# Differential Equations and Modelling 2

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# Contents

# Part I
# Symmetry Methods

## 1 Symmetry Transformations

Consider a partial differential equation for $u(x,\, t)$ whose domain lies in $\mathbb{R}^2$. Such a problem typically does not have any natural length or time scales associated with it's fundamental solution. Thus, let us transform the independent and dependent variables through the mapping $(x,\, t,\, u) \mapsto (X,\, T,\, U)$ where $X = X(x,\, t,\, u)$, $T = T(x,\, t,\, u)$, and $U = U(x,\, t,\, u)$, such that the PDE is invariant under this transformation, that is, the transformed PDE in $U$ has the same form as the original PDE in $u$.
The choice of mapping may consist of dilations and translations of the independent and dependent variables by some constant factors, where each constant is enforced by the invariance condition. To do this, we must ensure that the function $U(x,\, t,\, u(X,\, T))$ satisfies the transformed PDE. When this is the case, the PDE is said to have a *symmetry transformation*.
The goal of these symmetry methods is to find an appropriate transformation that simplies the PDE, often reducing it to an ODE.

### 1.1 Dilation Symmetry

A dilation symmetry is a transformation of the form

$$X = ax, \quad T = a^\beta t, \quad U = a^\gamma u,$$

where $a$, $\beta$, and $\gamma$ are constants. Note that we do not express the transformation for $x$ as $X = a^\alpha x$ since it does not provide any additional information.

**Example 1.1.** Consider the transport equation

$$\frac{\partial u(x,\, t)}{\partial t} + c\frac{\partial u(x,\, t)}{\partial x} = 0$$

and the mapping

$$X = ax, \quad T = a^\beta t, \quad U = u.$$

After substituting the transformed solution into the PDE, we find

$$\begin{aligned}
\frac{\partial u(X,\, T)}{\partial t} + c\frac{\partial u(X,\, T)}{\partial x} &= 0\\
\frac{\partial u(X,\, T)}{\partial T}\frac{\mathrm{d}T}{\mathrm{d}t} + c\frac{\partial u(X,\, T)}{\partial X}\frac{\mathrm{d}X}{\mathrm{d}x} &= 0\\
a^\beta\frac{\partial u(X,\, T)}{\partial T} + ac\frac{\partial u(X,\, T)}{\partial X} &= 0.
\end{aligned}$$

For this PDE to be invariant under a dilation transformation, we must factor out the constant $a$ from the equation. This implies that:

$$a^\beta = a \implies \beta = 1,$$

so that

$$\frac{\partial u\left(X,\ T\right)}{\partial T}+c\frac{\partial u\left(X,\ T\right)}{\partial X}=0.$$

Here we have shown that the transport equation is invariant under the above dilation transformation when $\beta = 1$.

**Example 1.2.** Consider the nonlinear PDE

$$\frac{\partial u\left(x,\ y\right)}{\partial x}+y^2\frac{\partial u\left(x,\ y\right)}{\partial y}=0$$

and the mapping

$$X = ax, \quad Y = a^\beta y, \quad U = u.$$

By substituting the transformed solution into the PDE, we find

$$\begin{aligned}
\frac{\partial u\left(X,\ Y\right)}{\partial x}+y^2\frac{\partial u\left(X,\ Y\right)}{\partial y}&=0\\
\frac{\partial u\left(X,\ Y\right)}{\partial X}\frac{\mathrm{d}X}{\mathrm{d}x}+y^2\frac{\partial u\left(X,\ Y\right)}{\partial Y}\frac{\mathrm{d}Y}{\mathrm{d}y}&=0\\
a\frac{\partial u\left(X,\ Y\right)}{\partial X}+a^\beta y^2\frac{\partial u\left(X,\ Y\right)}{\partial Y}&=0\\
a\frac{\partial u\left(X,\ Y\right)}{\partial X}+a^\beta\left(Y/a^\beta\right)^2\frac{\partial u\left(X,\ Y\right)}{\partial Y}&=0\\
a\frac{\partial u\left(X,\ Y\right)}{\partial X}+a^{-\beta}Y^2\frac{\partial u\left(X,\ Y\right)}{\partial Y}&=0.
\end{aligned}$$

For this PDE to be invariant under a dilation transformation, we must again factor out $a$ from the equation so that:

$$a = a^{-\beta} \implies 1 = -\beta \implies \beta = -1,$$

and

$$\frac{\partial u\left(X,\ Y\right)}{\partial X}+Y^2\frac{\partial u\left(X,\ Y\right)}{\partial Y}=0.$$

Here we have shown that this nonlinear PDE is invariant under the above dilation transformation when $\beta = -1$.

**Example 1.3.** Consider the nonlinear convection diffusion equation

$$\frac{\partial u\left(x,\ t\right)}{\partial t}=u\left(x,\ t\right)\frac{\partial^2 u\left(x,\ t\right)}{\partial x^2}-\frac{\partial u\left(x,\ t\right)}{\partial x}$$

and the mapping

$$X = ax, \quad T = a^\beta t, \quad U = a^\gamma u.$$

We will once again substitute the transformed solution into the PDE to find

$$\begin{aligned}
a^{\gamma}\frac{\partial u\left(X,\ T\right)}{\partial t} &= a^{2\gamma}u\left(X,\ T\right)\frac{\partial^2 u\left(X,\ T\right)}{\partial x^2} - a^{\gamma}\frac{\partial u\left(X,\ T\right)}{\partial x} \\
a^{\gamma}\frac{\partial u\left(X,\ T\right)}{\partial T}\frac{\mathrm{d}T}{\mathrm{d}t} &= a^{2\gamma}u\left(X,\ T\right)\frac{\partial}{\partial x}\left[\frac{\partial u\left(X,\ T\right)}{\partial X}\frac{\mathrm{d}X}{\mathrm{d}x}\right] - a^{\gamma}\frac{\partial u\left(X,\ T\right)}{\partial X}\frac{\mathrm{d}X}{\mathrm{d}x} \\
a^{\beta+\gamma}\frac{\partial u\left(X,\ T\right)}{\partial T} &= a^{1+2\gamma}u\left(X,\ T\right)\frac{\partial}{\partial X}\left[\frac{\mathrm{d}X}{\mathrm{d}x}\frac{\partial u\left(X,\ T\right)}{\partial X}\right] - a^{1+\gamma}\frac{\partial u\left(X,\ T\right)}{\partial X} \\
a^{\beta+\gamma}\frac{\partial u\left(X,\ T\right)}{\partial T} &= a^{2+2\gamma}u\left(X,\ T\right)\frac{\partial^2 u\left(X,\ T\right)}{\partial X^2} - a^{1+\gamma}\frac{\partial u\left(X,\ T\right)}{\partial X}.
\end{aligned}$$

For this PDE to be invariant under a dilation transformation, we must factor out $a$ from the equation so that:

$$a^{\beta+\gamma} = a^{2+2\gamma} = a^{1+\gamma} \implies \beta+\gamma = 2+2\gamma = 1+\gamma \implies \beta = 1, \gamma = -1,$$

and

$$\frac{\partial u\left(X,\ T\right)}{\partial T} = u\left(X,\ T\right)\frac{\partial^2 u\left(X,\ T\right)}{\partial X^2} - \frac{\partial u\left(X,\ T\right)}{\partial X}.$$

Here we have shown that this nonlinear PDE is invariant under the above dilation transformation when $\beta = 1$ and $\gamma = -1$.

## 1.2 Translation Symmetry

A translation symmetry is a transformation of the form

$$X = x - x_0, \quad T = t - t_0, \quad U = u - u_0,$$

where $x_0$, $t_0$, and $u_0$ are constants.

**Example 1.4.** Consider the heat equation

$$\frac{\partial u\left(x,\ t\right)}{\partial t} = D\frac{\partial^2 u\left(x,\ t\right)}{\partial x^2}$$

and the mapping

$$X = x - x_0, \quad T = t - t_0, \quad U = u - u_0.$$

By substituting the transformed solution into the PDE, we find

$$\begin{aligned}
\frac{\partial}{\partial t}\left(u\left(X,\ T\right) - u_0\right) &= D\frac{\partial^2}{\partial x^2}\left(u\left(X,\ T\right) - u_0\right) \\
\frac{\partial u\left(X,\ T\right)}{\partial T}\frac{\mathrm{d}T}{\mathrm{d}t} &= D\frac{\partial}{\partial x}\left(\frac{\partial u\left(X,\ T\right)}{\partial X}\frac{\mathrm{d}X}{\mathrm{d}x}\right) \\
\frac{\partial u\left(X,\ T\right)}{\partial T} &= D\frac{\partial}{\partial X}\frac{\mathrm{d}X}{\mathrm{d}x}\left(\frac{\partial u\left(X,\ T\right)}{\partial X}\right) \\
\frac{\partial u\left(X,\ T\right)}{\partial T} &= D\frac{\partial^2 u\left(X,\ T\right)}{\partial X^2}.
\end{aligned}$$

Here we have shown that the heat equation is invariant under the above translation transformation.

## 2 Similarity Solutions

In the previous section, we considered dilation transformations where both independent variables were scaled by some power of $a$. Notice that the product $\eta = xt^{-1/\beta}$ is invariant under the same dilation transformation:

$$\eta = xt^{-1/\beta} \implies H = \left(\frac{x}{a}\right)\left(\frac{T}{a^{\beta}}\right)^{-1/\beta} = XT^{-1/\beta}.$$

Let us therefore consider transformations of the form

$$X = ax, \quad T = a^{\beta}t, \quad U = t^{-\alpha} f(\eta),$$

where $a$, $\alpha$, and $\beta$ are constant and $f$ is an arbitrary function to be determined. $f$ is called a *similarity solution* (with *similarity variable* $\eta$) and our goal is to transform the PDE into an ODE with a single independent variable $\eta$. To do this:

1. Substitute $u(X, T)$ into the PDE to solve for $\alpha$ and $\beta$ while maintaining invariance.
2. Obtain an ODE in terms of $f(\eta)$ by substituting the similarity solution into the PDE.
3. Obtain boundary conditions in terms of $\eta$ using the same transformation.
4. Solve the ODE to find $f(\eta)$.
5. Transform back to the original variables to find a solution to the PDE.

## 3 Travelling Wave Solutions

Another type of solution is the *travelling wave solution*, where the solution to a PDE appears to move at a constant velocity when after a long period of time. This is similar to a steady state solution where the solution does not change after a long period of time. In this problem, we assume solutions of the form

$$u(x, t) = f(z), \quad z = x - ct,$$

where $c$ is the speed at which the solution travels. The travelling wave speed is often determined using analysis, boundary conditions, numerical methods, or physical constraints. This method is only applicable to PDEs that are invariant under translations to both independent variables. As such, the PDE must not contain any explicit dependence on $x$ or $t$.

# Part II
# Method of Characteristics

The method of characteristics is a technique used to solve nonlinear PDEs by reducing them to a system of ODEs. It considers parametrisations of the solution through a set of curves in the solution space along which the solution is constant.

# 4 First Order PDEs

## 4.1 Linearity

Consider the general first order PDE in two variables:

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = c.$$

This PDE has four classifications based on the value of the variables $a$, $b$, and $c$:

- **Linear**:
$$a\left(x,\ y\right)\frac{\partial u}{\partial x} + b\left(x,\ y\right)\frac{\partial u}{\partial y} = c\left(x,\ y\right).$$
- **Semi-Linear**:
$$a\left(x,\ y\right)\frac{\partial u}{\partial x} + b\left(x,\ y\right)\frac{\partial u}{\partial y} = c\left(x,\ y,\ u\right).$$
- **Quasi-Linear**:
$$a\left(x,\ y,\ u\right)\frac{\partial u}{\partial x} + b\left(x,\ y,\ u\right)\frac{\partial u}{\partial y} = c\left(x,\ y,\ u\right).$$
- **Nonlinear**: Otherwise.

## 4.2 Solution Method

Consider a quasi-linear PDE of the form:

$$a\left(x,\ y,\ u\right)\frac{\partial u}{\partial x} + b\left(x,\ y,\ u\right)\frac{\partial u}{\partial y} = c\left(x,\ y,\ u\right)$$

with initial data $u_0\left(x\right)$ and initial condition $u\left(x_0,\ y_0\right) = u_0\left(x\right)$, where $a$, $b$, and $c$ are continuous functions in $x$, $y$, and $u$. To reduce this problem into a family of ODEs, we will consider the solution surface $u = u\left(x,\ y\right)$ on which the PDE is satisfied. If we express is surface implicitly as $f\left(x,\ y,\ u\right) = u\left(x,\ y\right) - u = 0$, we find the following normal vector:

$$\mathbf{n} = \boldsymbol{\nabla} f = \begin{bmatrix} u_x \\ u_y \\ -1 \end{bmatrix}.$$

We will also define the vector field $\mathbf{v}$ consisting of the coefficients of this PDE:

$$\mathbf{v} = \begin{bmatrix} a\left(x,\ y,\ u\right) \\ b\left(x,\ y,\ u\right) \\ c\left(x,\ y,\ u\right) \end{bmatrix}$$

so that we express the PDE as $\mathbf{v}\cdot\mathbf{n} = 0$. As $\mathbf{n}$ is normal to the solution surface $u\left(x,\ y\right)$, $\mathbf{v}$ must lie on the tangent plane to the solution surface. This vector field is therefore known as the *characteristic direction* of the PDE. Let us now consider a parametric curve defined as:

$$\mathbf{r}\left(s\right) = \begin{bmatrix} x\left(s\right) \\ y\left(s\right) \\ u\left(s\right) \end{bmatrix},$$

and let us impose that it's tangent vector is equal to $\mathbf{v}$ so that the curve also lies on the solution surface:

$$\mathbf{r}'(s) = \mathbf{v}.$$

Doing so yields the following system of ODEs:

$$\frac{\partial x}{\partial s} = a(x,\ y,\ u),$$
$$\frac{\partial y}{\partial s} = b(x,\ y,\ u),$$
$$\frac{\partial u}{\partial s} = c(x,\ y,\ u).$$

We can solve these parametric curves through integration and solve for all integration constants $x_0$, $y_0$, and $u_0$ using the parametric initial condition:

$$u(0) = u_0(\xi) \quad \text{on} \quad x(0) = x_0(\xi) \quad \text{and} \quad y(0) = y_0(\xi).$$

where $\xi$ parametrises the initial condition. The resulting parametric equations $x(s)$ and $y(s)$ form *characteristics* for the PDE, which allows us to determine the value of $u(x,\ y)$ along the curve for specific values of $\xi$. The solution $u(x,\ y)$ can then be found by eliminating $s$ and $\xi$ from the three parametric equations.

## 4.3 Characteristic Curves

After determining characteristics for a PDE, we can plot these parametric curves in the $x$-$y$ plane to determine regions where the solution is constant. We do so by considering the *domain of dependence* for points in each region to determine the value of the solution $u$ by walking backward along these curves. Doing so allows us to understand how the solution behaves along different characteristics without needing to plot the entire solution surface.

### 4.3.1 Expansion Waves

Consider a quasi-linear PDE of the form

$$a(x,\ t,\ u)\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = c(x,\ t,\ u)$$

with some initial condition. If this initial condition causes characteristics to change slope when $x_0 = 0$, so that they fan outwards from this point, the region is called an *expansion wave*. For the solution to be defined at all points, we assume that $u(x_0,\ 0)$ takes all values between $u(x_0^-,\ 0)$ and $u(x_0^+,\ 0)$, on $x(s)$. This means that as time increases, the discontinuity in the initial condition spreads along the $x$-axis.

### 4.3.2 Shock Waves

Consider a quasi-linear PDE of the form

$$a(x,\ t,\ u)\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = c(x,\ t,\ u)$$

with some initial condition. If this initial condition causes characteristics to change slope at the point $x_0 = 0$, where the slopes of characteristics for $x_0 < 0$ are steeper than those when $x_0 > 0$, we have an intersection of characteristics. This results in a *shock* in the solution, so that the solution overtakes itself as time increases, leading to a multi-valued solution.

# 5 System of First Order PDEs

Consider an $n \times n$ coupled system of first order PDEs:

$$\mathbf{A}\frac{\partial \mathbf{u}}{\partial x} + \mathbf{B}\frac{\partial \mathbf{u}}{\partial y} = \mathbf{c}, \quad \text{with} \quad \mathbf{u}\left(x_0,\, y_0\right) = \mathbf{u}_0\left(x\right),$$

where $\mathbf{A} = \mathbf{A}\left(x,\, y,\, \mathbf{u}\right)$ and $\mathbf{B} = \mathbf{B}\left(x,\, y,\, \mathbf{u}\right)$ are $n \times n$ matrix functions, and $\mathbf{c} = \mathbf{c}\left(x,\, y,\, \mathbf{u}\right)$ is an $n \times 1$ vector function. In this case, we want to find characteristic directions $\mathbf{v}$ that will allow us to decouple the system into first order ODEs. Let us try to decompose $\mathbf{A}$ and $\mathbf{B}$ into a diagonal form by assuming the following relationship holds for some $\mathbf{m}$:

$$\mathbf{v}^\top \left(\mathbf{A}\frac{\partial \mathbf{u}}{\partial x} + \mathbf{B}\frac{\partial \mathbf{u}}{\partial y}\right) = \mathbf{m}^\top \left(\alpha\frac{\partial \mathbf{u}}{\partial x} + \beta\frac{\partial \mathbf{u}}{\partial y}\right),$$

where $\mathbf{v}$ and $\mathbf{m}$ are $n \times 1$ vector functions, and $\alpha = \frac{\mathrm{d}x}{\mathrm{d}s}$ and $\beta = \frac{\mathrm{d}y}{\mathrm{d}s}$ are scalar parametric functions. For this to hold, we must have:

$$\mathbf{v}^\top \mathbf{A} = \mathbf{m}^\top \alpha \quad \text{and} \quad \mathbf{v}^\top \mathbf{B} = \mathbf{m}^\top \beta.$$

By eliminating $\mathbf{m}^\top$, we find the following relationship between $\mathbf{A}$ and $\mathbf{B}$:

$$\begin{aligned}
\frac{1}{\alpha}\mathbf{v}^\top \mathbf{A} &= \frac{1}{\beta}\mathbf{v}^\top \mathbf{B} \\
\mathbf{v}^\top \mathbf{A} &= \frac{\alpha}{\beta}\mathbf{v}^\top \mathbf{B} \\
\mathbf{v}^\top \mathbf{A} &= \lambda \mathbf{v}^\top \mathbf{B}.
\end{aligned}$$

This is precisely the left-generalised eigenvalue problem for the matrix pair $(\mathbf{A},\, \mathbf{B})$ where the eigenvalues $\lambda_i$ are found by solving the characteristic equation:

$$\det\left(\mathbf{A} - \lambda \mathbf{B}\right) = 0.$$

Assuming a diagonalisable system, we have the following matrix decomposition:

$$\mathbf{A} = \mathbf{B}\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1},$$

where $\mathbf{V}$ is the matrix of eigenvectors and $\mathbf{\Lambda}$ is the diagonal matrix of eigenvalues. If we assume a solution of the form $\mathbf{u} = \mathbf{V}\mathbf{w}$, we can transform the system of PDEs into a diagonal form:

$$\begin{aligned}
\mathbf{A}\frac{\partial \mathbf{u}}{\partial x} + \mathbf{B}\frac{\partial \mathbf{u}}{\partial y} &= \mathbf{c} \\
\left(\mathbf{B}\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}\right)\mathbf{V}\frac{\partial \mathbf{w}}{\partial x} + \mathbf{B}\mathbf{V}\frac{\partial \mathbf{w}}{\partial y} &= \mathbf{c} \\
\mathbf{B}\mathbf{V}\mathbf{\Lambda}\frac{\partial \mathbf{w}}{\partial x} + \mathbf{B}\mathbf{V}\frac{\partial \mathbf{w}}{\partial y} &= \mathbf{c} \\
\mathbf{\Lambda}\frac{\partial \mathbf{w}}{\partial x} + \frac{\partial \mathbf{w}}{\partial y} &= \mathbf{V}^{-1}\mathbf{B}^{-1}\mathbf{c},
\end{aligned}$$

where we assume $\mathbf{B}$ is invertible. Here the initial conditions for $\mathbf{w}$ are determined by the initial conditions for $\mathbf{u}$:

$$\mathbf{w}\left(x_0,\, y_0\right) = \mathbf{V}^{-1}\mathbf{u}_0\left(x\right).$$

As $\mathbf{\Lambda}$ is diagonal, we can form a system of $n$ first order PDEs which can be solved independently to find $w_i\left(x,\, y\right)$, allowing us to find the solution $\mathbf{u}\left(x,\, y\right)$.

# 6 Second Order PDEs

Consider the general linear second order PDE in two variables:

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x\, \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} = f.$$

with the initial conditions $u\left(x_0,\, y_0\right) = f\left(x\right)$ and $u_y\left(x_0,\, y_0\right) = g\left(x\right)$ where variables $a$ through $f$ are functions of $x$ and $y$ only. We will begin by factoring this PDE into it's two first derivatives $u_x = \frac{\partial u}{\partial x}$ and $u_y = \frac{\partial u}{\partial y}$ to convert the second order PDE into a system of first order PDEs:

$$\begin{aligned}
a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial y\, \partial x} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} &= f \\
a\frac{\partial u_x}{\partial x} + 2b\frac{\partial u_x}{\partial y} + c\frac{\partial u_y}{\partial y} + du_x + eu_y &= f \\
a\frac{\partial u_x}{\partial x} + 2b\frac{\partial u_x}{\partial y} + c\frac{\partial u_y}{\partial y} &= f - du_x - eu_y.
\end{aligned}$$

Additionally, we will assume that Schwarz's theorem holds:

$$\frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial^2 u}{\partial x\, \partial y} \iff \frac{\partial u_x}{\partial y} = \frac{\partial u_y}{\partial x}.$$

This allows us to form the following system of first order PDEs:

$$\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix} u_x \\ u_y \end{bmatrix} + \begin{bmatrix} 2b & c \\ -1 & 0 \end{bmatrix}\frac{\partial}{\partial y}\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} f - du_x - eu_y \\ 0 \end{bmatrix},$$

which can be solved using the process described in the previous section, where the initial condition for $u_x$ is simply $u_x\left(x_0,\, y_0\right) = f'\left(x\right)$.

### 6.1 Classification of Second Order PDEs

Using the above form, we can determine the characteristic equation for a second order PDE:

$$\begin{vmatrix} a - 2b\lambda & -c \\ 1 & -\lambda \end{vmatrix} = a\lambda^2 - 2b\lambda + c = 0,$$

where the eigenvalues $\lambda$ are given by:

$$\lambda = \frac{2b \pm \sqrt{4b^2 - 4ac}}{2a} = \frac{b \pm \sqrt{b^2 - ac}}{a}.$$

We can classify this second order PDE into 3 types based on the sign of the discriminant $b^2 - ac$:

- **Hyperbolic**: (two real distinct eigenvalues)

$$b^2 - ac > 0.$$

- **Parabolic**: (two real equal eigenvalues)

$$b^2 - ac = 0.$$

- **Elliptic**: (two complex eigenvalues)

$$b^2 - ac < 0.$$

### 6.2 D'Alembert's Solution

D'Alembert's solution is a general solution to the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0,$$

with initial conditions $u(x,\, 0) = f(x)$ and $u_t(x,\, 0) = g(x)$. Using the techniques above, it can be shown that the general solution to this PDE has the form:

$$u(x,\, t) = \frac{1}{2}\left[f(x + ct) + f(x - ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s.$$

## 7 Conservation of Mass

Consider the flow of some mass in a one-dimensional channel between $x = a$ and $x = b$. Let $v = v(x,\, t)$ be the velocity of the flow, and $\rho = \rho(x,\, t)$ be the density of the flow, where velocity has units: length per time, and density has units: mass over volume. Then, the mass flux $q = q(x,\, t)$ is defined as $q = \rho v$, which has units: mass per unit time per unit area.
The mass of the fluid per unit area between $x = a$ and $x = b$ is given by the integral,

$$\int_a^b \rho\,\mathrm{d}x.$$

Additionally, this mass is a dynamic quantity that depends on the flux in at $x = a$ given by:

$$q\big|_{x=a}$$

and the mass flux out at $x = b$ given by:

$$q\big|_{x=b}\,.$$

The conservation of mass argument for this mass is a relationship between these fluxes and the rate of change of mass in the channel:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \rho \,\mathrm{d}x &= q\big|_{x=a} - q\big|_{x=b} \\
\int_a^b \frac{\partial \rho}{\partial t}\,\mathrm{d}x &= \int_b^a \frac{\partial q}{\partial x}\,\mathrm{d}x \\
\int_a^b \frac{\partial \rho}{\partial t}\,\mathrm{d}x &= -\int_a^b \frac{\partial q}{\partial x}\,\mathrm{d}x \\
\frac{\partial \rho}{\partial t} &= -\frac{\partial q}{\partial x} \\
\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} &= 0 \\
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) &= 0.
\end{aligned}$$

We can also allow $a$ and $b$ to be functions of time, where we can apply Leibniz integral rule to arrive at the same result on the interval $a(t) < x < b(t)$.

## 7.1 Traffic Flow

Consider a simple model for traffic flow where $\rho$ is the density of the flow of cars (number of cars per unit length), and $q$ is the flux of cars (number of cars per unit time), where $q = \rho v$ ($v$ velocity). The number of cars between $x = a$ and $x = b$ is given by:

$$\int_a^b \rho \,\mathrm{d}x,$$

the flux of cars in at $x = a$ is given by $q\big|_{x=a}$, and the flux of cars out at $x = b$ is given by $q\big|_{x=b}$. Then, for the conservation of the number of cars, we must have that:

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0.$$

This model is a PDE with two unknown variables, $\rho$ and $q$. To close this system, we must introduce a constitutive relationship between $\rho$ and $q$ (or between $\rho$ and $v$), which is determined through empirical analysis.

## 7.2 Shock Solutions

If a quasi-linear PDE inhibits a shock wave, and we desire a single-valued solution instead, we can introduce a discontinuity in the solution. To do so, we must assume that the total flow $q = uv$ is conserved across the *shock wave* $x_s(t)$, so that the flow from the left into the moving shock wave equals the flow to the right away from the shock wave.
Suppose conservation of mass holds between $x = a$ and $x = b$, except at a shock wave at $x = x_s(t)$, across which both $u$ and $q$ are discontinuous. We can still expect a weak form of conservation of mass to hold across the shock wave,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u\,\mathrm{d}x = q\big|_{x=a} - q\big|_{x=b}.$$

Consider dividing this integral into two regions that do not contain the shock wave. Then, using Leibniz's rule, we find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u\,\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t}\int_a^{x_s^-(t)} u\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_{x_s^+(t)}^b u\,\mathrm{d}x \\ &= \int_a^{x_s^-(t)} \frac{\partial u}{\partial t}\,\mathrm{d}x + u\big|_{x=x^-}\frac{\mathrm{d}x_s}{\mathrm{d}t} + \int_{x_s^+(t)}^b \frac{\partial u}{\partial t}\,\mathrm{d}x - u\big|_{x=x^+}\frac{\mathrm{d}x_s}{\mathrm{d}t}\end{aligned}$$

For the RHS, we can use the fundamental theorem of calculus to find:

$$\begin{aligned}q\big|_{x=a} - q\big|_{x=b} &= q\big|_{x=a} - q\big|_{x=x^-} + q\big|_{x=x^+} - q\big|_{x=b} + q\big|_{x=x^-} - q\big|_{x=x^+} \\ &= q\big|_{x=x^-}^{x=a} + q\big|_{x=b}^{x=x^+} + q\big|_{x=x^-} - q\big|_{x=x^+} \\ &= \int_{x_s^-(t)}^a \frac{\partial q}{\partial x}\,\mathrm{d}x + \int_b^{x_s^+(t)} \frac{\partial q}{\partial x}\,\mathrm{d}x + q\big|_{x=x^-} - q\big|_{x=x^+} \\ &= -\int_a^{x_s^-(t)} \frac{\partial q}{\partial x}\,\mathrm{d}x - \int_{x_s^+(t)}^b \frac{\partial q}{\partial x}\,\mathrm{d}x + q\big|_{x=x^-} - q\big|_{x=x^+}.\end{aligned}$$

By equating the LHS and RHS, we find

$$\begin{aligned}\int_a^{x_s^-(t)} \frac{\partial u}{\partial t} + \frac{\partial q}{\partial x}\,\mathrm{d}x + \int_{x_s^+(t)}^b \frac{\partial u}{\partial t} + \frac{\partial q}{\partial x}\,\mathrm{d}x + u\big|_{x=x^-}\frac{\mathrm{d}x_s}{\mathrm{d}t} - u\big|_{x=x^+}\frac{\mathrm{d}x_s}{\mathrm{d}t} &= q\big|_{x=x^-} - q\big|_{x=x^+} \\ \frac{\mathrm{d}x_s}{\mathrm{d}t}\left[u\big|_{x=x^-} - u\big|_{x=x^+}\right] &= q\big|_{x=x^-} - q\big|_{x=x^+} \\ \frac{\mathrm{d}x_s}{\mathrm{d}t} &= \frac{q\big|_{x=x^-} - q\big|_{x=x^+}}{u\big|_{x=x^-} - u\big|_{x=x^+}}.\end{aligned}$$

Or, more compactly,

$$x_s'(t) = \frac{q^+ - q^-}{u^+ - u^-} = \frac{[q]}{[u]} = \frac{\text{jump in flow}}{\text{jump in value}}.$$

By solving for the shock wave $x_s(t)$, we can divide the solution into two regions, one on either side of the shock wave. Characteristics will then meet at this shock wave leading to a single-valued solution. Note this solution does not have a derivative at the shock wave, and the shock solution is not unique.

### 7.3 Caustics

If the characteristics of a PDE intersect along some curve not arising from the $x$-axis, we have a *caustic*. Such characteristics may lead to more than 2 values for the solution. When determining a shock solution for such cases, we must consider the time $t_s$ at which the solution first becomes multi-valued. $t_s$ is the time at which the slope:

$$\frac{\partial u}{\partial x} = \frac{\frac{\partial u}{\partial \xi}}{\frac{\partial x(s)}{\partial \xi}}$$

becomes infinite. This can be found by solving for the value of $s$ at which the denominator of the above expression becomes zero. If we call this value $s_{\text{caustic}}$, we can parametrically define the caustic as:

$$\mathbf{x}_{\text{caustic}}\left(\xi\right) = \begin{bmatrix} x\left(s_{\text{caustic}}\right) \\ t\left(s_{\text{caustic}}\right) \end{bmatrix} = \begin{bmatrix} x_{\text{caustic}}\left(\xi\right) \\ t_{\text{caustic}}\left(\xi\right) \end{bmatrix}$$

We can then use the tangent vector $\mathbf{x}'_{\text{caustic}}\left(\xi\right)$ to determine the minimum of this curve to find the first time $t_s$ at which the solution becomes multi-valued. When solving for the shock $x_s\left(t\right)$ as outlined in the previous section, we can use this initial value to find the constant of integration.

### 7.4 Higher-Order Terms

If we wish for a shock to be resolved without a jump discontinuity, we can introduce higher-order terms into the PDE that will smooth out the solution.

# Part III
# Incompressible Fluid Flow

In the following sections, we will discuss the flow of incompressible Newtonian fluids. We will begin by deriving the governing equations for fluid flow using methods from continuum mechanics. Here we treat fluids as continuous media, where fluid properties are described by fields that vary continuously in space and time. We will then introduce ideal fluid flow, where we assume a fluid is inviscid and incompressible, and derive the Euler equations.

## 8 Continuum Mechanics

### 8.1 Material Derivative

Consider the temporal rate of change of a quantity which follows the motion of a fluid particle at position $\mathbf{r}\left(x,\, y,\, z,\, t\right)$. We cannot simply take the partial derivative of the field with respect to time as it is not fixed in space. Instead, we will introduce the *material derivative*, which accounts for the change in a field as a fluid particle moves through space. We will motivate this definition by deriving the acceleration of a fluid particle. Using the limit definition of the derivative, we find:

$$\mathbf{a}\left(x,\, y,\, z,\, t\right) = \lim_{\Delta t \to 0} \frac{\mathbf{q}\left(x + \Delta x,\, y + \Delta y,\, z + \Delta z,\, t + \Delta t\right) - \mathbf{q}\left(x,\, y,\, z,\, t\right)}{\Delta t},$$

where $\mathbf{q}$ is the velocity field of the fluid, defined:

$$\mathbf{q}(x,\ y,\ z,\ t) = \begin{bmatrix} u(x,\ y,\ z,\ t) \\ v(x,\ y,\ z,\ t) \\ w(x,\ y,\ z,\ t) \end{bmatrix} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}.$$

Notice that a small step in time has also resulted in a small step in space. Let us therefore use the Taylor series expansion of the velocity field to simplify the numerator of the above expression to:

$$\mathbf{q}(x+\Delta x,\ y+\Delta y,\ z+\Delta z,\ t+\Delta t) = \mathbf{q}(x,\ y,\ z,\ t) + \frac{\partial \mathbf{q}}{\partial x}\Delta x + \frac{\partial \mathbf{q}}{\partial y}\Delta y + \frac{\partial \mathbf{q}}{\partial z}\Delta z + \frac{\partial \mathbf{q}}{\partial t}\Delta t + \dots,$$

where we are evaluating the velocity field at the point $(x,\ y,\ z,\ t)$. Substituting this into the numerator, we find:

$$\begin{aligned}
\mathbf{a}(x,\ y,\ z,\ t) &= \lim_{\Delta t\to 0}\left[\frac{\partial \mathbf{q}}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial \mathbf{q}}{\partial y}\frac{\Delta y}{\Delta t} + \frac{\partial \mathbf{q}}{\partial z}\frac{\Delta z}{\Delta t} + \frac{\partial \mathbf{q}}{\partial t} + \dots\right] \\
&= \frac{\partial \mathbf{q}}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \mathbf{q}}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial \mathbf{q}}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{\partial \mathbf{q}}{\partial t} \\
&= \frac{\partial \mathbf{q}}{\partial x}u + \frac{\partial \mathbf{q}}{\partial y}v + \frac{\partial \mathbf{q}}{\partial z}w + \frac{\partial \mathbf{q}}{\partial t} \\
&= \frac{\partial \mathbf{q}}{\partial t} + u\frac{\partial \mathbf{q}}{\partial x} + v\frac{\partial \mathbf{q}}{\partial y} + w\frac{\partial \mathbf{q}}{\partial z} \\
&= \frac{\partial \mathbf{q}}{\partial t} + \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)\mathbf{q} \\
&= \frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}\cdot\boldsymbol{\nabla})\,\mathbf{q}.
\end{aligned}$$

**Definition 8.1** (Material Derivative)**.** The rate of change of a quantity as observed from a moving reference frame following a fluid particle can be expressed as the material derivative of the quantity. This operator is denoted as:

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{q}\cdot\boldsymbol{\nabla}.$$

Therefore, we can express the acceleration of a fluid particle in a moving reference frame as:

$$\mathbf{a} = \frac{\mathrm{D}\mathbf{q}}{\mathrm{D}t}.$$

## 8.2 Conservation of Mass

Let us now consider what a conservation of mass argument would look like for a fluid. For a control volume $V$, the total mass of the fluid is given by the integral:

$$M = \iiint_V \rho\,\mathrm{d}V,$$

where $\rho = \rho(x,\ y,\ z,\ t)$ is the density of the fluid. As we saw in the previous section, we want the rate of change of mass to be equal to the net flux of fluid into the control volume. This can be expressed using a surface integral over the surface of the control volume:

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\oiint_{\partial V} \rho\left(\mathbf{q}\cdot\mathbf{n}\right)\mathrm{d}\boldsymbol{\sigma},$$

where $\mathbf{n}$ is the outward unit normal vector to the surface of the control volume, and $\mathrm{d}\boldsymbol{\sigma}$ is the differential area element. By applying the divergence theorem, we can rewrite this surface integral as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \rho\,\mathrm{d}V = -\iiint_V \boldsymbol{\nabla}\cdot\left(\rho\mathbf{q}\right)\mathrm{d}V.$$

As the control volume is arbitrary and fixed, we can apply Leibniz's integral rule to find:

$$\begin{aligned}
\iiint_V \frac{\partial\rho}{\partial t}\,\mathrm{d}V + \iiint_V \boldsymbol{\nabla}\cdot\left(\rho\mathbf{q}\right)\mathrm{d}V &= 0\\
\iiint_V \left(\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\left(\rho\mathbf{q}\right)\right)\mathrm{d}V &= 0\\
\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\left(\rho\mathbf{q}\right) &= 0.
\end{aligned}$$

We can alternatively express this in terms of the material derivative, by expanding the divergence term:

$$\begin{aligned}
\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\left(\rho\mathbf{q}\right) &= 0\\
\frac{\partial\rho}{\partial t} + \frac{\mathrm{d}\rho u}{\mathrm{d}x} + \frac{\mathrm{d}\rho v}{\mathrm{d}y} + \frac{\mathrm{d}\rho w}{\mathrm{d}z} &= 0\\
\frac{\partial\rho}{\partial t} + \frac{\mathrm{d}\rho}{\mathrm{d}x}u + \rho\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}\rho}{\mathrm{d}y}v + \rho\frac{\mathrm{d}v}{\mathrm{d}y} + \frac{\mathrm{d}\rho}{\mathrm{d}z}w + \rho\frac{\mathrm{d}w}{\mathrm{d}z} &= 0\\
\frac{\partial\rho}{\partial t} + \frac{\mathrm{d}\rho}{\mathrm{d}x}u + \frac{\mathrm{d}\rho}{\mathrm{d}y}v + \frac{\mathrm{d}\rho}{\mathrm{d}z}w + \rho\frac{\mathrm{d}u}{\mathrm{d}x} + \rho\frac{\mathrm{d}v}{\mathrm{d}y} + \rho\frac{\mathrm{d}w}{\mathrm{d}z} &= 0\\
\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\rho\cdot\mathbf{q} + \rho\boldsymbol{\nabla}\cdot\mathbf{q} &= 0\\
\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\boldsymbol{\nabla}\cdot\mathbf{q} &= 0.
\end{aligned}$$

This result is known as the *continuity equation* and expresses the conservation of mass for a fluid.

## 8.3 Conservation of Momentum

Let us now consider the conservation of momentum for a fluid. Using Newton's second law of motion, we can express the rate of change in momentum as the sum of the forces acting on the fluid. For a control volume $V$, the rate of change in momentum of a fluid is given by the integral:

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \rho\mathbf{q}\,\mathrm{d}V = \mathbf{F}.$$

Here, $\mathbf{F}$ is the total force acting on the fluid, per unit volume. In a fluid, we can decompose this force into the sum of **body forces** $\mathbf{F}_b$, and **surface forces** $\mathbf{F}_s$, that act on some differential fluid element $\mathrm{d}V$. Body forces are uniformally distributed through an element and can include gravitational forces, electromagnetic forces, coriolis forces, etc. Here we will denote this generally as:

$$\mathrm{d}\mathbf{F}_b = \rho\mathbf{g}\,\mathrm{d}V \implies \mathbf{F}_b = \iiint_V \rho\mathbf{g}\,\mathrm{d}V,$$

where $\mathbf{g}$ is the acceleration due to some external body force. Note that we do not multiply this by the mass of the fluid element, as we are considering the force per unit volume. Surface forces act on the surface of the fluid element and include pressure and viscous forces. Here we will introduce the **total stress tensor $\boldsymbol{\sigma}$** which is the sum of **viscous stresses $\boldsymbol{\tau}$** that act both tangentially and normally to the surface of the fluid element, and hydrostatic **pressures** $p$ that act normally to the surface of the fluid element. We can express this in component form as:

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij},$$

where $\delta_{ij}$ is the Kronecker delta, or, using matrix notation:

$$\begin{bmatrix} \sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} -p+\tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & -p+\tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & -p+\tau_{zz} \end{bmatrix}$$
$$\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\tau}.$$

The subscript $\sigma_{ij}$ denotes a stress in the $j$ direction acting on a surface element with normal in the $i$ direction. To find the total surface force acting on a fluid element, consider all the forces acting in the $x$-direction:

$$\mathrm{d}\mathbf{F}_{sx} = \left(-\sigma_{xx} + \sigma_{x+\Delta x,x}\right)\Delta y\,\Delta z + \left(-\sigma_{yx} + \sigma_{y+\Delta y,x}\right)\Delta x\,\Delta z + \left(-\sigma_{zx} + \sigma_{z+\Delta z,x}\right)\Delta x\,\Delta y,$$

If we use the first-order Taylor expansion on these terms, this simplifies to:

$$\begin{aligned} \mathrm{d}\mathbf{F}_{sx} &= \frac{\partial\sigma_{xx}}{\partial x}\,\Delta x\,\Delta y\,\Delta z + \frac{\partial\sigma_{yx}}{\partial y}\,\Delta y\,\Delta x\,\Delta z + \frac{\partial\sigma_{zx}}{\partial z}\,\Delta z\,\Delta x\,\Delta y \\ &= \left(\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{yx}}{\partial y} + \frac{\partial\sigma_{zx}}{\partial z}\right)\Delta V. \end{aligned}$$

Therefore, for all three directions, we find:

$$\begin{aligned} \mathrm{d}\mathbf{F}_{sx} &= \left(\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{yx}}{\partial y} + \frac{\partial\sigma_{zx}}{\partial z}\right)\Delta V, \\ \mathrm{d}\mathbf{F}_{sy} &= \left(\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{zy}}{\partial z}\right)\Delta V, \\ \mathrm{d}\mathbf{F}_{sz} &= \left(\frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z}\right)\Delta V, \end{aligned}$$

written compactly as:

$$\mathrm{d}\mathbf{F}_{si} = \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_i^\top\,\mathrm{d}V, \quad \text{where } \boldsymbol{\sigma}_i = -p\mathbf{e}_i^\top + \boldsymbol{\tau}_i, \quad \text{for } i = x,\ y,\ z.$$

Here $\boldsymbol{\sigma}_i$ is the $i$-th row of the total stress tensor, $\mathbf{e}_i$ is the $i$-th unit vector, and $\boldsymbol{\tau}_i$ is the $i$-th row of the viscous stress tensor. For Newtonian fluids, the viscous stress tensor can be related to the velocity field of a fluid, by the *Newtonian law of viscosity*, which states that the viscous stress is proportional to the rate of strain of the fluid. This can be expressed as:

$$\tau_{ij} = \mu\left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i}\right) \implies \boldsymbol{\tau}_i = \mu\begin{bmatrix}\frac{\partial q_i}{\partial x} + \frac{\partial u}{\partial x_i} & \frac{\partial q_i}{\partial y} + \frac{\partial v}{\partial x_i} & \frac{\partial q_i}{\partial z} + \frac{\partial w}{\partial x_i}\end{bmatrix},$$

where $\mu$ is the dynamic viscosity of the fluid and $q_i$ is the $i$-th component of the velocity field. This allows us to express the total surface force acting on the fluid element in the $i$-th direction as:

$$\begin{aligned}
\mathrm{d}\mathbf{F}_{si} &= \boldsymbol{\nabla}\cdot\left(-p\mathbf{e}_i + \boldsymbol{\tau}_i^\top\right)\mathrm{d}V \\
&= \left(-\frac{\partial p}{\partial x_i} + \boldsymbol{\nabla}\cdot\boldsymbol{\tau}_i^\top\right)\mathrm{d}V \\
&= \left(-\frac{\partial p}{\partial x_i} + \mu\boldsymbol{\nabla}\cdot\begin{bmatrix}\frac{\partial q_i}{\partial x} + \frac{\partial u}{\partial x_i} \\ \frac{\partial q_i}{\partial y} + \frac{\partial v}{\partial x_i} \\ \frac{\partial q_i}{\partial z} + \frac{\partial w}{\partial x_i}\end{bmatrix}\right)\mathrm{d}V \\
&= \left(-\frac{\partial p}{\partial x_i} + \mu\left(\frac{\partial^2 q_i}{\partial x^2} + \frac{\partial^2 u}{\partial x\,\partial x_i} + \frac{\partial^2 q_i}{\partial y^2} + \frac{\partial^2 v}{\partial y\,\partial x_i} + \frac{\partial^2 q_i}{\partial z^2} + \frac{\partial^2 w}{\partial z\,\partial x_i}\right)\right)\mathrm{d}V \\
&= \left(-\frac{\partial p}{\partial x_i} + \mu\left(\frac{\partial^2 q_i}{\partial x^2} + \frac{\partial^2 q_i}{\partial y^2} + \frac{\partial^2 q_i}{\partial z^2} + \frac{\partial^2 u}{\partial x\,\partial x_i} + \frac{\partial^2 v}{\partial y\,\partial x_i} + \frac{\partial^2 w}{\partial z\,\partial x_i}\right)\right)\mathrm{d}V \\
&= \left(-\frac{\partial p}{\partial x_i} + \mu\nabla^2 q_i + \mu\boldsymbol{\nabla}\cdot\frac{\partial\mathbf{q}}{\partial x_i}\right)\mathrm{d}V \\
&= \left(-\frac{\partial p}{\partial x_i} + \mu\nabla^2 q_i + \mu\frac{\partial}{\partial x_i}\left(\boldsymbol{\nabla}\cdot\mathbf{q}\right)\right)\mathrm{d}V.
\end{aligned}$$

By combining all three directions, we can write this using the gradient operator:

$$\mathrm{d}\mathbf{F}_s = \left(-\boldsymbol{\nabla}p + \mu\nabla^2\mathbf{q} + \mu\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{q}\right)\right)\mathrm{d}V.$$

By integrating this over the control volume, we find that the total surface force acting on the fluid element is:

$$\mathbf{F}_s = \iiint_V \left(-\boldsymbol{\nabla}p + \mu\nabla^2\mathbf{q} + \mu\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{q}\right)\right)\mathrm{d}V.$$

Therefore, the conservation of momentum in a fluid can be expressed as:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \rho\mathbf{q}\,\mathrm{d}V &= \mathbf{F}_b + \mathbf{F}_s \\
\iiint_V \frac{\mathrm{D}\rho\mathbf{q}}{\mathrm{D}t}\,\mathrm{d}V &= \iiint_V \left(\rho\mathbf{g} - \boldsymbol{\nabla}p + \mu\nabla^2\mathbf{q} + \mu\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{q}\right)\right)\mathrm{d}V \\
\frac{\mathrm{D}\rho}{\mathrm{D}t}\mathbf{q} + \rho\frac{\mathrm{D}\mathbf{q}}{\mathrm{D}t} &= \rho\mathbf{g} - \boldsymbol{\nabla}p + \mu\nabla^2\mathbf{q} + \mu\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{q}\right)
\end{aligned}$$

We can simplify the continuity equations by assuming that the density of the fluid is constant so that $\dfrac{\mathrm{D}\rho}{\mathrm{D}t} = 0$. Therefore, given the following constitutive relationship for an incompressible fluid:

$$\boldsymbol{\nabla} \cdot \mathbf{q} = 0,$$

he governing equations for the flow of an incompressible Newtonian fluid are:

$$\rho \frac{\mathrm{D}\mathbf{q}}{\mathrm{D}t} = \rho \mathbf{g} - \boldsymbol{\nabla} p + \mu \nabla^2 \mathbf{q}.$$

These results are known as the *Navier-Stokes equations* for an incompressible fluid.

# 9 Unidirectional Flow

The Navier-Stokes equations are a nonlinear system of four PDEs that are extremely difficult to solve both analytically and numerically. However, by considering fluid flow in a single direction, we can simplify this system greatly. Here we will orient our coordinate system with the flow of the fluid, for example, in the $x$-direction. This simplifies the velocity vector to:

$$u \neq 0, \quad v = 0, \quad w = 0.$$

The continuity equation then also simplifies to:

$$\boldsymbol{\nabla} \cdot \mathbf{q} = \frac{\partial u}{\partial x} = 0 \implies u = u\left(y,\ z,\ t\right).$$

We can also simplify the material derivative to be used in the momentum equation:

$$\begin{aligned}
\frac{\mathrm{D}\mathbf{q}}{\mathrm{D}t} &= \frac{\partial \mathbf{q}}{\partial t} + \left(\mathbf{q} \cdot \boldsymbol{\nabla}\right) \mathbf{q} \\
&= \begin{bmatrix} \dfrac{\partial u}{\partial t} \\ 0 \\ 0 \end{bmatrix} + \left(u \frac{\partial}{\partial x}\right) \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{\partial u}{\partial t} + u \dfrac{\partial u}{\partial x} \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{\partial u}{\partial t} \\ 0 \\ 0 \end{bmatrix}.
\end{aligned}$$

We can now consider each component of the Navier-Stokes equations using these simplifications. The $x$-component of the Navier-Stokes equations lets us cancel the derivatives of $u$ in the $x$ direction:

$$\begin{aligned}
\rho \frac{\partial u}{\partial t} &= \rho g_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \\
\rho \frac{\partial u}{\partial t} &= \rho g_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right).
\end{aligned}$$

In the $y$-component, we can cancel all derivatives of $v$, as it is zero:

$$
\begin{aligned}
0 &= \rho g_y - \frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \\
0 &= \rho g_y - \frac{\partial p}{\partial y}.
\end{aligned}
$$

Finally, in the $z$-component, we can cancel all derivatives of $w$, as it is also zero:

$$
\begin{aligned}
0 &= \rho g_z - \frac{\partial p}{\partial z} + \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) \\
0 &= \rho g_z - \frac{\partial p}{\partial z}.
\end{aligned}
$$

Therefore, the governing equations for unidirectional flow in the $x$-direction is given by the following system of coupled PDEs:

$$
\begin{cases}
\rho \dfrac{\partial u}{\partial t} = \rho g_x - \dfrac{\partial p}{\partial x} + \mu \left( \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} \right) \\
0 = \rho g_y - \dfrac{\partial p}{\partial y} \\
0 = \rho g_z - \dfrac{\partial p}{\partial z}
\end{cases}
$$

Depending on the problem, we can further simplify these equations if:

- the body force acts in one direction, or if it is negligible.
- the flow is steady, so that all time derivatives are zero.
- the flow is in 2D, so that there is no variation in the $z$-direction.
- there is no pressure gradient driving the flow, so that $\dfrac{\partial p}{\partial x} = 0$.

After simplifying these equations, we can often solve this system of PDEs through the method of separation of variables.

## 9.1 Boundary Conditions

A common boundary condition for fluid flow is the no-slip condition, which states that the velocity of the fluid at the boundary of a solid object is equal to the velocity of the object. This can be expressed as:

$$u = U \quad \text{at } y = 0$$

where $U$ is the velocity of the object. When the object is stationary, this condition reduces to:

$$u = 0 \quad \text{at } y = 0.$$

# 10 Dimensionless Form

Consider a non-dimensional form of the Navier-Stokes equations, where all variables are scaled by their dimensional counterparts. Here we will introduce the following dimensionless variables:

$$\hat{\mathbf{x}} = \frac{1}{\mathsf{L}}\mathbf{x}, \quad \hat{t} = \frac{1}{\mathsf{T}}t, \quad \hat{\mathbf{q}} = \frac{1}{\mathsf{Q}}\mathbf{q}, \quad \hat{\mathbf{p}} = \frac{1}{\mathsf{P}}\mathbf{p},$$

where $\mathsf{L}$, $\mathsf{T}$, $\mathsf{Q}$, and $\mathsf{P}$ are characteristic length, time, velocity, and pressure scales, respectively. By substituting these into the Navier-Stokes equations, we can find the dimensionless form of the governing equations of incompressible fluid flow:

$$\rho\left(\frac{\mathsf{Q}}{\mathsf{T}}\frac{\partial\hat{\mathbf{q}}}{\partial t} + \frac{\mathsf{Q}^2}{\mathsf{L}}\left(\hat{\mathbf{q}}\cdot\hat{\boldsymbol{\nabla}}\right)\hat{\mathbf{q}}\right) = \rho\mathbf{g} - \frac{\mathsf{P}}{\mathsf{L}}\hat{\boldsymbol{\nabla}}\hat{p} + \frac{\mu\mathsf{Q}}{\mathsf{L}^2}\hat{\nabla}^2\hat{\mathbf{q}}$$

Let us then define the velocity scale as the ratio of the characteristic length to the characteristic time:

$$\mathsf{Q} = \frac{\mathsf{L}}{\mathsf{T}}.$$

This lets us factor the dimensions on the material derivative:

$$\frac{\rho\mathsf{Q}^2}{\mathsf{L}}\left(\frac{\partial\hat{\mathbf{q}}}{\partial t} + \left(\hat{\mathbf{q}}\cdot\hat{\boldsymbol{\nabla}}\right)\hat{\mathbf{q}}\right) = \rho g\hat{\mathbf{g}} - \frac{\mathsf{P}}{\mathsf{L}}\hat{\boldsymbol{\nabla}}\hat{p} + \frac{\mu\mathsf{Q}}{\mathsf{L}^2}\hat{\nabla}^2\hat{\mathbf{q}},$$

where $g$ is the magnitude of $\mathbf{g}$. The magnitudes of the terms in the Navier-Stokes equations can be compared by considering the order of magnitude of each term:

- **Inertial Force**: $\mathcal{O}\left(\frac{\rho\mathsf{Q}^2}{\mathsf{L}}\right)$ from $\partial\hat{\mathbf{q}}/\partial t + \left(\hat{\mathbf{q}}\cdot\hat{\boldsymbol{\nabla}}\right)\hat{\mathbf{q}}$.
- **Gravity Force**: $\mathcal{O}(\rho g)$ from $\hat{\mathbf{g}}$.
- **Pressure Gradient Force**: $\mathcal{O}\left(\frac{\mathsf{P}}{\mathsf{L}}\right)$ from $\hat{\boldsymbol{\nabla}}\hat{p}$.
- **Viscous Force**: $\mathcal{O}\left(\frac{\mu\mathsf{Q}}{\mathsf{L}^2}\right)$ from $\hat{\nabla}^2\hat{\mathbf{q}}$.

## 10.1 Reynolds and Froude Numbers

If we consider the ratio of the inertial force to the viscous force, we derive the *Reynolds number*:

$$\mathrm{Re} = \frac{\rho\mathsf{Q}\mathsf{L}}{\mu} = \frac{\text{Inertial Force}}{\text{Viscous Force}}.$$

This number is a measure of the relative importance of inertial forces to viscous forces in a fluid flow.

- For $\mathrm{Re} \ll 1$, the flow is dominated by viscous forces, chracterised by low velocity or high viscosity. This regmine is known as **viscous flow** or **Stokes flow**.
- For $\mathrm{Re} \gg 1$, the flow is dominated by inertial forces, characterised by high velocity or low viscosity. This is referred to as **inviscid flow** or **ideal flow**.

We can also consider the ratio of the inertial force to the body force to derive the *Froude number* when considering the effects of gravity:

$$\mathrm{Fr} = \frac{\mathsf{Q}}{\sqrt{g\mathsf{L}}} = \sqrt{\frac{\text{Inertial Force}}{\text{Gravity Force}}}.$$

This number is a measure of the relative importance of inertial forces to gravitational forces in a fluid flow.

- For $\mathrm{Fr} < 1$, the flow is dominated by gravitational forces and disturbances propogate upstream. This is known as **subcritical flow**.
- For $\mathrm{Fr} > 1$, the flow is dominated by inertial forces and disturbances propogate downstream. This is known as **supercritical flow**.
- For $\mathrm{Fr} = 1$, the flow is in a state of **critical flow** where the flow velocity matches the wave speed.

We can express the Navier-Stokes equations in dimensionless form using these dimensionless variables:

$$\frac{\mathrm{D}\hat{\mathbf{q}}}{\mathrm{D}t} = \frac{1}{\mathrm{Fr}^2}\hat{\mathbf{g}} - \frac{\mathsf{P}}{\rho\mathsf{Q}^2}\hat{\boldsymbol{\nabla}}\hat{p} + \frac{1}{\mathrm{Re}}\hat{\nabla}^2\hat{\mathbf{q}},$$

with the constitutive relationship:

$$\hat{\boldsymbol{\nabla}} \cdot \hat{\mathbf{q}} = 0,$$

for incompressible flow.

# 11 Inviscid Flow

If we suppose that flow is inviscid, $\mathrm{Re} \gg 1$, and we can simplify the Navier-Stokes by removing the viscous term:

$$\frac{\mathrm{D}\hat{\mathbf{q}}}{\mathrm{D}t} = \frac{1}{\mathrm{Fr}^2}\hat{\mathbf{g}} - \frac{\mathsf{P}}{\rho\mathsf{Q}^2}\hat{\boldsymbol{\nabla}}\hat{p}.$$

Notice all terms are dimensionless and suggests that the pressure scale is proportional to the square of the velocity scale:

$$\mathsf{P} = \rho\mathsf{Q}^2.$$

This allows us to simplify the governing equations to[1]:

$$\frac{\mathrm{D}\mathbf{q}}{\mathrm{D}t} = \frac{1}{\mathrm{Fr}^2}\mathbf{g} - \boldsymbol{\nabla}p,$$

with the constitutive relationship:

$$\boldsymbol{\nabla} \cdot \mathbf{q} = 0.$$

This system of equations is known as the *Euler equations* and describes the flow of an inviscid fluid.

[1] Note we have dropped the hats on the variables for brevity.

## 11.1 Irrotational Flow

Consider a flow whose vorticity (tendancy to rotate) is zero. This type of flow is known as *irrotational flow* or *potential flow*, where we assume the velocity field is conservative:

$$\mathbf{q} = \boldsymbol{\nabla}\phi.$$

Here $\phi = \phi(x, y, z, t)$ is a velocity potential with continuous second derivatives. When flow is both inviscid and irrotational, it is called *ideal flow.* Using this restriction, we can further simplify the Euler equations by reducing the material derivative as follows:

$$\begin{aligned}\frac{\mathrm{D}\mathbf{q}}{\mathrm{D}t} &= \frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}\cdot\boldsymbol{\nabla})\,\mathbf{q} \\ &= \frac{\partial \mathbf{q}}{\partial t} + \frac{1}{2}\boldsymbol{\nabla}(\mathbf{q}\cdot\mathbf{q}) - \mathbf{q}\times(\boldsymbol{\nabla}\times\mathbf{q}) \\ &= \frac{\partial \mathbf{q}}{\partial t} + \frac{1}{2}\boldsymbol{\nabla}\|\mathbf{q}\|^2 \\ &= \frac{\partial \boldsymbol{\nabla}\phi}{\partial t} + \frac{1}{2}\boldsymbol{\nabla}\|\boldsymbol{\nabla}\phi\|^2 \\ &= \boldsymbol{\nabla}\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}\|\boldsymbol{\nabla}\phi\|^2\right).\end{aligned}$$

Then, if we assume the body forces only consist of gravitational forces that act in the $z$-direction, we can express this force as:

$$\mathbf{g} = -\mathbf{k} = -\boldsymbol{\nabla}(z).$$

So, the Euler equations for inviscid, irrotational flow are:

$$\begin{aligned}\boldsymbol{\nabla}\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}\|\boldsymbol{\nabla}\phi\|^2\right) &= -\boldsymbol{\nabla}p - \frac{1}{\mathrm{Fr}^2}\boldsymbol{\nabla}(z) \\ \boldsymbol{\nabla}\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}\|\boldsymbol{\nabla}\phi\|^2\right) &= -\boldsymbol{\nabla}\left(p + \frac{1}{\mathrm{Fr}^2}z\right) \\ \boldsymbol{\nabla}\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}\|\boldsymbol{\nabla}\phi\|^2 + p + \frac{1}{\mathrm{Fr}^2}z\right) &= 0 \\ \frac{\partial \phi}{\partial t} + \frac{1}{2}\|\boldsymbol{\nabla}\phi\|^2 + p + \frac{1}{\mathrm{Fr}^2}z &= c(t).\end{aligned}$$

Note that if we choose $\phi = \bar{\phi} + \int c(t)\,\mathrm{d}t$, then:

$$\begin{aligned}\frac{\partial \bar{\phi}}{\partial t} + c(t) + \frac{1}{2}\|\boldsymbol{\nabla}\bar{\phi}\|^2 + p + \frac{1}{\mathrm{Fr}^2}z &= c(t) \\ \frac{\partial \bar{\phi}}{\partial t} + \frac{1}{2}\|\boldsymbol{\nabla}\bar{\phi}\|^2 + p + \frac{1}{\mathrm{Fr}^2}z &= 0.\end{aligned}$$

Therefore, as $c(t)$ plays no role in this equation we can choose $c(t) = 0$. Therefore, for inviscid, irrotational flow, the governing equation is given by:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\|\boldsymbol{\nabla}\phi\|^2 + p + \frac{1}{\mathrm{Fr}^2}z = 0.$$

This result is known as **Bernoulli's equation**. Additionally, the continuity equation becomes:

$$\boldsymbol{\nabla} \cdot \mathbf{q} = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\phi = \nabla^2\phi = 0.$$

This is precisely Laplace's equation, a linear second-order elliptic PDE that can be solved using separation of variables. Thus, by solving this equation, we can find the velocity potential $\phi$ which satisfies Bernoulli's equation.

## 11.2 Boundary Conditions

For inviscid, irrotational flow, we can no longer use the no-slip condition as the flow is inviscid, so the fluid can slip past the boundary. Instead, we enforce the condition that the normal velocity to the boundary is zero:

$$\mathbf{q} \cdot \mathbf{n} = 0 \implies \boldsymbol{\nabla}\phi \cdot \mathbf{n} = 0,$$

for a solid boundary with normal vector $\mathbf{n}$. This ensures that flow does not penetrate the boundary. For a solid boundary that is moving with velocity $\mathbf{U}$, this condition becomes:

$$(\mathbf{q} - \mathbf{U}) \cdot \mathbf{n} = 0.$$