Partial Differential Equations

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Contents

1 Fourier Series

Definition 1.1 (Fourier series expansion)**.** The **Fourier series expansion** of f represents f by a periodic function using trigonometric (sine and cosine) terms. Suppose a function $f(x)$ is defined on an interval $[-L, L]$, then the Fourier series expansion of f is given by:

$$f_F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \tag{1}$$

so that $f = f_F$ on $[-L, L]$. Note that $f = f_F$ may not hold for all x as f_F is periodic and the convergence of the series is not guaranteed.

To determine the coefficients a_n and b_n, let us look at some useful integral properties.

1.1 Integral Relationships

1.1.1 Sine and Cosine

For $n \in \mathbb{Z}$:

$$\begin{aligned}
\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x &= \frac{L}{n\pi}\left[\sin\left(\frac{n\pi x}{L}\right)\right]_{-L}^{L} \\
&= \frac{L}{n\pi}\left[\sin(n\pi) - \sin(-n\pi)\right] \\
&= \frac{L}{n\pi}\left[0 - 0\right] \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x &= -\frac{L}{n\pi}\left[\cos\left(\frac{n\pi x}{L}\right)\right]_{-L}^{L} \\
&= \frac{L}{n\pi}\left[\cos(n\pi) - \cos(-n\pi)\right] \\
&= \frac{L}{n\pi}\left[1 - 1\right] \\
&= 0.
\end{aligned}$$

1.1.2 Combinations of Sine and Cosine

Recall the Werner formulas:

$$\begin{aligned}
2\cos(\alpha)\cos(\beta) &= \cos(\alpha-\beta) + \cos(\alpha+\beta) \\
2\sin(\alpha)\sin(\beta) &= \cos(\alpha-\beta) - \cos(\alpha+\beta) \\
2\sin(\alpha)\cos(\beta) &= \sin(\alpha-\beta) + \sin(\alpha+\beta)
\end{aligned}$$

For n, $m \in \mathbb{N}$,

Product of two cosine functions:

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\mathrm{d}x = \frac{1}{2}\int_{-L}^{L}\cos\left(\frac{(n-m)\,\pi x}{L}\right)\mathrm{d}x + \frac{1}{2}\int_{-L}^{L}\cos\left(\frac{(n+m)\,\pi x}{L}\right)\mathrm{d}x$$

When:

- $n = m$: $n - m = 0$ and $(n+m) \in \mathbb{Z}$, so that the second term is 0, and the first term is L.
- $n \neq m$: $(n-m)\,,\ (n+m) \in \mathbb{Z}$ so that both terms evaluate to 0.

Therefore,

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\mathrm{d}x = \begin{cases} 0, & n \neq m \\ L, & n = m \end{cases}$$

Product of two sine functions:

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x = \frac{1}{2}\int_{-L}^{L}\cos\left(\frac{(n-m)\,\pi x}{L}\right)\mathrm{d}x - \frac{1}{2}\int_{-L}^{L}\cos\left(\frac{(n+m)\,\pi x}{L}\right)\mathrm{d}x$$

By the same argument,

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x = \begin{cases} 0, & n \neq m \\ L, & n = m \end{cases}$$

Product of sine and cosine functions:

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\mathrm{d}x = \frac{1}{2}\int_{-L}^{L}\sin\left(\frac{(n-m)\,\pi x}{L}\right)\mathrm{d}x + \frac{1}{2}\int_{-L}^{L}\sin\left(\frac{(n+m)\,\pi x}{L}\right)\mathrm{d}x$$

When:

- $n = m$: $n - m = 0$ and $(n+m) \in \mathbb{Z}$, so that the integral reduces to 0.
- $n \neq m$: $(n-m)\,,\ (n+m) \in \mathbb{Z}$ so that both terms evaluate to 0 when integrated separately.

Therefore,

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\mathrm{d}x = 0$$

In summary:

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x = 0 \tag{2}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x = 0 \tag{3}$$

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\mathrm{d}x = \begin{cases} 0, & n \neq m \\ L, & n = m \end{cases} \tag{4}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x = \begin{cases} 0, & n \neq m \\ L, & n = m \end{cases} \tag{5}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\mathrm{d}x = 0 \tag{6}$$

1.2 Coefficients of the Fourier Series

1.2.1 For a_0

For a_0 consider integrating Equation 1 from $-L$ to L.

$$\begin{aligned}
\int_{-L}^{L} f(x)\,\mathrm{d}x &= \int_{-L}^{L} a_0\,\mathrm{d}x + \sum_{n=1}^{\infty} a_n \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x + \sum_{n=1}^{\infty} b_n \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x \\
\int_{-L}^{L} f(x)\,\mathrm{d}x &= 2a_0 L \\
a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,\mathrm{d}x
\end{aligned}$$

so that a_0 represents the average value of f on $[-L, L]$.

1.2.2 For a_n

For coefficients a_m, multiply the equation by $\cos\left(\frac{m\pi x}{L}\right)$ before integrating.

$$\begin{aligned}
f(x)\cos\left(\frac{m\pi x}{L}\right) = a_0 \cos\left(\frac{m\pi x}{L}\right) &+ \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) \\
&+ \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)
\end{aligned}$$

$$\begin{aligned}
\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)\mathrm{d}x &= a_0 \cancelto{0}{\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)}\mathrm{d}x + \sum_{n=1}^{\infty} a_n \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\mathrm{d}x \\
&\quad + \sum_{n=1}^{\infty} b_n \cancelto{0}{\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)}\mathrm{d}x \\
\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)\mathrm{d}x &= a_m L \\
a_m &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)\mathrm{d}x
\end{aligned}$$

1.2.3 For b_n

For coefficients b_m, multiply the equation by $\sin\left(\frac{m\pi x}{L}\right)$ before integrating.

$$\begin{aligned} f(x)\sin\left(\frac{m\pi x}{L}\right) = a_0\sin\left(\frac{m\pi x}{L}\right) &+ \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) \\ &+ \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) \end{aligned}$$

$$\begin{aligned} \int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x = a_0\cancelto{0}{\int_{-L}^{L}\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x} &+ \sum_{n=1}^{\infty} a_n\cancelto{0}{\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x} \\ &+ \sum_{n=1}^{\infty} b_n\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x \end{aligned}$$

$$\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x = b_m L$$

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x$$

To summarise,

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,\mathrm{d}x \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x \end{aligned}$$

for $n \in \mathbb{N}$.

Definition 1.2 (Piecewise smooth)**.** A function $f : [a, b] \to \mathbb{R}$, is **piecewise smooth** if each component f_i of f has a bounded derivative f_i' which is continuous everywhere in $[a, b]$, except at a finite number of points at which left- and right-sided derivatives exist.

Theorem 1.2.1 (Convergence of piecewise smooth functions)**.** *If f is a periodic piecewise smooth function on $[-L, L]$, f_F will converge to*

$$f_F(x) = \lim_{\epsilon\to 0^+}\frac{f(x+\epsilon)+f(x-\epsilon)}{2}$$

that is, $f = f_F$, except at discontinuities, where f_F is equal to the point halfway between the left- and right-hand limits.

Corollary 1.2.1.1 (Dirichlet conditions)**.** *The Dirichlet conditions provide sufficient conditions for a real-valued function f to be equal to its Fourier series f_F on $[-L, L]$, at each point where f is continuous. The conditions are:*

1. *f has a finite number of maxima and minima over $[-L, L]$.*
2. *f has a finite number of discontinuities, in each of which the derivative f' exists and does not change sign.*
3. *$\int_{-L}^{L} |f(x)| \,\mathrm{d}x$ exists.*

Definition 1.3 (Gibbs phenomenon)**.** If f_F does not converge to f at discontinuities x_i, then the f_F converges non-uniformly. For Fourier series expansions, this property is known as the *Gibbs phenomenon*.

Note 1.2.1. When f is non-periodic, f_F converges to the periodic extension of f. The endpoints may converge non-uniformly, corresponding to jump discontinuities in the periodic extension of f.

1.3 Sine and Cosine Series

Definition 1.4 (Odd function)**.** f is an *odd* function if it satisfies

$$f(-x) = -f(x)$$

Definition 1.5 (Even function)**.** f is an *even* function if it satisfies

$$f(-x) = f(x)$$

If f is an odd function on $[-L, L]$, then the coefficients corresponding to the cosine terms will be zero. The Fourier series simplifies to

$$f_F = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where $b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x$. Likewise, if f is an even function on $[-L, L]$, then the coefficients corresponding to the sine terms will be zero. The Fourier series simplifies to

$$f_F = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where $a_0 = \frac{1}{L}\int_0^L f(x) \,\mathrm{d}x$ and $a_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x$. These special cases are known as the sine and cosine series expansions respectively, resulting in the **odd** or **even** periodic extension of f.

2 Partial Differential Equations

A partial differential equation (PDE) is a differential equation that must be solved for an unknown function of at least two independent variables, where the equation contains partial derivatives of the unknown function. PDEs are characterised by several properties:

- The **order** of the PDE is the order of the highest derivative in the equation. Furthermore, each independent variable can be described by its order.
- A PDE is **linear** if it is linear in its unknown function and its derivatives.

- A linear PDE has **constant** coefficients if the coefficients of the linear terms do not depend on the independent variables, and has **variable** coefficients otherwise.
- A linear PDE is **homogeneous** if all terms depend on the unknown function, and **nonhomogeneous** otherwise.

2.1 Initial Boundary Value Problems

As with ODEs, we can find the general solution to a PDE and then use initial/boundary conditions to solve for arbitrary constants. The number of conditions for each independent variable depends on the order of that variable in the PDE. Problems with initial and boundary conditions are called **initial boundary value problems** and are often referred to as **IBVPs**.

2.1.1 Boundary Condition Classification

Boundary conditions may depend on u, the gradient $\frac{\partial u}{\partial x}$, or both, depending on the situation being modelled. The following is a list of the different types of boundary conditions:

Dirichlet $u(a,\, t) = C$

Neumann $\frac{\partial u}{\partial x}(a,\, t) = C$

Robin $Au(a,\, t) + B\frac{\partial u}{\partial x}(a,\, t) = C$

where in each classification, the boundary condition is homogeneous iff $C = 0$.

2.2 Linear Operators and Superposition

By linearity, we can write a PDE in terms of linear operators. For example, we can write the PDE

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}$$

as

$$L(u) = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} \iff L = \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} - \frac{\partial}{\partial t}.$$

Similarly, we can describe initial/boundary conditions as linear or homogeneous.

Theorem 2.2.1 (Superposition)**.** *If u_n, $n = 1, \ldots, N$ are solutions to the homogeneous PDE $L(u) = 0$, then any linear combination of these solutions is a solution to the PDE*

$$u = \sum_{n=1}^{N} c_n u_n$$

where c_n are constants.

2.3 Heat Equation

Consider the temperature $u(x,\ t)$ of a 1d metal rod of length L with an initial temperature $u(x,\ 0) = f(x)$ and boundary conditions $u(0,\ t) = T_1$ and $u(L,\ t) = T_2$. If we consider a small section $[x_1, x_2] \in [0, L]$, then the rate of change of heat $H(x,\ t)$ in this section is given by

$$\text{Rate of change of heat energy} = \text{Flow in} - \text{Flow out}$$
$$\int_{x_1}^{x_2} \frac{\partial H}{\partial t}\,\mathrm{d}x = Q(x_1,\ t) - Q(x_2,\ t)$$

where $Q(x,\ t)$ is the heat flux at time t. By making the following assumptions, we can formulate a relationship for the temperature in the rod at position x at time t.

1. No energy is lost in the rod.
2. The change in heat energy is proportional to the change in temperature (i.e., no phase changes are present) so that the specific heat equation applies.
$$\Delta H = \rho c\,\Delta u \iff \frac{\partial H}{\partial t} = \rho c \frac{\partial u}{\partial t}$$
where ρ is the density of the rod and c is the specific heat of the rod.
3. The material of the rod is homogeneous, and Fourier's law of conduction applies.
$$\boldsymbol{Q} = -\kappa \nabla \boldsymbol{u} \implies Q = -\kappa \frac{\partial u}{\partial x}$$
where $Q = Q(x,\ t)$ is the heat flux at time t, and κ is the thermal conductivity of the rod.

Using these assumptions, we find

$$\begin{aligned}
\int_{x_1}^{x_2} \frac{\partial H}{\partial t}\,\mathrm{d}x &= Q(x_1,\ t) - Q(x_2,\ t) \\
\int_{x_1}^{x_2} \rho c \frac{\partial u}{\partial t}\,\mathrm{d}x &= \left[-\kappa \frac{\partial u}{\partial x}\right]_{x_1} - \left[-\kappa \frac{\partial u}{\partial x}\right]_{x_2} \\
\int_{x_1}^{x_2} \rho c \frac{\partial u}{\partial t}\,\mathrm{d}x &= \left[\kappa \frac{\partial u}{\partial x}\right]_{x_2} - \left[\kappa \frac{\partial u}{\partial x}\right]_{x_1} \\
\int_{x_1}^{x_2} \rho c \frac{\partial u}{\partial t}\,\mathrm{d}x &= \int_{x_1}^{x_2} \frac{\partial}{\partial x} \kappa \frac{\partial u}{\partial x}\,\mathrm{d}x \\
\rho c \frac{\partial u}{\partial t} &= \frac{\partial}{\partial x} \kappa \frac{\partial u}{\partial x} \\
\frac{\partial u}{\partial t} &= \frac{\kappa}{\rho c} \frac{\partial^2 u}{\partial x^2}
\end{aligned}$$

where k is the thermal diffusivity of the rod:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}.$$

More generally, we can write the PDE as

$$\frac{\partial \boldsymbol{u}}{\partial t} = k\Delta \boldsymbol{u}$$

for multiple spatial dimensions. This PDE is called the heat equation. The heat equation is first order w.r.t. time and second order w.r.t. space.

2.4 Wave Equation

Consider an elastic string that is stretched tightly with its two ends fixed at $x = 0$ and $x = L$ where the vertical displacement of the string is given by $u(x, t)$, and the initial displacement is arbitrary: $u(x, 0) = f(x)$. Let $\theta(x, t)$ be the angle of the string from the horizontal with tension $T(x, t)$ (magnitude). We can then apply the law of conservation. In the horizontal direction, assume equilibrium:

$$T(x_1, t)\cos(\theta(x_1, t)) = T(x_2, t)\cos(\theta(x_2, t))$$

In the vertical direction, assume no external forces:

$$\begin{aligned} ma &= \sum F \\ \int_{x_1}^{x_2} \rho \frac{\partial^2 u}{\partial t^2}\,\mathrm{d}S &= -T(x_1, t)\sin(\theta(x_1, t)) + T(x_2, t)\sin(\theta(x_2, t)) \end{aligned}$$

where ρ is the linear density of the string, and the integral is defined along the arc $\mathrm{d}S$. If we assume that the magnitude of the rate of displacement is small, then

$$\begin{aligned} \theta \approx \sin(\theta) \approx \tan(\theta) &= \frac{\partial u}{\partial x} \\ \cos(\theta) &\approx 1 \end{aligned}$$

therefore in the horizontal direction,

$$T(x_1, t) = T(x_2, t)$$

the tension is independent of x. In the vertical direction,

$$\begin{aligned} \int_{x_1}^{x_2} \rho \frac{\partial^2 u}{\partial t^2}\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2}\,\mathrm{d}x &= -T(x_1, t)\left[\frac{\partial u}{\partial x}\right]_{x_1} + T(x_2, t)\left[\frac{\partial u}{\partial x}\right]_{x_2} \\ \int_{x_1}^{x_2} \rho \frac{\partial^2 u}{\partial t^2}\,\mathrm{d}x &= T\int_{x_1}^{x_2} \frac{\partial^2 u}{\partial x^2}\,\mathrm{d}x \\ \frac{\partial^2 u}{\partial t^2} &= \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2} \\ \frac{\partial^2 u}{\partial t^2} &= c^2\frac{\partial^2 u}{\partial x^2} \end{aligned}$$

where $c = \sqrt{\frac{T}{\rho}}$ is known as the *wave speed*. This PDE is known as the wave equation. As this PDE is second order w.r.t. time, the second initial condition is

$$\frac{\partial u}{\partial t}(x, 0) = g(x)$$

where $g(x)$ is an initial velocity applied to the string.

2.5 Laplace's Equation

By considering higher spatial dimensions, we can model the temperature of a plate $u(x,\ y,\ t)$ with:

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

and similarly the displacement of an elastic membrane $u(x,\ y,\ t)$:

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).$$

The time-independent or **steady-state** case of these equations yields

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

which is known as Laplace's equation. Commonly, this equation is written using the Laplacian operator,

$$\Delta u = \nabla^2 u = \nabla \cdot \nabla u = 0.$$

When this equation is nonhomogeneous, the PDE is known as Poisson's equation.

2.6 Classification of Linear Second Order PDEs

All second order, linear partial differential equations in two dimensions (either space and time or space and space) may be written in the following way:

$$A(x,\ y)\frac{\partial^2 u}{\partial x^2} + B(x,\ y)\frac{\partial^2 u}{\partial x\,\partial y} + C(x,\ y)\frac{\partial^2 u}{\partial y^2} + D(x,\ y)\frac{\partial u}{\partial x} + E(x,\ y)\frac{\partial u}{\partial y} + F(x,\ y)\,u = G(x,\ y).$$

We classify the equation as follows:

- Hyperbolic: $B^2 - 4AC > 0$,
- Parabolic: $B^2 - 4AC = 0$,
- Elliptical: $B^2 - 4AC < 0$.

It follows that the heat equation is parabolic, the wave equation is hyperbolic and the Laplace equation is elliptical.

3 Separation of Variables

To solve an IBVP consider the following:

1. Assume a set of solutions of the form

$$u_n(x,\ t) = X_n(x)\,T_n(t).$$

2. Substitute u_n into the homogeneous PDE and separate

$$f_1\left(x,\ X,\ X',\ \ldots\right) = f_2\left(t,\ T,\ T',\ \ldots\right).$$

3. As each term depends on a different variable, each f_i must be a scalar α_n.

$$\begin{aligned} f_1\left(x,\ X,\ X',\ \ldots\right) &= \alpha_n \\ f_2\left(t,\ T,\ T',\ \ldots\right) &= \alpha_n. \end{aligned}$$

4. Solve the ODEs with boundary conditions while selecting appropriate values of α_n (i.e., negative, zero, positive) that produce non-trivial solutions.

5. Solve the remaining ODEs using α_n from the previous step.

6. Use the principle of superposition to construct a general solution:

$$u\left(x,\ t\right) = \sum_{n=1}^{\infty} u_n\left(x,\ t\right).$$

7. Calculate any remaining constants using initial conditions.

3.1 Separation of Variables: Heat Equation

Assuming the following conditions:

$$u\left(x,\ 0\right) = f\left(x\right), \qquad u\left(0,\ t\right) = u\left(L,\ t\right) = 0.$$

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} \\ \frac{\partial XT}{\partial t} &= k\frac{\partial^2 XT}{\partial x^2} \\ XT' &= kX''T \\ \frac{1}{k}\frac{T'}{T} &= \frac{X''}{X} = \alpha_n \end{aligned}$$

This results in the following two ODEs

$$\begin{aligned} T' - \alpha_n kT &= 0 \\ X'' - \alpha_n X &= 0 \end{aligned}$$

3.1.1 Spatial Dimension

Case 1. $\alpha_n > 0$.

$$\begin{aligned} m^2 - \alpha_n &= 0 \\ m &= \pm\sqrt{\alpha_n} \end{aligned}$$

Therefore
$$X_n(x) = c_1 e^{\sqrt{\alpha_n}x} + c_2 e^{-\sqrt{\alpha_n}x}.$$
Applying the BCs gives
$$\begin{aligned} X_n(0) &= c_1 + c_2 = 0 \\ X_n(L) &= c_1 e^{\sqrt{\alpha_n}L} + c_2 e^{-\sqrt{\alpha_n}L} = 0 \end{aligned}$$
so that
$$\begin{bmatrix} 1 & 1 \\ e^{\sqrt{\alpha_n}L} & e^{-\sqrt{\alpha_n}L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
This homogeneous equation has non-trivial solutions iff the determinant is zero.
$$\begin{aligned} \begin{vmatrix} 1 & 1 \\ e^{\sqrt{\alpha_n}L} & e^{-\sqrt{\alpha_n}L} \end{vmatrix} &= 0 \\ e^{-\sqrt{\alpha_n}L} - e^{\sqrt{\alpha_n}L} &= 0 \\ -2\sinh\left(\sqrt{\alpha_n}L\right) &= 0 \\ \alpha_n &= 0 \end{aligned}$$
but as $\alpha_n > 0$, no solutions exist.

Case 2. $\alpha_n = 0$.
$$X_n(x) = c_1 x + c_2.$$
Applying the BCs gives
$$\begin{aligned} X_n(0) &= c_2 = 0 \\ X_n(L) &= c_1(L) = 0 \implies c_1 = 0 \end{aligned}$$
hence there are no non-trivial solutions as $X_n \equiv 0$.

Case 3. $\alpha_n < 0$.
$$\begin{aligned} m^2 + \alpha_n &= 0 \\ m &= \pm\sqrt{-\alpha_n}i \end{aligned}$$
therefore
$$X_n(x) = c_1 \cos\left(\sqrt{-\alpha_n}x\right) + c_2 \sin\left(\sqrt{-\alpha_n}x\right).$$
Applying the BCs gives
$$\begin{aligned} X_n(0) &= c_1 = 0 \\ X_n(L) &= c_2 \sin\left(\sqrt{-\alpha_n}L\right) = 0 \end{aligned}$$
therefore
$$\begin{aligned} \sqrt{-\alpha_n}L &= n\pi \\ \alpha_n &= -\frac{n^2\pi^2}{L^2} \end{aligned}$$
which gives the following family of solutions:
$$X_n(x) = c_2 \sin\left(\frac{n\pi}{L}x\right)$$

3.1.2 Time Dimension

$$\begin{aligned} m - \alpha_n k &= 0 \\ m &= \alpha_n k \end{aligned}$$

which gives

$$T_n\left(t\right) = c_3 e^{\alpha_n k t} = c_3 e^{-\frac{n^2\pi^2}{L^2}kt}.$$

3.1.3 General Solution

Given these two functions, we can solve for u_n as

$$u_n\left(x,\ t\right) = B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}kt}$$

then by applying superposition, we find the general solution to the PDE:

$$u\left(x,\ t\right) = \sum_{n=1}^{\infty} u_n\left(x,\ t\right) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}kt}.$$

Applying the initial conditions gives

$$u\left(x,\ 0\right) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) = f\left(x\right)$$

so that the coefficients B_n are given by the Fourier sine coefficients of the initial condition $f\left(x\right)$. Therefore, the general solution to the PDE is given by

$$u\left(x,\ t\right) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}kt}, \qquad B_n = \frac{2}{L}\int_0^L f\left(x\right) \sin\left(\frac{n\pi}{L}x\right) \mathrm{d}x.$$

In this solution, as time tends to infinity, the exponential forces the solution to tend toward 0. We also observe that for large n, the sum produces very small values, and hence we can say

$$u\left(x,\ t\right) \approx B_1 \sin\left(\frac{\pi}{L}x\right) e^{-\frac{\pi^2}{L^2}kt}.$$

For large t

$$u\left(x,\ t\right) \approx B_1 \sin\left(\frac{\pi}{L}x\right).$$

3.2 Separation of Variables: Wave Equation

Assume that the initial velocity is 0 and that the ends of the string can move freely in the direction of the string, so that the conditions are given by

$$u\left(x,\ 0\right) = f\left(x\right), \qquad \frac{\partial u}{\partial t}\left(x,\ 0\right) = 0, \qquad \frac{\partial u}{\partial x}\left(0,\ t\right) = \frac{\partial u}{\partial x}\left(L,\ t\right) = 0.$$

Then by using the ansatz

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2}\\ \frac{\partial^2 XT}{\partial t^2} &= c^2 \frac{\partial^2 XT}{\partial x^2}\\ XT'' &= c^2 X'' T\\ \frac{1}{c^2}\frac{T''}{T} &= \frac{X''}{X} = \alpha_n\end{aligned}$$

This results in the following two ODEs

$$\begin{aligned}T'' - \alpha_n c^2 T &= 0\\ X'' - \alpha_n X &= 0\end{aligned}$$

3.2.1 Spatial Dimension

Case 1. $\alpha_n > 0$.

$$\begin{aligned}m^2 - \alpha_n &= 0\\ m &= \pm\sqrt{\alpha_n}\end{aligned}$$

Therefore

$$X_n(x) = c_1 e^{\sqrt{\alpha_n}x} + c_2 e^{-\sqrt{\alpha_n}x}$$

with

$$X_n'(x) = c_1\sqrt{\alpha_n}e^{\sqrt{\alpha_n}x} - c_2\sqrt{\alpha_n}e^{-\sqrt{\alpha_n}x}.$$

Applying the BCs gives

$$\begin{aligned}X_n'(0) &= c_1\sqrt{\alpha_n} - c_2\sqrt{\alpha_n} = 0\\ X_n'(L) &= c_1\sqrt{\alpha_n}e^{\sqrt{\alpha_n}L} - c_2\sqrt{\alpha_n}e^{-\sqrt{\alpha_n}L} = 0\end{aligned}$$

so that

$$\begin{bmatrix}\sqrt{\alpha_n} & -\sqrt{\alpha_n}\\ \sqrt{\alpha_n}e^{\sqrt{\alpha_n}L} & -\sqrt{\alpha_n}e^{-\sqrt{\alpha_n}L}\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix} = \begin{bmatrix}1 & -1\\ e^{\sqrt{\alpha_n}L} & -e^{-\sqrt{\alpha_n}L}\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

This homogeneous equation has non-trivial solutions iff the determinant is zero.

$$\begin{aligned}\begin{vmatrix}1 & -1\\ e^{\sqrt{\alpha_n}L} & -e^{-\sqrt{\alpha_n}L}\end{vmatrix} &= 0\\ -e^{-\sqrt{\alpha_n}L} + e^{\sqrt{\alpha_n}L} &= 0\\ 2\sinh\left(\sqrt{\alpha_n}L\right) &= 0\\ \alpha_n &= 0\end{aligned}$$

but as $\alpha_n > 0$, no solutions exist.

Case 2. $\alpha_n = 0$.

$$X_n(x) = c_1 x + c_2$$

with

$$X_n'(x) = c_1.$$

Applying the BCs gives

$$\begin{aligned} X_n'(0) &= 0 = 0 \\ X_n'(L) &= c_1 = 0 \end{aligned}$$

therefore

$$X_n(x) = c_2$$

is a solution.

Case 3. $\alpha_n < 0$.

$$\begin{aligned} m^2 + \alpha_n &= 0 \\ m &= \pm\sqrt{-\alpha_n}i \end{aligned}$$

therefore

$$X_n(x) = c_1 \cos\left(\sqrt{-\alpha_n}x\right) + c_2 \sin\left(\sqrt{-\alpha_n}x\right)$$

with

$$X_n'(x) = -c_1\sqrt{-\alpha_n} \sin\left(\sqrt{-\alpha_n}x\right) + c_2\sqrt{-\alpha_n} \cos\left(\sqrt{-\alpha_n}x\right).$$

Applying the BCs gives

$$\begin{aligned} X_n'(0) &= c_2\sqrt{-\alpha_n} = 0 \implies c_2 = 0 \\ X_n'(L) &= -c_1\sqrt{-\alpha_n} \sin\left(\sqrt{-\alpha_n}L\right) = 0 \end{aligned}$$

therefore

$$\begin{aligned} \sqrt{-\alpha_n}L &= n\pi \\ \alpha_n &= -\frac{n^2\pi^2}{L^2} \end{aligned}$$

which gives the following family of solutions:

$$X_n(x) = c_1 \cos\left(\frac{n\pi}{L}x\right)$$

3.2.2 Time Dimension

As we found two cases for α_n, we must do the same for T_n.

Case 1. $\alpha_n < 0$.

$$\begin{aligned} m^2 - \alpha_n c^2 &= 0 \\ m^2 &= \alpha_n c^2 \\ m &= \pm\sqrt{\alpha_n}c \\ m &= \pm\sqrt{-\alpha_n}ci \end{aligned}$$

which gives

$$T_n(t) = c_3 \cos\left(\sqrt{-\alpha_n} ct\right) + c_4 \sin\left(\sqrt{-\alpha_n} ct\right) = c_3 \cos\left(\frac{n\pi}{L} ct\right) + c_4 \sin\left(\frac{n\pi}{L} ct\right).$$

Case 2. $\alpha_n = 0$.

$$\begin{aligned} m^2 &= 0 \\ m &= 0 \end{aligned}$$

which gives

$$T_n(t) = c_3 t + c_4.$$

3.2.3 General Solution

Given these two functions, we find two solutions for u_n

$$u_n(x,\, t) = \cos\left(\frac{n\pi}{L} x\right)\left[A_n \cos\left(\frac{n\pi}{L} ct\right) + B_n \sin\left(\frac{n\pi}{L} ct\right)\right].$$

for $\alpha_n < 0$, and also

$$u_0(x,\, t) = A_0 + B_0 t$$

for $\alpha_n = 0$, where u_0 does not depend on n. By applying superposition, we find the general solution to the PDE:

$$u(x,\, t) = u_0(x,\, t) + \sum_{n=1}^{\infty} u_n(x,\, t) = A_0 + B_0 t + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{L} x\right)\left[A_n \cos\left(\frac{n\pi}{L} ct\right) + B_n \sin\left(\frac{n\pi}{L} ct\right)\right].$$

Applying the initial conditions gives

$$u_n(x,\, 0) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L} x\right) = f(x)$$

so that the coefficients A_n are given by the Fourier cosine coefficients of the initial condition $f(x)$. Applying the second initial condition requires the first derivative w.r.t. x:

$$\frac{\partial u(x,\, t)}{\partial x} = B_0 + \sum_{n=1}^{\infty} \frac{n\pi}{L} c \cos\left(\frac{n\pi}{L} x\right)\left[B_n \cos\left(\frac{n\pi}{L} ct\right) - A_n \sin\left(\frac{n\pi}{L} ct\right)\right]$$

so that

$$\frac{\partial u}{\partial x}(x,\, 0) = B_0 + \sum_{n=1}^{\infty} \frac{n\pi}{L} c B_n \cos\left(\frac{n\pi}{L} x\right) = 0.$$

In this case, a zero initial velocity requires $B_0 = B_n = 0$. Therefore, the solution to the IBVP is given by

$$u(x,\, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L} x\right) \cos\left(\frac{n\pi}{L} ct\right),$$

where

$$A_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, \mathrm{d}x, \qquad A_n = \frac{2}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi}{L} x\right) \mathrm{d}x.$$

3.3 Separation of Variables: Laplace's Equation

Let us assume the following boundary conditions for Laplace's equation:

$$u\,(x,\ 0) = 0, \qquad u\,(x,\ 1) = x^2, \qquad u\,(0,\ y) = u\,(1,\ y) = 0$$

so that our region of interest is given by the unit square. Then by using the ansatz

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0\\ \frac{\partial^2 XY}{\partial x^2} + \frac{\partial^2 XY}{\partial y^2} &= 0\\ X''Y + XY'' &= 0\\ \frac{X''}{X} &= -\frac{Y''}{Y} = \alpha_n\end{aligned}$$

This results in the following two ODEs

$$\begin{aligned}X'' - \alpha_n X &= 0\\ Y'' + \alpha_n Y &= 0\end{aligned}$$

3.3.1 Problem for X

Let us first consider the problem for X as a boundary condition for Y is nonhomogeneous. From the heat equation, we know that the only nontrivial solutions to this ODE occur when

$$\alpha_n = -n^2\pi^2 < 0, \qquad X\,(x) = X_n\,(x) = c\sin\,(n\pi x), \qquad n = 1,\ 2,\ 3,\ \ldots$$

for constant c.

3.3.2 Problem for Y

The problem for Y yields the following solution:

$$Y\,(y) = Y_n\,(y) = A_n\cosh\,(n\pi y) + B_n\sinh\,(n\pi y).$$

3.3.3 General Solution

Given these two functions, we can use superposition to find

$$u\,(x,\ y) = \sum_{n=1}^{\infty} X_n\,(x)\,Y_n\,(y) = \sum_{n=1}^{\infty}\left[A_n\cosh\,(n\pi y) + B_n\sinh\,(n\pi y)\right]\sin\,(n\pi x).$$

We can now apply the boundary conditions in y. At $y = 0$

$$u\,(x,\ 0) = \sum_{n=1}^{\infty} A_n\sin\,(n\pi x) = 0 \implies A_n = 0$$

At $y = 1$:

$$u(x,\ 1) = \sum_{n=1}^{\infty} B_n \sinh(n\pi) \sin(n\pi x) = x^2$$

Here we can use the sine series expansion of x^2 where the coefficient is now multiplied by $\sinh(n\pi)$. Therefore,

$$u(x,\ y) = \sum_{n=1}^{\infty} B_n \sinh(n\pi y) \sin(n\pi x)$$

with

$$B_n \sinh(n\pi) = 2 \int_0^1 x^2 \sin(n\pi x)\,\mathrm{d}x.$$

4 Sturm-Liouville Theory

Sturm-Liouville theory is used to solve real second-order linear ODEs of the form:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right] + q(x)\,y + \lambda w(x)\,y = 0,$$

with the boundary conditions

$$\begin{aligned} -l_1 y'(a) + h_1 y(a) &= 0 \\ l_2 y'(b) + h_2 y(b) &= 0 \end{aligned}$$

where both boundary conditions must be non-trivial (l or h is non-zero). A Sturm-Liouville problem is **regular** when $p(x), w(x) > 0$, and $p(x), p'(x), q(x), w(x)$ are continuous over the interval $[a, b]$. A second-order ODE of the form

$$a_2(x)\,y''(x) + a_1(x)\,y'(x) + a_0(x)\,y(x) = \lambda y(x)$$

can be converted into SL form by multiplying the ODE by the integrating factor

$$\mu = \frac{1}{a_2}\exp\left(\int \frac{a_1}{a_2}\,\mathrm{d}x\right).$$

4.1 Weighted Inner Product

The function $w(x) > 0$ is known as the **weight function** with which we can define the inner product:

$$\langle f,\ g\rangle_w = \int_a^b f(x)\,g(x)\,w(x)\,\mathrm{d}x.$$

4.2 Eigenvalue Problem

By defining the mapping:

$$u \mapsto -\frac{1}{w(x)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right]+q(x)\,u\right)$$

with the linear operator L, we can consider the associated eigenvalue problem of the Sturm-Liouville system:

$$Lu = \lambda u.$$

4.3 Self-Adjointness

Here we recognise that L is a **self-adjoint** operator, such that:

$$\langle Lf,\ g\rangle_w = \langle f,\ Lg\rangle_w.$$

4.4 Orthogonality

It then follows that all solutions to this ODE produce an infinite number of real **eigenvalues** λ_i, where $\lambda_n \to \infty$ as $n \to \infty$, where the corresponding **eigenfunctions** u_i of L are **orthogonal** with respect to the weighted inner product. Taking the weighted inner product between normalised eigenfunctions shows that

$$\langle y_n,\ y_m\rangle = \int_a^b y_n(x)\,y_m(x)\,w(x)\,\mathrm{d}x = \delta_{mn}$$

where δ_{mn} is the Kronecker delta.

4.5 Sign of Eigenvalues

A **proper** Sturm-Liouville system is a system in which $q(x) \leqslant 0$ on $[a,b]$, with $l_1h_1 \geqslant 0$ and $l_2h_2 \geqslant 0$. All eigenvalues of a proper Sturm-Liouville system are non-negative.

4.6 Singular and Periodic Sturm-Liouville Systems

- When $p(a) = 0$, and the BC at $x = a$ is replaced by the condition that y remain bounded; the system is **singular**[1].
- If instead of the BCs we have:

  $$p(a) = p(b) \quad \text{and} \quad p'(a) = p'(b)$$

  then we have a **periodic** system, where y must also be periodic.

[1] The same applies with the boundary condition at $x = b$

4.7 Eigenfunction Expansions

If we treat the set of eigenfunctions of a Sturm-Liouville system as a **basis**, we can write a given function f as a linear combination of eigenfunctions. Given an orthogonal basis $\{y_n(x) : n \in \mathbb{Z}^+\}$, the eigenfunction expansion of f is given by

$$f_E(x) = \sum_{n=1}^{\infty} c_n y_n(x)$$

with

$$c_n = \frac{\langle f_E,\, y_n \rangle_w}{\langle y_n,\, y_n \rangle_w} = \frac{\langle f_E,\, y_n \rangle_w}{\|y_n\|^2}.$$

where the usual definition of the norm applies:

$$\|y_n\| = \sqrt{\langle y_n,\, y_n \rangle}.$$

To prove this, consider the inner product of the function f with a particular eigenfunction y_m:

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} c_n y_n(x) \\ \langle f,\, y_m \rangle_w &= \sum_{n=1}^{\infty} c_n \langle y_n,\, y_m \rangle_w \\ \langle f,\, y_m \rangle_w &= c_m \langle y_m,\, y_m \rangle_w \\ c_m &= \frac{\langle f,\, y_m \rangle_w}{\langle y_m,\, y_m \rangle_w}. \end{aligned}$$

This result generalises the Fourier series expansion introduced in Section 1, as our basis is no longer restricted to trigonometric functions.

4.7.1 Convergence

As with Fourier series, f_E does not necessarily converge to f. For instance, if f is piecewise smooth,

$$f_E(x) = \lim_{\epsilon \to 0^+} \frac{f(x+\epsilon) + f(x-\epsilon)}{2}$$

for $a < x < b$.

5 Polar Coordinates

When considering problems posed on circular regions, we can apply coordinate transformation to solve problems using polar coordinates. Recall that we can write x and y in terms of r and θ:

$$x = r\cos(\theta), \quad y = r\sin(\theta) \quad \iff \quad r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right).$$

This allows us to express partial derivatives of $u\left(x,\, y\right)$ in terms of r and θ using the multivariable chain rule:

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} \\ &= \frac{2x}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial r} + \frac{-y/x^2}{1+\left(y/x\right)^2}\frac{\partial u}{\partial \theta} \\ &= \frac{2x}{r}\frac{\partial u}{\partial r} + \frac{-y}{r^2}\frac{\partial u}{\partial \theta}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} \\ &= \frac{2y}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial r} + \frac{1/x}{1+\left(y/x\right)^2}\frac{\partial u}{\partial \theta} \\ &= \frac{2y}{r}\frac{\partial u}{\partial r} + \frac{x}{r^2}\frac{\partial u}{\partial \theta}.\end{aligned}$$

5.1 The Laplacian in Polar Coordinates

Recall Laplace's equation in Cartesian coordinates

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

By using the multivariable chain rule, we can express the Laplacian in terms of r and θ:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

5.2 Laplace's Equation in Polar Coordinates

Consider the following example of Laplace's equation on a disk of radius a:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \qquad 0 \leqslant r < a, -\pi < \theta \leqslant \pi,$$

for $u = u\left(r,\, \theta\right)$ with the following boundary condition on $r = a$:

$$u\left(a,\, \theta\right) = f\left(\theta\right).$$

Here consider a solution of the form

$$u\left(r,\, \theta\right) = R\left(r\right)\Theta\left(\theta\right)$$

so that

$$
\begin{aligned}
u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= 0 \\
R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' &= 0 \\
\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} &= 0 \\
\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} &= -\frac{1}{r^2}\frac{\Theta''}{\Theta} \\
r^2\frac{R''}{R} + r\frac{R'}{R} &= -\frac{\Theta''}{\Theta} = -\alpha
\end{aligned}
$$

therefore

$$
\begin{aligned}
r^2R'' + rR' + \alpha R &= 0 \\
\Theta'' - \alpha\Theta &= 0.
\end{aligned}
$$

5.2.1 Periodicity

To identify the Sturm-Liouville problem, we require a homogeneous equation with homogeneous boundary conditions. As u is defined on a circular disk, it must satisfy the following condition of periodicity:

$$u\left(r,\ \theta\right) = u\left(r,\ \theta + 2\pi\right).$$

Additionally, as u is defined in terms of two separable functions, $u = R\Theta$, this periodicity must also hold in Θ:

$$\Theta\left(\theta\right) = \Theta\left(\theta + 2\pi\right).$$

Solving for Θ reveals that only $\alpha_n \leqslant 0$ has periodic solutions, therefore for $n = 1, 2, \ldots$ we have

$$\Theta_0 = 1, \qquad \Theta_{n1} = \cos\left(n\theta\right), \qquad \Theta_{n2} = \sin\left(n\theta\right).$$

where $\alpha_n = -n^2$. Therefore, unlike the previous examples, each eigenvalue has two linearly independent eigenfunctions. Solving the problem in R yields the Cauchy-Euler equation:

$$r^2R'' + rR' - n^2R = 0.$$

By assuming $R = r^m$, we find $m = \pm n$. If we consider positive values of n:

$$R_n = c_1r^n + c_2r^{-n}.$$

However, when $n = 0$,

$$R_0 = c_1\ln\left(r\right) + c_2.$$

5.2.2 Boundedness

As u is defined on a circular disk, it must be bounded at $r = 0$. Therefore, R_n must be bounded at $r = 0$, which implies that in the first solution, $c_2 = 0$, and in the second solution, $c_1 = 0$. This results in

$$R_0 = c_2, \qquad R_n = c_1r^n.$$

By applying superposition,

$$u\left(r,\ \theta\right)=A_0+\sum_{n=1}^{\infty}\left[A_n r_n\cos\left(n\theta\right)+B_n r_n\sin\left(n\theta\right)\right].$$

5.2.3 Boundary Conditions

Assuming $u\left(a,\ \theta\right)=f\left(\theta\right)$, we have

$$u\left(a,\ \theta\right)=f\left(\theta\right)=A_0+\sum_{n=1}^{\infty}\left[A_n a^n\cos\left(n\theta\right)+B_n a^n\sin\left(n\theta\right)\right]$$

where A_0, $A_n a^n$, and $B_n a^n$ are the Fourier coefficients of $f\left(\theta\right)$.

6 Nonhomogeneous Problems

Nonhomogeneous PDEs are problems that involve a linear PDE of the form:

$$Lu=F$$

possibly with a boundary condition of the form:

$$u\left(c,\ t\right)=a\left(t\right).$$

6.1 Steady-State and Transient Solutions

For **time-independent** non-homogeneities (i.e., when $F=F\left(x\right)$), we can separate the solution of a nonhomogeneous PDE into a **steady-state solution**, which is found by setting the time derivative to zero, and a **transient solution**, which will satisfy the homogeneous PDE and boundary conditions. The steady-state solution takes the form,

$$u\left(x,\ t\right)=U\left(x\right),$$

where $U\left(x\right)$ satisfies the PDE and boundary conditions, but not the initial condition for $u\left(x,\ t\right)$. To find the evolution of a system from an initial condition $u\left(x,\ 0\right)=f\left(x\right)$, we need to find a transient solution $v\left(x,\ t\right)$:

$$v\left(x,\ t\right)=u\left(x,\ t\right)-U\left(x\right).$$

This solution satisfies the general solution

$$u\left(x,\ t\right)=v\left(x,\ t\right)+U\left(x\right).$$

After substitution, $u\left(x,\ t\right)$ will be transformed into an homogeneous PDE that can be solved using the methods described in the previous sections.

6.2 Eigenfunction Expansion

For a general nonhomogeneous term $F = F(x,\ t)$, we assume that the solution will take the form of an eigenfunction expansion in one variable, where the eigenfunctions are those that come from the homogeneous version of the problem, and the unknown coefficients are functions of the other variable. Here the boundary conditions must be homogeneous, and therefore, the PDE must be transformed (i.e., via a subtraction), to cancel any nonhomogeneous terms. This can be done by choosing an appropriate $v(x,\ t)$ which yields a new PDE.

7 Integral Transforms

7.1 The Fourier Transform

The Fourier series approximation is defined on the finite domain $[-L, L]$, where the approximation f_F is periodically extended outside of this domain. The Fourier transform considers the limiting process $L \to \infty$ to obtain a transform defined on the infinite domain $(-\infty, \infty)$. Let $\omega_n = \frac{n\pi}{L}$ for $n \in \mathbb{Z}$ so that $\delta\omega = \omega_{n+1} - \omega_n = \frac{\pi}{L}$. Then

$$\begin{aligned} f_F(x) = \left[\frac{1}{2\pi}\int_{-L}^{L} f(x)\,\mathrm{d}x\right]\delta\omega + \frac{1}{\pi}\sum_{n=1}^{\infty}\Bigg(\left[\int_{-L}^{L} f(z)\cos(\omega_n z)\,\mathrm{d}z\right]\cos(\omega_n x) \\ + \left[\int_{-L}^{L} f(z)\sin(\omega_n z)\,\mathrm{d}z\right]\sin(\omega_n x)\Bigg)\delta\omega. \end{aligned}$$

Taking the limit $L \to \infty$ results in $\delta\omega \to 0$, so that

$$f_F(x) = \frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{\infty} f(z)\cos(\omega z)\,\mathrm{d}z\right]\cos(\omega x)\,\mathrm{d}\omega + \frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{\infty} f(z)\sin(\omega z)\,\mathrm{d}z\right]\sin(\omega x)\,\mathrm{d}\omega.$$

As the two integrands are even and odd respectively, we can use Euler's identity to simplify the expression. First, let us define $g(\omega) = \int_{-\infty}^{\infty} f(z)\cos(\omega z)\,\mathrm{d}z$ and $h(\omega) = \int_{-\infty}^{\infty} f(z)\sin(\omega z)\,\mathrm{d}z$. Then,

$$\begin{aligned} f_F(x) &= \frac{1}{\pi}\int_0^{\infty} g(\omega)\cos(\omega x)\,\mathrm{d}\omega + \frac{1}{\pi}\int_0^{\infty} h(\omega)\sin(\omega x)\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_0^{\infty} g(\omega)\left(e^{i\omega x} + e^{-i\omega x}\right)\mathrm{d}\omega + \frac{1}{2i\pi}\int_0^{\infty} h(\omega)\left(e^{i\omega x} - e^{-i\omega x}\right)\mathrm{d}\omega \\ &= \frac{1}{2\pi}\left[\int_0^{\infty} g(\omega)e^{i\omega x}\,\mathrm{d}\omega + \int_0^{\infty} g(\omega)e^{-i\omega x}\,\mathrm{d}\omega - i\int_0^{\infty} h(\omega)e^{i\omega x}\,\mathrm{d}\omega + i\int_0^{\infty} h(\omega)e^{-i\omega x}\,\mathrm{d}\omega\right] \end{aligned}$$

Using the substitution $u = -\omega$ for the second and fourth integrals, the RHS becomes

$$\frac{1}{2\pi}\left[\int_0^{\infty} g(\omega)e^{i\omega x}\,\mathrm{d}\omega - \int_0^{-\infty} g(-u)e^{iux}\,\mathrm{d}u - i\int_0^{\infty} h(\omega)e^{i\omega x}\,\mathrm{d}\omega - i\int_0^{-\infty} h(-u)e^{iux}\,\mathrm{d}u\right].$$

Reverting this variable back to ω allows for the following simplification:

$$\begin{aligned}
f_F(x) &= \frac{1}{2\pi}\left[\int_0^\infty g(\omega)\,e^{i\omega x}\,\mathrm{d}\omega + \int_{-\infty}^0 g(\omega)\,e^{i\omega x}\,\mathrm{d}\omega - i\int_0^\infty h(\omega)\,e^{i\omega x}\,\mathrm{d}\omega - i\int_{-\infty}^0 h(\omega)\,e^{i\omega x}\,\mathrm{d}\omega\right] \\
&= \frac{1}{2\pi}\left[\int_{-\infty}^\infty g(\omega)\,e^{i\omega x}\,\mathrm{d}\omega - i\int_{-\infty}^\infty h(\omega)\,e^{i\omega x}\,\mathrm{d}\omega\right] \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty\left[\int_{-\infty}^\infty f(z)\left(\cos(\omega z) - i\sin(\omega z)\right)\mathrm{d}z\right]e^{i\omega x}\,\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty\left[\int_{-\infty}^\infty f(z)\,e^{-i\omega z}\,\mathrm{d}z\right]e^{i\omega x}\,\mathrm{d}\omega.
\end{aligned}$$

Definition 7.1 (Fourier Transform)**.** The Fourier transform of a function $f(x)$ is denoted $\hat{f}(\omega)$ and is defined:

$$\hat{f}(\omega) = \mathscr{F}\{f(x)\} = \int_{-\infty}^\infty f(x)\,e^{-i\omega x}\,\mathrm{d}x.$$

Definition 7.2 (Inverse Fourier Transform)**.** The inverse Fourier transform of a function $\hat{f}(\omega)$ is denoted $f_F(x)$ and is defined:

$$f_F(x) = \mathscr{F}^{-1}\{f(x)\} = \frac{1}{2\pi}\int_{-\infty}^\infty \hat{f}(\omega)\,e^{i\omega x}\,\mathrm{d}\omega.$$

PDEs that are defined on infinite domains can utilise the Fourier transform to transform the problem to an ODE in ω. In such problems, it is required the solution be bounded as the spatial variable approaches $\pm\infty$.

7.2 The Laplace Transform

The Laplace transform is a generalisation of the Fourier transform that is defined on the semi-infinite domain $(0, \infty)$. The

Definition 7.3 (Laplace Transform)**.** The Laplace transform of a function $f(t)$ is denoted $\mathscr{L}\{f(t)\}$ and is defined:

$$F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty f(t)\,e^{-st}\,\mathrm{d}t.$$

where s is a complex variable. Note that for this integral to exist, f must not grow faster than e^{st} as $t \to \infty$.

PDEs that are defined on semi-infinite domains can utilise the Laplace transform to transform the problem to an ODE of the remaining variable. Once solved, the inverse Laplace transform can be used to find the solution in the original domain.

7.3 The Convolution

The convolution of two functions $f(x)$ and $g(x)$ is defined:

$$(f * g)(x) = \int_{-\infty}^\infty f(x-z)\,g(z)\,\mathrm{d}z.$$

Theorem 7.3.1 (Convolution Theorem for Fourier Transforms)**.**

$$\mathscr{F}\left\{(f * g)(x)\right\} = \hat{f}(\omega)\,\hat{g}(\omega).$$

$$\frac{1}{2\pi}\left(\hat{f} * \hat{g}\right)(\omega) = \mathscr{F}\left\{f(x)\,g(x)\right\}.$$

Theorem 7.3.2 (Convolution Theorem for Laplace Transforms)**.**

$$\mathscr{L}\left\{(f * g)(t)\right\} = F(s)\,G(s).$$

$$f(t)\,g(t) = \mathscr{L}^{-1}\left\{F(s) * G(s)\right\}.$$

8 Complex Analysis

8.1 Complex Numbers

8.1.1 Definition

A complex number is any number of the form $x + iy$ where $x, y \in \mathbb{R}$ and $i^2 = -1$. Here, i is known as the imaginary unit. The set of complex numbers $\mathbb{C}$ can therefore be defined as

$$\mathbb{C} = \left\{x + iy : x, y \in \mathbb{R}\right\}.$$

The real part of a complex number $z = x + iy$ is denoted $\mathfrak{R}(z) = x$ and the imaginary part is denoted $\mathfrak{I}(z) = y$.

8.1.2 Operations on Complex Numbers

Addition and multiplication of two complex numbers is performed on the real and imaginary components separately. All axioms for addition and multiplication in $\mathbb{R}$ are preserved in $\mathbb{C}$:

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i\,(y_1 + y_2) \\ z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i\,(x_1 y_2 + x_2 y_1). \end{aligned}$$

8.1.3 Polar Form

A complex number $z = x + iy$ can be expressed in polar form as

$$z = r\left(\cos(\theta) + i\sin(\theta)\right)$$

where r is the modulus of z defined

$$|z| = r = \sqrt{x^2 + y^2},$$

and θ is the argument of z defined

$$\operatorname{Arg}(z) = \theta = \arctan\left(\frac{y}{x}\right).$$

Euler's formula states that

$$e^{i\theta} = \cos(\theta) + i\sin(\theta),$$

this allows us to express the polar form of a complex number using the exponential function:

$$z = re^{i\theta}.$$

8.1.4 Complex Conjugate

The complex conjugate of a complex number $z = x + iy$ is denoted $\overline{z} = x - iy$. This operation can be used to find the modulus of a complex number:

$$|z| = \sqrt{z\overline{z}} \iff |z|^2 = z\overline{z}.$$

8.1.5 Theorems on the Complex Plane

Theorem 8.1.1 (De Moivre's formula)**.** *For any complex number $z = re^{i\theta}$ and integer n,*

$$z^n = r^n \left(\cos(n\theta) + i\sin(n\theta)\right).$$

Proof. Using Euler's formula, we have

$$\begin{aligned} z^n &= \left(re^{i\theta}\right)^n = r^n e^{in\theta} = r^n \left(\cos(n\theta) + i\sin(n\theta)\right) \\ &= r^n \cos(n\theta) + ir^n \sin(n\theta). \end{aligned}$$

□

Theorem 8.1.2 (Triangle inequality for complex numbers)**.** *For any two complex numbers z_1 and z_2:*

$$|z_1 + z_2| \leqslant |z_1| + |z_2|.$$

Proof. We will prove this theorem by showing that this inequality is a tautology. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then,

$$\begin{aligned} |z_1 + z_2| &\leqslant |z_1| + |z_2| \\ |z_1 + z_2|^2 &\leqslant \left(|z_1| + |z_2|\right)^2 \\ (z_1 + z_2)\,\overline{(z_1 + z_2)} &\leqslant |z_1|^2 + 2|z_1||z_2| + |z_2|^2 \\ (x_1 + x_2)^2 + (y_1 + y_2)^2 &\leqslant x_1^2 + y_1^2 + 2|z_1||z_2| + x_2^2 + y_2^2 \\ x_1^2 + 2x_1x_2 + x_2^2 + y_1^2 + 2y_1y_2 + y_2^2 &\leqslant x_1^2 + y_1^2 + 2|z_1||z_2| + x_2^2 + y_2^2 \\ 2x_1x_2 + 2y_1y_2 &\leqslant 2|z_1||z_2| \\ x_1x_2 + y_1y_2 &\leqslant \left(x_1^2 + y_1^2\right)^{1/2} \left(x_2^2 + y_2^2\right)^{1/2} \\ (x_1x_2 + y_1y_2)^2 &\leqslant \left(x_1^2 + y_1^2\right)\left(x_2^2 + y_2^2\right) \\ x_1^2x_2^2 + 2x_1x_2y_1y_2 + y_1^2y_2^2 &\leqslant x_1^2x_2^2 + x_1^2y_2^2 + x_2^2y_1^2 + y_1^2y_2^2 \\ 2x_1x_2y_1y_2 &\leqslant x_1^2y_2^2 + x_2^2y_1^2 \\ 0 &\leqslant x_1^2y_2^2 - 2x_1x_2y_1y_2 + x_2^2y_1^2 \\ 0 &\leqslant (x_1y_2 - x_2y_1)^2. \end{aligned}$$

□

Theorem 8.1.3 (Reverse triangle inequality for complex numbers)**.** *For any two complex numbers z_1 and z_2:*

$$\left||z_1| - |z_2|\right| \leqslant |z_1 - z_2|.$$

Proof.
$$|z_1| = |(z_1 - z_2) + z_2| \leqslant |z_1 - z_2| + |z_2| \implies |z_1| - |z_2| \leqslant |z_1 - z_2|,$$
and
$$|z_2| = |(z_2 - z_1) + z_1| \leqslant |z_2 - z_1| + |z_1| \implies |z_2| - |z_1| \leqslant |z_2 - z_1|.$$
But
$$\begin{aligned} |z_1 - z_2| &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= |z_2 - z_1|. \end{aligned}$$
Hence,
$$||z_1| - |z_2|| \leqslant |z_1 - z_2|.$$
□

Corollary 8.1.3.1 (Triangle inequality for multiple complex numbers)**.** *The triangle inequality can be extended to any number of complex numbers:*
$$\left|\sum_{i=1}^{n} z_i\right| \leqslant \sum_{i=1}^{n} |z_i|.$$

Proof. We will prove this corollary by induction. The base case with $n = 1$ complex numbers is trivially true:
$$|z_1| \leqslant |z_1|.$$
If we assume that this corollary holds for $n = k$ complex numbers:
$$\left|\sum_{i=1}^{k} z_i\right| \leqslant \sum_{i=1}^{k} |z_i|.$$
we can show that this corollary also holds for $n = k + 1$ complex numbers:
$$\left|\sum_{i=1}^{k+1} z_i\right| = \left|\sum_{i=1}^{k} z_i + z_{k+1}\right| \leqslant \left|\sum_{i=1}^{k} z_i\right| + |z_{k+1}| \leqslant \sum_{i=1}^{k} |z_i| + |z_{k+1}| = \sum_{i=1}^{k+1} |z_i|.$$
Since this corollary holds for the base case, and also for the inductive step whenever the inductive hypothesis is true, the corollary holds for all $n \in \mathbb{N}$. □

8.2 Complex-Valued Functions

A complex-valued function f maps the complex values z in a set S to a unique set of complex values w in a set T. The function f is defined:
$$w = f(z).$$
Separating the real and imaginary components of z and w as $z = x + iy$ and $w = u + iv$, shows that u and v are functions of x and y:
$$w = f(z) = u(x,\ y) + iv(x,\ y).$$
Therefore f is a mapping of values from the $(x,\ y)$ plane to the $(u,\ v)$ plane.

8.2.1 Complex Exponentials

The complex exponential is defined

$$w = e^z = \exp(z) = e^x (\cos(y) + i \sin(y)).$$

Using this form, it can be shown that the complex exponential maintains several properties of the real exponential:

1. $e^{z_1+z_2} = e^{z_1} e^{z_2}$
2. $e^{nz} = (e^z)^n$ for $n \in \mathbb{Z}$

8.2.2 Trigonometric and Hyperbolic Functions

Using Euler's formula, we can derive the exponential form of the following trigonometric functions:

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$
$$\cosh(z) = \frac{e^{z} + e^{-z}}{2} \qquad \sinh(z) = \frac{e^{z} - e^{-z}}{2}$$

from this, it can be deduced that

$$\cos(z) = \cosh(iz) \quad \text{and} \quad i \sin(z) = \sinh(iz)$$

or similarly,

$$\cosh(z) = \cos(iz) \quad \text{and} \quad i \sinh(z) = \sin(iz).$$

8.2.3 Complex Logarithms

The natural logarithm of a complex number z, is denoted

$$w = \log(z)$$

and is defined as the solution to $e^w = z$ for all $z \neq 0$. Using polar form, we can use the periodicity of the complex exponential to show that there are infinitely many solutions to this equation.

$$\begin{aligned} \log(z) &= \ln\left(|z| e^{i \arg(z)}\right) \\ &= \ln\left(|z| e^{i(\operatorname{Arg}(z) + 2\pi n)}\right) \\ &= \ln|z| + i(\operatorname{Arg}(z) + 2\pi n), \end{aligned}$$

where $\arg(z)$ is the argument of z, defined using the principal value of the argument, $\operatorname{Arg}(z)$, and $n \in \mathbb{Z}$. Here the function ln corresponds to the real-valued natural logarithm. If we wish to define a principal value $\operatorname{Log}(z)$ whose imaginary part lies in the interval $(-\pi, \pi]$, we can use the principal value of the argument:

$$\operatorname{Log}(z) = \ln|z| + i \operatorname{Arg}(z).$$

This lets us also define the complex logarithm in terms of the principal logarithm:

$$\log(z) = \operatorname{Log}(z) + 2\pi i n.$$

8.2.4 Analytic Functions

In this section, we discuss the idea of differentiability for complex functions. Let $w = f(z)$ be defined in a neighbourhood $0 < |z - z_0| < \delta$ of z_0, except possibly at some isolated points. We say

$$\lim_{z \to z_0} f(z) = w_0$$

if for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(z) - w_0| < \epsilon$$

for all $0 < |z - z_0| < \delta$[2]. As with real functions, this limit must be independent of the path taken to approach z_0. The derivative of a complex function $f(z)$ is defined as

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z},$$

provided this limit exists. It follows that all rules of differentiation for real functions apply to complex functions. Recall that we can express a complex function in terms of its real and imaginary parts:

$$f(z) = u(x,\ y) + iv(x,\ y)\,.$$

Therefore, let us decompose the derivative of f into its real and imaginary parts, using $\Delta z = \Delta x + i\,\Delta y$:

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x + \Delta x,\ y + \Delta y) + iv(x + \Delta x,\ y + \Delta y) - u(x,\ y) - iv(x,\ y)}{\Delta x + i\,\Delta y}.$$

For this limit to exist, the value must be independent of which variable approaches zero first. If we consider the limit as $\Delta y \to 0$ first, we find

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}z} &= \lim_{\Delta x \to 0} \frac{u(x + \Delta x,\ y) - u(x,\ y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x + \Delta x,\ y) - v(x,\ y)}{\Delta x} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}. \end{aligned}$$

Similarly, if we consider the limit as $\Delta x \to 0$ first, we find

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}z} &= \lim_{\Delta y \to 0} \frac{u(x,\ y + \Delta y) - u(x,\ y)}{i\,\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x,\ y + \Delta y) - v(x,\ y)}{i\,\Delta y} \\ &= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \\ &= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}. \end{aligned}$$

Therefore, the complex derivative of a function is given by:

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

[2]Note that it is not necessary for the function to be defined at z_0 itself: $f(z_0) = w_0$.

This implies that the following relationships hold between u and v:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

These equations are known as the **Cauchy-Riemann equations**, and are necessary conditions for the complex derivative to exist at a point, provided that the partial derivatives of u and v are continuous. A function that is complex differentiable in the **neighbourhood** of a point is said to be **analytic** at that point. Such functions are infinitely smooth (differentiable) and have a convergent Taylor series expansion about that point. A function that is analytic on the entire complex plane is said to be **entire**.

8.2.5 Common Analytic Functions

Using the definition of the complex derivative, we can show that the following functions are analytic in the complex plane:

- Power functions z^n for $n \in \mathbb{Z}$
- Exponential functions e^z
- Trigonometric functions $\cos(z)$ and $\sin(z)$
- Hyperbolic functions $\cosh(z)$ and $\sinh(z)$
- Rational functions $f(z) = \frac{P(z)}{Q(z)}$ (except when $Q(z) = 0$)
- Any linear combination or composition of the above functions
- Logarithmic, non-integer power, and inverse trigonometric functions (except at branch points or branch cuts)

Additionally, all rules of differentiation for real functions also apply to complex functions.

8.2.6 Harmonic Functions

The Cauchy-Riemann equations allow us to relate a complex function to Laplace's equation. Consider a complex function $f(z) = u(x, y) + iv(x, y)$ that is analytic in a domain $\boldsymbol{D}$. Then, we can show that both u and v satisfy Laplace's equation using the Cauchy-Riemann equations:

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}, & \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} & & \\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 v}{\partial x\,\partial y}, & \frac{\partial^2 u}{\partial y^2} &= -\frac{\partial^2 v}{\partial y\,\partial x} & \implies \frac{\partial^2 u}{\partial x^2} &= -\frac{\partial^2 u}{\partial y^2} \implies \nabla^2 u = 0. \\
\frac{\partial^2 u}{\partial y\,\partial x} &= \frac{\partial^2 v}{\partial y^2}, & \frac{\partial^2 u}{\partial x\,\partial y} &= -\frac{\partial^2 v}{\partial x^2} & \implies \frac{\partial^2 v}{\partial y^2} &= -\frac{\partial^2 v}{\partial x^2} \implies \nabla^2 v = 0.
\end{aligned}$$

Here we use Clairaut's theorem to show that the mixed partial derivatives are equal. If u and v satisfy Laplace's equations and the Cauchy-Riemann equations, they are said to be **harmonic conjugates**.

8.3 Contour Integrals

Consider the oriented curve $\mathcal{C}$ in the complex plane, which is parameterised by $z = z(t)$ for $a \leqslant t \leqslant b$:

$$z(t) = x(t) + iy(t),$$

where x and y are real continuous functions. A curve is said to be:

- (piecewise) **smooth** if $\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} + i\frac{\mathrm{d}y}{\mathrm{d}t}$ is (piecewise) continuous and nonzero for all t
- **closed** if $z(a) = z(b)$
- **simple** if it does not intersect itself: $z(t_1) \neq z(t_2)$ for $t_1 \neq t_2$

The line integral of a complex function $f(z)$ along a curve $\mathcal{C}$ is defined as a contour integral:

$$\int_{\mathcal{C}} f(z)\,\mathrm{d}z = \int_a^b f(z(t))\frac{\mathrm{d}z}{\mathrm{d}t}\,\mathrm{d}t.$$

A useful result in complex analysis is the closed line integral of the complex power function:

$$\oint_{\mathcal{C}_R} z^n\,\mathrm{d}z$$

where $\mathcal{C}_R$ is a circle of radius R oriented anticlockwise. We can solve this integral using the parametrisation $z(t) = Re^{it}$ for $0 \leqslant t \leqslant 2\pi$:

$$\begin{aligned}\oint_{\mathcal{C}_R} z^n\,\mathrm{d}z &= \int_0^{2\pi} \left(Re^{it}\right)^n iRe^{it}\,\mathrm{d}t \\ &= iR^{n+1}\int_0^{2\pi} e^{i(n+1)t}\,\mathrm{d}t \\ &= iR^{n+1}\left[\frac{e^{i(n+1)t}}{i(n+1)}\right]_0^{2\pi}.\end{aligned}$$

When $n \neq -1$, the integral evaluates to zero, as the exponential function is periodic with period 2π. When $n = -1$, the second line evaluates to:

$$iR^0\int_0^{2\pi}\mathrm{d}t = 2\pi i.$$

Therefore,

$$\oint_{\mathcal{C}_R} z^n\,\mathrm{d}z = \begin{cases} 2\pi i & n = -1, \\ 0 & n \neq -1. \end{cases}$$

8.3.1 Contour Integral Theorems

Theorem 8.3.1 (Cauchy's integral theorem)**.** *Let $f(z)$ be analytic on and within a simple closed curve $\mathcal{C}$:*

$$\oint_{\mathcal{C}} f(z)\,\mathrm{d}z = 0.$$

Proof. Let $f(z) = u(x,\ y) + iv(x,\ y)$ be analytic on and within $\mathcal{C}$. Then,

$$\begin{aligned}\oint_{\mathcal{C}} f(z)\,\mathrm{d}z &= \oint_{\mathcal{C}} (u(x,\ y) + iv(x,\ y))\,(\mathrm{d}x + i\,\mathrm{d}y) \\ &= \oint_{\mathcal{C}} (u(x,\ y)\,\mathrm{d}x - v(x,\ y)\,\mathrm{d}y) + i\oint_{\mathcal{C}} (v(x,\ y)\,\mathrm{d}x + u(x,\ y)\,\mathrm{d}y)\,.\end{aligned}$$

By Green's theorem, this becomes,

$$\oint_{\mathcal{C}} f(z)\,\mathrm{d}z = \iint_{\mathcal{D}} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y + i\iint_{\mathcal{D}} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y,$$

where $\mathcal{D}$ is the region enclosed by $\mathcal{C}$. As f is analytic, it satisfies the Cauchy-Riemann equations, so that both integrands are zero. Therefore, the integral is zero:

$$\oint_{\mathcal{C}} f(z)\,\mathrm{d}z = \iint_{\mathcal{D}} (0)\,\mathrm{d}x\,\mathrm{d}y + i\iint_{\mathcal{D}} (0)\,\mathrm{d}x\,\mathrm{d}y = 0.$$

□

Corollary 8.3.1.1 (Path independence)**.** *The contour integral between any two points z_1 and z_2 is independent of path, provided that the path is in a region where the integrand is analytic.*

Proof. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two curves connecting the points z_1 and z_2 in a region where the function $f(z)$ is analytic. Furthermore, the closed curve $\mathcal{C} = \mathcal{C}_1 - \mathcal{C}_2$ is a simple closed curve which encloses no singularities. By Cauchy's integral theorem,

$$\begin{aligned}\oint_{\mathcal{C}} f(z)\,\mathrm{d}z &= 0 \\ \oint_{\mathcal{C}_1} f(z)\,\mathrm{d}z - \oint_{\mathcal{C}_2} f(z)\,\mathrm{d}z &= 0 \\ \oint_{\mathcal{C}_1} f(z)\,\mathrm{d}z &= \oint_{\mathcal{C}_2} f(z)\,\mathrm{d}z.\end{aligned}$$

□

Corollary 8.3.1.2 (Equivalence of homotopic contours)**.** *The contour integral over any two closed contours in a region where the integrand is analytic are equal, provided that the contours can be continuously deformed into each other without crossing any singularities. Such contours are said to be* ***homotopic****.*

Proof. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two closed contours defined in a region where the function $f(z)$ is analytic and assume $\mathcal{C}_1$ is enclosed by $\mathcal{C}_2$. Then, we can define a closed curve $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_3 - \mathcal{C}_2 - \mathcal{C}_3$, where $\mathcal{C}_3$ is a curve that connects the endpoints of $\mathcal{C}_1$ and $\mathcal{C}_2$ in both directions. By Cauchy's integral theorem,

$$\oint_{\mathcal{C}} f(z)\,\mathrm{d}z = 0.$$

The contribution of $\mathcal{C}_3$ and $-\mathcal{C}_3$ cancel out, which leaves

$$\int_{\mathcal{C}_1-\mathcal{C}_2} f(z)\,\mathrm{d}z = 0$$
$$\int_{\mathcal{C}_1} f(z)\,\mathrm{d}z = \int_{\mathcal{C}_2} f(z)\,\mathrm{d}z.$$

□

Theorem 8.3.2 (Cauchy's integral formula for functions)**.** *Let $f(z)$ be analytic on and within a simple closed curve $\mathcal{C}$, and let z_0 be a point within $\mathcal{C}$. Then,*

$$f(z_0) = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(z)}{z-z_0}\,\mathrm{d}z.$$

Note we may alternatively write this as

$$f(z) = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(w)}{w-z}\,\mathrm{d}w$$

for an arbitrary point z within $\mathcal{C}$.

Proof. The function

$$\frac{f(z)}{z-z_0}$$

is analytic on and within $\mathcal{C}$, except at $z = z_0$. By Cauchy's integral theorem, we can deform $\mathcal{C}$ into a circle $\mathcal{C}_R$ of radius R centred at z_0, oriented in the same direction as $\mathcal{C}$:

$$\oint_{\mathcal{C}} \frac{f(z)}{z-z_0}\,\mathrm{d}z = \oint_{\mathcal{C}_R} \frac{f(z)}{z-z_0}\,\mathrm{d}z.$$

As f is analytic, it must be continuous, and $f(z_0)$ must be well-defined. Therefore, by taking the limit $R \to 0$, we have concluded our proof:

$$\oint_{\mathcal{C}_R} \frac{f(z)}{z-z_0}\,\mathrm{d}z = \lim_{R\to 0}\oint_{\mathcal{C}_R} \frac{f(z)}{z-z_0}\,\mathrm{d}z = \oint_{\mathcal{C}_R} \frac{f(z_0)}{z-z_0}\,\mathrm{d}z = f(z_0)\oint_{\mathcal{C}_R} \frac{1}{z-z_0}\,\mathrm{d}z = f(z_0)\,2\pi i.$$

□

Theorem 8.3.3 (Cauchy's integral formula for derivatives)**.** *Let $f(z)$ be analytic on and within a simple closed curve $\mathcal{C}$, then,*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_{\mathcal{C}} \frac{f(w)}{(w-z)^{n+1}}\,\mathrm{d}w.$$

Proof. We will prove this result using induction. For the base case, we will consider the limit definition of the first derivative of f:

$$f'(z) = \lim_{\Delta z\to 0}\frac{f(z+\Delta z) - f(z)}{\Delta z}.$$

We can express both terms in the numerator using the Cauchy integral formula:

$$f(z+\Delta z) = \frac{1}{2\pi i}\oint_C \frac{f(w)}{w-z-\Delta z}\,\mathrm{d}w, \qquad f(z) = \frac{1}{2\pi i}\oint_C \frac{f(w)}{w-z}\,\mathrm{d}w,$$

so that

$$\begin{aligned}
f'(z) &= \lim_{\Delta z\to 0}\frac{1}{\Delta z}\left[\frac{1}{2\pi i}\oint_C \frac{f(w)}{w-z-\Delta z}\,\mathrm{d}w - \frac{1}{2\pi i}\oint_C \frac{f(w)}{w-z}\,\mathrm{d}w\right] \\
&= \lim_{\Delta z\to 0}\frac{1}{2\pi i\,\Delta z}\oint_C f(w)\left[\frac{1}{w-z-\Delta z} - \frac{1}{w-z}\right]\mathrm{d}w \\
&= \lim_{\Delta z\to 0}\frac{1}{2\pi i\,\Delta z}\oint_C \frac{f(w)\,\Delta z}{(w-z-\Delta z)(w-z)}\,\mathrm{d}w \\
&= \lim_{\Delta z\to 0}\frac{1}{2\pi i}\oint_C \frac{f(w)}{(w-z)^2-\Delta z\,(w-z)}\,\mathrm{d}w \\
&= \frac{1}{2\pi i}\oint_C \frac{f(w)}{(w-z)^2}\,\mathrm{d}w.
\end{aligned}$$

We will now assume that this theorem holds for $n = k$,

$$f^{(k)}(z) = \frac{k!}{2\pi i}\oint_C \frac{f(w)}{(w-z)^{k+1}}\,\mathrm{d}w,$$

and show that it also holds when $n = k+1$. We will again use the limit definition of the derivative for this step:

$$f^{(k+1)}(z) = \lim_{\Delta z\to 0}\frac{f^{(k)}(z+\Delta z) - f^{(k)}(z)}{\Delta z}.$$

Using the induction hypothesis, we can express the terms in the numerator as:

$$f^{(k)}(z+\Delta z) = \frac{k!}{2\pi i}\oint_C \frac{f(w)}{(w-z-\Delta z)^{k+1}}\,\mathrm{d}w, \qquad f^{(k)}(z) = \frac{k!}{2\pi i}\oint_C \frac{f(w)}{(w-z)^{k+1}}\,\mathrm{d}w.$$

Then,

$$\begin{aligned}
f^{(k+1)}(z) &= \lim_{\Delta z\to 0}\frac{1}{\Delta z}\left[\frac{k!}{2\pi i}\oint_C \frac{f(w)}{(w-z-\Delta z)^{k+1}}\,\mathrm{d}w - \frac{k!}{2\pi i}\oint_C \frac{f(w)}{(w-z)^{k+1}}\,\mathrm{d}w\right] \\
&= \lim_{\Delta z\to 0}\frac{k!}{2\pi i\,\Delta z}\oint_C f(w)\left[\frac{1}{(w-z-\Delta z)^{k+1}} - \frac{1}{(w-z)^{k+1}}\right]\mathrm{d}w \\
&= \lim_{\Delta z\to 0}\frac{k!}{2\pi i\,\Delta z}\oint_C f(w)\,\frac{(w-z)^{k+1} - (w-z-\Delta z)^{k+1}}{(w-z-\Delta z)^{k+1}(w-z)^{k+1}}\,\mathrm{d}w \\
&= \lim_{\Delta z\to 0}\frac{k!}{2\pi i\,\Delta z}\oint_C f(w)\,\frac{(w-z)^{k+1} - (w-z-\Delta z)^{k+1}}{\left((w-z)^2 - \Delta z\,(w-z)\right)^{k+1}}\,\mathrm{d}w.
\end{aligned}$$

To simplify the numerator of the integrand, we will use difference of nth powers formula:

$$a^n - b^n = (a-b)\sum_{j=0}^{n-1} a^{n-1-j}b^j,$$

which greatly simplifies the numerator:

$$\begin{aligned}
(w-z)^{k+1} - (w-z-\Delta z)^{k+1} &= (w-z-w+z+\Delta z)\sum_{j=0}^{k}(w-z)^{k-j}(w-z-\Delta z)^j \\
&= \Delta z\sum_{j=0}^{k}(w-z)^{k-j}(w-z-\Delta z)^j \\
&= \Delta z\left[(w-z)^k + (w-z)^{k-1}(w-z-\Delta z) + \cdots + (w-z-\Delta z)^k\right].
\end{aligned}$$

Let us then substitute this result into the integrand:

$$\begin{aligned}
f^{(k+1)}(z) &= \lim_{\Delta z\to 0}\frac{k!}{2\pi i\,\Delta z}\oint_C \frac{f(w)\,\Delta z\left[(w-z)^k + \cdots + (w-z-\Delta z)^k\right]}{\left((w-z)^2 - \Delta z\,(w-z)\right)^{k+1}}\,\mathrm{d}w \\
&= \frac{k!}{2\pi i}\oint_C \frac{f(w)\left[(w-z)^k + (w-z)^{k-1}(w-z) + \cdots + (w-z)^k\right]}{\left((w-z)^2\right)^{k+1}}\,\mathrm{d}w \\
&= \frac{k!}{2\pi i}\oint_C \frac{f(w)\,(k+1)\,(w-z)^k}{(w-z)^{2k+2}}\,\mathrm{d}w \\
&= \frac{(k+1)!}{2\pi i}\oint_C \frac{f(w)}{(w-z)^{(k+1)+1}}\,\mathrm{d}w.
\end{aligned}$$

Since the theorem holds for the base case, and also for the inductive step whenever the inductive hypothesis is true, the theorem holds for all $n \in \mathbb{N}$. $\square$

Corollary 8.3.3.1 (Differentiability of analytic functions)**.** *If $f(z)$ is analytic on and within a simple closed curve C, then f is infinitely differentiable on C.*

Proof. Cauchy's integral formula for derivatives provides an explicit expression for $f^{(n)}(z)$ in terms of a contour integral, for all $n \in \mathbb{N}$. $\square$

Theorem 8.3.4 (Triangle inequality for contour integrals)**.** *Let $f(z)$ be analytic on and within a simple closed curve C, parametrised by $z = z(t)$ for $a \leqslant t \leqslant b$. Then,*

$$\left|\int_C f(z)\,\mathrm{d}z\right| \leqslant \int_C |f(z)||\mathrm{d}z|,$$

where $\mathrm{d}z = z'(t)\,\mathrm{d}t$ and $|\mathrm{d}z| = |z'(t)|\,\mathrm{d}t$.

Proof. Consider the Riemann sum of the parametrised integral:

$$\sum_{i=1}^{n} f\left(z\left(t_i\right)\right) z'\left(t_i\right) \Delta t,$$

where $\Delta t = \frac{b-a}{n}$. By the triangle inequality for n complex numbers, we have

$$\left|\sum_{i=1}^{n} f\left(z\left(t_i\right)\right) z'\left(t_i\right) \Delta t\right| \leqslant \sum_{i=1}^{n}\left|f\left(z\left(t_i\right)\right) z'\left(t_i\right) \Delta t\right| = \sum_{i=1}^{n}\left|f\left(z\left(t_i\right)\right)\right|\left|z'\left(t_i\right)\right| \Delta t.$$

If we consider the limit as $n \to \infty$, we find

$$\left|\int_a^b f\left(z\left(t\right)\right) z'\left(t\right) \mathrm{d}t\right| \leqslant \int_a^b \left|f\left(z\left(t\right)\right)\right|\left|z'\left(t\right)\right| \mathrm{d}t \iff \left|\int_C f\left(z\right) \mathrm{d}z\right| \leqslant \int_C \left|f\left(z\right)\right|\left|\mathrm{d}z\right|.$$

□

Corollary 8.3.4.1 (Estimation of contour integrals)**.** *If $\left|f\left(z\right)\right| \leqslant M$ for all z on and within a simple closed curve C parametrised by $z = z\left(t\right)$ for $a \leqslant t \leqslant b$, then*

$$\left|\int_C f\left(z\right) \mathrm{d}z\right| \leqslant M \int_C \left|\mathrm{d}z\right| = ML,$$

where L is the length of the path C:

$$L = \int_a^b \left|z'\left(t\right)\right| \mathrm{d}t.$$

Proof. By the triangle inequality for contour integrals, we have

$$\left|\int_C f\left(z\right) \mathrm{d}z\right| \leqslant \int_C \left|f\left(z\right)\right|\left|\mathrm{d}z\right| \leqslant M \int_C \left|\mathrm{d}z\right| = ML.$$

□

8.4 Complex Series

Theorem 8.4.1 (Cauchy's root test)**.** *Given a series of complex numbers*

$$\sum_{n=0}^{\infty} a_n,$$

consider the limit superior

$$C = \limsup_{n\to\infty} \left|a_n\right|^{1/n}.$$

If

- $C < 1$*: the series converges absolutely, and*
- $C > 1$*: the series diverges.*

Note that this test is inconclusive when $C = 1$.

Proof. We will consider each case separately. When $C < 1$,

$$\limsup_{n\to\infty} |a_n|^{1/n} < 1.$$

Now consider some $s \in \mathbb{R}$, where

$$\limsup_{n\to\infty} |a_n|^{1/n} < s < 1.$$

As a consequence of the definition of the limit superior, $|a_n|^{1/n}$ will be less than s for most values of n:

$$|a_n|^{1/n} < s \implies |a_n| < s^n.$$

Note that the series $\sum_{n=0}^{\infty} s^n$ converges by the geometric series test as $s < 1$. Therefore, by the comparison test, the series $\sum_{n=0}^{\infty} |a_n|$ also converges. Thus, the series $\sum_{n=0}^{\infty} a_n$ converges absolutely. When $C > 1$, we can use a similar argument and choose some $s \in \mathbb{R}$ such that

$$\limsup_{n\to\infty} |a_n|^{1/n} > s > 1.$$

Here, we can show that $|a_n|^{1/n}$ will be greater than s for most values of n:

$$|a_n|^{1/n} > s \implies |a_n| > s^n.$$

As $\sum_{n=0}^{\infty} s^n$ diverges the series $\sum_{n=0}^{\infty} a_n$ also diverges. □

Theorem 8.4.2 (Cauchy-Hadamard theorem)**.** *Given a complex power series*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

consider the limit superior

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n},$$

where we say $R = \infty$ if the limit superior is zero.

- *For all $0 < r < R$, the series converges absolutely and uniformly to a function $f(z)$ on the disk $|z - z_0| < r$.*
- *For all z outside this disk, $|z - z_0| > R$, the series diverges.*

The convergence of the series is unknown on the boundary of the disk $|z - z_0| = R$. Here R is known as the ***radius of convergence*** *and can take any value in $\mathbb{R} \cup \{\infty\}$.*

Proof. We will first show that this series converges absolutely and uniformly for all $0 < r < R$ inside the disk $|z - z_0| < r$. As $r < R$, we have $1/R < 1/r$, so that

$$\limsup_{n\to\infty} |a_n|^{1/n} < \frac{1}{r}.$$

Consider some $s \in \mathbb{R}$, where

$$\limsup_{n\to\infty} |a_n|^{1/n} < s < \frac{1}{r}.$$

As a consequence of the definition of the limit superior, $|a_n|^{1/n}$ will be less than s for most values of n:

$$|a_n|^{1/n} < s \implies |a_n| < s^n.$$

If we multiply this inequality by $|z - z_0|^n$, we find

$$\left|a_n (z - z_0)^n\right| < s^n |z - z_0|^n < s^n r^n = (sr)^n.$$

As $s < 1/r$, we have $sr < 1$, so that the series $\sum_{n=0}^{\infty} (sr)^n$ converges by the geometric series test. Therefore, by the Weierstrass M-test, the series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges absolutely and uniformly on the disk $|z - z_0| < r$. For values of z that lie outside the disk $|z - z_0| > R$, we can use the Cauchy root test to show that the series diverges. Consider the limit superior of the series:

$$\begin{aligned}
\limsup_{n\to\infty} \left|a_n (z - z_0)^n\right|^{1/n} &= \limsup_{n\to\infty} |a_n|^{1/n} |z - z_0| \\
&= |z - z_0| \limsup_{n\to\infty} |a_n|^{1/n} \\
&= |z - z_0| \frac{1}{R} \\
&> 1.
\end{aligned}$$

As the limit superior is greater than one, by the Cauchy root test, the series diverges for all z outside the disk $|z - z_0| = R$. □

8.4.1 Taylor Series

The Taylor series expansion of a real-valued function $f(x)$ about a point x_0 is defined as

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

where through repeated differentiation, we can show

$$a_n = \frac{f^{(n)}(x_0)}{n!}.$$

Here, $f^{(n)}(x_0)$ is the nth derivative of $f(x)$ evaluated at x_0. Let us propose that we can extend this definition to a complex-valued function $f(z)$, such that we can define the Taylor series about a point z_0 as a complex power series:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

with a radius of convergence R by the Cauchy-Hadamard theorem.

Theorem 8.4.3 (Analytic functions have a Taylor series expansion)**.** *Let $f(z)$ be an analytic function on and within a simple closed curve $\mathcal{C}$, then $f(z)$ has a Taylor series expansion about any point z_0 within $\mathcal{C}$:*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

with radius of convergence R, which is the largest disk from z_0 that is contained within $\mathcal{C}$. The coefficients of this series are given by:

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(w)}{(w - z_0)^{n+1}} \, \mathrm{d}w.$$

Proof. Consider Cauchy's integral formula for derivatives, which states that

$$f(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(w)}{w - z} \, \mathrm{d}w.$$

We will express the denominator term using a geometric series:

$$\begin{aligned}
\frac{1}{w - z} &= \frac{1}{w - z_0 + z_0 - z} \\
&= \frac{1}{w - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{w - z_0}} \\
&= \frac{1}{w - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^n, \qquad \left| \frac{z - z_0}{w - z_0} \right| < 1 \\
&= \sum_{n=0}^{\infty} \frac{1}{(w - z_0)^{n+1}} (z - z_0)^n.
\end{aligned}$$

Therefore,

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \oint_{\mathcal{C}} f(w) \left[\sum_{n=0}^{\infty} \frac{1}{(w - z_0)^{n+1}} (z - z_0)^n \right] \mathrm{d}w \\
&= \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(w)}{(w - z_0)^{n+1}} \, \mathrm{d}w \right] (z - z_0)^n \\
&= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n
\end{aligned}$$

within the disk $|z - z_0| < R$, where R is the radius of convergence. □

Theorem 8.4.4 (A function with a Taylor series expansion is analytic)**.** *Let $f(z)$ be a function with a Taylor series expansion about a point z_0 with radius of convergence R:*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

Then, $f(z)$ is analytic on and within the disk $|z-z_0|<R$, and its derivatives are given by termwise differentiation:

$$f^{(n)}(z)=\sum_{k=0}^{\infty}a_k\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left[(z-z_0)^k\right]=\sum_{k=n}^{\infty}a_k\frac{k!}{(k-n)!}(z-z_0)^{k-n}.$$

Proof. As the Taylor series is a sum of analytic functions, it is also analytic on the disk $|z-z_0|<R$. Proving the form of the derivative is trivial. □

8.4.2 Singularities

There are three types of **isolated singularities** that can occur in complex functions. Suppose $f(z)$ is analytic on a punctured neighbourhood $0<|z-z_0|<R$, but not at z_0.

- **Removable singularities**: $f(z)$ has a removable singularity at z_0 if the limit

  $$L=\lim_{z\to z_0}f(z)$$

  exists and is finite. In this case, we can define a new function $g(z)$

  $$g(z)=\begin{cases}f(z), & z\neq z_0\\ L, & z=z_0\end{cases}$$

  that is analytic at z_0.
- **Poles**: $f(z)$ has a pole of order $m\in\mathbb{N}$ at z_0 if there exists a function $g(z)$ that is analytic at z_0 such that

  $$f(z)=\frac{g(z)}{(z-z_0)^m},$$
- **Essential singularities**: $f(z)$ has an essential singularity at z_0 if the singularity is neither removable nor a pole.

8.4.3 Laurent Series

While Taylor series represent analytic functions within a disk centred at a point z_0, they fail to converge if the function is not analytic at z_0. To describe the behaviour of functions near isolated singularities, we will generalise the Taylor series by allowing negative powers of $(z-z_0)$. The Laurent series expansion of a function $f(z)$ about a point z_0 is defined as

$$f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n,$$

where the coefficients a_n are given by the contour integral:

$$a_n=\frac{1}{2\pi i}\oint_C\frac{f(w)}{(w-z_0)^{n+1}}\,\mathrm{d}w.$$

This series converges in an annular region defined by the radii of convergence r and R such that

$$r<|z-z_0|<R.$$