

# Modelling with Differential Equations 1

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# 1 Differential Equations

The study of differential equations (DE) finds techniques to solve equations relating unknown functions with their derivatives. These equations allow us to model and predict the behaviour of systems evolving over time or any other dimension.

## 1.1 Ordinary and Partial Differential Equations

Differential equations fall into one of two categories:

- *Ordinary differential equations* (ODE) — derivatives are taken with respect to only one variable.
- *Partial differential equations* (PDE) — derivatives are taken with respect to several variables.

Some examples of ODEs are shown below:

$$\begin{aligned}\frac{dy}{dt} &= ky && \text{(Exponential growth)} \\ \frac{dy}{dt} &= k(A - y) && \text{(Newton's law of cooling)} \\ m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx &= f(t) && \text{(Mechanical vibrations)}\end{aligned}$$

where  $x$ ,  $y$ , and  $f$  are functions of time -  $t$ . Below are some examples of PDEs:

$$\begin{aligned}\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} &= 0 && \text{(Transport equation)} \\ \frac{\partial u}{\partial t} &= k \frac{\partial^2 u}{\partial x^2} && \text{(Heat equation)} \\ \Delta u &= 0 && \text{(Laplace's equation)}\end{aligned}$$

where  $u$  is typically a function of space ( $x$ ,  $y$ ,  $z$ ) or space-time ( $x$ ,  $y$ ,  $z$ ,  $t$ ).

## 1.2 Order

The order of a differential equation refers to the highest derivative term that appears in the equation.

## 1.3 Linearity

A linear ODE is defined by a linear polynomial in the unknown function and its derivatives:

$$a_n(t)y^{(n)} + \dots + a_2(t)y'' + a_1(t)y' + a_0(t)y = f(t).$$

Here the notation  $y^{(n)}$  denotes the  $n$ th derivative of  $y$ , and we use this in-place of the apostrophe (') for compactness. Note that the function  $y$  itself, can be thought of as the “0”th derivative of  $y$ . Any ODE that cannot be expressed in the above form is nonlinear.

We can also classify PDEs in a similar manner. We will only consider a second-order linear PDE for brevity:

$$A(x, y)u_{xx} + B(x, y)u_{xy} + C(x, y)u_{yy} + D(x, y)u_x + E(x, y)u_y + F(x, y)u = f(x, y)$$

where we have a linear combination of second-order partial derivatives of  $u$ . Due to the complexity of PDEs, other classifications also exist, but these will not be discussed.

## 1.4 Homogeneity

Finally, a differential equation is homogeneous when it is an equation consisting only of the unknown function and its derivatives. In other words, there are no constant terms (or functions).

Both ODEs and PDEs are classified by the same rules, below is an example of homogeneous DEs:

$$ay'' + by' + cy = 0$$

$$\frac{\partial^2 u}{\partial x^2} + k \frac{\partial^2 u}{\partial t^2} = 0.$$

As we will see in further sections, differential equations are often simpler to solve when they are homogeneous.

## 1.5 Useful Theorems

**Theorem 1.5.1** (Existence and Uniqueness Theorem<sup>1</sup>). *Let the functions  $f$  and  $\frac{\partial f}{\partial y}$  be Lipschitz continuous<sup>2</sup> on the closed rectangle  $D \subseteq \mathbb{R}$  with  $(t_0, y_0) \in D$ . Then there exists some interval  $I \subseteq D$ , centred on  $t_0$  such that there is a unique solution to the initial value problem:*

$$\begin{cases} y'(t) = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

**Corollary 1.5.1.1** (Existence and Uniqueness Theorem for Higher-Order ODEs). *Let the functions  $\frac{\partial^i f}{\partial y^i}$  for all  $i = 0, 1, \dots, n-1$ , be Lipschitz continuous on the closed  $(n+1)$ -dimensional hyperrectangle  $D \subseteq \mathbb{R} \times \mathbb{R}^n$  with  $(t_0, y_0, y'_0, y''_0, \dots, y_0^{(n-1)}) \in D$ . Then there exists some interval  $I \subseteq D$  containing  $t_0$ , such that there is a unique solution to the initial value problem:*

$$\begin{cases} y^{(n)}(t) = f(t, y, y', y'', \dots, y^{(n-1)}) \\ y^{(i)}(t_0) = y_0^{(i)} \end{cases}$$

for all  $i = 0, 1, \dots, n-1$ . Note that  $y_0^{(i)}$  is the constant value associated with the initial value of the  $i$ th derivative of  $y$  at  $t_0$ .

**Theorem 1.5.2** (Principle of Superposition). *If  $y_1$  is a solution to the equation*

$$ay'' + by' + cy = f_1(t),$$

and  $y_2$  are solutions to

$$ay'' + by' + cy = f_2(t),$$

then for any constants  $c_1$  and  $c_2$ , the linear combination  $y = c_1 y_1 + c_2 y_2$  is a solution to the differential equation

$$ay'' + by' + cy = c_1 f_1(t) + c_2 f_2(t).$$

<sup>1</sup>Also known as the Picard-Lindelöf Theorem

<sup>2</sup>When  $f$  and  $\frac{\partial f}{\partial y}$  are continuous, then a solution exists with no information about its uniqueness. See the Peano Existence Theorem for more information.

*Proof.* We can prove the above result through substitution,

$$\begin{aligned} ay'' + by' + cy &= a(c_1y_1 + c_2y_2)'' + b(c_1y_1 + c_2y_2)' + c(c_1y_1 + c_2y_2) \\ &= ac_1y_1'' + ac_2y_2'' + bc_1y_1' + bc_2y_2' + cc_1y_1 + cc_2y_2 \\ &= c_1(ay_1'' + by_1' + cy_1) + c_2(ay_2'' + by_2' + cy_2) \\ &= c_1f_1(t) + c_2f_2(t) \end{aligned}$$

A similar proof demonstrates that this principle also applies for higher-order linear ODEs. □

## 2 Constant Coefficient ODEs

A constant coefficient ODE is a linear ODE with coefficients that do not depend on the independent variable:

$$a_n y^{(n)} + \dots + a_2 y'' + a_1 y' + a_0 y = f(t).$$

This type of ODE is typically solved via *Undetermined Coefficients* or *Variation of Parameters*.

### 2.1 Undetermined Coefficients

Consider the second-order constant coefficient ODE:

$$ay'' + by' + cy = f(t).$$

Recall the superposition principle which allows us to express solutions as

$$y = c_1 y_1 + c_2 y_2.$$

Let us assume that  $y_1$  is a solution to the *homogeneous* ODE, that is, when  $f(t) = 0$ , denoted  $y_h$ , for the “homogeneous solution”.

Let us also assume that  $y_2$  is a solution to the *nonhomogeneous* ODE, denoted  $y_p$ , for the “particular solution”.

By solving each part individually, we can construct a *general* solution for this ODE via the superposition principle

$$y = c_1 y_h + c_2 y_p.$$

### 2.2 Homogeneous Solution

We will assume<sup>3</sup> that the solution  $y_h$  has the form

$$y_h(t) = e^{rt}.$$

Let us then substitute this into the homogeneous ODE:

$$\begin{aligned} ay_h'' + by_h' + cy_h &= 0 \\ a(e^{rt})'' + b(e^{rt})' + ce^{rt} &= 0 \\ ar^2 e^{rt} + bre^{rt} + ce^{rt} &= 0 \\ (ar^2 + br + c)e^{rt} &= 0 \\ ar^2 + br + c &= 0 \end{aligned}$$

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<sup>3</sup>A common term for such assumptions is also *ansatz*.

This result is known as the **characteristic equation** (or **auxiliary equation**) of our ODE. Solving the *roots* of this algebraic equation allows us to find  $y_h$ .

As this is a polynomial expression, we expect three kinds of solutions:

- Real distinct roots:  $r_1, r_2 \in \mathbb{R} \rightarrow \Delta > 0$
- Complex conjugate roots:  $r_1, r_2 \in \mathbb{C} \rightarrow \Delta < 0$
- Real repeated roots:  $r \in \mathbb{R} \rightarrow \Delta = 0$

where  $\Delta$  is the discriminant given by  $\Delta = b^2 - 4ac$ .

### 2.2.1 Real Distinct Roots

Given real distinct roots  $r_1$  and  $r_2$ , the homogeneous solution is given by

$$y_h = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

where we use the principle of superposition to combine the solution from both roots.

### 2.2.2 Complex Conjugate Roots

Given complex conjugate roots  $r_1 = \alpha + \beta i$  and  $r_2 = r_1^* = \alpha - \beta i$ , the homogeneous solution is given by

$$y_h = k_1 e^{(\alpha + \beta i)t} + k_2 e^{(\alpha - \beta i)t}.$$

We can then simplify this solution using Euler's identity.

$$\begin{aligned} y_h &= e^{\alpha t} (k_1 e^{\beta i t} + k_2 e^{-\beta i t}) \\ &= e^{\alpha t} [k_1 (\cos(\beta t) + i \sin(\beta t)) + k_2 (\cos(\beta t) - i \sin(\beta t))] \\ &= e^{\alpha t} [k_1 \cos(\beta t) + i k_1 \sin(\beta t) + k_2 \cos(\beta t) - i k_2 \sin(\beta t)] \\ &= e^{\alpha t} [(k_1 + k_2) \cos(\beta t) + i (k_1 - k_2) \sin(\beta t)] \\ &= e^{\alpha t} [c_1 \cos(\beta t) + c_2 \sin(\beta t)] \end{aligned}$$

which demonstrates that complex conjugate roots correspond to oscillatory solutions.

### 2.2.3 Real Repeated Roots

When  $r = -\frac{b}{2a}$  is a repeated root, one solution is given by

$$y_{h,1} = e^{rt}.$$

However, this solution lacks linear independence<sup>4</sup> from the other solution as  $r$  has multiplicity 2. In the following section, we will use the method of *Reduction of Order* to find the second solution.

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<sup>4</sup>We will discuss linear independence in a later section.

### 2.3 Reduction of Order

The reduction of order method is a method for converting any linear ODE to another linear ODE of lower order. If the solution  $y_1$  is known in advance, we can construct additional solutions via the following substitution:

$$y_2 = u(t) y_1.$$

#### 2.3.1 Finding the Missing Solutions of the Repeated Roots Problem

In the previous section, we were unable to find a complete set of homogeneous solutions to our ODE. Therefore let us use this new method to find a linearly independent solution to  $y_{h,1}$ :

$$y_{h,2} = u(t) y_{h,1}.$$

Before we can substitute this equation into the ODE, let us find its first and second derivatives:

$$\begin{aligned} y'_{h,2} &= u' y_{h,1} + u y'_{h,1} \\ y''_{h,2} &= u'' y_{h,1} + u' y'_{h,1} + u' y'_{h,1} + u y''_{h,1} = u'' y_{h,1} + 2u' y'_{h,1} + u y''_{h,1} \end{aligned}$$

We can now substitute this equation into the homogeneous ODE:

$$\begin{aligned} a y''_{h,2} + b y'_{h,2} + c y_{h,2} &= 0 \\ a(u'' y_{h,1} + 2u' y'_{h,1} + u y''_{h,1}) + b(u' y_{h,1} + u y'_{h,1}) + c u y_{h,1} &= 0 \\ a u'' y_{h,1} + 2a u' y'_{h,1} + a u y''_{h,1} + b u' y_{h,1} + b u y'_{h,1} + c u y_{h,1} &= 0 \\ a u'' e^{rt} + 2a u' r e^{rt} + a u r^2 e^{rt} + b u' e^{rt} + b u r e^{rt} + c u e^{rt} &= 0 \\ [a u'' + 2a r u' + a r^2 u + b u' + b r u + c u] e^{rt} &= 0 \\ a u'' + (2a r + b) u' + (a r^2 + b r + c) u &= 0 \end{aligned}$$

As  $r = -\frac{b}{2a}$ , the coefficients of both  $u'$  and  $u$  become zero, resulting in:

$$a u'' = 0$$

so that  $u'' = 0$  because  $a \neq 0$ . Integrating twice gives the solution:

$$u(t) = k_1 t + k_2.$$

Here we may say  $y_{h,2} = k_1 t e^{rt} + k_2 e^{rt}$ , however, as  $k_2 e^{rt}$  is linearly dependent to the first solution, we must let  $k_2 = 0$  so that:

$$y_{h,2} = t e^{rt}.$$

Combining both solutions yields the general homogeneous solution:

$$y = c_1 e^{rt} + c_2 t e^{rt}.$$

It follows that this result applies to higher-order linear ODEs with repeated roots of any multiplicity.

**Proposition 1.** *A linear homogeneous ODE of order  $n$  with the root  $r$  of multiplicity  $k$  has the basis function:*

$$y = e^{rt} (1 + t + t^2 + \dots + t^k).$$

## 2.4 Linear Independence

The existence and uniqueness theorem implies that an  $n$ th order homogeneous linear ODE has  $n$  linearly independent solutions that form a general solution under the principle of superposition:

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n.$$

However, when given an arbitrary function, as seen in the *Reduction of Order* method, we must construct a test for whether two functions are linearly independent using a definitive method.

## 2.5 Test for Linear Independence

The set of  $(n-1)$ -times differentiable functions  $S = \{y_1, y_2, \dots, y_n\}$  is linearly independent iff

$$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0$$

for all  $t$ , with nonzero  $c_1, \dots, c_n$ . Let us then differentiate this equation  $n-1$  times to find the following system of equations:

$$\begin{aligned} c_1 y_1 + c_2 y_2 + \cdots + c_n y_n &= 0 \\ c_1 y_1' + c_2 y_2' + \cdots + c_n y_n' &= 0 \\ c_1 y_1'' + c_2 y_2'' + \cdots + c_n y_n'' &= 0 \\ &\vdots \\ c_1 y_1^{(n-1)} + c_2 y_2^{(n-1)} + \cdots + c_n y_n^{(n-1)} &= 0 \end{aligned}$$

which we can express as

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Recall from Linear Algebra that the linear system:

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

has a unique solution when  $\text{rank}(\mathbf{A}) = \text{rank}([\mathbf{A} \mid \mathbf{b}]) = n$ , where  $\mathbf{b} = \mathbf{0}$ . This is true when the matrix  $\mathbf{A}$  is invertible, so that

$$\begin{aligned} \mathbf{A}\mathbf{c} &= \mathbf{0} \\ \mathbf{A}^{-1}\mathbf{A}\mathbf{c} &= \mathbf{A}^{-1}\mathbf{0} \\ \mathbf{c} &= \mathbf{0} \end{aligned}$$

as required.

Therefore we can conclude that when the determinant of  $\mathbf{A}$  is nonzero, the set  $S$  is linearly independent. Let us summarise these findings in the following section and introduce the Wronskian matrix.



## 2.6 The Wronskian

**Definition 2.1** (Wronskian Matrix). Given the set of  $(n-1)$ -times differentiable functions  $S = \{y_1, y_2, \dots, y_n\}$ , the Wronskian matrix  $\mathbf{W}$  is defined as

$$\mathbf{W} = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix}$$

The “Wronskian” of these functions is given by

$$W[y_1, y_2, \dots, y_n] = \det(\mathbf{W}).$$

**Theorem 2.6.1** (Linear Independence and the Wronskian). *The set of  $(n-1)$ -times differentiable functions  $S = \{y_1, y_2, \dots, y_n\}$  is linearly independent if*

$$W[y_1, y_2, \dots, y_n] \neq 0.$$

*Note that  $W[y_1, y_2, \dots, y_n] = 0$  does not imply that  $S$  is linearly dependent.*

**Corollary 2.6.1.1.** *If the set  $S$  is linearly dependent, then*

$$W[y_1, y_2, \dots, y_n] = 0.$$