Modelling with Differential Equations 1

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Contents

1 Differential Equations

The study of differential equations (DE) finds techniques to solve equations relating unknown functions with their derivatives. These equations allow us to model and predict the behaviour of systems evolving over time or any other dimension.

1.1 Ordinary and Partial Differential Equations

Differential equations fall into one of two categories:

- *Ordinary differential equations* (ODE) — derivatives are taken with respect to only one variable.
- *Partial differential equations* (PDE) — derivatives are taken with respect to several variables.

Some examples of ODEs are shown below:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky \qquad \text{(Exponential growth)}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = k\left(A - y\right) \qquad \text{(Newton's law of cooling)}$$

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + c\frac{\mathrm{d}x}{\mathrm{d}t} + kx = f\left(t\right) \qquad \text{(Mechanical vibrations)}$$

where x, y, and f are functions of time - t. Below are some examples of PDEs:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \qquad \text{(Transport equation)}$$

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad \text{(Heat equation)}$$

$$\boldsymbol{\Delta} u = 0 \qquad \text{(Laplace's equation)}$$

where u is typically a function of space $(x,\ y,\ z)$ or space-time $(x,\ y,\ z,\ t)$.

1.2 Order

The order of a differential equation refers to the highest derivative term that appears in the equation.

1.3 Linearity

A linear ODE is defined by a linear polynomial in the unknown function and its derivatives:

$$a_n\left(t\right)y^{(n)} + \cdots + a_2\left(t\right)y'' + a_1\left(t\right)y' + a_0\left(t\right)y = f\left(t\right).$$

Here the notation $y^{(n)}$ denotes the nth derivative of y, and we use this in-place of the apostrophe ($'$) for compactness. Note that the function y itself, can be thought of as the "0"th derivative of y. Any ODE that cannot be expressed in the above form is nonlinear.

We can also classify PDEs in a similar manner. We will only consider a second-order linear PDE for brevity:

$$A\left(x,\ y\right)u_{xx} + B\left(x,\ y\right)u_{xy} + C\left(x,\ y\right)u_{yy} + D\left(x,\ y\right)u_x + E\left(x,\ y\right)u_y + F\left(x,\ y\right)u = f\left(x,\ y\right)$$

where we have a linear combination of second-order partial derivatives of u. Due to the complexity of PDEs, other classifications also exist, but these will not be discussed.

1.4 Homogeneity

Finally, a differential equation is homogeneous when it is an equation consisting only of the unknown function and its derivatives. In other words, there are no constant terms (or functions).
Both ODEs and PDEs are classified by the same rules, below is an example of homogeneous DEs:

$$ay'' + by' + cy = 0$$
$$\frac{\partial^2 u}{\partial x^2} + k\frac{\partial^2 u}{\partial t^2} = 0.$$

As we will see in further sections, differential equations are often simpler to solve when they are homogeneous.

1.5 Useful Theorems

Theorem 1.5.1 (Existence and Uniqueness Theorem[1])**.** *Let the functions f and $\frac{\partial f}{\partial y}$ be Lipschitz continuous[2] on the closed rectangle $D \subseteq \mathbb{R}$ with $(t_0,\ y_0) \in D$. Then there exists some interval $I \subseteq D$, centred on t_0 such that there is a unique solution to the initial value problem:*

$$\begin{cases} y'(t) = f(t,\ y) \\ y(t_0) = y_0 \end{cases}$$

Corollary 1.5.1.1 (Existence and Uniqueness Theorem for Higher-Order ODEs)**.** *Let the functions $\frac{\partial^i f}{\partial y^i}$ for all $i = 0, 1, \dots, n-1$, be Lipschitz continuous on the closed $(n+1)$-dimensional hyperrectangle $D \subseteq \mathbb{R}\times\mathbb{R}^n$ with $\left(t_0,\ y_0,\ y_0',\ y_0'',\ \dots,\ y_0^{(n-1)}\right) \in D$. Then there exists some interval $I \subseteq D$ containing t_0, such that there is a unique solution to the initial value problem:*

$$\begin{cases} y^{(n)}(t) = f\left(t,\ y,\ y',\ y'',\ \dots,\ y^{n-1}\right) \\ y^{(i)}(t_0) = y_0^{(i)} \end{cases}$$

for all $i = 0,\ 1,\ \dots,\ n-1$. Note that $y_0^{(i)}$ is the constant value associated with the initial value of the ith derivative of y at t_0.

Theorem 1.5.2 (Principle of Superposition)**.** *If y_1 is a solution to the equation*

$$ay'' + by' + cy = f_1(t),$$

and y_2 are solutions to

$$ay'' + by' + cy = f_2(t),$$

then for any constants c_1 and c_2, the linear combination $y = c_1y_1 + c_2y_2$ is a solution to the differential equation

$$ay'' + by' + cy = c_1f_1(t) + c_2f_2(t).$$

[1] Also known as the Picard-Lindelöf Theorem

[2] When f and $\frac{\partial f}{\partial y}$ are *continuous*, then a solution exists with no information about its uniqueness. See the Peano Existence Theorem for more information.

Proof. We can prove the above result through substitution,

$$\begin{aligned} ay'' + by' + cy &= a\left(c_1 y_1 + c_2 y_2\right)'' + b\left(c_1 y_1 + c_2 y_2\right)' + c\left(c_1 y_1 + c_2 y_2\right) \\ &= ac_1 y_1'' + ac_2 y_2'' + bc_1 y_1' + bc_2 y_2' + cc_1 y_1 + cc_2 y_2 \\ &= c_1\left(ay_1'' + by_1' + cy_1\right) + c_2\left(ay_2'' + by_2' + cy_2\right) \\ &= c_1 f_1(t) + c_2 f_2(t) \end{aligned}$$

A similar proof demonstrates that this principle also applies for higher-order linear ODEs. □

2 Constant Coefficient ODEs

A constant coefficient ODE is a linear ODE with coefficients that do not depend on the independent variable:

$$a_n y^{(n)} + \cdots + a_2 y'' + a_1 y' + a_0 y = f(t).$$

Using the superposition principle, we can express solutions as

$$y = c_1 y_h + c_2 y_p.$$

where:

- y_h is a solution to the *homogeneous* ODE, that is, when $f(t) = 0$, and is known as the "homogeneous solution".
- y_p is a solution to the *nonhomogeneous* ODE, and is known as the "particular solution".

2.1 Homogeneous Solution

As the homogeneous ODE is linear, we will assume[3] that the solution y_h has the form

$$y_h(t) = e^{rt}.$$

Let us then substitute this into the homogeneous ODE:

$$\begin{aligned} a_n y_h^{(n)} + \cdots + a_2 y_h'' + a_1 y_h' + a_0 y_h &= 0 \\ a_n r^n e^{rt} + \cdots + a_2 r^2 e^{rt} + a_1 r e^{rt} + a_0 e^{rt} &= 0 \\ \left(a_n r^n + \cdots + a_2 r^2 + a_1 r + a_0\right) e^{rt} &= 0 \\ a_n r^n + \cdots + a_2 r^2 + a_1 r + a_0 &= 0 \end{aligned}$$

This result is known as the **characterisic equation** (also **complementary equation** or **auxiliary equation**) of our ODE. Finding the *roots* of this algebraic equation allows us to find y_h. As this is a polynomial expression, we expect three kinds of solutions:

- Real distinct roots: $r_1, r_2, \ldots, r_k \in \mathbb{R}$
- Complex conjugate roots: $r, r^* \in \mathbb{C}$
- Real repeated roots: $r \in \mathbb{R}$

For higher order ODEs that may have real, repeated, and complex roots simultaneously, we can take the linear combination of the solutions discussed below.

[3] A common term for such assumptions is also *ansatz*.

2.1.1 Real Distinct Roots

Given real distinct roots $r_1,\ r_2,\ \dots,\ r_k$, the homogeneous solution is given by

$$y_h = c_1 e^{r_1 t} + c_2 e^{r_2 t} \cdots + c_k e^{r_k t}$$

for $k \leq n$.

2.1.2 Complex Conjugate Roots

Given complex conjugate roots $r = \alpha + \beta i$ and $r^* = \alpha - \beta i$, the homogeneous solution is given by

$$y_h = k_1 e^{(\alpha+\beta i)t} + k_2 e^{(\alpha-\beta i)t}.$$

We can then simplify this solution using Euler's identity.

$$\begin{aligned} y_h &= e^{\alpha t}\left(k_1 e^{\beta i t} + k_2 e^{-\beta i t}\right) \\ &= e^{\alpha t}\left[k_1\left(\cos\left(\beta t\right) + i\sin\left(\beta t\right)\right) + k_2\left(\cos\left(\beta t\right) - i\sin\left(\beta t\right)\right)\right] \\ &= e^{\alpha t}\left[k_1\cos\left(\beta t\right) + ik_1\sin\left(\beta t\right) + k_2\cos\left(\beta t\right) - ik_2\sin\left(\beta t\right)\right] \\ &= e^{\alpha t}\left[\left(k_1 + k_2\right)\cos\left(\beta t\right) + i\left(k_1 - k_2\right)\sin\left(\beta t\right)\right] \\ &= e^{\alpha t}\left[c_1\cos\left(\beta t\right) + c_2\sin\left(\beta t\right)\right] \end{aligned}$$

which demonstrates that complex conjugate roots correspond to oscillatory solutions.

2.1.3 Real Repeated Roots

When $r = -\frac{a_1}{2a_2}$ is a repeated root with multiplicity k, a solution is given by

$$y_{h,1} = e^{rt}.$$

However, as we will see later, this is not sufficient.

2.2 Reduction of Order

The reduction of order method is a method for converting any linear ODE to another linear ODE of lower order. If the solution y_1 is known in advance, we can construct additional solutions via the following substitution:

$$y_2 = u\left(t\right) y_1.$$

2.2.1 Obtaining the General Solution in a Repeated Roots Problem

In the previous section, we were unable to find a complete set of homogeneous solutions to our ODE. Therefore let us use this new method to find other solutions that satisfy the homogeneous equation.

$$y_{h,2} = u\left(t\right) y_{h,1}.$$

The following example uses a second-order ODE, but the process is applicable in general. Before we can substitute this equation into the ODE, let us find its first and second derivatives:

$$y'_{h,2} = u' y_{h,1} + u y'_{h,1}$$
$$y''_{h,2} = u'' y_{h,1} + u' y'_{h,1} + u' y'_{h,1} + u y''_{h,1} = u'' y_{h,1} + 2u' y'_{h,1} + u y''_{h,1}$$

We can now substitute this equation into the homogeneous ODE:

$$\begin{aligned}
a_2 y''_{h,2} + a_1 y'_{h,2} + a_0 y_{h,2} &= 0 \\
a_2 \left(u'' y_{h,1} + 2u' y'_{h,1} + u y''_{h,1}\right) + a_1 \left(u' y_{h,1} + u y'_{h,1}\right) + a_0 u y_{h,1} &= 0 \\
a_2 u'' y_{h,1} + 2a_2 u' y'_{h,1} + a_2 u y''_{h,1} + a_1 u' y_{h,1} + a_1 u y'_{h,1} + a_0 u y_{h,1} &= 0 \\
a_2 u'' e^{rt} + 2a_2 u' r e^{rt} + a_2 u r^2 e^{rt} + a_1 u' e^{rt} + a_1 u r e^{rt} + a_0 u e^{rt} &= 0 \\
\left[a_2 u'' + 2a_2 r u' + a_2 r^2 u + a_1 u' + a_1 r u + a_0 u\right] e^{rt} &= 0 \\
a_2 u'' + \left(2a_2 r + a_1\right) u' + \left(a_2 r^2 + a_1 r + a_0\right) u &= 0
\end{aligned}$$

As $r = -\frac{a_1}{2a_2}$, the coefficients of both u' and u become zero, resulting in:

$$a_2 u'' = 0$$

so that $u'' = 0$ because $a_2 \neq 0$. Integrating twice gives the solution:

$$u\left(t\right) = k_1 t + k_2.$$

Let $k_2 = 0$ so that:

$$y_{h,2} = t e^{rt}.$$

Finally, combining both solutions yields the general homogeneous solution:

$$y = c_1 e^{rt} + c_2 t e^{rt}.$$

If follows that this result applies to higher-order linear ODEs with repeated roots of any multiplicity.

Proposition 1. *A linear homogeneous ODE of order n with the root r of multiplicity $k \leq n$ has the fundamental form:*

$$y = e^{rt}\left(1 + t + t^2 + \cdots + t^k\right).$$

2.3 Fundamental Set of Solutions

When determining the solutions of an nth order linear homogeneous ODE, we require a **general** solution y_h which consists of n linearly independent functions $y_{h,i}$ (for $i = 1, 2, \ldots, n$) such that the set of solutions form a **fundamental set of solutions** over the problem domain.
As was alluded to before, this is done by taking the linear combination of each "basis" function $y_{h,i}$, to find the general homogeneous solution:

$$y_h = c_1 y_{h,1} + c_2 y_{h,2} + \cdots + c_n y_{h,n}.$$

This is
To show that a set of functions are linearly independent, we must construct a definitive test, therefore, consider the following.

2.3.1 Test for Linear Independence

The set of $(n-1)$-times differentiable functions $S = \{y_1,\ y_2,\ \dots,\ y_n\}$ is linearly independent iff

$$c_1y_1 + c_2y_2 + \dots + c_ny_n = 0$$

has the trivial solution $c_1 = \dots = c_n = 0$, for all t. Let us then differentiate this equation $n-1$ times to find the following system of equations:

$$\begin{aligned}
c_1y_1 + c_2y_2 + \dots + c_ny_n &= 0 \\
c_1y_1' + c_2y_2' + \dots + c_ny_n' &= 0 \\
c_1y_1'' + c_2y_2'' + \dots + c_ny_n'' &= 0 \\
&\vdots \\
c_1y_1^{(n-1)} + c_2y_2^{(n-1)} + \dots + c_ny_n^{(n-1)} &= 0
\end{aligned}$$

which we can express as

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Recall from Linear Algebra that the linear system:

$$\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$$

has a unique solution when $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}([\mathbf{A} \mid \boldsymbol{b}]) = n$, where $\boldsymbol{b} = \mathbf{0}$. This is true when the matrix $\mathbf{A}$ is invertible, so that

$$\begin{aligned}
\mathbf{A}\boldsymbol{c} &= \mathbf{0} \\
\mathbf{A}^{-1}\mathbf{A}\boldsymbol{c} &= \mathbf{A}^{-1}\mathbf{0} \\
\boldsymbol{c} &= \mathbf{0}
\end{aligned}$$

as required.
Therefore we can conclude that when the determinant of $\mathbf{A}$ is nonzero, the set S is linearly independent. Let us summarise these findings in the following section and introduce the Wronskian matrix.

2.3.2 The Wronskian

Definition 2.1 (Wronskian Matrix)**.** Given the set of $(n-1)$-times differentiable functions $S = \{y_1,\ y_2,\ \dots,\ y_n\}$, the Wronskian matrix $\mathbf{W}$ is defined as

$$\mathbf{W} = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix}.$$

The "Wronskian" of these functions is given by

$$W = W\left[y_1,\, y_2,\, \dots,\, y_n\right] = \det\left(\mathbf{W}\right).$$

Theorem 2.3.1 (Linear Independence and the Wronskian)**.** *The set of* $(n-1)$*-times differentiable functions* $S = \{y_1,\, y_2,\, \dots,\, y_n\}$ *is linearly independent if*

$$W\left[y_1,\, y_2,\, \dots,\, y_n\right] \neq 0.$$

Note that the converse does not apply: $W\left[y_1,\, y_2,\, \dots,\, y_n\right] = 0$ *does not imply that* S *is linearly dependent.*

Corollary 2.3.1.1. *If the set* S *is linearly dependent, then*

$$W\left[y_1,\, y_2,\, \dots,\, y_n\right] = 0.$$

2.4 Particular Solution

As with the homogeneous solution, the method of *Undetermined Coefficients* requires us to guess a solution y_p based on the form of $f(t)$[4].

2.4.1 Choosing a Particular Solution

Consider the vector space of particularly nice functions W with the basis:

for all $n \in \mathbb{N}_{>0}$ and $r \in \mathbb{R}_{>0}$.
If f can be written as a linear combination or product of functions in W, then we can construct a particular solution y_p that satisfies the nonhomogeneous ODE. The table below shows the choice of y_p for various f.

Form of f	**Particular Solution** y_p
$P_n(t)$	$\sum_{i=0}^{n} A_i t^i$
e^{rt}	Ce^{rt}
$T(\omega t)$	$K\cos(\omega t) + M\sin(\omega t)$
$P_n(t)\,e^{rt}$	$\left(\sum_{i=0}^{n} A_i t^i\right) e^{rt}$
$e^{rt}T(\omega t)$	$e^{rt}\left[K\cos(\omega t) + M\sin(\omega t)\right]$
$P_n(t)\,e^{rt}T(\omega t)$	$e^{rt}\left[\left(\sum_{i=0}^{n} K_i t^i\right)\cos(\omega t) + \left(\sum_{i=0}^{n} M_i t^i\right)\sin(\omega t)\right]$

Table 1: Choice of particular solution based on the form of the nonhomogeneous term where $n \in \mathbb{N}_{\geq 0}$, $r,\, \omega \in \mathbb{R}_{>0}$, and $A,\, C,\, K,\, M \in \mathbb{R}$. Any constant coefficients in f are to be ignored.

[4] $f(t)$ is known as the **forcing function** or **input** in the system, and only depends on t.

In the above table, $P_n(t)$ is any polynomial function of order n, $T(t)$ is used to denote any of the following trigonometric functions:

$$T(t) = \begin{cases} \cos(t) \\ \sin(t) \\ \cos(t) \pm \sin(t) \end{cases}$$

For more complex forms of f as shown in the last three rows of the table above, we can use the following facts:

- If f can be expressed as the linear combination of two forms f_1 and f_2, then the particular solution is also a linear combination of the particular solutions associated with f_1 and f_2.
- If f can be expressed as a product of two forms f_1 and f_2, then the particular solution will also be the product of the particular solutions associated with f_1 and f_2. In this situation, it is important to always simplify all arbitrary constants as shown in the table above.

In general, it is sufficient to only memorise the first three rows of the table above.

2.4.2 Undetermined Coefficients

Equipped with a particular solution y_p, we can substitute it into the LHS of the nonhomogeneous ODE to obtain:

$$a_n y_p^{(n)} + \cdots + a_2 y_p'' + a_1 y_p' + a_0 y_p = f(t).$$

We then simplify the LHS and factor out any terms that are terms that depend on t. For example:

$$(\dots) f_1 + (\dots) f_2 + \cdots + (\dots) f_3 f_4 + \cdots = f.$$

The purpose of doing so is to construct a system of equations that allows us to solve for the undetermined coefficients in y_p.

2.4.3 The Resonance Case

In some problems, the forcing term $f(t)$ may have a form similar to y_h leading to a linearly dependent solution y_p. In this case, we can use the *Reduction of Order* method, as we did for repeated roots, and multiply y_p by t until it is linearly independent to y_h.

2.5 Initial Value Problems

Real-world systems modelled using differential equations are accompanied by initial conditions that allow us to eliminate any remaining constants in our general solution *after* we have found **both** the homogeneous and particular solutions (where applicable).

For an nth order ODE, we require n initial conditions y_0, corresponding to the first n derivatives of y:

$$\begin{aligned} y(t_0) &= y_0 \\ y'(t_0) &= y_0' \\ &\vdots \\ y^{(n-1)}(t_0) &= y_0^{(n-1)} \end{aligned}$$

note that y_0 is a constant and we are using the dash ($'$) for notational convenience.

3 Nonconstant Coefficient ODEs

A nonconstant coefficient ODE (or a linear ODE) has the following form:

$$a_n(t)\,y^{(n)} + \cdots + a_2(t)\,y'' + a_1(t)\,y' + a_0(t)\,y = f(t)\,.$$

where a_i depends on the independent variable. In the previous section, we found that the method of *Undetermined Coefficients* was insufficient for ODEs with more complex forcing functions. As we will see soon, we can construct a different particular solution where our guess depends on the homogeneous solutions.

3.1 Variation of Parameters

Given a set of homogeneous solutions y_1, y_2, ..., and y_n to the above linear ODE, consider a particular solution of the form

$$y_p = v_1(t)\,y_1 + v_2(t)\,y_2 + \cdots + v_n(t)\,y_n$$

where v_i are now functions of the independent variable.
As before, we want to substitute this function into the linear ODE above, which requires us to compute the derivatives of y_p. Because y_p has several products, we will impose additional constraints such that we eliminate any derivatives of v in our calculations (except for in the final derivative)[5]. This will give:

$$\begin{aligned} y_p^{(k)} &= v_1 y_1^{(k)} + v_2 y_2^{(k)} + \cdots + v_n y_n^{(k)} && k = 0,\ 1,\ \ldots,\ n-1 \\ y_p^{(n)} &= v_1 y_1^{(n)} + v_2 y_2^{(n)} + \cdots + v_n y_n^{(n)} + v_1' y_1^{(n-1)} + v_2' y_2^{(n-1)} + \cdots + v_n' y_n^{(n-1)} \end{aligned}$$

where we make the following constraints:

$$0 = v_1' y_1^{(k)} + v_2' y_2^{(k)} + \cdots + v_n' y_n^{(k)} \qquad k = 0,\ 1,\ \ldots,\ n-2.$$

Explicitly this looks like the following:

$$\begin{cases} y_p = v_1 y_1 + v_2 y_2 + \cdots + v_n y_n \\ y_p' = v_1 y_1' + v_2 y_2' + \cdots + v_n y_n' \\ \vdots \\ y_p^{(n-1)} = v_1 y_1^{(n-1)} + v_2 y_2^{(n-1)} + \cdots + v_n y_n^{(n-1)} \\ y_p^{(n)} = v_1 y_1^{(n)} + v_2 y_2^{(n)} + \cdots + v_n y_n^{(n)} + v_1' y_1^{(n-1)} + v_2' y_2^{(n-1)} + \cdots + v_n' y_n^{(n-1)} \end{cases}$$

with the constraints:

$$\begin{cases} 0 = v_1' y_1 + v_2' y_2 + \cdots + v_n' y_n \\ 0 = v_1' y_1' + v_2' y_2' + \cdots + v_n' y_n' \\ \vdots \\ 0 = v_1' y_1^{(n-2)} + v_2' y_2^{(n-2)} + \cdots + v_n' y_n^{(n-2)} \end{cases}$$

[5] See the appendix for an example using a second-order linear ODE.

We can now substitute y_p into the ODE to find

$$a_n y_p^{(n)} + a_{n-1} y_p^{(n-1)} + \cdots + a_1 y_p' + a_0 y_p = f(t)$$

$$\begin{aligned}
a_0 \left[v_1 y_1 + v_2 y_2 + \cdots + v_n y_n\right] &+ \\
a_1 \left[v_1 y_1' + v_2 y_2' + \cdots + v_n y_n'\right] &+ \\
&\vdots \\
a_n \left[v_1 y_1^{(n)} + v_2 y_2^{(n)} + \cdots + v_n y_n^{(n)}\right] &+ \\
a_n \left[v_1' y_1^{(n-1)} + v_2' y_2^{(n-1)} + \cdots + v_n' y_n^{(n-1)}\right] &= f(t)
\end{aligned}$$

$$\begin{aligned}
v_1 \left[a_n y_1^{(n)} + a_{n-1} y_1^{(n-1)} + \cdots + a_1 y_1' + a_0 y_1\right] &+ \\
v_2 \left[a_n y_2^{(n)} + a_{n-1} y_2^{(n-1)} + \cdots + a_1 y_2' + a_0 y_2\right] &+ \\
&\vdots \\
v_n \left[a_n y_n^{(n)} + a_{n-1} y_n^{(n-1)} + \cdots + a_1 y_n' + a_0 y_n\right] &+ \\
a_n \left[v_1' y_1^{(n-1)} + v_2' y_2^{(n-1)} + \cdots + v_n' y_n^{(n-1)}\right] &= f(t)
\end{aligned}$$

This simplifies greatly and results in:

$$v_1' y_1^{(n-1)} + v_2' y_2^{(n-1)} + \cdots + v_n' y_n^{(n-1)} = g(t)$$

where $g(t) = f(t)/a_n$. Using the constraints stated previously, we reach the following system of equations:

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \\ \vdots \\ v_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(t) \end{bmatrix}.$$

Using Cramer's rule, we find that the solution to this system is given by

$$v_i' = \frac{W_i}{W}$$

where W is the Wronskian, and W_i is the determinant of the Wronskian matrix where the ith column is replaced with the RHS vector $\boldsymbol{b}$. Integration[6] yields:

$$y_p = y_1 \int \frac{W_1}{W}\,\mathrm{d}t + y_2 \int \frac{W_2}{W}\,\mathrm{d}t + \cdots + y_n \int \frac{W_n}{W}\,\mathrm{d}t.$$

In this section, we built our particular solution from solutions to the homogeneous ODE; let us now discuss two methods to determine the homogeneous solution for linear ODEs.

[6]Due to the rational form of the integrand, a closed-form solution may not exist.

4 Power Series Solutions

Consider the following linear ODE:

$$y'' + P(t)\, y' + Q(t)\, y = f(t)\,.$$

let us consider a power series solution of the form

$$y = \sum_{n=0}^{\infty} c_n t^n.$$

This allows us to form the following derivatives:

$$y' = \sum_{n=1}^{\infty} c_n n t^{n-1}, \qquad y'' = \sum_{n=2}^{\infty} c_n n\,(n-1)\, t^{n-2} \qquad \ldots$$

As with constant coefficient ODEs, we can determine c_n through substitution:

$$\sum_{n=2}^{\infty} c_n n\,(n-1)\, t^{n-2} + P(t) \sum_{n=1}^{\infty} c_n n t^{n-1} + Q(t) \sum_{n=0}^{\infty} c_n t^n = f(t)\,.$$

Here we must use the following steps to simplify this equation:

1. Shift the index n of any summation so that the exponent of t is the same in each summation.
2. Extract lower order terms from summations so that all summations start from the same index.
3. Factor all summations together.

This results in an equation of the form:

$$\mathrm{RR}_0 t^p + \sum_{n=n_0}^{\infty} \mathrm{RR} t^{n+q} = f(t)\,.$$

where p and q are some integer value, and RR and RR_0 depend on n and c_n.
We can solve this expression by equating linearly dependent terms w.r.t. t. As $f(t)$ is not guaranteed to have a simple Taylor series expansion, we will typically consider the homogeneous problem first. This forces all coefficients of t to be equal to 0, and hence allows us to form a **recurrence relationship** using $\mathrm{RR} = 0$. This relationship will be of the form:

$$c_{n+u} = v(n)\, c_n$$

where higher order terms depend on lower order terms ($u > 0$ and v may depend on n). By substituting $n = n_0,\ n_0 + 1,\ n_0 + 2,\ \ldots$, we can substitute these coefficients back into the original power series, and, if possible, find an explicit form of the coefficients c_n.

4.1 Classification of Points

Values of $t \in \mathbb{C}$ can be classified into **ordinary points**, at which the ODE's coefficients are analytic, and **singular points** at which some coefficient has a singularity (i.e., division by 0).
A singular point $t = t_0$ is a **regular singular point** if

$$(t - t_0)\, P(t) \qquad \text{and} \qquad (t - t_0)^2\, Q(t)$$

are analytic at t_0. A singular point that is not regular is called an **irregular singular point**.

4.2 Frobenius Method

When $t = t_0$ forms a regular singular point, there exists at least one series solution of the form

$$y = \sum_{n=0}^{\infty} c_n \left(t - t_0\right)^{n+r}.$$

where r is some real number to be determined. This gives us the following derivatives:

$$y' = \sum_{n=0}^{\infty} c_n \left(n + r\right) t^{n+r-1}, \qquad y' = \sum_{n=0}^{\infty} c_n \left(n + r\right)\left(n + r - 1\right) t^{n+r-2} \qquad \ldots$$

Using a similar approach, we can substitute these expressions into our ODE to find a second relationship between the roots r, this is known as the **indicial equation**.

$$\mathrm{IE} t^{r+p} + \sum_{n=n_0}^{\infty} \mathrm{RR} t^{n+r+q} = f\left(t\right).$$

where again, p and q are some integers, and IE and RR depend on n, r, and c_n.

4.2.1 Cases of Indicial Roots

In the event that the indicial equation yields several **indicial roots**, we must take the following into account. Given real roots r_1 and r_2 with $r_1 > r_2$:

1. When $r_1 - r_2 \notin \mathbb{N}$:

$$y_1\left(t\right) = \sum_{n=0}^{\infty} c_n t^{n+r_1}$$
$$y_2\left(t\right) = \sum_{n=0}^{\infty} b_n t^{n+r_2}$$

   with $c_0 \neq 0$ and $b_0 \neq 0$.

2. When $r_1 - r_2 \in \mathbb{N}$:

$$y_1\left(t\right) = \sum_{n=0}^{\infty} c_n t^{n+r_1}$$
$$y_2\left(t\right) = c y_1\left(t\right) \ln\left(t\right) + \sum_{n=0}^{\infty} b_n t^{n+r_2}$$

   with $c_0 \neq 0$ and $b_0 \neq 0$, where c may be 0.

3. When $r_1 = r_2$:

$$y_1\left(t\right) = \sum_{n=0}^{\infty} c_n t^{n+r_1}$$
$$y_2\left(t\right) = y_1\left(t\right) \ln\left(t\right) + \sum_{n=1}^{\infty} b_n t^{n+r_2}$$

   with $c_0 \neq 0$.

When roots differ by an integer, notice that solutions may be linearly dependent. In this case, we can strategically choose roots that allow us to form a complete set of solutions. For instance, in the following indicial equation,

$$c_0 r\,(r+1)\,t^{r-1} + a_1\,(r+1)\,(r+2)\,t^r = 0$$

we can choose:

- $r = -1$, where c_0 and c_1 are arbitrary,
- $r = 0$, where $c_1 = 0$ but c_0 is arbitrary,
- $r = -2$, where $c_0 = 0$ but c_1 is arbitrary.

By choosing $r = -1$, we mitigate the risk of reducing higher order terms that may depend on c_0 or c_1 to 0.

5 Laplace Transforms

The Laplace transfrom is a type of integral transform that maps a time-dependent problem in t to the complex s-domain, it is defined:

$$F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, \mathrm{d}t.$$

In the context of ODEs, this transformation allows us to take difficult nonlinear problems and solve them using the inverse Laplace transformation.

5.1 Partial Fraction Decomposition

In this unit, we will use tables to find the inverse Laplace transform, rather than using the explicit definition. To do so, we must make use of partial fraction decomposition (PFD).
Given the fraction

$$\frac{P(t)}{Q(t)}$$

where P and Q are polynomials in t and the degree of P is less than the degree of Q, we can decompose a fraction by considering the following cases:
Assume Q is fully factored.

Case I: Q has distinct polynomial factors:

$$Q(t) = q_1(t)\, q_2(t) \cdots q_k(t)$$

then,

$$\frac{P(t)}{Q(t)} = \frac{A_1(t)}{q_1(t)} + \frac{A_2(t)}{q_2(t)} + \cdots + \frac{A_k(t)}{q_k(t)}.$$

where A_i has the form:

$$A_i(t) = a_{i,n-1}t^{n-1} + a_{i,n-2}t^{n-2} + \cdots + a_{i,1}t + a_{i,0}$$

where the order of A_i will be one less than the order of the denominator q_i.

Case II: Q has repeated polynomial factors:

$$Q(t) = q(t)^r$$

then,

$$\frac{P(t)}{Q(t)} = \frac{A_1(t)}{q(t)} + \frac{A_2(t)}{q(t)^2} + \cdots + \frac{A_r(t)}{q(t)^r}.$$

where the denominator will have increasing powers of q.

Case III: Q has both distinct and repeated polynomial factors,
the fraction will contain a combination of the above two cases.

5.2 Heaviside Step Function

The (Heaviside) step function (also unit step function) is defined:

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 \le t \end{cases}$$

along with the shifted step function:

$$u_c(t) = u(t-c) = \begin{cases} 0 & t < c \\ 1 & t \ge c. \end{cases}$$

These special functions allow us to express piecewise functions more compactly. For example,

$$f(t) = \begin{cases} f_1(t) & 0 \le t < a \\ f_2(t) & t \ge a \end{cases}$$

can be expressed as

$$\begin{aligned} f(t) &= f_1(t)\,u(t-0) \underbrace{-f_1(t)\,u(t-a)}_{\text{turn } f_1 \text{ off after } t=a} + f_2(t)\,u(t-a) \\ &= f_1(t)\left[u(t) - u(t-a)\right] + f_2(t)\,u(t-a). \end{aligned}$$

Additionally,

$$f(t) = \begin{cases} 0 & 0 \le t < a \\ f_p(t) & a \le t < b \\ 0 & t \ge b \end{cases}$$

can be expressed as

$$\begin{aligned} f(t) &= f_p(t)\,u(t-a) - f_p(t)\,u(t-b) \\ &= f_p(t)\left[u(t-a) - u(t-b)\right] \end{aligned}$$

To summarise

$$f_1(t)\left[u(t) - u(t-a)\right] + f_2(t)\,u(t-a) \equiv \begin{cases} f_1(t) & 0 \le t < a \\ f_2(t) & t \ge a \end{cases}$$

and

$$f(t)\left[u(t-a) - u(t-b)\right] \equiv \begin{cases} 0 & 0 \le t < a \\ f(t) & a \le t < b \\ 0 & t \ge b \end{cases}$$

5.3 Properties of Laplace Transforms

5.3.1 First Translation Theorem

$$\mathscr{L}\left\{e^{at} f(t)\right\} = F(s-a)$$

5.3.2 Second Translation Theorem

$$\mathscr{L}\left\{f\left(t-a\right)u\left(t-a\right)\right\} = e^{-as}F\left(s\right)$$

5.4 Linear Systems

A linear system of ODEs is a system of the form

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

where $\mathbf{x}$ is an $n \times 1$ vector of functions $x_1\left(t\right), x_2\left(t\right), \dots, x_n\left(t\right)$. So far we have looked at cases where $n = 1$ and $n = 2$, however the methods we have used can be extended to higher order systems.
To convert the nth order constant coefficient linear differential equation

$$a_n y^{(n)} + \dots + a_2 y'' + a_1 y' + a_0 y = f\left(t\right)$$

into a first-order linear system, define the state variables $x_1 = y$, $x_2 = y'$, ..., $x_n = y^{(n-1)}$, so that $\mathbf{x} = \langle x_1,\ x_2,\ \dots,\ x_n\rangle$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0/a_n & -a_1/a_n & -a_2/a_n & \cdots & a_{n-1}/a_n \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

The solution cases for systems of ODEs will be discussed in the next section.

6 Phase Plane Analysis

When a differential equation cannot be solved analytically, we often wish to draw conclusions about its qualitative behaviour. This is particularly useful for nonlinear differential equations, where the methods discussed so far cannot be used to find an explicit solution.
The phase space method is a graphical method for analysing a system through a phase portait, where the flow of states is represented by a vector field. The phase space is a space of the state variables of the system (i.e., the derivatives of each dependent variable). In 1 dimension, this is known as a phase line, and in 2 dimensions, a phase plane.
This plot allows us to visualise limiting behaviour in various regions of the phase space.

6.1 Autonomous Systems

For each of the systems discussed in this section, consider an autonomous system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = g\left(x\right)$$

that does not have explicit dependence on t.

6.1.1 Equilibrium Points

The point where $g\left(x\right) = 0$ is known as **equilibrium point** x_e, such that if $x\left(0\right) = x_e$ is an initial condition, the system has a trivial solution: $x\left(t\right) = 0$.

6.1.2 Stability

The stability of each equilibrium point can be determined by considering the behaviour of the system in the vicinity of the equilibrium point. An equilibrium point is **stable** if the system returns to the equilibrium point after a small perturbation, and **unstable** if the system diverges from the equilibrium point after a small perturbation.

- When $g'\left(x_e\right) < 0$, the equilibrium point is **stable**.
- When $g'\left(x_e\right) > 0$, the equilibrium point is **unstable**.

To understand why this is the case, consider the Taylor series expansion of $g\left(x\right)$ about x_e.

$$g\left(x\right) = g\left(x_e\right) + g'\left(x_e\right)\left(x - x_e\right) + \mathcal{O}\left(x^2\right)$$

such that when x is close to x_e, the behaviour of the system is approximately linear:

$$g\left(x\right) \approx g'\left(x_e\right)\left(x - x_e\right).$$

By solving this linear system, we find

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &\approx g'\left(x_e\right)\left(x - x_e\right) \\ \frac{1}{x - x_e}\frac{\mathrm{d}x}{\mathrm{d}t} &\approx g'\left(x_e\right) \\ \int \frac{1}{x - x_e}\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t &\approx \int g'\left(x_e\right)\mathrm{d}t \\ \ln\left(x - x_e\right) &\approx g'\left(x_e\right)t + c \\ x - x_e &\approx Ae^{g'(x_e)t} \\ x &\approx x_e + Ae^{g'(x_e)t}\end{aligned}$$

near the equilibrium point. Therefore,

- If $g'\left(x_e\right) < 0$: $x \to x_e$ as $t \to \infty$.
- If $g'\left(x_e\right) > 0$: x diverges from x_e as $t \to \infty$.

When the derivative is unobtainable, we can plot the function $g\left(x\right)$ w.r.t. x. Any intersections with the x-axis are equilibrium points, and the stability of each point can be determined by considering the sign of $g\left(x\right)$ around each equilibrium point.
By determining local behaviour around each equilibrium point, we can infer the global behaviour of the system.

6.2 Linear Systems

Given the linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x},$$

solutions are superpositions of the form:

$$\mathbf{x} = e^{\lambda t}\mathbf{v}$$

where $(\lambda, \mathbf{v})$ are eigenpairs of $\mathbf{A}$. For a 2D system, the characteristic polynomial for $\mathbf{A}$ is given by

$$P\left(\lambda\right) = \lambda^2 - \tau\lambda + \Delta$$

where $\tau = \mathrm{Tr}\left(\mathbf{A}\right)$ and $\Delta = \det\left(\mathbf{A}\right)$. The eigenvalues λ are the roots of $P\left(\lambda\right)$.

6.3 Stability

In a linear system, we can construct phase portaits about the equilibrium point at the origin. The following sections consider each case of eigenvalues.

6.3.1 Real Distinct Eigenvalues

The following case is possible when $\tau^2 - 4\Delta > 0$.

When λ_1 and λ_2 are real and distinct eigenvalues, with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, then the general solution is given by

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2.$$

The stability of this system is determined by the sign of λ_1 and λ_2.

- When $\lambda_1, \lambda_2 < 0$, the origin is a **stable node**.
- When $\lambda_1, \lambda_2 > 0$, the origin is an **unstable node**.
- When $\lambda_1 < 0 < \lambda_2$, the origin is a **saddle point**.

To draw a phase portait, draw two lines spanned by the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, where the direction of the arrows on each line is determined by the sign of the corresponding eigenvalue.

6.3.2 Real Repeated Eigenvalues

The following case is possible when $\tau^2 - 4\Delta = 0$.

When λ is a real repeated eigenvalue, and if λ has two linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, the general solution is given by

$$\mathbf{x}(t) = c_1 e^{\lambda t}\mathbf{v}_1 + c_2 e^{\lambda t}\mathbf{v}_2 = e^{\lambda t}\left(c_1\mathbf{v}_1 + c_2\mathbf{v}_2\right).$$

If λ only has one linearly independent eigenvector $\mathbf{v}$, the general solution is given by

$$\mathbf{x}(t) = c_1 e^{\lambda t}\mathbf{v} + c_2\left(te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{w}\right).$$

The stability of this system is determined by the sign of λ.

- When $\lambda < 0$, the origin is a **degenerate stable node**.
- When $\lambda > 0$, the origin is a **degenerate unstable node**.

To draw a phase portait, draw one/two lines spanned by the eigenvector(s) $\mathbf{v}$, where the direction of the arrows on each line is determined by the sign of λ.

6.3.3 Complex Conjugate Eigenvalues

The following case is possible when $\tau^2 - 4\Delta < 0$.

When $\lambda = \sigma \pm \omega i$ are complex conjugate eigenvalues, with corresponding eigenvectors $\mathbf{v} = \mathbf{a} \pm \mathbf{b}i$, then the general solution is given by

$$\mathbf{x}(t) = e^{\sigma t}\left[c_1\left(\mathbf{a}\cos(\omega t) - \mathbf{b}\sin(\omega t)\right) + c_2\left(\mathbf{a}\sin(\omega t) + \mathbf{b}\cos(\omega t)\right)\right].$$

The stability of this system is determined by the real part of λ.

- When $\alpha < 0$, the origin is a **stable spiral**.
- When $\alpha > 0$, the origin is an **unstable spiral**.
- When $\alpha = 0$, the origin is a **centre**.

To draw a phase portait, choose an arbitary initial state $\mathbf{x}_0$ (i.e., $\mathbf{x}_0 = \langle 1,\ 0 \rangle$), the direction that this vector points in will determine the orientation of the spiral. The same applies for a centre, however a centre will be composed of closed ellipses.

Condition	Behaviour	Solution
Real distinct eigenvalues $\lambda_1,\ \lambda_2$ with eigenvectors $\mathbf{v}_1,\ \mathbf{v}_2$ $(\tau^2 - 4\Delta > 0)$		
$\lambda_1, \lambda_2 < 0$ $(\tau < 0, \Delta > 0)$ $\lambda_1, \lambda_2 > 0$ $(\tau > 0, \Delta > 0)$ $\lambda_1 < 0 < \lambda_2$ $(\Delta < 0)$	Stable node Unstable node Saddle point	$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$
Real repeated eigenvalues λ with eigenvectors $\mathbf{v}_1,\ \mathbf{v}_2$ $(\tau^2 - 4\Delta = 0)$		
$\lambda > 0$ $\lambda < 0$	Degenerate stable node Degenerate unstable node	$\mathbf{x}(t) = e^{\lambda t}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2)$
Real repeated eigenvalues λ with eigenvector $\mathbf{v}$. $\mathbf{w}$ satisfies $(\lambda I - \mathbf{A})\mathbf{w} = \mathbf{v}$ $(\tau^2 - 4\Delta = 0)$		
$\lambda < 0$ $\lambda > 0$	Degenerate stable node Degenerate unstable node	$\mathbf{x}(t) = c_1 e^{\lambda t}\mathbf{v} + c_2\left(te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{w}\right)$
Complex conjugate eigenvalues $\lambda = \sigma \pm \omega i$ with eigenvectors $\mathbf{v} = \mathbf{a} \pm \mathbf{b}i$ $(\tau^2 - 4\Delta < 0)$		
$\sigma < 0$ $(\tau \neq 0)$ $\sigma > 0$ $(\tau \neq 0)$ $\sigma = 0$ $(\tau = 0)$	Stable spiral Unstable spiral Centre	$x_1(t) = e^{\sigma t}(\mathbf{a}\cos(\omega t) - \mathbf{b}\sin(\omega t))$ $x_2(t) = e^{\sigma t}(\mathbf{a}\sin(\omega t) + \mathbf{b}\cos(\omega t))$

6.4 Nonlinear Systems

Consider the 2D nonlinear system

$$\begin{aligned} \dot{x_1} &= F(x,\ y) \\ \dot{x_2} &= G(x,\ y) \end{aligned}$$

critical points exist where $F(x,\ y) = G(x,\ y) = 0$. If a system has a critical point at $\mathbf{x}_e = \langle x_e, y_e \rangle$, we can find a local linearisation around that point.

Definition 6.1 (Jacobian)**.** The Jacobian of a system is defined as

$$\mathbf{J} = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}$$

The local behaviour of a nonlinear system can therefore be approximated by analysing the following linear system:

$$\dot{\mathbf{h}} = \mathbf{J}\big|_{(x_e,\, y_e)}\,\mathbf{h}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x - x_e \\ y - y_e \end{bmatrix} = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}\Bigg|_{(x_e,\, y_e)} \begin{bmatrix} x - x_e \\ y - y_e \end{bmatrix}$$

An alternate approach is to use the chain rule directly, $\dfrac{\frac{\mathrm{d}x}{\mathrm{d}t}}{\frac{\mathrm{d}y}{\mathrm{d}t}} = \dfrac{\mathrm{d}x}{\mathrm{d}y} = \dfrac{F(x,y)}{G(x,y)}$.

6.5 Nullclines

The curve along which $x' = 0$ is called the x-nullcline, and the curve along which $y' = 0$ is called the y-nullcline. The points where these nullclines intersect are called **steady states**.

6.6 Parameterised Systems

When a system is parametrised by an unknown constant μ, multiple phase portraits can be drawn for different values of μ. Consider the 1D system

$$\dot{x} = f\,(\mu,\, x)$$

By observing the phase plane plot of the function $f\,(\mu,\, x)$ against x for different values of μ, we can plot the behaviour of f on a **bifurcation diagram**.
In this diagram, the horizontal axis contains the values of μ, and the vertical axis contains the values of x. By plotting each equilibrium point as a function of μ, we can indicate the stability along each equilibrium point by drawing arrows pointing upwards and downwards depending on the phase plane from earlier.
A saddle-node **bifurcation** occurs at the parameter value $\mu = \mu_0$ if as μ passes through μ_0, the system goes from no equilibrium points to one or more equilibrium points. Here μ_0 is called a **bifurcation point**.
If by increasing μ past μ_0, the system goes from one equilibrium point to three equilibrium points, then the system has a **pitchfork bifurcation**.

A Second-Order Variation of Parameters

Consider the following second-order linear ODE

$$a_2(t)\,y'' + a_1(t)\,y' + a_0(t)\,y = f(t)\,.$$

The particular solution is given by

$$y_p = v_1(t)\,y_1 + v_2(t)\,y_2$$

where y_1 and y_2 are solutions to the homogeneous problem. Differentiating y_p once gives us

$$y_p' = v_1'y_1 + v_1y_1' + v_2'y_2 + v_2y_2'$$

where we will use the assumption

$$v_1'y_1 + v_2'y_2 = 0$$

to arrive at

$$y_p' = v_1y_1' + v_2y_2'.$$

The second derivative is then

$$y_p'' = v_1'y_1' + v_1y_1'' + v_2'y_2' + v_2y_2''.$$

If we then substitute these values into our ODE, and factor in terms of v, we find

$$\begin{aligned}
a_2y_p'' + a_1y_p' + a_0y_p &= f(t)\\
a_2\left[v_1'y_1' + v_1y_1'' + v_2'y_2' + v_2y_2''\right] + a_1\left[v_1y_1' + v_2y_2'\right] + a_0\left[v_1y_1 + v_2y_2\right] &= f(t)\\
v_1\left[a_2y_1'' + a_1y_1' + a_0y_1\right] + v_2\left[a_2y_2'' + a_1y_2' + a_0y_2\right] + a_2v_1'y_1' + a_2v_2'y_2' &= f(t)\\
a_2v_1'y_1' + a_2v_2'y_2' &= f(t)\\
v_1'y_1' + v_2'y_2' &= g(t)
\end{aligned}$$

where $g(t) = f(t)/a_2(t)$. This gives us a system of equations where we can solve for v_1' and v_2', using the assumption from earlier:

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g(t) \end{bmatrix}.$$

where the coefficient matrix is simply the Wronskian matrix $\mathbf{W}$. Using Cramer's rule, we can conclude that

$$v_1' = \frac{W_1}{W} \qquad \text{and} \qquad v_2' = \frac{W_2}{W}$$

where W_i are the determinants of the Wronskian with the ith columns replaced with the RHS vector $\boldsymbol{b}$:

$$W_1 = \begin{vmatrix} 0 & y_2 \\ g(t) & y_2' \end{vmatrix} \qquad \text{and} \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g(t) \end{vmatrix}$$

Integrating these results gives

$$v_1 = \int \frac{W_1}{W}\,\mathrm{d}t \qquad \text{and} \qquad v_2 = \int \frac{W_2}{W}\,\mathrm{d}t$$

so that the particular solution is given by

$$y_p = y_1\int \frac{W_1}{W}\,\mathrm{d}t + y_2\int \frac{W_2}{W}\,\mathrm{d}t.$$

Note that it may not be possible to compute these integrals.