Fourier Series

Approximate f on $[-L, L]$ by

$$f_F(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(\omega_n x) + b_n \sin(\omega_n x)\right]$$

where $\omega_n = \frac{n\pi}{L}$ and $f = f_F$ on $[-L, L]$ and **periodically extended** elsewhere.

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \,\mathrm{d}x$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos(\omega_n x) \,\mathrm{d}x$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin(\omega_n x) \,\mathrm{d}x$$

for $n \in \mathbb{N}$.

Integral Relationships

$$\int_{-L}^{L} \cos(\omega_n x) \,\mathrm{d}x = 0$$

$$\int_{-L}^{L} \sin(\omega_n x) \,\mathrm{d}x = 0$$

$$\int_{-L}^{L} \sin(\omega_n x) \cos(\omega_m x) \,\mathrm{d}x = 0$$

for $n, m \in \mathbb{N}$

$$\int_{-L}^{L} \cos(\omega_n x) \cos(\omega_m x) \,\mathrm{d}x = L$$

$$\int_{-L}^{L} \sin(\omega_n x) \sin(\omega_m x) \,\mathrm{d}x = L$$

when $n = m$, and 0 otherwise.

Cosine (Even) Series

When f is even, $b_n = 0$, and

$$a_0 = \frac{1}{L} \int_{0}^{L} f(x) \,\mathrm{d}x$$

$$a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos(\omega_n x) \,\mathrm{d}x$$

Sine (Odd) Series

When f is odd, $a_0 = a_n = 0$, and

$$b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin(\omega_n x) \,\mathrm{d}x$$

Both expansions result in even/odd periodic extensions of f.

Partial Differential Equations

- **Dirichlet** $u(a,\, t) = C$
- **Neumann** $\frac{\partial u}{\partial x}(a,\, t) = C$
- **Robin** $Au(a,\, t) + B\frac{\partial u}{\partial x}(a,\, t) = C$

Separation of Variables

$$u_n(x,\, t) = X_n(x) T_n(t)$$

on some finite interval $[a, b]$ with $t \geqslant 0$. Substitute and separate into two ODEs:

$$f_1(x,\, X,\, X',\, \dots) = \alpha_n$$

$$f_2(t,\, T,\, T',\, \dots) = \alpha_n$$

Solve ODE with BCs to find eigenvalues α_n and eigenfunctions X_n. Solve other ODE to find $u_n(t)$. Apply superposition and solve ICs to find $u(x,\, t)$.

Given two spatial dimensions, consider the ODE with homogeneous BCs first.

Polar Coordinates

$$u(r,\, \theta) = R(r)\,\Theta(\theta)$$

with periodicity: $\Theta(\theta) = \Theta(\theta + 2\pi)$.

For radially symmetric problems

$$u(r,\, t) = R(r)\,T(t)$$

with $\frac{\partial u}{\partial \theta} = 0$.

Solutions require **boundedness** in r.

Sturm-Liouville Theory

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right] + q(x)\,y + \lambda w(x)\,y = 0$$

with two non-trivial homogeneous BCs:

$$-l_1 y'(a) + h_1 y(a) = 0$$

$$l_2 y'(b) + h_2 y(b) = 0$$

have infinitely many λ_n and y_n, where $\lambda_n \to \infty$ as $n \to \infty$. $\{y_n : n \in \mathbb{Z}^+\}$ form an orthogonal basis that satisfy the BCs.

$$y \mapsto -\frac{1}{w(x)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right] + q(x)\,y\right)$$

- **Regular** when $p, w > 0$, and p, p', q, w are continuous over the interval $[a, b]$.
- **Proper** when $q(x) \leqslant 0$ on $[a, b]$, with $l_1 h_1 \geqslant 0$ and $l_2 h_2 \geqslant 0$. All eigenvalues are non-negative.
- **Singular** when $p(a) = 0$, and $x = a$ is replaced by the condition that y remain bounded.
- **Periodic** when instead of BCs we have, $p(a) = p(b)$ and $p'(a) = p'(b)$. y is then also periodic.

Transform the ODE

$$a_2 y'' + a_1 y' + a_0 y = 0$$

with the integrating factor:

$$\mu = \frac{1}{a_2}\exp\left(\int \frac{a_1}{a_2}\,\mathrm{d}x\right).$$

Weighted Inner-Product

$$\langle y_n,\, y_m\rangle = \int_a^b y_n y_m w \,\mathrm{d}x = \delta_{mn}$$

Eigenfunction Expansion

Approximate f on $[a, b]$ by

$$f_E = \sum_{n=1}^{\infty} c_n y_n = \sum_{n=1}^{\infty} \frac{\langle f,\, y_n\rangle_w}{\langle y_n,\, y_n\rangle_w} y_n.$$

where c_m is found via the inner-product:

$$\langle f,\, y_m\rangle_w = \sum_{n=1}^{\infty} c_n \langle y_n,\, y_m\rangle_w$$

Nonhomogeneous Problems

For time-dependent problems, separate solution into **steady-state** part

$$U(x)$$

which is found by setting $u_t = 0$, and **transient** part

$$v(x,\, t) = u(x,\, t) - U(x)$$

and solve via substitution.

Eigenfunction Expansion

Assume the solution takes the form of an eigenfunction expansion in one variable. Here the boundary conditions must be homogeneous.

Integral Transforms

Fourier Transform $\mathscr{F}$

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x} \,\mathrm{d}x$$

$$f_F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega x} \,\mathrm{d}\omega$$

Solve PDEs on infinite domains where u is bounded at $\pm\infty$.

Convolution Theorem

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x - z)\, g(z) \,\mathrm{d}z$$

$$\mathscr{F}\{fg\} = \frac{1}{2\pi}\left(\hat{f} * \hat{g}\right)(\omega)$$

$$\mathscr{F}^{-1}\left\{\hat{f}\hat{g}\right\} = (f * g)(x)$$

Laplace Transform $\mathscr{L}$

$$F(s) = \int_0^{\infty} f(t)\, e^{-st} \,\mathrm{d}t$$

$$f(t) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} F(s)\, e^{st} \,\mathrm{d}s$$

for sufficiently large σ so that $f(t)\, e^{\sigma t} \to 0$ as $t \to \infty$. σ must be to the right of all singularities.

Convolution Theorem

$$(f * g)(t) = \int_0^{t} f(t - \tau)\, g(\tau) \,\mathrm{d}\tau$$

$$\mathscr{L}\{fg\} = (F * G)(s)$$

$$\mathscr{L}^{-1}\{FG\} = (f * g)(t)$$

Specific PDE Problems

Heat Equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

Laplace's Equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Common Taylor Series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, |z| < \infty$$

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}, |z| < \infty$$

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}, |z| < \infty$$

$$\cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}, |z| < \infty$$

$$\sinh(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}, |z| < \infty$$

$$\frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n, |z| < 1$$

Complex Analysis

Complex-Valued Functions

$$w = f(z) = u(x,\, y) + iv(x,\, y)$$

where $w = u + iv$ and $z = x + iy$.

Analytic Functions

f satisfies **Cauchy-Riemann** equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

The derivative is given by

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

As z is a complex number, the limit

$$\lim_{z \to z_0} f(z) = L$$

must be path independent. If a function is differentiable at a point and in its neighbourhood, it is analytic at that point. Analytic functions are infinitely differentiable and have convergent Taylor series expansions near that point.

Complex Differentiation

Polynomials, rational functions (except at singularities), and exponentials follow familiar rules. As do any sums, products, or compositions of these functions.
Logarithms, non-integer powers, and inverse trigonometric functions behave similarly, except at branch points and branch cuts.

Laplace's Equation

If f is analytic in a region $\mathcal{D}$, then u and v both satisfy Laplace's equations $\nabla^2 u = 0$, $\nabla^2 v = 0$ in $\mathcal{D}$. u and v are **harmonic** functions and v is the **harmonic conjugate** of u.

Complex Integration

Compute line integrals in the complex plane, where an oriented curve C is parametrised by

$$z(t) = x(t) + iy(t)$$

for $t \in [a, b]$. A curve is:

- **Smooth** if $\frac{\partial z}{\partial t}$ is piecewise continuous and nonzero for all t.
- **Closed** if $z(a) = z(b)$.
- **Simple** if it does not cross itself: $z(t_1) \neq z(t_2)$ for $t_1 \neq t_2$, $a < t_1$, $t_2 < b$.

Complex Line Integrals

$$\int_C f(z)\,\mathrm{d}z = \int_a^b f(z(t))\,\frac{\mathrm{d}z}{\mathrm{d}t}\,\mathrm{d}t$$

Cauchy's Integral Theorem

If f is analytic in a region $\mathcal{D}$, then the contour integral along any simple closed curve C in $\mathcal{D}$ is zero

$$\oint_C f(z)\,\mathrm{d}z = 0.$$

For any two points $z_1, z_2 \in \mathcal{D}$,

$$\int_{C_1} f(z)\,\mathrm{d}z = \int_{C_2} f(z)\,\mathrm{d}z$$

where C_1 and C_2 are any two paths from z_1 to z_2. This is because the integral along the closed curve $C_1 - C_2$ is zero by CIT.
The same holds for an annulus where two simple closed curves C_1 and C_2 have nonzero integrals, but the integral along $C_1 + C_3 - C_2 + C_4$ is zero, where $C_3 = -C_4$ are paths connecting C_1 and C_2.

Cauchy's Integral Formula

If f is analytic on and within a simple closed curve C, then for any point z_0 within C

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,\mathrm{d}z$$

Laurent Series Expansion

If f is analytic on $0 < |z - z_0| < d$, but contains an **isolated singularity** at z_0, then f can be represented by

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where $a_n \in \mathbb{C}$.

Isolated Singularities

Suppose f is analytic in $V = U \setminus \{z_0\}$, then z_0 is a **singularity** of f. Assume the existence of g such that g is analytic in U. Then the singularity of f at z_0 is:

- **Removable** (no negative powers)
  $$\forall z \in U,\ \exists g : f(z) = g(z)$$
- **Pole** (finitely many negative powers)
  $$\forall z \in V : \exists g : g(z) = (z - z_0)^n f(z)$$
  where $g(z_0) \neq 0$. The **order** of a pole is the largest value of n (smallest power in Laurent series). When $n = 1$, z_0 is a **simple pole**.
- **Essential** (infinitely many negative powers)

Residues (a_{-1} term)

For a simple pole,

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} (z - z_0) f(z).$$

For a pole of order n,

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\left[(z - z_0)^n f(z)\right].$$

Residue Theorem

If f be analytic on and within a simple closed curve C, except for a finite number of isolated singularities $z_1, \dots, z_n \in C$

$$\oint_C f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

Jordan's Lemma

Consider a large semi-circular curve C_R centred at $s = \sigma$, extending toward the left hand plane: $s(\theta) = \sigma + Re^{i\theta}$ with $\pi/2 < \theta < 3\pi/2$. If $F(s) \to 0$ as $|s| \to \infty$ for all s on C_R, then

$$\lim_{R \to \infty} \int_{C_R} F(s)\, e^{st}\,\mathrm{d}s = 0.$$

Adding this integral to the inverse Laplace transform integral creates a closed curve, so by the residue theorem

$$f(t) = \sum_{k=1}^{n} \operatorname*{Res}_{s=s_k} F(s)\, e^{st}$$

Fourier Transform

Integrating in the complex x/ω plane, along the real axis, consider a semi-circle in the upper/lower half plane, where the direction depends on the sign of ω/x.

- **Forward** ($e^{-i\omega x}$), $\omega \in \mathbb{R}$, $x \in \mathbb{C}$:
  - If $\omega < 0$, $\left|e^{-i\omega x}\right| = e^{\omega \Im(x)} \to 0$ as $\Im(x) \to \infty$. (upper half x-plane).
  - If $\omega > 0$, $\left|e^{-i\omega x}\right| = e^{\omega \Im(x)} \to 0$ as $\Im(x) \to -\infty$. (lower half x-plane).
- **Inverse** ($e^{i\omega x}$), $\omega \in \mathbb{C}$, $x \in \mathbb{R}$:
  - If $x < 0$, $\left|e^{i\omega x}\right| = e^{-x\Im(\omega)} \to 0$ as $\Im(\omega) \to -\infty$. (lower half ω-plane).
  - If $x > 0$, $\left|e^{i\omega x}\right| = e^{-x\Im(\omega)} \to 0$ as $\Im(\omega) \to \infty$. (upper half ω-plane).

Useful Results

$$\oint_C z^n\,\mathrm{d}z = \int_0^{2\pi} \left(Re^{it}\right)^n iRe^{it}\,\mathrm{d}t = 2\pi i$$

when $n = -1$, and 0 otherwise, for a circle of radius R oriented anti-clockwise.

Hyperbolic Functions

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$
$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$
$$\cosh(z) = \frac{e^{z} + e^{-z}}{2}$$
$$\sinh(z) = \frac{e^{z} - e^{-z}}{2}$$
$$\cosh(iz) = \cos(z)$$
$$\sinh(iz) = i\sin(z)$$

$\sinh(z) = 0$ when $z = n\pi i$ for $n \in \mathbb{Z}$. $\cosh(z) = 0$ when $z = n\pi i + \pi i/2$ for $n \in \mathbb{Z}$.