Differential Equations and Modelling 2

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Part I
Symmetry Methods

1 Symmetry Transformations

Consider a partial differential equation for $u(x, t)$ whose domain lies in $\mathbb{R}^2$. Such a problem typically does not have any natural length or time scales associated with it's fundamental solution. Thus, let us transform the independent and dependent variables through the mapping $(x, t, u) \mapsto (X, T, U)$ where $X = X(x, t, u)$, $T = T(x, t, u)$, and $U = U(x, t, u)$, such that the PDE is invariant under this transformation, that is, the transformed PDE in U has the same form as the original PDE in u.
The choice of mapping may consist of dilations and translations of the independent and dependent variables by some constant factors, where each constant is enforced by the invariance condition. To do this, we must ensure that the function $U(x, t, u(X, T))$ satisfies the transformed PDE. When this is the case, the PDE is said to have a *symmetry transformation*.
The goal of these symmetry methods is to find an appropriate transformation that simplies the PDE, often reducing it to an ODE.

1.1 Dilation Symmetry

A dilation symmetry is a transformation of the form

$$X = ax, \quad T = a^{\beta}t, \quad U = a^{\gamma}u,$$

where a, β, and γ are constants. Note that we do not express the transformation for x as $X = a^{\alpha}x$ since it does not provide any additional information.

Example 1.1. Consider the transport equation

$$\frac{\partial u(x, t)}{\partial t} + c\frac{\partial u(x, t)}{\partial x} = 0$$

and the mapping

$$X = ax, \quad T = a^{\beta}t, \quad U = u.$$

After substituting the transformed solution into the PDE, we find

$$\begin{aligned}
\frac{\partial u(X, T)}{\partial t} + c\frac{\partial u(X, T)}{\partial x} &= 0 \\
\frac{\partial u(X, T)}{\partial T}\frac{\mathrm{d}T}{\mathrm{d}t} + c\frac{\partial u(X, T)}{\partial X}\frac{\mathrm{d}X}{\mathrm{d}x} &= 0 \\
a^{\beta}\frac{\partial u(X, T)}{\partial T} + ac\frac{\partial u(X, T)}{\partial X} &= 0.
\end{aligned}$$

For this PDE to be invariant under a dilation transformation, we must factor out the constant a from the equation. This implies that:

$$a^{\beta} = a \implies \beta = 1,$$

so that

$$\frac{\partial u\left(X,\,T\right)}{\partial T}+c\frac{\partial u\left(X,\,T\right)}{\partial X}=0.$$

Here we have shown that the transport equation is invariant under the above dilation transformation when $\beta=1$.

Example 1.2. Consider the nonlinear PDE

$$\frac{\partial u\left(x,\,y\right)}{\partial x}+y^2\frac{\partial u\left(x,\,y\right)}{\partial y}=0$$

and the mapping

$$X=ax,\quad Y=a^{\beta}y,\quad U=u.$$

By substituting the transformed solution into the PDE, we find

$$\begin{aligned}
\frac{\partial u\left(X,\,Y\right)}{\partial x}+y^2\frac{\partial u\left(X,\,Y\right)}{\partial y}&=0\\
\frac{\partial u\left(X,\,Y\right)}{\partial X}\frac{\mathrm{d}X}{\mathrm{d}x}+y^2\frac{\partial u\left(X,\,Y\right)}{\partial Y}\frac{\mathrm{d}Y}{\mathrm{d}y}&=0\\
a\frac{\partial u\left(X,\,Y\right)}{\partial X}+a^{\beta}y^2\frac{\partial u\left(X,\,Y\right)}{\partial Y}&=0\\
a\frac{\partial u\left(X,\,Y\right)}{\partial X}+a^{\beta}\left(Y/a^{\beta}\right)^2\frac{\partial u\left(X,\,Y\right)}{\partial Y}&=0\\
a\frac{\partial u\left(X,\,Y\right)}{\partial X}+a^{-\beta}Y^2\frac{\partial u\left(X,\,Y\right)}{\partial Y}&=0.
\end{aligned}$$

For this PDE to be invariant under a dilation transformation, we must again factor out a from the equation so that:

$$a=a^{-\beta}\implies 1=-\beta\implies\beta=-1,$$

and

$$\frac{\partial u\left(X,\,Y\right)}{\partial X}+Y^2\frac{\partial u\left(X,\,Y\right)}{\partial Y}=0.$$

Here we have shown that this nonlinear PDE is invariant under the above dilation transformation when $\beta=-1$.

Example 1.3. Consider the nonlinear convection diffusion equation

$$\frac{\partial u\left(x,\,t\right)}{\partial t}=u\left(x,\,t\right)\frac{\partial^2 u\left(x,\,t\right)}{\partial x^2}-\frac{\partial u\left(x,\,t\right)}{\partial x}$$

and the mapping

$$X=ax,\quad T=a^{\beta}t,\quad U=a^{\gamma}u.$$

We will once again substitute the transformed solution into the PDE to find

$$\begin{aligned}
a^{\gamma}\frac{\partial u(X,\,T)}{\partial t} &= a^{2\gamma}u(X,\,T)\,\frac{\partial^2 u(X,\,T)}{\partial x^2} - a^{\gamma}\frac{\partial u(X,\,T)}{\partial x}\\
a^{\gamma}\frac{\partial u(X,\,T)}{\partial T}\frac{\mathrm{d}T}{\mathrm{d}t} &= a^{2\gamma}u(X,\,T)\,\frac{\partial}{\partial x}\left[\frac{\partial u(X,\,T)}{\partial X}\frac{\mathrm{d}X}{\mathrm{d}x}\right] - a^{\gamma}\frac{\partial u(X,\,T)}{\partial X}\frac{\mathrm{d}X}{\mathrm{d}x}\\
a^{\beta+\gamma}\frac{\partial u(X,\,T)}{\partial T} &= a^{1+2\gamma}u(X,\,T)\,\frac{\partial}{\partial X}\left[\frac{\mathrm{d}X}{\mathrm{d}x}\frac{\partial u(X,\,T)}{\partial X}\right] - a^{1+\gamma}\frac{\partial u(X,\,T)}{\partial X}\\
a^{\beta+\gamma}\frac{\partial u(X,\,T)}{\partial T} &= a^{2+2\gamma}u(X,\,T)\,\frac{\partial^2 u(X,\,T)}{\partial X^2} - a^{1+\gamma}\frac{\partial u(X,\,T)}{\partial X}.
\end{aligned}$$

For this PDE to be invariant under a dilation transformation, we must factor out a from the equation so that:

$$a^{\beta+\gamma} = a^{2+2\gamma} = a^{1+\gamma} \implies \beta+\gamma = 2+2\gamma = 1+\gamma \implies \beta = 1, \gamma = -1,$$

and

$$\frac{\partial u(X,\,T)}{\partial T} = u(X,\,T)\,\frac{\partial^2 u(X,\,T)}{\partial X^2} - \frac{\partial u(X,\,T)}{\partial X}.$$

Here we have shown that this nonlinear PDE is invariant under the above dilation transformation when $\beta = 1$ and $\gamma = -1$.

1.2 Translation Symmetry

A translation symmetry is a transformation of the form

$$X = x - x_0, \quad T = t - t_0, \quad U = u - u_0,$$

where x_0, t_0, and u_0 are constants.

Example 1.4. Consider the heat equation

$$\frac{\partial u(x,\,t)}{\partial t} = D\frac{\partial^2 u(x,\,t)}{\partial x^2}$$

and the mapping

$$X = x - x_0, \quad T = t - t_0, \quad U = u - u_0.$$

By substituting the transformed solution into the PDE, we find

$$\begin{aligned}
\frac{\partial}{\partial t}\left(u(X,\,T) - u_0\right) &= D\frac{\partial^2}{\partial x^2}\left(u(X,\,T) - u_0\right)\\
\frac{\partial u(X,\,T)}{\partial T}\frac{\mathrm{d}T}{\mathrm{d}t} &= D\frac{\partial}{\partial x}\left(\frac{\partial u(X,\,T)}{\partial X}\frac{\mathrm{d}X}{\mathrm{d}x}\right)\\
\frac{\partial u(X,\,T)}{\partial T} &= D\frac{\partial}{\partial X}\frac{\mathrm{d}X}{\mathrm{d}x}\left(\frac{\partial u(X,\,T)}{\partial X}\right)\\
\frac{\partial u(X,\,T)}{\partial T} &= D\frac{\partial^2 u(X,\,T)}{\partial X^2}.
\end{aligned}$$

Here we have shown that the heat equation is invariant under the above translation transformation.

2 Similarity Solutions

In the previous section, we considered dilation transformations where both independent variables were scaled by some power of a. Notice that the product $\eta = xt^{-1/\beta}$ is invariant under the same dilation transformation:

$$\eta = xt^{-1/\beta} \implies H = \left(\frac{x}{a}\right)\left(\frac{T}{a^\beta}\right)^{-1/\beta} = XT^{-1/\beta}.$$

Let us therefore consider transformations of the form

$$X = ax, \quad T = a^\beta t, \quad U = t^{-\alpha} f(\eta),$$

where a, α, and β are constant and f is an arbitrary function to be determined. f is called a *similarity solution* (with *similarity variable* η) and our goal is to transform the PDE into an ODE with a single independent variable η. To do this:

1. Substitute $u(X, T)$ into the PDE to solve for α and β while maintaining invariance.
2. Obtain an ODE in terms of $f(\eta)$ by substituting the similarity solution into the PDE.
3. Obtain boundary conditions in terms of η using the same transformation.
4. Solve the ODE to find $f(\eta)$.
5. Transform back to the original variables to find a solution to the PDE.

3 Travelling Wave Solutions

Another type of solution is the *travelling wave solution*, where the solution to a PDE appears to move at a constant velocity when after a long period of time. This is similar to a steady state solution where the solution does not change after a long period of time. In this problem, we assume solutions of the form

$$u(x, t) = f(z), \quad z = x - ct,$$

where c is the speed at which the solution travels. The travelling wave speed is often determined using analysis, boundary conditions, numerical methods, or physical constraints. This method is only applicable to PDEs that are invariant under translations to both independent variables. As such, the PDE must not contain any explicit dependence on x or t.

Part II
Method of Characteristics

The method of characteristics is a technique used to solve nonlinear PDEs by reducing them to a system of ODEs. It considers parametrisations of the solution through a set of curves in the solution space along which the solution is constant.

4 First Order PDEs

4.1 Linearity

Consider the general first order PDE in two variables:

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = c.$$

This PDE has four classifications based on the value of the variables a, b, and c:

- **Linear**:
$$a\left(x,\ y\right)\frac{\partial u}{\partial x} + b\left(x,\ y\right)\frac{\partial u}{\partial y} = c\left(x,\ y\right).$$
- **Semi-Linear**:
$$a\left(x,\ y\right)\frac{\partial u}{\partial x} + b\left(x,\ y\right)\frac{\partial u}{\partial y} = c\left(x,\ y,\ u\right).$$
- **Quasi-Linear**:
$$a\left(x,\ y,\ u\right)\frac{\partial u}{\partial x} + b\left(x,\ y,\ u\right)\frac{\partial u}{\partial y} = c\left(x,\ y,\ u\right).$$
- **Nonlinear**: Otherwise.

4.2 Solution Method

Consider a quasi-linear PDE of the form:

$$a\left(x,\ y,\ u\right)\frac{\partial u}{\partial x} + b\left(x,\ y,\ u\right)\frac{\partial u}{\partial y} = c\left(x,\ y,\ u\right)$$

with initial data $u_0\left(x\right)$ and initial condition $u\left(x_0,\ y_0\right) = u_0\left(x\right)$, where a, b, and c are continuous functions in x, y, and u. To reduce this problem into a family of ODEs, we will consider the solution surface $u = u\left(x,\ y\right)$ on which the PDE is satisfied. If we express is surface implicitly as $f\left(x,\ y,\ u\right) = u\left(x,\ y\right) - u = 0$, we find the following normal vector:

$$\mathbf{n} = \boldsymbol{\nabla} f = \begin{bmatrix} u_x \\ u_y \\ -1 \end{bmatrix}.$$

We will also define the vector field $\mathbf{v}$ consisting of the coefficients of this PDE:

$$\mathbf{v} = \begin{bmatrix} a\left(x,\ y,\ u\right) \\ b\left(x,\ y,\ u\right) \\ c\left(x,\ y,\ u\right) \end{bmatrix}$$

so that we express the PDE as $\mathbf{v}\cdot\mathbf{n} = 0$. As $\mathbf{n}$ is normal to the solution surface $u\left(x,\ y\right)$, $\mathbf{v}$ must lie on the tangent plane to the solution surface. This vector field is therefore known as the *characteristic direction* of the PDE. Let us now consider a parametric curve defined as:

$$\mathbf{r}\left(s\right) = \begin{bmatrix} x\left(s\right) \\ y\left(s\right) \\ u\left(s\right) \end{bmatrix},$$

and let us impose that it's tangent vector is equal to $\mathbf{v}$ so that the curve also lies on the solution surface:

$$\mathbf{r}'(s) = \mathbf{v}.$$

Doing so yields the following system of ODEs:

$$\begin{aligned}
\frac{\partial x}{\partial s} &= a(x,\ y,\ u),\\
\frac{\partial y}{\partial s} &= b(x,\ y,\ u),\\
\frac{\partial u}{\partial s} &= c(x,\ y,\ u).
\end{aligned}$$

We can solve these parametric curves through integration and solve for all integration constants x_0, y_0, and u_0 using the parametric initial condition:

$$u(0) = u_0(\xi) \quad \text{on} \quad x(0) = x_0(\xi) \quad \text{and} \quad y(0) = y_0(\xi).$$

where ξ parametrises the initial condition. The resulting parametric equations $x(s)$ and $y(s)$ form *characteristics* for the PDE, which allows us to determine the value of $u(x,\ y)$ along the curve for specific values of ξ. The solution $u(x,\ y)$ can then be found by eliminating s and ξ from the three parametric equations.

4.3 Characteristic Curves

After determining characteristics for a PDE, we can plot these parametric curves in the x-y plane to determine regions where the solution is constant. We do so by considering the *domain of dependence* for points in each region to determine the value of the solution u by walking backward along these curves. Doing so allows us to understand how the solution behaves along different characteristics without needing to plot the entire solution surface.

4.3.1 Expansion Waves

Consider a quasi-linear PDE of the form

$$a(x,\ t,\ u)\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = c(x,\ t,\ u)$$

with the initial condition:

$$u(x_0,\ 0) = \begin{cases} u_0, & x_0 < 0,\\ u_1, & x_0 > 0, \end{cases}$$

where $u_0 < u_1$. The nature of this initial condition causes characteristics to change slope when $x_0 = 0$, so that they fan outwards from this point, resulting in an *expansion wave*. For the solution to be defined at all points, we assume that $u(x_0,\ 0)$ takes all values between u_0 and u_1, on $x(s)$. This means that as time increases, the discontinuity in the initial condition spreads along the x-axis.

4.3.2 Shock Waves

Consider a quasi-linear PDE of the form

$$a\left(x,\ t,\ u\right)\frac{\partial u}{\partial x}+\frac{\partial u}{\partial t}=c\left(x,\ t,\ u\right)$$

with the initial condition:

$$u\left(x_0,\ 0\right)=\begin{cases}u_0, & x_0<0,\\ u_1, & x_0>0,\end{cases}$$

where $u_0 > u_1$ instead. The nature of this initial condition causes characteristics to change slope at the point $x_0 = 0$, where the slopes of characteristics for $x_0 < 0$ are steeper than those when $x_0 > 0$, resulting in an intersection of characteristics. This results in a *shock* in the solution, so that the value u_0 overtakes u_1 as time increases, leading to a multi-valued solution.
If the solution must be single-valued, we must assume that the total flow $q = uv$ is conserved across the *shock wave* $x_s\left(s\right)$, so that the flow from the left into the moving shock wave equals the flow to the right away from the shock wave. In this form, v is the velocity of the function $u\left(x,\ t\right)$, which we can find by expressing the PDE as

$$\frac{\partial u}{\partial t}+\frac{\partial q}{\partial x}=c\left(x,\ t,\ u\right).$$

Mathematically,

$$\begin{aligned}\left[v\left(x_s^-,\ t\right)-x_s'\left(s\right)\right]u\left(x_s^-,\ t\right)&=\left[v\left(x_s^+,\ t\right)-x_s'\left(s\right)\right]u\left(x_s^+,\ t\right)\\ v\left(x_s^-,\ t\right)u\left(x_s^-,\ t\right)-x_s'\left(s\right)u\left(x_s^-,\ t\right)&=v\left(x_s^+,\ t\right)u\left(x_s^+,\ t\right)-x_s'\left(s\right)u\left(x_s^+,\ t\right)\\ x_s'\left(s\right)u\left(x_s^+,\ t\right)-x_s'\left(s\right)u\left(x_s^-,\ t\right)&=q\left(x_s^+,\ t\right)-q\left(x_s^-,\ t\right)\\ x_s'\left(s\right)&=\frac{q\left(x_s^+,\ t\right)-q\left(x_s^-,\ t\right)}{u\left(x_s^+,\ t\right)-u\left(x_s^-,\ t\right)}.\end{aligned}$$

Or, more compactly,

$$x_s'\left(s\right)=\frac{q^+-q^-}{u^+-u^-}=\frac{[q]}{[u]}=\frac{\text{jump in flow}}{\text{jump in value}}.$$

By solving for the shock wave $x_s\left(s\right)$ at $x_0 = 0$, it follows that the characteristics with constant values will appear on either side of the shock wave, where the solution is single-valued, meeting at the shock wave.

5 System of First Order PDEs

Consider an $n \times n$ coupled system of first order PDEs:

$$\mathbf{A}\frac{\partial\mathbf{u}}{\partial x}+\mathbf{B}\frac{\partial\mathbf{u}}{\partial y}=\mathbf{c},\quad\text{with}\quad\mathbf{u}\left(x_0,\ y_0\right)=\mathbf{u}_0\left(x\right),$$

where $\mathbf{A}=\mathbf{A}\left(x,\ y,\ \mathbf{u}\right)$ and $\mathbf{B}=\mathbf{B}\left(x,\ y,\ \mathbf{u}\right)$ are $n \times n$ matrix functions, and $\mathbf{c}=\mathbf{c}\left(x,\ y,\ \mathbf{u}\right)$ is an $n \times 1$ vector function. In this case, we want to find characteristic directions $\mathbf{v}$ that will allow us to decouple the system into first order ODEs. Let us try to decompose **A** and **B** into a diagonal form by assuming the following relationship holds for some **m**:

$$\mathbf{v}^\top \left(\mathbf{A}\frac{\partial \mathbf{u}}{\partial x} + \mathbf{B}\frac{\partial \mathbf{u}}{\partial y}\right) = \mathbf{m}^\top \left(\alpha\frac{\partial \mathbf{u}}{\partial x} + \beta\frac{\partial \mathbf{u}}{\partial y}\right),$$

where **v** and **m** are $n \times 1$ vector functions, and $\alpha = \frac{\mathrm{d}x}{\mathrm{d}s}$ and $\beta = \frac{\mathrm{d}y}{\mathrm{d}s}$ are scalar parametric functions. For this to hold, we must have:

$$\mathbf{v}^\top \mathbf{A} = \mathbf{m}^\top \alpha \quad \text{and} \quad \mathbf{v}^\top \mathbf{B} = \mathbf{m}^\top \beta.$$

By eliminating $\mathbf{m}^\top$, we find the following relationship between **A** and **B**:

$$\begin{aligned}
\frac{1}{\alpha}\mathbf{v}^\top \mathbf{A} &= \frac{1}{\beta}\mathbf{v}^\top \mathbf{B} \\
\mathbf{v}^\top \mathbf{A} &= \frac{\alpha}{\beta}\mathbf{v}^\top \mathbf{B} \\
\mathbf{v}^\top \mathbf{A} &= \lambda \mathbf{v}^\top \mathbf{B}.
\end{aligned}$$

This is precisely the left-generalised eigenvalue problem for the matrix pair $(\mathbf{A},\, \mathbf{B})$ where the eigenvalues λ_i are found by solving the characteristic equation:

$$\det\left(\mathbf{A} - \lambda \mathbf{B}\right) = 0.$$

Assuming a diagonalisable system, we have the following matrix decomposition:

$$\mathbf{A} = \mathbf{B}\mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1},$$

where **V** is the matrix of eigenvectors and $\boldsymbol{\Lambda}$ is the diagonal matrix of eigenvalues. If we assume a solution of the form $\mathbf{u} = \mathbf{V}\mathbf{w}$, we can transform the system of PDEs into a diagonal form:

$$\begin{aligned}
\mathbf{A}\frac{\partial \mathbf{u}}{\partial x} + \mathbf{B}\frac{\partial \mathbf{u}}{\partial y} &= \mathbf{c} \\
\left(\mathbf{B}\mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1}\right)\mathbf{V}\frac{\partial \mathbf{w}}{\partial x} + \mathbf{B}\mathbf{V}\frac{\partial \mathbf{w}}{\partial y} &= \mathbf{c} \\
\mathbf{B}\mathbf{V}\boldsymbol{\Lambda}\frac{\partial \mathbf{w}}{\partial x} + \mathbf{B}\mathbf{V}\frac{\partial \mathbf{w}}{\partial y} &= \mathbf{c} \\
\boldsymbol{\Lambda}\frac{\partial \mathbf{w}}{\partial x} + \frac{\partial \mathbf{w}}{\partial y} &= \mathbf{V}^{-1}\mathbf{B}^{-1}\mathbf{c},
\end{aligned}$$

where we assume **B** is invertible. Here the initial conditions for **w** are determined by the initial conditions for **u**:

$$\mathbf{w}\left(x_0,\, y_0\right) = \mathbf{V}^{-1}\mathbf{u}_0\left(x\right).$$

As $\boldsymbol{\Lambda}$ is diagonal, we can form a system of n first order PDEs which can be solved independently to find $w_i\left(x,\, y\right)$, allowing us to find the solution $\mathbf{u}\left(x,\, y\right)$.

6 Second Order PDEs

Consider the general linear second order PDE in two variables:

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x\,\partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} = f.$$

with the initial conditions $u\left(x_0,\, y_0\right) = f\left(x\right)$ and $u_y\left(x_0,\, y_0\right) = g\left(x\right)$ where variables a through f are functions of x and y only. We will begin by factoring this PDE into it's two first derivatives $u_x = \frac{\partial u}{\partial x}$ and $u_y = \frac{\partial u}{\partial y}$ to convert the second order PDE into a system of first order PDEs:

$$\begin{aligned} a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial y\,\partial x} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} &= f \\ a\frac{\partial u_x}{\partial x} + 2b\frac{\partial u_x}{\partial y} + c\frac{\partial u_y}{\partial y} + du_x + eu_y &= f \\ a\frac{\partial u_x}{\partial x} + 2b\frac{\partial u_x}{\partial y} + c\frac{\partial u_y}{\partial y} &= f - du_x - eu_y. \end{aligned}$$

Additionally, we will assume that Schwarz's theorem holds:

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial^2 u}{\partial x\,\partial y} \iff \frac{\partial u_x}{\partial y} = \frac{\partial u_y}{\partial x}.$$

This allows us to form the following system of first order PDEs:

$$\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} u_x \\ u_y \end{bmatrix} + \begin{bmatrix} 2b & c \\ -1 & 0 \end{bmatrix} \frac{\partial}{\partial y} \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} f - du_x - eu_y \\ 0 \end{bmatrix},$$

which can be solved using the process described in the previous section, where the initial condition for u_x is simply $u_x\left(x_0,\, y_0\right) = f'\left(x\right)$.

6.1 Classification of Second Order PDEs

Using the above form, we can determine the characteristic equation for a second order PDE:

$$\begin{vmatrix} a - 2b\lambda & -c \\ 1 & -\lambda \end{vmatrix} = a\lambda^2 - 2b\lambda + c = 0,$$

where the eigenvalues λ are given by:

$$\lambda = \frac{2b \pm \sqrt{4b^2 - 4ac}}{2a} = \frac{b \pm \sqrt{b^2 - ac}}{a}.$$

We can classify this second order PDE into 3 types based on the sign of the discriminant $b^2 - ac$:

- **Hyperbolic**: (two real distinct eigenvalues)
$$b^2 - ac > 0.$$
- **Parabolic**: (two real equal eigenvalues)
$$b^2 - ac = 0.$$
- **Elliptic**: (two complex eigenvalues)
$$b^2 - ac < 0.$$

6.2 D'Alembert's Solution

D'Alembert's solution is a general solution to the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0,$$

with initial conditions $u\left(x,\ 0\right) = f\left(x\right)$ and $u_t\left(x,\ 0\right) = g\left(x\right)$. Using the techniques above, it can be shown that the general solution to this PDE has the form:

$$u\left(x,\ t\right) = \frac{1}{2}\left[f\left(x+ct\right) + f\left(x-ct\right)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g\left(s\right) \mathrm{d}s.$$