Advanced Calculus

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Contents

1 Euclidean Space

The Euclidean space $\mathbb{R}^n$ is an n-dimensional vector space of real numbers. This space is closed under addition and scalar multiplication.

1.1 Operations

1.1.1 Addition

The sum of two vectors $\mathbf{x}$ and $\mathbf{y}$ is defined element-wise

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

In a coordinate system, the vectors $\mathbf{x}$ and $\mathbf{y}$ are added tip-to-tail.

1.1.2 Scalar Multiplication

The scalar multiplication of a vector $\mathbf{x}$ by a scalar $\lambda \in \mathbb{R}$ is defined element-wise

$$\lambda\mathbf{x} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \end{bmatrix}$$

In a coordinate system, λ scales the vector $\mathbf{x}$ along the same line.

1.1.3 Norm

The norm (length) of a vector $\mathbf{x}$ is defined as

$$\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

The norm of a vector $\mathbf{x}$ is the distance from the origin to the tip of the vector. This allows us to define the unit vector $\hat{\mathbf{x}}$ as

$$\hat{\mathbf{x}} = \frac{\mathbf{x}}{\|\mathbf{x}\|}$$

which is a vector of length 1 in the same direction as $\mathbf{x}$.

1.1.4 Scalar Product

The scalar product (dot product) of two vectors $\mathbf{x}$ and $\mathbf{y}$ is defined as

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i$$

The scalar product allows us to define the angle θ between two vectors $\mathbf{x}$ and $\mathbf{y}$ as

$$\cos(\theta) = \hat{\mathbf{x}} \cdot \hat{\mathbf{y}}$$

where we use the unit vectors of $\mathbf{x}$ and $\mathbf{y}$, as the angle between two vectors is invariant under scaling. Additionally, we can determine the projection of the vector $\mathbf{x}$ onto the vector $\mathbf{y}$ using trigonometry

$$\text{proj}_{\mathbf{y}}(\mathbf{x}) = (\|\mathbf{x}\| \cos(\theta))\,\hat{\mathbf{y}} = (\|\mathbf{x}\|\,(\hat{\mathbf{x}} \cdot \hat{\mathbf{y}}))\,\hat{\mathbf{y}} = (\mathbf{x} \cdot \hat{\mathbf{y}})\,\hat{\mathbf{y}}$$

where $\mathbf{x} \cdot \hat{\mathbf{y}}$ is the norm of the projection vector.

1.2 Normed Vector Space Axioms

1.2.1 Triangle Inequality

$$\|\mathbf{x} + \mathbf{y}\| \leqslant \|\mathbf{x}\| + \|\mathbf{y}\|$$

1.2.2 Inverse Triangle Inequality

$$\|\mathbf{x} - \mathbf{y}\| \geqslant |\|\mathbf{x}\| - \|\mathbf{y}\||$$

1.2.3 Cauchy-Schwarz Inequality

$$|\mathbf{x} \cdot \mathbf{y}| \leqslant \|\mathbf{x}\|\|\mathbf{y}\|$$

1.3 Topological Properties

Definition 1.1. An open ball of radius $r > 0$ centred at a point $\mathbf{p} \in \mathbb{R}^n$ is denoted $B_r(\mathbf{p})$, and is defined as

$$B_r(\mathbf{p}) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{p}\| < r\}.$$

This region includes all points less than a distance r from the vector $\mathbf{p}$, where the distance is typically defined by the L_2-norm:

$$\|\mathbf{x} - \mathbf{p}\|_2 = \left(\sum_{i=1}^{n} (x_i - p_i)^2\right)^{1/2}.$$

Definition 1.2 (Open Set)**.** A set $S \subset \mathbb{R}^n$ is open if for every point $\mathbf{x} \in S$, there exists $\delta > 0$, such that the open ball $B_\delta(\mathbf{x}) \in S$.

Definition 1.3 (Closed Set)**.** A set $S \subset \mathbb{R}^n$ is closed if its complement $\mathbb{R}^n \setminus S$ is open.

Definition 1.4 (Connected Set)**.** A set $S \subset \mathbb{R}^n$ is connected if it cannot be represented as the union of two or more disjoint non-empty open subsets.

Definition 1.5 (Path Connected Set)**.** A set $S \subset \mathbb{R}^n$ is path connected if for every pair of points $\mathbf{x}, \mathbf{y} \in S$, there exists a continuous path $\mathbf{r}(t)$ from $\mathbf{x}$ to $\mathbf{y}$.

Definition 1.6 (Simply Connected Set)**.** A set $S \subset \mathbb{R}^n$ is simply connected if it is path-connected and if every closed path in S can be continuously contracted to a point in S.

1.4 Parametrisations of Curves

Definition 1.7 (Path)**.** A path is a continuous function

$$\mathbf{r} : [a, b] \subset \mathbb{R} \to \mathscr{C} \subset \mathbb{R}^n$$

where $t \mapsto \mathbf{r}(\mathbf{x}(t))$ is the parameter of the path.

Definition 1.8 (Curve)**.** A curve $\mathscr{C}$ is the set of points in $\mathbb{R}^n$ corresponding to the range of a path $\mathbf{r}$.

$$\mathscr{C} = \{\mathbf{r}(t) : t \in [a, b)\}$$

A curve $\mathscr{C}$ is **parametrised** by $\mathbf{r}(t)$, and a path $\mathbf{r}(t)$ is a **parametrisation** of $\mathscr{C}$.

Definition 1.9 (Closed Path)**.** A path is closed if it starts and ends at the same point:

$$\mathbf{r}(a) = \mathbf{r}(b).$$

Definition 1.10 (Simple Path)**.** A path is simple if the map $t \mapsto \mathbf{r}(t)$ is injective. That is, the path does not intersect itself.

Definition 1.11 (Regular Path)**.** A path is regular (or smooth) if has nonzero continuous first derivatives:

$$\mathbf{r}(t) \in C^1 \quad \text{and} \quad \mathbf{r}'(t) \neq \mathbf{0}$$

for all t. This restriction ensures that the path does not have any cusps and allows a unit tangent vector to be defined.

Definition 1.12 (Piecewise Regular Path)**.** A path is piecewise regular if it can be divided into a finite number of regular paths.

Definition 1.13 (Path Concatenation)**.** The path concatenation of two paths $\mathbf{r}_1 : [a, b] \subset \mathbb{R} \to \mathbb{R}^n$ and $\mathbf{r}_2 : [b, c] \subset \mathbb{R} \to \mathbb{R}^n$ is defined as

$$\mathbf{r}(t) = (\mathbf{r}_1 \vee \mathbf{r}_2)(t) = \begin{cases} \mathbf{r}_1(t) & t \in [a, b] \\ \mathbf{r}_2(t) & t \in [b, c] \end{cases}$$

where $a < b < c$.

Definition 1.14 (Travelling Direction)**.** The travelling direction of a path $\mathbf{r}(t)$ is the direction of increasing t. A regular curve can be oriented by choosing one of the two travelling directions.

1.4.1 Remarks

- A curve is closed/simple/(piecewise) regular if it has a (closed/simple)/(piecewise) regular parametrisation.
- The implicit/explicit Cartesian representation of a curve is a curve, as it describes a set of points.
- The parametric representation of a curve is a path, as it includes the timing of the points.

- Converting from a curve to a path introduces a parameter.
- Converting from a path to a curve eliminates a parameter.
- $\mathbf{r}'(t)$ is the velocity vector of a path.
- $\|\mathbf{r}'(t)\|$ is the speed of a path.
- $\mathbf{r}''(t)$ is the acceleration vector of a path.
- Parametrisations are not unique.

1.4.2 Reparametrisation

Let $\mathscr{C}$ be a curve parametrised by $\mathbf{r}(t)$ with $t \in [a,b]$. If there exists a bijective map $t = \theta(u)$, defined by $\theta : [c,d] \to [a,b]$ such that

$$\mathbf{r}(\theta(u)) = \tilde{\mathbf{r}}(u),$$

where

- $\theta(u)$ is continuously differentiable, and
- $\theta'(u) \neq 0$ for all $u \in [c,d]$,

then $\tilde{\mathbf{r}}(u)$ is a reparametrisation of $\mathbf{r}(t)$ for $u \in [c,d]$, and a parametrisation of $\mathscr{C}$. The map $\theta(u)$ guarantees that the simple/regular properties of $\mathbf{r}(t)$ are preserved in $\tilde{\mathbf{r}}(u)$.

- If $\theta'(u) > 0$, then $\mathbf{r}(t)$ and $\tilde{\mathbf{r}}(u)$ are **equivalent** parametrisations.
- If $\theta'(u) < 0$, then $\mathbf{r}(t)$ and $\tilde{\mathbf{r}}(u)$ are **opposite** parametrisations.

The unit tangent vectors of $\mathbf{r}(t)$ and $\tilde{\mathbf{r}}(u)$ are related by the chain rule:

$$\tilde{\mathbf{r}}'(u) = \mathbf{r}'(\theta(u))\,\theta'(u) \implies \|\tilde{\mathbf{r}}'(u)\| = \|\mathbf{r}'(\theta(u))\|\,|\theta'(u)|$$

so that by dividing the first result by the second, we obtain

$$\begin{aligned}
\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} &= \frac{\tilde{\mathbf{r}}'(u)}{\|\tilde{\mathbf{r}}'(u)\|}\frac{\theta'(u)}{|\theta'(u)|} \\
&= \operatorname{sgn}(\theta'(u))\frac{\tilde{\mathbf{r}}'(u)}{\|\tilde{\mathbf{r}}'(u)\|} \\
&= \begin{cases} \dfrac{\tilde{\mathbf{r}}'(u)}{\|\tilde{\mathbf{r}}'(u)\|} & \text{if } \theta'(u) > 0 \\ -\dfrac{\tilde{\mathbf{r}}'(u)}{\|\tilde{\mathbf{r}}'(u)\|} & \text{if } \theta'(u) < 0 \end{cases}
\end{aligned}$$

1.5 Common Parametrisations of Curves

To parametrise a curve, consider the following strategies:

- For a closed curve, consider the polar parametrisation in terms of the angle θ:

$$\mathbf{r}(\theta) = \langle R(\theta)\cos(\theta),\ R(\theta)\sin(\theta)\rangle.$$

- For a curve that is the intersection of two surfaces, consider one of the following mappings:

$$x \mapsto \begin{bmatrix} x \\ y(x) \\ z(x) \end{bmatrix} \qquad y \mapsto \begin{bmatrix} x(y) \\ y \\ z(y) \end{bmatrix} \qquad z \mapsto \begin{bmatrix} x(z) \\ y(z) \\ z \end{bmatrix}$$

- Otherwise, consider a vector construction.

1.5.1 Line Segments

To parametrise a line segment from point A to B, define the parameter $t \in [0,1]$. Then, consider the vectors $\mathbf{a} = \overline{OA}$ and $\mathbf{b} = \overline{OB}$. By scaling the vector from A to B by t, we can parametrise the line segment as

$$\mathbf{r}(t) = \mathbf{a} + t(\mathbf{b} - \mathbf{a}) = \mathbf{a}(1-t) + \mathbf{b}t.$$

1.5.2 Circles

To parametrise a circle of radius R centred at the $\langle x_0,\ y_0\rangle$, first parametrise the curve in terms of the angle θ, then shift the curve by $\langle x_0,\ y_0\rangle$.

$$\mathbf{r}(\theta) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} R\cos(\theta) \\ R\sin(\theta) \end{bmatrix} = \begin{bmatrix} x_0 + R\cos(\theta) \\ y_0 + R\sin(\theta) \end{bmatrix}$$

1.5.3 Velocity Vectors

The velocity vector of a parametrised curve $\mathbf{r}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ is defined as

$$\mathbf{v}(t) = \mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}$$

where $\mathbf{v}(t)$ is a tangent vector to the curve at the point $\mathbf{r}(t)$, for all t.

1.5.4 Tangent Vectors

Following from the definition of the velocity vector, the tangent vectors of a parametrised curve are unit vectors in the direction of the velocity vector.

$$\hat{\boldsymbol{\tau}}(t) = \pm\frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \pm\hat{\mathbf{v}}(t)$$

For a curve given in explicit form $y = f(x)$, the tangent vectors are given by

$$\hat{\boldsymbol{\tau}}(x) = \pm\frac{1}{\sqrt{1+(f'(x))^2}}\begin{bmatrix}1\\ f'(x)\end{bmatrix}$$

1.6 Parametrisations of Surfaces

Definition 1.15 (Homeomorphism)**.** A homeomorphism is a bijective map between two topological spaces T and S that preserves the continuity of both the function and its inverse.

Definition 1.16 (Parametric Surface)**.** A parametric surface $S \subset \mathbb{R}^3$ is the set of points corresponding to the range of the parametric function

$$\mathbf{r} : T \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$$

where T is open and connected

$$S = \{\mathbf{r}(s,\, t) : \langle s,\, t\rangle \in T\}.$$

Here $\mathbf{r}$ is a homeomorphism between T and S of class C^1, and the Jacobian matrix of $\mathbf{r}$ has rank 2 for all $\langle s,\, t\rangle \in T$. In other words, the vectors $\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}$ and $\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}$ are linearly independent so that

$$\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}.$$

This condition is necessary to ensure that the tangent plane and to the surface is well-defined, and allows for the definition of the unit normal vector

$$\hat{\mathbf{n}} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}.$$

Definition 1.17 (Chart)**.** The inverse of a parametric surface $\mathbf{r}$ is called a chart:

$$\mathbf{r}^{-1} : S \subset \mathbb{R}^3 \to T \subset \mathbb{R}^2$$

Definition 1.18 (Surface)**.** $\mathcal{S}$ is a surface if it is the union of parametric surfaces S_i such that the intersection of any two surfaces $S_i \cap S_j$ is empty, a point, or a curve.

Definition 1.19 (Orientable Surface)**.** A surface $\mathcal{S}$ is orientable if there exists a continuous unit normal vector $\hat{\mathbf{n}}$ defined everywhere on $\mathcal{S}$.

1.7 Common Parametrisations of Surfaces

To parametrise an implicit surface $z = f(x,\, y)$, consider the parametrisation

$$\mathbf{r}(x,\, y) = \langle x,\, y,\, f(x,\, y)\rangle.$$

1.7.1 Planes

To parametrise a plane that passes through the points A, B, and C, define a vector from the origin to C, and two vectors from C to A and B:

$$\mathbf{c} = \overline{OC},\ \mathbf{a} = \overline{CA},\ \mathbf{b} = \overline{CB}.$$

Then, consider the parametrisation

$$\mathbf{r}\,(s,\ t) = \mathbf{c} + s\mathbf{a} + t\mathbf{b}.$$

The unit normal vector of the plane is given by

$$\hat{\mathbf{n}} = \pm\frac{\mathbf{r}_s \times \mathbf{r}_t}{\|\mathbf{r}_s \times \mathbf{r}_t\|} = \pm\frac{\mathbf{a} \times \mathbf{b}}{\|\mathbf{a} \times \mathbf{b}\|}.$$

1.7.2 Cylindrical Shells

A cylindrical shell of radius R is parametrised by

$$\mathbf{r}\,(\theta,\ z) = \begin{bmatrix} R\cos\theta \\ R\sin\theta \\ z \end{bmatrix}$$

for $\theta \in [0, 2\pi)$ and $z \in \mathbb{R}$. The unit normal vector is given by

$$\hat{\mathbf{n}} = \pm\frac{\mathbf{r}_\theta \times \mathbf{r}_z}{\|\mathbf{r}_\theta \times \mathbf{r}_z\|} = \pm\begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$$

1.7.3 Spherical Shells

A spherical shell of radius R is parametrised by

$$\mathbf{r}\,(\phi,\ \theta) = \begin{bmatrix} R\sin\phi\cos\theta \\ R\sin\phi\sin\theta \\ R\cos\phi \end{bmatrix}$$

for $\phi \in (0, \pi)$ and $\theta \in (0, 2\pi)$. The unit normal vector is given by

$$\hat{\mathbf{n}} = \pm\frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{\|\mathbf{r}_\phi \times \mathbf{r}_\theta\|} = \pm\begin{bmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{bmatrix}$$

1.7.4 Conical Shells

A conical shell with opening angle α is parametrised by

$$\mathbf{r}\,(r,\ \theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ r\cot\alpha \end{bmatrix}$$

for $r \in (0, \infty)$ and $\theta \in [0, 2\pi)$. The unit normal vector is given by

$$\hat{\mathbf{n}} = \pm \frac{\mathbf{r}_r \times \mathbf{r}_\theta}{\|\mathbf{r}_r \times \mathbf{r}_\theta\|} = \pm \begin{bmatrix} -\cos\alpha\cos\theta \\ -\cos\alpha\sin\theta \\ \sin\alpha \end{bmatrix}$$

1.7.5 Explicit Surfaces

A general surface $z = f(x,\ y)$ can be parametrised by

$$\mathbf{r}(x,\ y) = \begin{bmatrix} x \\ y \\ f(x,\ y) \end{bmatrix}$$

The unit normal vector is given by

$$\hat{\mathbf{n}} = \pm \frac{\mathbf{r}_x \times \mathbf{r}_y}{\|\mathbf{r}_x \times \mathbf{r}_y\|} = \pm \frac{1}{\sqrt{f_x^2 + f_y^2 + 1}} \begin{bmatrix} -f_x \\ -f_y \\ 1 \end{bmatrix}$$

1.8 Mathematical Representation of Curves and Surfaces

Definition 1.20 (Degree of Freedom)**.** The degree of freedom of a curve is the difference between the number of variables and the number of equations.

1.8.1 Curves in Explicit Form (2D)

A curve $\mathscr{C} \in \mathbb{R}^2$ can be represented explicitly as

$$y = f(x)$$

When $n > 2$, it is not possible to describe a curve explicitly. The degree of freedom for a 2D explicit curve is 1.

1.8.2 Curves in Implicit Form

A curve $\mathscr{C} \in \mathbb{R}^n$ can be represented implicitly as the intersection of $n - 1$ surfaces.

$$\begin{cases} F_1(\mathbf{x},\ z) = 0 \\ F_2(\mathbf{x},\ z) = 0 \\ \vdots \\ F_{n-1}(\mathbf{x},\ z) = 0 \end{cases}$$

The degree of freedom is 1.

1.8.3 Curves in Parametric Form

A curve $\mathscr{C} \in \mathbb{R}^n$ can be represented parametrically as

$$\mathbf{r}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

where t is the parameter. The degree of freedom is $n-1$.

1.8.4 Surfaces in Explicit Form

A surface $\mathscr{S} \in \mathbb{R}^3$ can be represented explicitly as

$$z = f(x,\, y)$$

The degree of freedom is 2.

1.8.5 Surfaces in Implicit Form

A surface $\mathscr{S} \in \mathbb{R}^3$ can be represented implicitly as

$$F(x,\, y,\, z) = 0$$

The degree of freedom is 2.

1.8.6 Surfaces in Parametric Form

A surface $\mathscr{S} \in \mathbb{R}^3$ can be represented parametrically as

$$\mathbf{r}(s,\, t) = \begin{bmatrix} x(s,\, t) \\ y(s,\, t) \\ z(s,\, t) \end{bmatrix}$$

where s and t are the parameters. The degree of freedom is 1.

1.9 Converting Between Representations

1.9.1 Explicit to Implicit

The equation $z = f(\mathbf{x})$ can always be converted to implicit form by rewriting it as

$$F(\mathbf{x},\, z) = z - f(\mathbf{x}) = 0.$$

1.9.2 Implicit to Explicit

The equation $F(\mathbf{x},\, z) = 0$ can be converted to explicit form if we can solve for z in terms of $\mathbf{x}$.

1.9.3 Parametric to Explicit/Implicit

The equation $\mathbf{r}(\mathbf{t}) = \langle x_1(\mathbf{t}), \ldots, x_n(\mathbf{t})\rangle$ can be written in explicit or implicit form, if the parameters $\mathbf{t}$ can be eliminated from the simultaneous equations.

1.9.4 Explicit to Parametric

The equation $z = f(\mathbf{x})$ can always be converted to parametric form by choosing the parameter $\mathbf{t} = \mathbf{x}$, so that

$$\mathbf{r}(\mathbf{t}) = \langle \mathbf{t},\ f(\mathbf{t})\rangle.$$

1.9.5 Implicit to Parametric

The equation $F(\mathbf{x}) = 0$ can be converted to parametric form if we can find $\mathbf{x} = \mathbf{r}(\mathbf{t})$ such that $F(\mathbf{r}(\mathbf{t})) = 0$, and

$$\mathbf{r}(\mathbf{t}) = \langle x_1(\mathbf{t}), \ldots, x_n(\mathbf{t})\rangle.$$

for all t.

2 Multivariable Calculus

2.1 Multivariable Functions

Definition 2.1 (Multivariable Function)**.** A multivariable function f maps several independent variables to a real number:

$$f : E \subset \mathbb{R}^n \to \mathbb{R}$$

where

$$\langle x_1, x_2, \ldots, x_n\rangle \mapsto z = f(x_1, x_2, \ldots, x_n)$$
$$\mathbf{x} \mapsto z = f(\mathbf{x})$$

Definition 2.2 (Domain)**.** The domain of f is the subset of $\mathbb{R}^n$ for which f is defined. It corresponds to the set of all possible inputs to f.

$$\mathscr{D}(f) = E$$

Definition 2.3 (Range)**.** The range of f is the image of the domain E under f:

$$\mathscr{R}(f) = \{f(\mathbf{x}) \in \mathbb{R} : \mathbf{x} \in E\}$$

Definition 2.4 (Graph)**.** The graph of f is defined as the set

$$G = \{\langle \mathbf{x},\ f(\mathbf{x})\rangle : \mathbf{x} \in E\} \subset \mathbb{R}^{n+1}$$

2.2 Curves of Intersection

Definition 2.5 (Curves of Intersection)**.** The curves of intersection of f with the plane $x_i = c$ are defined as

$$\{\langle x_1,\, x_2,\, \ldots,\, x_{i-1},\, c,\, x_{i+1},\, \ldots,\, x_n\rangle : \langle x_1,\, x_2,\, \ldots,\, x_n\rangle \in E\}$$

In 3D, curves of intersection of $z = f(x,\, y)$ with the planes perpendicular to the x, y, or z-axes allow us to represent the function in 3D.

$$\begin{array}{ccc} \perp x\,(x = c) & \perp y\,(y = c) & \perp z\,(z = c) \\ z = f(c,\, y) = g(y) & z = f(x,\, c) = g(x) & c = f(x,\, y) \implies y = g(x) \end{array}$$

Definition 2.6 (Level Set)**.** The level sets of f are the set of all points in the domain of f that map to a given value c.

$$\{\mathbf{x} \in E : f(\mathbf{x}) = c\}$$

In 2D, level sets are called **level curves**.

Definition 2.7 (Contour Map)**.** The projection of all level curves onto the xy-plane is called the **contour map** of f. The lines of a contour map are called **contours**.

Theorem 2.2.1 (Level Sets and Gradients)**.** *The level set of a function f is perpendicular to the gradient of f.*

2.3 Derivatives

Definition 2.8 (Continuity)**.** A function f is continuous at a point $\mathbf{x}_0$ if

$$\forall \varepsilon > 0,\, \exists \delta > 0 : \|\mathbf{x} - \mathbf{x}_0\| < \delta \implies \|f(\mathbf{x}) - f(\mathbf{x}_0)\| < \varepsilon$$

Equivalently, f is continuous at $\mathbf{x}_0$ when

$$\lim_{n\to\infty} f(\mathbf{x}_n) = f(\mathbf{x}_0)$$

for all sequences $\{\mathbf{x}_n\}$ where $\mathbf{x}_n \to \mathbf{x}_0$.
A function f is continuous on a set E if it is continuous at every point in E.

Definition 2.9 (Partial Derivatives)**.** Partial derivatives represent the rate of change of a function with respect to one of its variables, holding all other variables constant.

$$\frac{\partial f}{\partial x_i} = f_{x_i} = \lim_{h\to 0} \frac{f(x_1,\, x_2,\, \ldots,\, x_i + h,\, \ldots,\, x_n) - f(x_1,\, x_2,\, \ldots,\, x_i,\, \ldots,\, x_n)}{h}$$

Definition 2.10 (Higher-order Partial Derivatives)**.** Higher-order partial derivatives are defined as

$$\frac{\partial^n f}{\partial x_i^n} = \frac{\partial}{\partial x_i}\left(\frac{\partial^{n-1} f}{\partial x_i^{n-1}}\right)$$

Definition 2.11 (Mixed Partial Derivatives)**.** Mixed partial derivatives are given the following notation

$$\frac{\partial^2 f}{\partial x_i\, \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) = f_{x_i x_j}$$

Theorem 2.3.1 (Schwarz's Theorem (or Clairaut's Theorem))**.** *For a function $f : E \subset \mathbb{R}^2 \to \mathbb{R}$ with $\mathbf{x}_a \in E$, if f_{xy} and f_{yx} are continuous on $\mathbf{x}_0$, then*

$$f_{xy} = f_{yx}$$

Remark 1. For an arbitrary n and $k \leqslant n$, if all partials of order $\leqslant k$ are continuous in the neighbourhood of $\mathbf{x}_0$, then mixed partials of order k are equal for any permutation of indices:

$$\frac{\partial^k f}{\partial x_{i_1} ... \partial x_{i_k}} = \frac{\partial^k f}{\partial x_{j_1} ... \partial x_{j_k}}$$

2.4 Chain Rule

For a function with multiple arguments, each argument of f that has an implicit dependence on the variable of differentiation must be differentiated using the chain rule, where all contributions are summed.

$$\frac{\mathrm{d}}{\mathrm{d}t} f\left(x\left(t\right),\, y\left(t\right)\right) = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$
$$\frac{\partial}{\partial u} f\left(x\left(u,\, v\right),\, y\left(u,\, v\right)\right) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

Definition 2.12 (Total Derivative)**.** The total derivative of a function $f : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\partial f}{\partial x_1}\frac{\mathrm{d}x_1}{\mathrm{d}t} + \frac{\partial f}{\partial x_2}\frac{\mathrm{d}x_2}{\mathrm{d}t} + \cdots + \frac{\partial f}{\partial x_n}\frac{\mathrm{d}x_n}{\mathrm{d}t}$$

Proof. Consider the case where $n = 2$. The total derivative can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t} f\left(x\left(t\right),\, y\left(t\right)\right) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[f\left(x\left(t+\Delta t\right),\, y\left(t+\Delta t\right)\right) - f\left(x\left(t\right),\, y\left(t\right)\right)\right]$$

Using the substitutions $x\left(t+\Delta t\right) = x\left(t\right) + \Delta x$ and $y\left(t+\Delta t\right) = y\left(t\right) + \Delta y$,

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}t} &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[f\left(x+\Delta x,\, y+\Delta y\right) - f\left(x,\, y\right)\right] \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[f\left(x+\Delta x,\, y+\Delta y\right) - f\left(x,\, y+\Delta y\right) + f\left(x,\, y+\Delta y\right) - f\left(x,\, y\right)\right] \\
&= \lim_{\Delta t \to 0}\left[\frac{\Delta x}{\Delta t}\frac{f\left(x+\Delta x,\, y+\Delta y\right) - f\left(x,\, y+\Delta y\right)}{\Delta x} + \frac{\Delta y}{\Delta t}\frac{f\left(x,\, y+\Delta y\right) - f\left(x,\, y\right)}{\Delta y}\right] \\
&= \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial f}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\partial f}{\partial y}
\end{aligned}$$

□

Definition 2.13 (Gradient)**.** The gradient is an operator that collects all partial derivatives of a function into a vector.

$$\boldsymbol{\nabla} = \begin{bmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \vdots \\ \partial_{x_n} \end{bmatrix} \implies \boldsymbol{\nabla} f = \begin{bmatrix} \partial_{x_1} f \\ \partial_{x_2} f \\ \vdots \\ \partial_{x_n} f \end{bmatrix}$$

The gradient allows us to define the chain rule using the dot product:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \boldsymbol{\nabla} f \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$$

2.5 Directional Derivatives

The directional derivative of a function $f : E \subset \mathbb{R}^n \to \mathbb{R}$ at a point $\mathbf{x}_0 \in E$ in the direction of a unit vector $\hat{\mathbf{u}} \in \mathbb{R}^n$ is the slope of f in the direction of $\mathbf{u}$:

$$D_{\mathbf{u}} f(\mathbf{x}_0) = \partial_{\mathbf{u}} f(\mathbf{x}_0) = \lim_{h \to 0} \frac{f(\mathbf{x}_0 + h\hat{\mathbf{u}}) - f(\mathbf{x}_0)}{h}$$

Proposition 2.5.1. *The directional derivative can be computed using the gradient of f*

$$D_{\mathbf{u}} f(\mathbf{x}_0) = \boldsymbol{\nabla} f(\mathbf{x}_0) \cdot \hat{\mathbf{u}}.$$

Proof. Consider the path in the output space $g(s) = f(\mathbf{x}_0 + s\hat{\mathbf{u}}) \in \mathbb{R}$ for $s \in \mathbb{R}$. The derivative of g w.r.t. s is given by:

$$\frac{\mathrm{d}g}{\mathrm{d}s} = \lim_{h \to 0} \frac{g(s+h) - g(s)}{h}$$

where if we evaluate g at $s = 0$, we get

$$\frac{\mathrm{d}g}{\mathrm{d}s}\Big|_{s=0} = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = \lim_{h \to 0} \frac{f(\mathbf{x}_0 + h\hat{\mathbf{u}}) - f(\mathbf{x}_0)}{h} = D_{\mathbf{u}} f(\mathbf{x}_0).$$

Using the derivative of the parametrised vector $\mathbf{x} = \mathbf{x}_0 + s\hat{\mathbf{u}} \in \mathbb{R}^n$:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} = \hat{\mathbf{u}}$$

we can evaluate the derivative of g w.r.t. s using the total derivative:

$$\begin{aligned}
\frac{\mathrm{d}g}{\mathrm{d}s}\Big|_{s=0} &= \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\right)_{s=0} \frac{\mathrm{d}x_i}{\mathrm{d}s}\Big|_{s=0} \\
&= \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\right)_{s=0} \hat{u}_i \\
&= \boldsymbol{\nabla} f(\mathbf{x})_{s=0} \cdot \hat{\mathbf{u}} \\
&= \boldsymbol{\nabla} f(\mathbf{x}_0) \cdot \hat{\mathbf{u}}
\end{aligned}$$

where $\hat{u}_i$ is the i-th component of $\hat{\mathbf{u}}$. Therefore $D_{\mathbf{u}} f(\mathbf{x}_0) = \boldsymbol{\nabla} f(\mathbf{x}_0) \cdot \hat{\mathbf{u}}$. □

Remark 1. Partial derivatives are directional derivatives in the direction of the canonical basis vectors $\hat{\mathbf{e}}_i$.

$$\frac{\partial f}{\partial x_i} = D_{\hat{\mathbf{e}}_i} f$$

We can therefore say that the directional derivative is a generalisation of the partial derivative for any direction $\hat{\mathbf{u}}$.

Proposition 2.5.2. *The gradient of a function $\boldsymbol{\nabla} f$ is orthogonal to the level curves of f.*

Proof. Consider the path $\mathbf{x}(s)$ on a contour of f. As f is constant on the contour,

$$\frac{\partial f}{\partial s} = 0.$$

Using the chain rule,

$$\frac{\partial f}{\partial s} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial s} = \boldsymbol{\nabla} f \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} = 0$$

therefore as the dot product is zero, $\boldsymbol{\nabla} f$ is orthogonal to the path $\mathbf{x}(s)$. □

Proposition 2.5.3. *The directional derivative is maximised when $\hat{\mathbf{u}}$ is parallel to $\boldsymbol{\nabla} f$.*

Proof. Using the angle definition of the dot product, the directional derivative is given by

$$D_{\mathbf{u}} f = \boldsymbol{\nabla} f \cdot \hat{\mathbf{u}} = \|\boldsymbol{\nabla} f\| \|\hat{\mathbf{u}}\| \cos\theta = \|\boldsymbol{\nabla} f\| \cos\theta$$

This expression is maximised when $\cos\theta = 1$, or when $\mathbf{u}$ is parallel to $\boldsymbol{\nabla} f$. □

Remark 2. The maximum slope of f at $\mathbf{x}_0$ is given by the magnitude of the gradient at $\mathbf{x}_0$:

$$\max_{\hat{\mathbf{u}}} D_{\mathbf{u}} f(\mathbf{x}_0) = \|\boldsymbol{\nabla} f(\mathbf{x}_0)\|$$

2.6 Normal Vectors to Curves

2.6.1 Parametric Curves

For a parametric curve $\mathbf{r}(t)$, we find a normal vector $\hat{\mathbf{n}}$ such that

$$\hat{\mathbf{n}} \cdot \mathbf{r}'(t) = 0$$

where $\mathbf{r}'(t)$ is a tangent vector to the curve.

2.6.2 Implicit Curves

For an implicit curve $F(x, y) = 0$, the normal vector is given by

$$\hat{\mathbf{n}} = \pm \frac{\boldsymbol{\nabla} F}{\|\boldsymbol{\nabla} F\|}$$

2.6.3 Explicit Curves

For an explicit curve $y = f(x)$, we must convert the curve to implicit or parametric form.

2.7 Normal Vectors to Surfaces

2.7.1 Parametric Surfaces

For a parametric surface $\mathbf{r}(s, t)$, the normal vector is given by

$$\hat{\mathbf{n}} = \pm \frac{\mathbf{r}_s \times \mathbf{r}_t}{\|\mathbf{r}_s \times \mathbf{r}_t\|}$$

where $\mathbf{r}_s$ and $\mathbf{r}_t$ are tangent vectors to the surface.

2.7.2 Implicit Surfaces

For an implicit surface $F(x,\ y,\ z) = 0$, the normal vector is given by

$$\hat{\mathbf{n}} = \pm\frac{\boldsymbol{\nabla} F}{\|\boldsymbol{\nabla} F\|}$$

2.7.3 Explicit Surfaces

For an explicit surface $z = f(x,\ y)$, we can either convert the surface to implicit form, or consider the tangents of two curves $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ on the surface. Then we can find the normal vector by taking the cross product of the tangent vectors:

$$\hat{\mathbf{n}} = \pm\frac{\mathbf{r}_1'(t) \times \mathbf{r}_2'(t)}{\|\mathbf{r}_1'(t) \times \mathbf{r}_2'(t)\|}$$

2.8 Tangent Vectors to Curves

2.8.1 Parametric Curves

For a parametric curve $\mathbf{r}(t)$, the tangent vector is given by

$$\hat{\boldsymbol{\tau}} = \pm\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

2.8.2 Implicit Curves in 2D

For an implicit curve $F(x,\ y) = 0$, the tangent vector can be found by first determining the normal vector $\hat{\mathbf{n}}$ which is proportional to the gradient of F:

$$\hat{\mathbf{n}} = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \propto \boldsymbol{\nabla} f$$

such that the tangent vector is given by

$$\hat{\boldsymbol{\tau}} = \pm\begin{bmatrix} -n_2 \\ n_1 \end{bmatrix}$$

2.8.3 Implicit Curves in 3D

Given the intersection of two implicit curves $F(x,\ y,\ z) = 0$ and $G(x,\ y,\ z) = 0$, the tangent vector along the intersection is given by

$$\hat{\boldsymbol{\tau}} = \pm\frac{\boldsymbol{\nabla} F \times \boldsymbol{\nabla} G}{\|\boldsymbol{\nabla} F \times \boldsymbol{\nabla} G\|}$$

2.8.4 Explicit Curves in 2D

For an explicit curve $y = f(x)$, we can either:

- convert the curve to parametric form: $\mathbf{r}(t) = \langle t,\ f(t)\rangle$
- convert the curve to implicit form: $F(x,\ y) = y - f(x) = 0$

2.9 Differentiability

2.10 Single Variable Functions

$f(x)$ is differentiable at a if $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists. This is equivalent to saying that the tangent line at $\langle a,\ f(a)\rangle$ is well-defined:

$$f(x) = \underbrace{f(a) + f'(a)\, f(x-a)}_{\text{tangent line}} + R(x-a) \qquad \text{with } \lim_{x\to a} \frac{R(x-a)}{x-a} = 0$$

2.11 Multivariable Functions

$f(\mathbf{x})$ is differentiable at $\mathbf{a} \in \Omega \subset \mathbb{R}^n$ if there exists a linear map $f'(\mathbf{a}) : \begin{cases} \mathbb{R}^n \to \mathbb{R} \\ \xi \mapsto f'(\mathbf{a}) \cdot \xi \end{cases}$ such that

$$f(\mathbf{x}) = \underbrace{f(\mathbf{a}) + f'(\mathbf{a}) \cdot (\mathbf{x}-\mathbf{a})}_{\text{tangent plane}} + R(\mathbf{x}-\mathbf{a}) \qquad \text{with } \lim_{\mathbf{x}\to\mathbf{a}} \frac{R(\mathbf{x}-\mathbf{a})}{\|\mathbf{x}-\mathbf{a}\|} = 0$$

The linear map $f'(\mathbf{a})$ is then the derivative of f at $\mathbf{a}$.

Theorem 2.11.1 (Derivative Equivalence with Gradient)**.** *If f is differentiable, then*

$$f'(\mathbf{a}) = \boldsymbol{\nabla} f(\mathbf{a})$$

Additionally, if $\boldsymbol{\nabla} f(\mathbf{a})$ exists $\forall \mathbf{a} \in \Omega$, and all partial derivatives are continuous (i.e., $\frac{\partial f}{\partial x_i} : \Omega \subset \mathbb{R}^n \to R$ are continuous), then f is differentiable everywhere in Ω.

The first result tells us that the derivative is unique and that f is differentiable at $\mathbf{a}$ if it has a tangent plane at $\mathbf{a}$. The second result gives us a sufficient condition for differentiability.

2.12 Taylor Series Expansion

The Taylor series expansion of a function f at a point $\mathbf{a}$ is given by

$$f(\mathbf{x}) = f(\mathbf{a}) + \sum_{i=0}^{n} \frac{\partial f(\mathbf{a})}{\partial x_i}(x_i - a_i) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f(\mathbf{a})}{\partial x_i\, \partial x_j}(x_i - a_i)(x_j - a_j) + \cdots$$
$$+ \frac{1}{k!}\sum_{i_1,\ldots,i_k}^{n} \frac{\partial^3 f(\mathbf{a})}{\partial x_{i_1}\, \partial \ldots \partial x_{i_k}} \left(x_{i_1} - a_{i_1}\right) \cdots \left(x_{i_k} - a_{i_k}\right) + \cdots$$

This allows us to compute the tangent plane, paraboloid, and so on, of a function f at $\mathbf{a}$, by increasing the order k of the Taylor series expansion.

3 Double Integrals

Integrals represent continuous sums of infinitesimal quantities, and allow us to measure extensive and average properties of objects, such as lengths, surface areas, volumes, masses, centres of mass, and so on.

3.1 Riemann Sums

The signed area under the function $f(x)$ is represented by

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x$$

Similarly, the signed volume under the surface $f(x,\ y)$ is represented by

$$\iint_R f(x,\ y)\,\mathrm{d}A = \lim_{\Delta A_i\to 0}\sum_{i=1}^{n} f(x_i,\ y_i)\,\Delta A_i$$

where R is the region of integration.

3.2 Lebesgue Integrals

Lebesgue integrals are a top down approach to multiple integrals that allow us to integrate a wider class of nonnegative functions. This includes improper integrals where f is discontinuous or singular, or when the domain of integration is unbounded.

Theorem 3.2.1 (Fubini's Theorem)**.** *For a nonnegative function $f : \mathbb{R}^2 \to [0,\infty)$, the following equality holds,*

$$0 \leqslant \iint_{\mathbb{R}^2} f(x,\ y)\,\mathrm{d}A = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x,\ y)\,\mathrm{d}x\right]\mathrm{d}y = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x,\ y)\,\mathrm{d}y\right]\mathrm{d}x \leqslant \infty$$

The above theorem holds for particular classes of functions. The following techniques can be used to integrate arbitrary functions.

3.2.1 Finite Domains

For double integrals over domains $R \subset \mathbb{R}^2$, we can define a new function $\tilde{f}$ such that

$$\iint_R f(x,\ y)\,\mathrm{d}A \equiv \iint_{\mathbb{R}^2} \tilde{f}(x,\ y)\,\mathrm{d}A, \qquad \text{where } \tilde{f}(x,\ y) = \begin{cases} f(x,\ y) & (x,\ y)\in R \\ 0 & \text{otherwise} \end{cases}$$

3.2.2 Nonpositive Functions

For nonpositive functions $f : \mathbb{R}^2 \to (-\infty,\infty)$, we can express f as $f = f^+ - f^-$, where f^+ and f^- are the positive and negative parts of f respectively:

$$f^+(x,\ y) = \max\{0,\ f(x,\ y)\} = \begin{cases} f(x,\ y) & f(x,\ y)\geqslant 0 \\ 0 & f(x,\ y) < 0 \end{cases}$$

$$f^-(x,\ y) = -\min\{0,\ f(x,\ y)\} = \begin{cases} -f(x,\ y) & f(x,\ y) < 0 \\ 0 & f(x,\ y)\geqslant 0 \end{cases}$$

Therefore, the double integral is defined

$$\iint_{\mathbb{R}^2} f(x,\ y)\,\mathrm{d}A = \iint_{\mathbb{R}^2} f^+(x,\ y)\,\mathrm{d}A - \iint_{\mathbb{R}^2} f^-(x,\ y)\,\mathrm{d}A$$

Definition 3.1 (Integrable Function)**.** A function $f : \mathbb{R}^n \to \mathbb{R}$ is **integrable** if

$$\int_{\mathbb{R}^n} |f(\mathbf{x})| \, \mathrm{d}\mathbf{x} < \infty$$

Theorem 3.2.2 (Fubini's Theorem for Integrable Functions)**.** *If $f : \mathbb{R}^2 \to \mathbb{R}$ is an integrable function*

$$\iint_{\mathbb{R}^2} f(x,\, y) \, \mathrm{d}A = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x,\, y) \, \mathrm{d}x \right] \mathrm{d}y = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x,\, y) \, \mathrm{d}y \right] \mathrm{d}x$$

3.3 Measures

Definition 3.2 (Indicator Function)**.** The indicator function of R is defined:

$$1_R(x,\, y) = \begin{cases} 1 & (x,\, y) \in R \\ 0 & \text{otherwise} \end{cases}$$

Definition 3.3 (Measure)**.** The measure of a region R is defined as

$$\mu(R) = \iint_{\mathbb{R}^2} 1_R \, \mathrm{d}A$$

In two dimensions, the measure is the area of the region.

3.3.1 Rectangular Regions

For a rectangular region $R = [a,\, b] \times [c,\, d]$, the indicator function is given by

$$\begin{aligned} 1_R(x,\, y) &= \begin{cases} 1 & (x,\, y) \in D \\ 0 & \text{otherwise} \end{cases} \\ &= 1_{[a,\, b]}(x) \, 1_{[c,\, d]}(y) \end{aligned}$$

so that the area of R is given by

$$\begin{aligned} \mu(R) = \iint_{\mathbb{R}^2} 1_R \, \mathrm{d}A &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} 1_{[a,\, b]}(x) \, 1_{[c,\, d]}(y) \, \mathrm{d}y \, \mathrm{d}x = \int_{-\infty}^{\infty} 1_{[a,\, b]}(x) \int_{-\infty}^{\infty} 1_{[c,\, d]}(y) \, \mathrm{d}y \, \mathrm{d}x \\ &= \int_a^b 1 \left[\int_c^d 1 \, \mathrm{d}y \right] \mathrm{d}x = \int_a^b (d - c) \, \mathrm{d}x = (b - a)(d - c) \end{aligned}$$

3.3.2 Integrating Functions over Regions

When integrating over a nonuniform density f, we can multiply f with the indicator function over R:

$$\iint_R f(x,\, y) \, \mathrm{d}A = \iint_{\mathbb{R}^2} f(x,\, y) \, 1_R \, \mathrm{d}A$$

3.4 Simple Domains

3.4.1 Type I — y-Simple

Let Y be defined as a region bounded by $x_1 \leqslant x \leqslant x_2$ and $y_1(x) \leqslant y \leqslant y_2(x)$. Y is called y-simple as it can be split into subdomains R_i with continuous lines **parallel to the y-axis**.
The double integral over Y can then be decomposed into the sum of integrals over each subdomain Y_i:

$$\iint_Y f(x,\, y)\,\mathrm{d}A = \sum_{i=1}^{n} \iint_{Y_i} f(x,\, y)\,\mathrm{d}A$$

which is equivalent to integrating between the y-bounds of each subdomain Y_i within the x-bounds of Y:

$$\iint_Y f(x,\, y)\,\mathrm{d}A = \int_{x_1}^{x_2} \left[\int_{y_1(x)}^{y_2(x)} f(x,\, y)\,\mathrm{d}y \right] \mathrm{d}x$$

3.4.2 Type II — x-Simple

Let X be defined as a region bounded by $x_1(y) \leqslant x \leqslant x_2(y)$ and $y_1 \leqslant y \leqslant y_2$. X is called x-simple as it can be split into subdomains R_i with continuous lines **parallel to the x-axis**.
The double integral over X can then be decomposed into the sum of integrals over each subdomain X_i:

$$\iint_X f(x,\, y)\,\mathrm{d}A = \sum_{i=1}^{n} \iint_{X_i} f(x,\, y)\,\mathrm{d}A$$

which is equivalent to integrating between the x-bounds of each subdomain X_i within the y-bounds of X:

$$\iint_X f(x,\, y)\,\mathrm{d}A = \int_{y_1}^{y_2} \left[\int_{x_1(y)}^{x_2(y)} f(x,\, y)\,\mathrm{d}x \right] \mathrm{d}y$$

3.5 Transformation of Coordinates

In single variable calculus, we can use the transformation $x = g(u)$ to change the variable of integration from x to u, using the chain rule, $\mathrm{d}x = g'(u)\,\mathrm{d}u$. For multiple integrals, the same can be accomplished using the Jacobian matrix.

Definition 3.4 (Jacobian Matrix)**.** Consider the transformation $\mathbf{x} = T(\mathbf{u})$, where $T : \mathbb{R}^m \to \mathbb{R}^n$ is once differentiable. The Jacobian matrix of T is defined as an $m \times n$ matrix, denoted $\mathbf{J}$, whose $(i,\, j)$-th entry is given by

$$\mathbf{J}_{i,\,j} = \frac{\partial x_i}{\partial u_j}.$$

Explicitly, the Jacobian matrix is given by

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x_1}{\partial u_1} & \cdots & \dfrac{\partial x_1}{\partial u_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial x_m}{\partial u_1} & \cdots & \dfrac{\partial x_m}{\partial u_n} \end{bmatrix}.$$

It may also be notated as

$$\mathbf{J} = \frac{\partial (x_1, \ldots, x_m)}{\partial (u_1, \ldots, u_n)}$$

Definition 3.5 (Jacobian)**.** When $m = n$, we can determine the ratio of the area of the original region R to the area of the transformed region $T(R)$ using the determinant of the Jacobian matrix:

$$|\mathbf{J}| = \det(\mathbf{J}) = \left| \frac{\partial (x_1, \ldots, x_n)}{\partial (u_1, \ldots, u_n)} \right|.$$

In the case of two variables, $(x, y) = T(u, v)$, and the Jacobian matrix is defined as

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix}.$$

The Jacobian is then given by

$$|\mathbf{J}| = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

3.5.1 General Transformation

Using this matrix determinant allows us to change the variables of integration from (x, y) to (u, v) using the following formula:

$$\iint_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{T(R)} f(T(u, v))\,|\det(\mathbf{J})|\,\mathrm{d}u\,\mathrm{d}v$$

where $f(T(u, v))$ is multiplied by the absolute value of the Jacobian.

3.5.2 Polar Coordinate Transformation

Definition 3.6 (Polar Coordinates)**.** A polar coordinate system is defined by the transformation

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

where $r > 0$ is the radius of the circle and $\theta \in [0, 2\pi)$ is the angle of rotation measured anticlockwise from the positive x-axis. The Jacobian is given by

$$\begin{aligned} \mathbf{J} &= \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \\ \det(\mathbf{J}) &= r\cos^2\theta + r\sin^2\theta \\ &= r \end{aligned}$$

The infinitesimal area element $\mathrm{d}A$ is therefore

$$\mathrm{d}A = |\det(\mathbf{J})|\,\mathrm{d}r\,\mathrm{d}\theta = r\,\mathrm{d}r\,\mathrm{d}\theta.$$

Using polar coordinates, the double integral over a region R is defined

$$\iint_R f(x,\ y)\,\mathrm{d}A = \int_0^{\infty}\int_0^{2\pi} f(r\cos\theta,\ r\sin\theta)\,r\,\mathrm{d}\theta\,\mathrm{d}r$$

3.6 Strategies for Evaluating Double Integrals

1. Decompose the region of integration into simple domains, and if this is not possible, divide the region into subregions.
2. Change the variables of integration to simplify the integral.
3. Swap the order of integration to simplify the integral.

4 Multiple Integrals

All results and properties for double integrals can be extended to triple integrals.

Theorem 4.0.1 (Fubini's Theorem in n-dimensions)**.** *If $f : \mathbb{R}^n \to \mathbb{R}^+$, or $f : \mathbb{R}^n \to \mathbb{R}$ is integrable, then all $n!$ permutations of integrals are equal:*

$$\int_{\mathbb{R}^n} f(x_1,\ \ldots,\ x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n = \int_{-\infty}^{\infty}\left[\cdots\left[\int_{-\infty}^{\infty} f(x_1,\ \ldots,\ x_n)\,\mathrm{d}x_1\right]\cdots\right]\mathrm{d}x_n$$

As with double integrals, we can evaluate integrals of nonpositive functions by using the positive and negative parts of the function.

4.1 Vector-Valued Functions

Definition 4.1 (Vector-Valued Function)**.** A vector-valued function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is a function that maps a vector $\mathbf{x} \in \mathbb{R}^n$ to a vector $\mathbf{f}(\mathbf{x}) \in \mathbb{R}^m$.

A vector-valued function is integrable if either:

- the norm of the function is integrable

$$\int_{\mathbb{R}^n} \|\mathbf{f}(\mathbf{x})\|\,\mathrm{d}\mathbf{x} < \infty$$

- or if all components of the function are integrable

$$\int_{\mathbb{R}^n} |f_i(\mathbf{x})|\,\mathrm{d}\mathbf{x} < \infty \qquad \forall i \in \{1,\ \ldots,\ m\}$$

The integral of a vector-valued function is defined as

$$\int_{\mathbb{R}^n} \mathbf{f}(\mathbf{x})\,\mathrm{d}\mathbf{x} = \left\langle \int_{\mathbb{R}^n} f_1(\mathbf{x})\,\mathrm{d}\mathbf{x},\ \ldots,\ \int_{\mathbb{R}^n} f_m(\mathbf{x})\,\mathrm{d}\mathbf{x} \right\rangle$$

4.2 Change of Variables

4.2.1 Cylindrical Coordinate Transformation

Definition 4.2 (Cylindrical Coordinates)**.** A cylindrical coordinate system is defined by the transformation

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

where $r > 0$ is the radius of the cylinder, $\theta \in [0, 2\pi)$ is the angle of rotation measured anticlockwise from the positive x-axis, and $z \in \mathbb{R}$ is the height of the cylinder. The Jacobian is given by

$$\begin{aligned} \mathbf{J} &= \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial z} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial z} \\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \theta} & \dfrac{\partial z}{\partial z} \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \det(\mathbf{J}) &= r\cos^2\theta + r\sin^2\theta \\ &= r \end{aligned}$$

The infinitesimal area element $\mathrm{d}V$ is therefore

$$\mathrm{d}V = |\det(\mathbf{J})|\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z.$$

Using cylindrical coordinates, the triple integral over a region R is defined

$$\iiint_R f(x,\ y,\ z)\,\mathrm{d}V = \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{2\pi} f(r\cos\theta,\ r\sin\theta,\ z)\,r\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}z$$

4.2.2 Spherical Coordinate Transformation

Definition 4.3 (Spherical Coordinates)**.** A spherical coordinate system is defined by the transformation

$$\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \end{aligned}$$

where $\rho > 0$ is the radius of the sphere, $\phi \in [0, \pi)$ is the polar angle measured down from the positive z-axis, and $\theta \in [0, 2\pi)$ is the azimuthal angle measured anticlockwise from the positive

x-axis. The Jacobian is given by

$$\begin{aligned}\mathbf{J} &= \begin{bmatrix} \dfrac{\partial x}{\partial \rho} & \dfrac{\partial x}{\partial \phi} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial \rho} & \dfrac{\partial y}{\partial \phi} & \dfrac{\partial y}{\partial \theta} \\ \dfrac{\partial z}{\partial \rho} & \dfrac{\partial z}{\partial \phi} & \dfrac{\partial z}{\partial \theta} \end{bmatrix} \\ &= \begin{bmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{bmatrix} \\ \det(\mathbf{J}) &= \rho^2\sin^3\phi\cos^2\theta + \rho^2\sin\phi\cos^2\phi\cos^2\theta - \rho\sin\phi\sin\theta\left(-\rho\sin^2\phi\sin\theta - \rho\cos^2\phi\sin\theta\right) \\ &= \rho^2\sin\phi\cos^2\theta\left(\sin^2\phi + \cos^2\phi\right) + \rho^2\sin\phi\sin^2\theta\left(\sin^2\phi + \cos^2\phi\right) \\ &= \rho^2\sin\phi\left(\sin^2\theta + \cos^2\theta\right) \\ &= \rho^2\sin\phi\end{aligned}$$

The infinitesimal area element $\mathrm{d}V$ is therefore

$$\mathrm{d}V = |\det(\mathbf{J})|\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta = \rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta.$$

Using spherical coordinates, the triple integral over a region R is defined

$$\iiint_R f(x,\,y,\,z)\,\mathrm{d}V = \int_0^{\infty}\int_0^{2\pi}\int_0^{\pi} f(\rho\sin\phi\cos\theta,\,\rho\sin\phi\sin\theta,\,\rho\cos\phi)\,\rho^2\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}\rho$$

4.3 Interpretations of Integrals

Measure

The measure of a region $R \in \mathbb{R}^n$ is given by:

$$\mu(R) = \int_R \mathrm{d}\mathbf{x}$$

In two dimensions, the measure is the area of the region:

$$\mu(R) = \iint_R \mathrm{d}A$$

In three dimensions, the measure is the volume of the region:

$$\mu(R) = \iiint_R \mathrm{d}V$$

Mass

The mass of a region $R \in \mathbb{R}^n$ with density function ρ is given by:

$$M = \int_R \rho(\mathbf{x})\,\mathrm{d}\mathbf{x}$$

Centroid

The average position of a region $R \in \mathbb{R}^n$ with uniform density is given by:

$$\langle \mathbf{r} \rangle = \frac{1}{\mu(R)} \int_R \mathbf{x} \, \mathrm{d}\mathbf{x}$$

This point is the geometric centre of the region where the region would balance if it were made of a uniform material.

Centre of Mass

The average position of a region $R \in \mathbb{R}^n$ with density function ρ is given by:

$$\langle \mathbf{r} \rangle_\rho = \frac{1}{M} \int_R \rho(\mathbf{x}) \, \mathbf{x} \, \mathrm{d}\mathbf{x}$$

This point is the centre of mass of the region where the region would balance if it were made of a material with density ρ.

Average Value

The average value of a function $f : \mathbb{R}^n \to \mathbb{R}$ over a region $R \in \mathbb{R}^n$ is given by:

$$\langle f(\mathbf{r}) \rangle = \frac{1}{\mu(R)} \int_R f(\mathbf{x}) \, \mathrm{d}\mathbf{x}$$

Expected Value

The average value of a function $f : \mathbb{R}^n \to \mathbb{R}$ over a region $R \in \mathbb{R}^n$ with density function p is given by:

$$\langle f(\mathbf{r}) \rangle_p = \int_R p(\mathbf{x}) \, f(\mathbf{x}) \, \mathrm{d}\mathbf{x}$$

where p is a probability density function satisfying

$$\int_R p(\mathbf{x}) \, \mathrm{d}\mathbf{x} = 1$$

This result is the expected value of f over R if the region is randomly sampled according to the probability density function p.

4.4 Properties of Integrals

Linearity of Integrals

If f and g are integrable functions, and $c \in \mathbb{R}$, then

$$\int_{\mathbb{R}^n} (f(\mathbf{x}) + g(\mathbf{x})) \, \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^n} f(\mathbf{x}) \, \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^n} g(\mathbf{x}) \, \mathrm{d}\mathbf{x}$$

and

$$\int_{\mathbb{R}^n} cf(\mathbf{x}) \, \mathrm{d}\mathbf{x} = c \int_{\mathbb{R}^n} f(\mathbf{x}) \, \mathrm{d}\mathbf{x}$$

Positivity of Integrals

If $f \geqslant 0$ for all $\mathbf{x} \in \mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \geqslant 0$$

Monotonicity of Integrals

If f and g are integrable functions, and $f(\mathbf{x}) \leqslant g(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \leqslant \int_{\mathbb{R}^n} g(\mathbf{x})\,\mathrm{d}\mathbf{x}$$

Triangle Inequality

If f is an integrable function, then

$$\left\| \int_{\mathbb{R}^n} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \right\| \leqslant \int_{\mathbb{R}^n} \|f(\mathbf{x})\|\,\mathrm{d}\mathbf{x}$$

Change of Variables

When f is positive or integrable, the bijective transformation $\mathbf{T} : R \subset \mathbb{R}^n \to R' \subset \mathbb{R}^n$ with continuous first derivative $\mathbf{T}'$ allows us to change the variables of integration from $\mathbf{x}$ to $\mathbf{u}$:

$$\int_R f(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{R'} f(\mathbf{T}(\mathbf{u}))\,|\det(\mathbf{J})|\,\mathrm{d}\mathbf{u}$$

where $\mathbf{J}$ is the Jacobian matrix of $\mathbf{T}$.

Measure Zero Integrals

If $\mu(R) = \int_R \mathrm{d}\mathbf{x} = 0$, then

$$\int_R f(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0$$

Almost Equal Functions

If f and g are integrable functions, and $f(\mathbf{x}) = g(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$ except for a set of measure zero, then

$$\int_{\mathbb{R}^n} f(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^n} g(\mathbf{x})\,\mathrm{d}\mathbf{x}$$

5 Vector Calculus

A vector-valued multivariable function is a function $\boldsymbol{f} : E \subset \mathbb{R}^n \to \mathbb{R}^m$ that maps a vector $\mathbf{x} \in \mathbb{R}^n$ to the vector $\boldsymbol{f}(\mathbf{x}) \in \mathbb{R}^m$. f is continuous if all components of $\boldsymbol{f}$ are continuous, and differentiable if all components of $\boldsymbol{f}$ are differentiable.

If $\boldsymbol{f}$ is differentiable at a point $\mathbf{x}_0$, then it's derivative $\boldsymbol{f}'(\mathbf{x}_0)$ is uniquely defined by the Jacobian matrix of $\boldsymbol{f}$ at $\mathbf{x}_0$:

$$\boldsymbol{f}'(\mathbf{x}_0) = \boldsymbol{\nabla} \boldsymbol{f}(\mathbf{x}_0) = \mathbf{J}^\top(\mathbf{x}_0)$$

Definition 5.1 (Scalar Field)**.** If $m = 1$, then $f : E \subset \mathbb{R}^n \to \mathbb{R}$ is a scalar field that associates a scalar value to each point $\mathbf{x} \in E$.

Definition 5.2 (Vector Field)**.** If $m = n$, then $\mathbf{F} : E \subset \mathbb{R}^n \to \mathbb{R}^n$ is a vector field that associates a vector to each point $\mathbf{x} \in E$. In this case, the Jacobian matrix is a square matrix and the Jacobian may be defined.

Definition 5.3 (Field Lines)**.** The field lines of a vector field $\mathbf{F}$ are a family of curves that are tangent to $\mathbf{F}$ for all $\mathbf{x}$. They are defined as the solutions to the differential equation

$$\mathbf{r}'(t) = \mathbf{F}(\mathbf{r}(t))$$

for all t. When $m = 2$,

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= F_1(x,\, y) \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= F_2(x,\, y) \end{aligned}$$

and a field line may be defined in Cartesian coordinates as the solution to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{F_2(x,\, y)}{F_1(x,\, y)}.$$

Lemma 5.0.0.1. *The field lines of a scalar multiple of a vector field are the same as the field lines of that vector field.*

5.1 Differential Operators on Scalar Fields

Definition 5.4 (Gradient)**.** The gradient of a scalar field is defined as the derivative of the scalar field in every direction:

$$\operatorname{grad} f = \boldsymbol{\nabla} f = \mathbf{J}^\top = \begin{bmatrix} \frac{\partial f}{\partial \mathbf{x}_1} \\ \vdots \\ \frac{\partial f}{\partial \mathbf{x}_n} \end{bmatrix}$$

The gradient measures the rate of change of a scalar field in all directions at a given point.

Definition 5.5 (Laplacian)**.** The Laplacian of a scalar field is defined as the divergence of the gradient:

$$\boldsymbol{\Delta} f = \operatorname{div}(\operatorname{grad} f) = \boldsymbol{\nabla}^2 f = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial \mathbf{x}_i^2}$$

The Laplacian measures the curvature or convexity of the surface $z = f(\mathbf{x})$.

5.2 Differential Operators on Vector Fields

Definition 5.6 (Divergence)**.** The divergence of a vector field is defined as the dot product of the gradient and the vector field:

$$\operatorname{div}\mathbf{F} = \boldsymbol{\nabla}\cdot\mathbf{F} = \frac{\partial \mathbf{F}_1}{\partial \mathbf{x}_1} + \cdots + \frac{\partial \mathbf{F}_n}{\partial \mathbf{x}_n} = \sum_{i=1}^{n} \frac{\partial \mathbf{F}_i}{\partial \mathbf{x}_i}.$$

Divergence measures the expansion of a vector field at a given point.

- $\operatorname{div}\mathbf{F}(\mathbf{x}_0) > 0$ indicates that $\mathbf{x}_0$ is a source, and the vector field is diverging out from $\mathbf{x}_0$.
- $\operatorname{div}\mathbf{F}(\mathbf{x}_0) < 0$ indicates that $\mathbf{x}_0$ is a sink, and the vector field is converging into $\mathbf{x}_0$.
- $\operatorname{div}\mathbf{F}(\mathbf{x}_0) = 0$ indicates that the net flow of the vector field at $\mathbf{x}_0$ is zero.

Definition 5.7 (Curl)**.** The curl of a vector field is defined as the cross product of the gradient and the vector field:

$$\operatorname{curl}\mathbf{F} = \boldsymbol{\nabla}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial \mathbf{x}_1} & \dfrac{\partial}{\partial \mathbf{x}_2} & \dfrac{\partial}{\partial \mathbf{x}_3} \\ \mathbf{F}_1 & \mathbf{F}_2 & \mathbf{F}_3 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial}{\partial \mathbf{x}_2} & \dfrac{\partial}{\partial \mathbf{x}_3} \\ \mathbf{F}_2 & \mathbf{F}_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \dfrac{\partial}{\partial \mathbf{x}_1} & \dfrac{\partial}{\partial \mathbf{x}_3} \\ \mathbf{F}_1 & \mathbf{F}_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \dfrac{\partial}{\partial \mathbf{x}_1} & \dfrac{\partial}{\partial \mathbf{x}_2} \\ \mathbf{F}_1 & \mathbf{F}_2 \end{vmatrix}\mathbf{k}$$

Curl measures the rotation of a vector field at a given point.

- $\operatorname{curl}\mathbf{F}(\mathbf{x}_0) > 0$ indicates that the vector field is rotating anticlockwise about $\mathbf{x}_0$.
- $\operatorname{curl}\mathbf{F}(\mathbf{x}_0) < 0$ indicates that the vector field is rotating clockwise about $\mathbf{x}_0$.
- $\operatorname{curl}\mathbf{F}(\mathbf{x}_0) = 0$ indicates that the net rotation about $\mathbf{x}_0$ is zero.

5.3 Conservative Fields

A vector field $\mathbf{F}$ is conservative if it is the gradient of a potential function ϕ:

$$\mathbf{F} = \boldsymbol{\nabla}\phi.$$

Such a vector field represents a force field in which the total energy is conserved.
The contours of the scalar field ϕ are called equipotential lines (ϕ is constant). These lines are perpendicular to the field lines of $\mathbf{F}$.

$$(\mathbf{F} = \boldsymbol{\nabla}\phi) \perp (\phi = \text{constant})$$

This is because the contours of a scalar field are defined to be perpendicular to the gradient of that scalar field.

5.4 Line Integrals

A line integral is an integral where the function to be integrated is evaluated along a curve. This function may be a scalar field or a vector field. Line integrals can be interpreted as a measure of the total effect of a function along a curve.
To evaluate line integrals, we must define a parametrisation of the arc length of the curve.

5.4.1 Arc Length

Arc length is the distance travelled along a path or curve. When $\mathbf{r}(t)$ is a path, the arc length between $\mathbf{r}(a)$ to $\mathbf{r}(t)$ is given by

$$s(t) = \int_a^t \|\mathbf{r}'(\tau)\| \, \mathrm{d}\tau$$

where the integrand can be interpreted as the product of the speed of along the path $\mathbf{r}(t)$ with a small time interval $\mathrm{d}\tau$. The length L of a path is therefore

$$L = s(b) = \int_a^b \|\mathbf{r}'(\tau)\| \, \mathrm{d}\tau.$$

Theorem 5.4.1 (Arc Length Reparametrisation)**.** *A (piecewise) regular curve $\mathscr{C}$ can always be reparametrised by the arc length parametrisation $s(t)$:*

$$\tilde{\mathbf{r}}(s) = \mathbf{r}(\theta(s))$$

for $s \in [0, L]$.

Proof. As $r'(t) \neq 0$, $\|r'(t)\| > 0$, and therefore, $s(t)$ is a strictly increasing function. This means that $s(t)$ is invertible, and therefore, $t = \theta(s)$ is a bijective map between $[0, L]$ and $[a, b]$. □

Remark 1. The rate of change of $s(t)$ is the speed of the path $\mathbf{r}(t)$:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_a^t \|\mathbf{r}'(\tau)\| \, \mathrm{d}\tau\right] = \|\mathbf{r}'(t)\|.$$

Remark 2. The speed of a path parametrised by arc length is always 1:

$$\|\tilde{\mathbf{r}}'(s)\| = \|\mathbf{r}'(\theta(s))\|\left|\frac{\mathrm{d}\theta(s)}{\mathrm{d}s}\right| = \|\mathbf{r}'(\theta(s))\|\frac{1}{\left|\frac{\mathrm{d}s}{\mathrm{d}\theta(s)}\right|} = \|\mathbf{r}'(\theta(s))\|\frac{1}{\|\mathbf{r}'(\theta(s))\|} = 1.$$

5.4.2 Line Integral of a Scalar Field

Let $\mathscr{C}$ be a simple and piecewise regular curve parametrised by $\mathbf{r}(t)$ with $t \in [a, b]$. The line integral of a scalar field f along $\mathscr{C}$ is defined as

$$\int_{\mathscr{C}} f \, \mathrm{d}s = \int_a^b f(\mathbf{r}(t)) \|\mathbf{r}'(t)\| \, \mathrm{d}t$$

where $\mathrm{d}s = \|\mathbf{r}'(t)\| \, \mathrm{d}t$ is the differential arc length element along $\mathscr{C}$.
This integral represents the weighted sum of f along $\mathscr{C}$, where the weight is the speed of the path $\mathbf{r}(t)$. This line integral can be interpreted as the area under the surface defined by f along $\mathscr{C}$.

Lemma 5.4.1.1 (Equivalence of Parametrisations)**.** *Let $\tilde{\mathbf{r}}(u)$ be a reparametrisation of $\mathbf{r}(t)$ with $u \in [c, d]$. Then*

$$\int_{\mathscr{C}} f \, \mathrm{d}s = \int_c^d f(\tilde{\mathbf{r}}(u)) \|\tilde{\mathbf{r}}'(u)\| \, \mathrm{d}u$$

so that the line integral of a scalar field is independent of the parametrisation of $\mathscr{C}$.

Proof. Consider the parametrisation $t = \theta(u)$, where the differential time element $\mathrm{d}t = |\theta'(u)|\,\mathrm{d}u$, by the Change of Variables property. Then

$$\begin{aligned}\int_{\mathscr{C}} f\,\mathrm{d}s &= \int_a^b f(\mathbf{r}(t))\,\|\mathbf{r}'(t)\|\,\mathrm{d}t \\ &= \int_c^d f(\mathbf{r}(\theta(u)))\,\|\mathbf{r}'(\theta(u))\||\theta'(u)|\,\mathrm{d}u \\ &= \int_c^d f(\tilde{\mathbf{r}}(u))\,\|\tilde{\mathbf{r}}'(u)\|\,\mathrm{d}u.\end{aligned}$$

□

5.4.3 Line Integral of a Vector Field

Let $\mathscr{C}$ be a simple and piecewise regular curve parametrised by $\mathbf{r}(t)$ with $t \in [a,b]$. The line integral of a vector field $\mathbf{F}$ along $\mathscr{C}$ is defined as

$$\int_{\mathscr{C}} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,\mathrm{d}t$$

where $\mathrm{d}\mathbf{r} = \mathbf{r}'(t)\,\mathrm{d}t$ is the differential path element along $\mathscr{C}$.
This integral represents the weighted sum of $\mathbf{F}$ along $\mathscr{C}$, where the weight is the component of the velocity of the path $\mathbf{r}(t)$ in the direction of $\mathbf{F}$. This line integral can be interpreted as the work done by the force field $\mathbf{F}$ along $\mathscr{C}$.

Lemma 5.4.1.2 (Equivalence of Parametrisations)**.** *Let $\tilde{\mathbf{r}}(u)$ be a reparametrisation of $\mathbf{r}(t)$ with $u \in [c,d]$. Then*

$$\int_{\mathscr{C}} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \begin{cases} \displaystyle\int_c^d \mathbf{F}(\tilde{\mathbf{r}}(u))\cdot\tilde{\mathbf{r}}'(u)\,\mathrm{d}u & \text{if } \theta'(u) > 0 \\ \displaystyle -\int_c^d \mathbf{F}(\tilde{\mathbf{r}}(u))\cdot\tilde{\mathbf{r}}'(u)\,\mathrm{d}u & \text{if } \theta'(u) < 0 \end{cases}$$

so that the line integral of a vector field is independent of the parametrisation of $\mathscr{C}$.

Proof. Consider the parametrisation $t = \theta(u)$, where the differential time element $\mathrm{d}t = |\theta'(u)|\,\mathrm{d}u$,

by the Change of Variables property. Then

$$\begin{aligned}\int_{\mathscr{C}} \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \int_a^b \mathbf{F}\left(\mathbf{r}\left(t\right)\right)\cdot\mathbf{r}'\left(t\right)\mathrm{d}t\\ &= \begin{cases}\displaystyle\int_c^d \mathbf{F}\left(\mathbf{r}\left(\theta\left(u\right)\right)\right)\cdot\mathbf{r}'\left(\theta\left(u\right)\right)\left|\theta'\left(u\right)\right|\mathrm{d}u & \text{if } \theta'\left(u\right)>0\\ \displaystyle-\int_c^d \mathbf{F}\left(\mathbf{r}\left(\theta\left(u\right)\right)\right)\cdot\mathbf{r}'\left(\theta\left(u\right)\right)\left|\theta'\left(u\right)\right|\mathrm{d}u & \text{if } \theta'\left(u\right)<0\end{cases}\\ &= \begin{cases}\displaystyle\int_c^d \mathbf{F}\left(\tilde{\mathbf{r}}\left(u\right)\right)\cdot\tilde{\mathbf{r}}'\left(u\right)\mathrm{d}u & \text{if } \theta'\left(u\right)>0\\ \displaystyle-\int_c^d \mathbf{F}\left(\tilde{\mathbf{r}}\left(u\right)\right)\cdot\tilde{\mathbf{r}}'\left(u\right)\mathrm{d}u & \text{if } \theta'\left(u\right)<0\end{cases}\end{aligned}$$

□

Corollary 5.4.1.1 (Line Integrals in the Reverse Direction)**.** *Taking the line integral of a path in the reverse direction is equivalent to negating the line integral over the original path.*

$$\int_{-\mathscr{C}} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = -\int_{\mathscr{C}} \mathbf{F}\cdot\mathrm{d}\mathbf{r}$$

where $-\mathscr{C}$ is the curve parametrised by $\mathbf{r}\left(a+b-t\right)$.

5.4.4 Relationship between Line Integrals over Scalar and Vector Fields

Line integrals over scalar fields may be evaluated with respect to a single variable x_i:

$$\int_{\mathscr{C}} f\,\mathrm{d}x_i = \int_a^b f\left(\mathbf{r}\left(t\right)\right)\mathbf{r}_i'\left(t\right)\mathrm{d}t$$

where $\mathrm{d}x_i = \mathbf{r}_i'\left(t\right)\mathrm{d}t$ is the differential element in the direction of x_i along $\mathscr{C}$. By adding the line integrals over all x_i, we obtain

$$\int_{\mathscr{C}} \left(\mathbf{F}_1\,\mathrm{d}x_1+\cdots+\mathbf{F}_n\,\mathrm{d}x_n\right) = \int_a^b \left(\mathbf{F}_1\mathbf{r}_1'\,\mathrm{d}t+\cdots+\mathbf{F}_n\mathbf{r}_n'\,\mathrm{d}t\right) = \int_a^b \mathbf{F}\left(\mathbf{r}\left(t\right)\right)\cdot\mathbf{r}'\left(t\right)\mathrm{d}t = \int_{\mathscr{C}} \mathbf{F}\cdot\mathrm{d}\mathbf{r}.$$

Line integrals over vector fields may be evaluated using the unit tangent vector of the path $\mathbf{r}\left(t\right)$, so that

$$\int_{\mathscr{C}} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}\left(\mathbf{r}\left(t\right)\right)\cdot\frac{\mathbf{r}'\left(t\right)}{\left\|\mathbf{r}'\left(t\right)\right\|}\left\|\mathbf{r}'\left(t\right)\right\|\mathrm{d}t = \int_a^b \underbrace{\mathbf{F}\left(\mathbf{r}\left(t\right)\right)\cdot\hat{\boldsymbol{\tau}}\left(t\right)}_{f}\left\|\mathbf{r}'\left(t\right)\right\|\mathrm{d}t = \int_{\mathscr{C}} f\,\mathrm{d}s.$$

5.4.5 Line Integrals over Multiple Paths

Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two simple and piecewise regular curves parametrised by $\mathbf{r}_1 : [a,b]\subset\mathbb{R}\to\mathbb{R}^n$ and $\mathbf{r}_2 : [b,c]\subset\mathbb{R}\to\mathbb{R}^n$ respectively, such that $\mathbf{r}_1\left(b\right) = \mathbf{r}_2\left(b\right)$. The line integrals over the

concatenation of $\mathscr{C}_1$ and $\mathscr{C}_2$ are defined as

$$\int_{\mathscr{C}_1+\mathscr{C}_2} f\,\mathrm{d}s = \int_{\mathscr{C}_1} f\,\mathrm{d}s + \int_{\mathscr{C}_2} f\,\mathrm{d}s$$
$$\int_{\mathscr{C}_1+\mathscr{C}_2} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_{\mathscr{C}_1} \mathbf{F}\cdot\mathrm{d}\mathbf{r} + \int_{\mathscr{C}_2} \mathbf{F}\cdot\mathrm{d}\mathbf{r}.$$

5.4.6 Fundamental Theorem of Line Integrals

Let $\mathbf{F} = \boldsymbol{\nabla}\phi$ be a conservative vector field and $\mathscr{C}$ be a simple and piecewise regular curve parametrised by $\mathbf{r}(t)$ with $t \in [a,b]$. Then

$$\int_{\mathscr{C}} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_{\mathscr{C}} \boldsymbol{\nabla}\phi\cdot\mathrm{d}\mathbf{r} = \phi(\mathbf{b}) - \phi(\mathbf{a})$$

where $\mathbf{a} = \mathbf{r}(a)$ and $\mathbf{b} = \mathbf{r}(b)$. This result demonstrates that a line integral in a conservative field is path independent, and depends only on the endpoints of the path.

5.4.7 Circulation

Let $\mathscr{C}$ be a simple closed curve parametrised by $\mathbf{r}(t)$ with $t \in [a,b]$, where $\mathbf{r}(a) = \mathbf{r}(b)$. The line integral of a vector field $\mathbf{F}$ around $\mathscr{C}$ is called the circulation of $\mathbf{F}$ around $\mathscr{C}$, and is notated by a circle inside the integral sign:

$$\Gamma = \oint_{\mathscr{C}} \mathbf{F}\cdot\mathrm{d}\mathbf{r}.$$

5.4.8 Theorems in Conservative Fields

Let $\mathbf{F}$ be a continuous conservative vector field in an open and connected region $\Omega \subset \mathbb{R}^n$. Then the following theorems hold:

Theorem 5.4.2 (Circulation in a Conservative Field)**.** *The circulation of* $\mathbf{F}$ *around any closed path* $\mathscr{C} \in \Omega$ *is zero:*

$$\oint_{\mathscr{C}} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = 0.$$

Proof. Let $\mathscr{C}$ be a closed path parametrised by $\mathbf{r}(t)$ with $t \in [a,b]$, where $\mathbf{r}(a) = \mathbf{r}(b)$. Then

$$\oint_{\mathscr{C}} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \oint_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,\mathrm{d}t = \oint_a^b \boldsymbol{\nabla}\phi(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,\mathrm{d}t = \phi(\mathbf{r}(b)) - \phi(\mathbf{r}(a)) = 0.$$

□

Theorem 5.4.3 (Path Independence in a Conservative Field)**.** *The line integral of* $\mathbf{F}$ *between two points* $\mathbf{a}$ *and* $\mathbf{b}$ *in* Ω *is independent of the path connecting* $\mathbf{a}$ *and* $\mathbf{b}$*:*

$$\int_{\mathscr{C}_1} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_{\mathscr{C}_2} \mathbf{F}\cdot\mathrm{d}\mathbf{r}.$$

Proof. Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two different paths parametrised by $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ respectively, with $t \in [a,b]$, where $\mathbf{r}_1(a) = \mathbf{r}_2(a)$ and $\mathbf{r}_1(b) = \mathbf{r}_2(b)$. Consider the opposite reparametrisation of $\mathbf{r}_2(t)$ such that the concatenation of $\mathbf{r}_1$ and $\tilde{\mathbf{r}}_2$ is a closed path. Then, by Theorem 5.4.2,

$$0 = \oint_{\mathscr{C}_1 - \mathscr{C}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{\mathscr{C}_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{-\mathscr{C}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{\mathscr{C}_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} - \int_{\mathscr{C}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

which implies that

$$\int_{\mathscr{C}_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{\mathscr{C}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

□

Using this theorem, it is possible to show that a conservative field is the gradient of a potential function.

Proof for the Definition of a Conservative Field. Consider the line integral of $\mathbf{F}$ from a fixed point $\mathbf{x}_0$ to a variable point $\mathbf{x}$ in Ω:

$$\phi(\mathbf{x}) = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

The derivative of $\phi(\mathbf{x})$ with respect to $\mathbf{x}_i$ is given by

$$\begin{aligned}
\frac{\partial \phi}{\partial \mathbf{x}_i} &= \lim_{h \to 0} \frac{1}{h} \left[\phi(\mathbf{x} + h\mathbf{e}_i) - \phi(\mathbf{x}) \right] \\
&= \lim_{h \to 0} \frac{1}{h} \left[\int_{\mathbf{x}_0}^{\mathbf{x}+h\mathbf{e}_i} \mathbf{F} \cdot \mathrm{d}\mathbf{r} - \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \right] \\
&= \lim_{h \to 0} \frac{1}{h} \int_{\mathbf{x}}^{\mathbf{x}+h\mathbf{e}_i} \mathbf{F} \cdot \mathrm{d}\mathbf{r}
\end{aligned}$$

The path of integration is a straight line from $\mathbf{x}$ to $\mathbf{x}+h\mathbf{e}_i$, and can be parametrised by $\mathbf{r}(t) = \mathbf{x}+th\mathbf{e}_i$ for $t \in [0,1]$. Then

$$\begin{aligned}
\frac{\partial \phi}{\partial \mathbf{x}_i} &= \lim_{h \to 0} \frac{1}{h} \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, \mathrm{d}t \\
&= \lim_{h \to 0} \frac{1}{h} \int_0^1 \mathbf{F}(\mathbf{x} + th\mathbf{e}_i) \cdot h\mathbf{e}_i \, \mathrm{d}t \\
&= \lim_{h \to 0} \int_0^1 \mathbf{F}(\mathbf{x} + th\mathbf{e}_i) \, \mathrm{d}t
\end{aligned}$$

As the integrand is a continuous function of t and $\mathbf{x}$, the mean value theorem for integrals implies that there exists a time $t_0 \in [0,1]$ where the integral is equal to the mean value of $\mathbf{F}$ on $[0,1]$:

$$\frac{\partial \phi}{\partial \mathbf{x}_i} = \lim_{h \to 0} \mathbf{F}(\mathbf{x} + t_0 h\mathbf{e}_i) \cdot \mathbf{e}_i = \mathbf{F}(\mathbf{x}) \cdot \mathbf{e}_i.$$

Without loss of generality, this can be extended to all $i \in [1,n]$, so that:

$$\mathbf{F}(\mathbf{x}) = \boldsymbol{\nabla} \phi(\mathbf{x}).$$

□

Theorem 5.4.4 (Antiderivative of a Conservative Field)**.** *The line integral of* $\mathbf{F}$ *from a fixed point* $\mathbf{x}_0$ *to a variable point* $\mathbf{x}$ *in* Ω *is precisely the potential function* $\phi(\mathbf{x})$ *evaluated at* $\mathbf{x}$*:*

$$\phi(\mathbf{x}) = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

Remark 3. When $\mathbf{F}$ is not a conservative field, the line integral may still be evaluated, but the result will not be path independent.

Theorem 5.4.5 (Curl Criterion for Conservative Fields)**.** *For* $n = 2$ *and* $n = 3$*, if* $\mathbf{F} \subset C^1$ *is a continuous vector field in an open and simply-connected region* $\Omega \subset \mathbb{R}^n$*, then*

$$(\exists \phi \subset C^2 : \mathbf{F} = \boldsymbol{\nabla}\phi) \iff \boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}.$$

Proof. The forward direction uses Schwartz theorem:

$$(\exists \phi \subset C^2 : \mathbf{F} = \boldsymbol{\nabla}\phi) \implies \boldsymbol{\nabla} \times \mathbf{F} = \boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \begin{bmatrix} \phi_x \\ \phi_y \\ \phi_z \end{bmatrix} = \begin{bmatrix} \phi_{zy} - \phi_{yz} \\ -(\phi_{zx} - \phi_{xz}) \\ \phi_{yx} - \phi_{xy} \end{bmatrix} = \mathbf{0}.$$

□

5.4.9 Energy Theorems

Theorem 5.4.6 (Work Energy Theorem)**.** *The total work done by a field* $\mathbf{F}$ *on a particle moving along a curve* $\mathscr{C}$ *from point* $\mathbf{a}$ *to* $\mathbf{b}$ *is equal to the change in kinetic energy* T *of the particle along* $\mathscr{C}$*:*

$$W = T_{\mathbf{b}} - T_{\mathbf{a}}.$$

Proof. From Newton's Second Law

$$\begin{aligned}
m\mathbf{r}''(t) &= \mathbf{F}(\mathbf{r}(t)) \\
m\mathbf{r}'(t) \cdot \mathbf{r}''(t) &= \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \\
\int_a^b m\mathbf{r}'(t) \cdot \mathbf{r}''(t)\,\mathrm{d}t &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t \\
\int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}m\mathbf{r}'(t) \cdot \mathbf{r}'(t)\right]\mathrm{d}t &= \int_{\mathscr{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \\
\int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}m\|\mathbf{v}(t)\|^2\right]\mathrm{d}t &= \int_{\mathscr{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \\
\frac{1}{2}m\|\mathbf{v}(b)\|^2 - \frac{1}{2}m\|\mathbf{v}(a)\|^2 &= \int_{\mathscr{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \\
T_{\mathbf{b}} - T_{\mathbf{a}} &= W
\end{aligned}$$

□

Definition 5.8 (Total Energy)**.** The total energy of a particle is given by $E = T + V$, where V is the potential energy of conservative forces acting on the particle: $\mathbf{F}_{\mathrm{c}} = -\boldsymbol{\nabla}V$.

Theorem 5.4.7 (Conservation of Energy)**.** *The work done by non-conservative forces on a particle moving along a curve $\mathscr{C}$ from point* $\mathbf{a}$ *to* $\mathbf{b}$ *is equal to the change in total energy of the particle along $\mathscr{C}$:*

$$W_{\text{nc}} = E_{\mathbf{b}} - E_{\mathbf{a}}.$$

Proof. From the Work Energy Theorem,

$$\begin{aligned}
T_{\mathbf{b}} - T_{\mathbf{a}} &= W \\
T_{\mathbf{b}} - T_{\mathbf{a}} &= \int_{\mathscr{C}} (\mathbf{F}_{\text{c}} + \mathbf{F}_{\text{nc}}) \cdot \mathrm{d}\mathbf{r} \\
T_{\mathbf{b}} - T_{\mathbf{a}} &= \int_{\mathscr{C}} (-\boldsymbol{\nabla} V + \mathbf{F}_{\text{nc}}) \cdot \mathrm{d}\mathbf{r} \\
T_{\mathbf{b}} - T_{\mathbf{a}} &= -\int_{\mathscr{C}} \boldsymbol{\nabla} V \cdot \mathrm{d}\mathbf{r} + \int_{\mathscr{C}} \mathbf{F}_{\text{nc}} \cdot \mathrm{d}\mathbf{r} \\
T_{\mathbf{b}} - T_{\mathbf{a}} &= -(V_{\mathbf{b}} - V_{\mathbf{a}}) + \int_{\mathscr{C}} \mathbf{F}_{\text{nc}} \cdot \mathrm{d}\mathbf{r} \\
(T_{\mathbf{b}} + V_{\mathbf{b}}) - (T_{\mathbf{a}} + V_{\mathbf{a}}) &= \int_{\mathscr{C}} \mathbf{F}_{\text{nc}} \cdot \mathrm{d}\mathbf{r} \\
E_{\mathbf{b}} - E_{\mathbf{a}} &= W_{\text{nc}}
\end{aligned}$$

□

5.5 Surface Integrals

A surface integral is a multiple integral where the function to be integrated is evaluated across a curved surface. This function may be a scalar field or a vector field.

5.5.1 Surface Area

The surface area of a surface $\mathscr{S}$ parametrised by $\mathbf{r}(s,t)$ is given by

$$A = \iint_{\mathscr{S}} \mathrm{d}\sigma = \iint_{\mathscr{S}} \|\mathbf{r}_s \times \mathbf{r}_t\| \,\mathrm{d}s\,\mathrm{d}t$$

where $\|\mathbf{r}_s \times \mathbf{r}_t\|$ represents the area of the parallelogram spanned by $\mathbf{r}_s$ and $\mathbf{r}_t$.

5.5.2 Surface Integral over a Scalar Field

Let $\mathscr{S}$ be a surface parametrised by $\mathbf{r}(s,t)$. The surface integral of a scalar field f over $\mathscr{S}$ is defined as

$$\iint_{\mathscr{S}} f \,\mathrm{d}\sigma = \iint_{\mathscr{S}} f(\mathbf{r}(s,t)) \,\|\mathbf{r}_s \times \mathbf{r}_t\| \,\mathrm{d}s\,\mathrm{d}t.$$

where $\mathrm{d}\sigma = \|\mathbf{r}_s \times \mathbf{r}_t\| \,\mathrm{d}s\,\mathrm{d}t$ is the differential surface element of $\mathscr{S}$. This integral represents the weighted sum of f over $\mathscr{S}$.

Lemma 5.5.0.1 (Equivalence of Parametrisations)**.** *Let $\mathcal{S}$ be a surface parametrised by $\mathbf{r}(s,t)$ and let $\tilde{\mathbf{r}}(u,v)$ be a reparametrisation of $\mathbf{r}(s,t)$. Then*

$$\iint_{\mathcal{S}} f \,\mathrm{d}\sigma = \iint_{\mathcal{S}} f\left(\tilde{\mathbf{r}}(u,\ v)\right) \|\tilde{\mathbf{r}}_u \times \tilde{\mathbf{r}}_v\| \,\mathrm{d}u \,\mathrm{d}v.$$

so that the surface integral of a scalar field is independent of the parametrisation of $\mathcal{S}$.

5.5.3 Surface Integral over a Vector Field

Let $\mathcal{S}$ be an orientable surface parametrised by $\mathbf{r}(s,t)$. The surface (flux) integral of a vector field $\mathbf{F}$ over $\mathcal{S}$ is defined as the flux Φ of $\mathbf{F}$ through $\mathcal{S}$:

$$\Phi = \iint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\boldsymbol{\sigma} = \pm \iint_{\mathcal{S}} \mathbf{F}\left(\mathbf{r}(s,t)\right) \cdot \left(\mathbf{r}_s \times \mathbf{r}_t\right) \mathrm{d}s \,\mathrm{d}t.$$

where $\hat{\mathbf{n}}$ is the unit normal vector to $\mathcal{S}$, and $\mathrm{d}\boldsymbol{\sigma} = (\mathbf{r}_s \times \mathbf{r}_t)\,\mathrm{d}s\,\mathrm{d}t$ is the differential vector surface element of $\mathcal{S}$, which is related to the differential surface element $\mathrm{d}\sigma$ by

$$\mathrm{d}\sigma = \|\mathrm{d}\boldsymbol{\sigma}\|$$
$$\mathrm{d}\boldsymbol{\sigma} = \hat{\mathbf{n}}\,\mathrm{d}\sigma$$

so that this integral may also be written as

$$\Phi = \iint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\boldsymbol{\sigma} = \pm \iint_{\mathcal{S}} \mathbf{F}\left(\mathbf{r}(s,t)\right) \cdot \hat{\mathbf{n}}\,\mathrm{d}\sigma.$$

This integral represents the outward or inward flow of $\mathbf{F}$ through $\mathcal{S}$, where the direction of flow depends on the sign of $\hat{\mathbf{n}}$.

Lemma 5.5.0.2 (Equivalence of Parametrisations)**.** *Let $\mathcal{S}$ be an orientable surface parametrised by $\mathbf{r}(s,t)$ and let $\tilde{\mathbf{r}}(u,v)$ be a reparametrisation of $\mathbf{r}(s,t)$. Then*

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\boldsymbol{\sigma} = \pm \iint_{\mathcal{S}} \mathbf{F}\left(\tilde{\mathbf{r}}(u,\ v)\right) \cdot \left(\tilde{\mathbf{r}}_u \times \tilde{\mathbf{r}}_v\right) \mathrm{d}u \,\mathrm{d}v.$$

so that the surface integral of a vector field is independent of the parametrisation of $\mathcal{S}$.