\theta) \\ R\sin(\theta) \end{bmatrix} = \begin{bmatrix} x_0 + R\cos(\theta) \\ y_0 + R\sin(\theta) \end{bmatrix}$$

1.7.3 Velocity Vectors

The velocity vector of a parametrised curve $\mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ is defined as

$$\mathbf{v}(t) = \mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$$

where $\mathbf{v}(t)$ is a tangent vector to the curve at the point $\mathbf{r}(t)$, for all t.

1.7.4 Tangent Vectors

Following from the definition of the velocity vector, the tangent vectors of a parametrised curve are unit vectors in the direction of the velocity vector.

$$\hat{\boldsymbol{\tau}}(t) = \pm \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \pm \mathbf{v}\hat{(t)}$$

For a curve given in explicit form $y = f(x)$, the tangent vectors are given by

$$\hat{\boldsymbol{\tau}}(x) = \pm \frac{1}{\sqrt{1 + (f'(x))^2}} \begin{bmatrix} 1 \\ f'(x) \end{bmatrix}$$