Differential Equations and Modelling 2

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Part I
Symmetry Methods

1 Symmetry Transformations

Consider a partial differential equation for $u(x,\, t)$ whose domain lies in $\mathbb{R}^2$. Such a problem typically does not have any natural length or time scales associated with it's fundamental solution. Thus, let us transform the independent and dependent variables through the mapping $(x,\, t,\, u) \mapsto (X,\, T,\, U)$ where $X = X(x,\, t,\, u)$, $T = T(x,\, t,\, u)$, and $U = U(x,\, t,\, u)$, such that the PDE is invariant under this transformation, that is, the transformed PDE in U has the same form as the original PDE in u.
The choice of mapping may consist of dilations and translations of the independent and dependent variables by some constant factors, where each constant is enforced by the invariance condition. To do this, we must ensure that the function $U(x,\, t,\, u(X,\, T))$ satisfies the transformed PDE. When this is the case, the PDE is said to have a *symmetry transformation*.
The goal of these symmetry methods is to find an appropriate transformation that simplies the PDE, often reducing it to an ODE.

1.1 Dilation Symmetry

A dilation symmetry is a transformation of the form

$$X = ax, \quad T = a^\beta t, \quad U = a^\gamma u,$$

where a, β, and γ are constants. Note that we do not express the transformation for x as $X = a^\alpha x$ since it does not provide any additional information.

Example 1.1. Consider the transport equation

$$\frac{\partial u(x,\, t)}{\partial t} + c\frac{\partial u(x,\, t)}{\partial x} = 0$$

and the mapping

$$X = ax, \quad T = a^\beta t, \quad U = u.$$

After substituting the transformed solution into the PDE, we find

$$\begin{aligned} \frac{\partial u(X,\, T)}{\partial t} + c\frac{\partial u(X,\, T)}{\partial x} &= 0 \\ \frac{\partial u(X,\, T)}{\partial T}\frac{\mathrm{d}T}{\mathrm{d}t} + c\frac{\partial u(X,\, T)}{\partial X}\frac{\mathrm{d}X}{\mathrm{d}x} &= 0 \\ a^\beta\frac{\partial u(X,\, T)}{\partial T} + ac\frac{\partial u(X,\, T)}{\partial X} &= 0. \end{aligned}$$

For this PDE to be invariant under a dilation transformation, we must factor out the constant a from the equation. This implies that:

$$a^\beta = a \implies \beta = 1,$$

so that
$$\frac{\partial u\left(X,\,T\right)}{\partial T}+c\frac{\partial u\left(X,\,T\right)}{\partial X}=0.$$
Here we have shown that the transport equation is invariant under the above dilation transformation when $\beta=1$.

Example 1.2. Consider the nonlinear PDE
$$\frac{\partial u\left(x,\,y\right)}{\partial x}+y^2\frac{\partial u\left(x,\,y\right)}{\partial y}=0$$
and the mapping
$$X=ax,\quad Y=a^{\beta}y,\quad U=u.$$
By substituting the transformed solution into the PDE, we find
$$\begin{aligned}
\frac{\partial u\left(X,\,Y\right)}{\partial x}+y^2\frac{\partial u\left(X,\,Y\right)}{\partial y}&=0\\
\frac{\partial u\left(X,\,Y\right)}{\partial X}\frac{\mathrm{d}X}{\mathrm{d}x}+y^2\frac{\partial u\left(X,\,Y\right)}{\partial Y}\frac{\mathrm{d}Y}{\mathrm{d}y}&=0\\
a\frac{\partial u\left(X,\,Y\right)}{\partial X}+a^{\beta}y^2\frac{\partial u\left(X,\,Y\right)}{\partial Y}&=0\\
a\frac{\partial u\left(X,\,Y\right)}{\partial X}+a^{\beta}\left(Y/a^{\beta}\right)^2\frac{\partial u\left(X,\,Y\right)}{\partial Y}&=0\\
a\frac{\partial u\left(X,\,Y\right)}{\partial X}+a^{-\beta}Y^2\frac{\partial u\left(X,\,Y\right)}{\partial Y}&=0.
\end{aligned}$$
For this PDE to be invariant under a dilation transformation, we must again factor out a from the equation so that:
$$a=a^{-\beta}\implies 1=-\beta\implies \beta=-1,$$
and
$$\frac{\partial u\left(X,\,Y\right)}{\partial X}+Y^2\frac{\partial u\left(X,\,Y\right)}{\partial Y}=0.$$
Here we have shown that this nonlinear PDE is invariant under the above dilation transformation when $\beta=-1$.

Example 1.3. Consider the nonlinear convection diffusion equation
$$\frac{\partial u\left(x,\,t\right)}{\partial t}=u\left(x,\,t\right)\frac{\partial^2 u\left(x,\,t\right)}{\partial x^2}-\frac{\partial u\left(x,\,t\right)}{\partial x}$$
and the mapping
$$X=ax,\quad T=a^{\beta}t,\quad U=a^{\gamma}u.$$

We will once again substitute the transformed solution into the PDE to find

$$\begin{aligned}
a^{\gamma}\frac{\partial u(X,\,T)}{\partial t} &= a^{2\gamma}u(X,\,T)\,\frac{\partial^2 u(X,\,T)}{\partial x^2} - a^{\gamma}\frac{\partial u(X,\,T)}{\partial x} \\
a^{\gamma}\frac{\partial u(X,\,T)}{\partial T}\frac{\mathrm{d}T}{\mathrm{d}t} &= a^{2\gamma}u(X,\,T)\,\frac{\partial}{\partial x}\left[\frac{\partial u(X,\,T)}{\partial X}\frac{\mathrm{d}X}{\mathrm{d}x}\right] - a^{\gamma}\frac{\partial u(X,\,T)}{\partial X}\frac{\mathrm{d}X}{\mathrm{d}x} \\
a^{\beta+\gamma}\frac{\partial u(X,\,T)}{\partial T} &= a^{1+2\gamma}u(X,\,T)\,\frac{\partial}{\partial X}\left[\frac{\mathrm{d}X}{\mathrm{d}x}\frac{\partial u(X,\,T)}{\partial X}\right] - a^{1+\gamma}\frac{\partial u(X,\,T)}{\partial X} \\
a^{\beta+\gamma}\frac{\partial u(X,\,T)}{\partial T} &= a^{2+2\gamma}u(X,\,T)\,\frac{\partial^2 u(X,\,T)}{\partial X^2} - a^{1+\gamma}\frac{\partial u(X,\,T)}{\partial X}.
\end{aligned}$$

For this PDE to be invariant under a dilation transformation, we must factor out a from the equation so that:

$$a^{\beta+\gamma} = a^{2+2\gamma} = a^{1+\gamma} \implies \beta+\gamma = 2+2\gamma = 1+\gamma \implies \beta = 1, \gamma = -1,$$

and

$$\frac{\partial u(X,\,T)}{\partial T} = u(X,\,T)\,\frac{\partial^2 u(X,\,T)}{\partial X^2} - \frac{\partial u(X,\,T)}{\partial X}.$$

Here we have shown that this nonlinear PDE is invariant under the above dilation transformation when $\beta = 1$ and $\gamma = -1$.

1.2 Translation Symmetry

A translation symmetry is a transformation of the form

$$X = x - x_0, \quad T = t - t_0, \quad U = u - u_0,$$

where x_0, t_0, and u_0 are constants.

Example 1.4. Consider the heat equation

$$\frac{\partial u(x,\,t)}{\partial t} = D\frac{\partial^2 u(x,\,t)}{\partial x^2}$$

and the mapping

$$X = x - x_0, \quad T = t - t_0, \quad U = u - u_0.$$

By substituting the transformed solution into the PDE, we find

$$\begin{aligned}
\frac{\partial}{\partial t}\left(u(X,\,T) - u_0\right) &= D\frac{\partial^2}{\partial x^2}\left(u(X,\,T) - u_0\right) \\
\frac{\partial u(X,\,T)}{\partial T}\frac{\mathrm{d}T}{\mathrm{d}t} &= D\frac{\partial}{\partial x}\left(\frac{\partial u(X,\,T)}{\partial X}\frac{\mathrm{d}X}{\mathrm{d}x}\right) \\
\frac{\partial u(X,\,T)}{\partial T} &= D\frac{\partial}{\partial X}\frac{\mathrm{d}X}{\mathrm{d}x}\left(\frac{\partial u(X,\,T)}{\partial X}\right) \\
\frac{\partial u(X,\,T)}{\partial T} &= D\frac{\partial^2 u(X,\,T)}{\partial X^2}.
\end{aligned}$$

Here we have shown that the heat equation is invariant under the above translation transformation.

2 Similarity Solutions

In the previous section, we considered dilation transformations where both independent variables were scaled by some power of a. Notice that the product $\eta = xt^{-1/\beta}$ is invariant under the same dilation transformation:

$$\eta = xt^{-1/\beta} \implies H = \left(\frac{x}{a}\right)\left(\frac{T}{a^{\beta}}\right)^{-1/\beta} = XT^{-1/\beta}.$$

Let us therefore consider transformations of the form

$$X = ax, \quad T = a^{\beta}t, \quad U = t^{-\alpha} f\left(\eta\right),$$

where a, α, and β are constant and f is an arbitrary function to be determined. f is called a *similarity solution* (with *similarity variable* η) and our goal is to transform the PDE into an ODE with a single independent variable η. To do this:

1. Substitute $u\left(X,\, T\right)$ into the PDE to solve for α and β while maintaining invariance.
2. Obtain an ODE in terms of $f\left(\eta\right)$ by substituting the similarity solution into the PDE.
3. Obtain boundary conditions in terms of η using the same transformation.
4. Solve the ODE to find $f\left(\eta\right)$.
5. Transform back to the original variables to find a solution to the PDE.

3 Travelling Wave Solutions

Another type of solution is the *travelling wave solution*, where the solution to a PDE appears to move at a constant velocity when after a long period of time. This is similar to a steady state solution where the solution does not change after a long period of time. In this problem, we assume solutions of the form

$$u\left(x,\, t\right) = f\left(z\right), \quad z = x - ct,$$

where c is the speed at which the solution travels. The travelling wave speed is often determined using analysis, boundary conditions, numerical methods, or physical constraints. This method is only applicable to PDEs that are invariant under translations to both independent variables. As such, the PDE must not contain any explicit dependence on x or t.

Part II

Method of Characteristics

The method of characteristics is a technique used to solve nonlinear PDEs by reducing them to a system of ODEs. It considers parametrisations of the solution through a set of curves in the solution space.

4 First Order PDEs

4.1 Classification

Consider the general first order PDE in two variables:

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = f.$$

We can classify this PDE into 3 types based on the value of the variables a, b, and f:

- **Linear**:
$$a\left(x,\ y\right)\frac{\partial u}{\partial x} + b\left(x,\ y\right)\frac{\partial u}{\partial y} = f\left(x,\ y\right).$$
- **Semi-Linear**:
$$a\left(x,\ y\right)\frac{\partial u}{\partial x} + b\left(x,\ y\right)\frac{\partial u}{\partial y} = f\left(x,\ y,\ u\right).$$
- **Quasi-Linear**:
$$a\left(x,\ y,\ u\right)\frac{\partial u}{\partial x} + b\left(x,\ y,\ u\right)\frac{\partial u}{\partial y} = f\left(x,\ y,\ u\right).$$
- **Nonlinear**: Otherwise.

4.2 Characteristics

Given a quasi-linear PDE of the form:

$$a\left(x,\ y,\ u\right)\frac{\partial u}{\partial x} + b\left(x,\ y,\ u\right)\frac{\partial u}{\partial y} = f\left(x,\ y,\ u\right)$$

with the initial condition $u\left(x,\ 0\right) = u_0\left(x\right)$, let us consider the surface $u = u\left(x,\ y\right)$, which has the normal vector:

$$\mathbf{n} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ -1 \end{bmatrix}.$$

We can find a tangent vector to $\mathbf{n}$ by using the orthogonality property of the dot product of the normal vector:

$$\begin{bmatrix} a\left(x,\ y,\ u\right) \\ b\left(x,\ y,\ u\right) \\ f\left(x,\ y,\ u\right) \end{bmatrix} \cdot \mathbf{n} = 0,$$

Let us call this vector $\mathbf{v}$:

$$\mathbf{v} = \begin{bmatrix} a\left(x,\ y,\ u\right) \\ b\left(x,\ y,\ u\right) \\ f\left(x,\ y,\ u\right) \end{bmatrix}.$$

Let us now consider a parametric curve defined:

$$\mathbf{r}\left(s\right) = \begin{bmatrix} x\left(s\right) \\ y\left(s\right) \\ u\left(s\right) \end{bmatrix}$$

and let us impose that $\mathbf{r}'(s) = \mathbf{v}$, so that the curve lies on the solution surface. This gives us the following system of equations:

$$\mathbf{r}'(s) = \mathbf{v} \iff \begin{cases} \dfrac{\partial x}{\partial s} = a(x,\ y,\ u), \\ \dfrac{\partial y}{\partial s} = b(x,\ y,\ u), \\ \dfrac{\partial u}{\partial s} = f(x,\ y,\ u). \end{cases}$$

These equations are called *characteristic curves* of the PDE. We can integrate each of these equations to find parametric curves in s, and solve for all integration constants using the initial condition:

$$u(0) = u_0(\xi).$$

where $x = \xi$[1] and $y = 0$. We can then solve for $u(x,\ y)$ by eliminating s and ξ from the parametric equations.

[1] ξ is introduced to avoid confusion with the variable x in the initial condition.