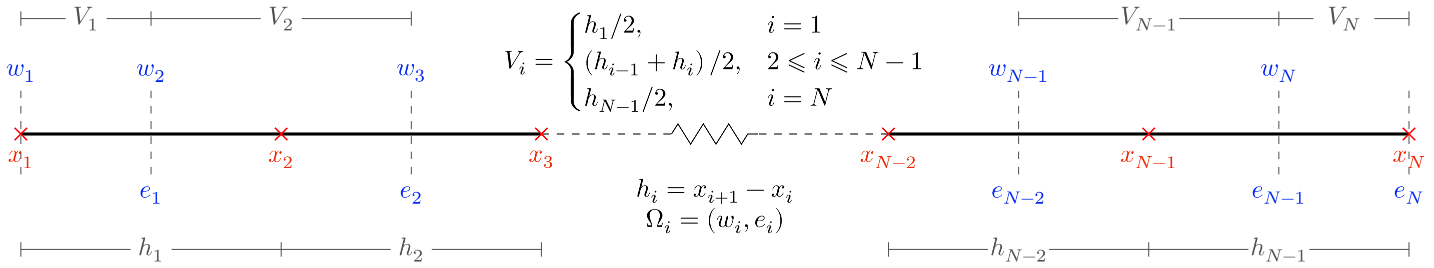

$$V_i = \begin{cases} h_1/2, & i = 1 \\ (h_{i-1}+h_i)/2, & 2 \leqslant i \leqslant N-1 \\ h_{N-1}/2, & i = N \end{cases}$$

$$h_i = x_{i+1} - x_i \qquad \Omega_i = (w_i, e_i)$$

Finite Differences

$$
\begin{array}{lll}
u(w_i,\, t) = (1-\sigma)\, u_{i-1} + \sigma u_i & \partial_x u(w_i,\, t) = (u_i - u_{i-1})/h_{i-1} & \text{(west node)} \\
u(e_i,\, t) = (1-\sigma)\, u_i + \sigma u_{i+1} & \partial_x u(e_i,\, t) = (u_{i+1} - u_i)/h_i & \text{(east node)} \\
\partial_x u(w_1,\, t) = \partial_x u(0,\, t) & \partial_x u(e_1,\, t) = (u_2 - u_1)/h_1 & \text{(node 1)} \\
\partial_x u(w_N,\, t) = (u_N - u_{N-1})/h_{N-1} & \partial_x u(e_N,\, t) = \partial_x u(L,\, t) & \text{(node } N\text{)}
\end{array}
$$

$D(u(x_i,\, t)) = D(u_i)$, $D(u(w_i,\, t)) = \frac{D(u_{i-1})+D(u_i)}{2}$, $D(u(e_i,\, t)) = \frac{D(u_i)+D(u_{i+1})}{2}$
Flow left to right ($v > 0$): $\sigma = 0$. Flow right to left ($v < 0$): $\sigma = 1$.

Time Discretisation (integrate between t_n and t_{n+1})

$$(\mathbf{I} - \delta t \theta_1 \mathbf{A})\, \mathbf{u}^{(n+1)} = [\mathbf{I} + \delta t (1-\theta_1)\, \mathbf{A}]\, \mathbf{u}^{(n)} + \delta t \left[\mathbf{b}_1 + (1-\theta_2)\, \mathbf{b}_2^{(n)} + \theta_2 \mathbf{b}_2^{(n+1)}\right]$$

using $\int_{t_n}^{t_{n+1}} f(t)\, \mathrm{d}t \approx \delta t\, [(1-\theta)\, f(t_n) + \theta f(t_{n+1})]$, $\tilde{\mathbf{A}}\mathbf{u}^{(n+1)} = \tilde{\mathbf{B}}\mathbf{u}^{(n)} + \tilde{\mathbf{c}} = \tilde{\mathbf{b}}$

FE ($\theta_1 = \theta_2 = 0$), **BE** ($\theta_1 = \theta_2 = 1$), **C-N** ($\theta_1 = \theta_2 = \frac{1}{2}$). **Dirichlet BCs** replace row of $\tilde{\mathbf{A}}$ with $\mathbf{e}_1$ or $\mathbf{e}_N \in \mathbb{R}^{1\times N}$ and row of $\tilde{\mathbf{b}}$ with Dirichlet BC.

Krylov Methods $\mathbf{K}_m(\mathbf{A},\, \mathbf{b}) = \mathbf{span}\left\{\mathbf{b},\ \mathbf{Ab},\ \mathbf{A}^2\mathbf{b},\ \ldots,\ \mathbf{A}^{m-1}\mathbf{b}\right\}$

Hessenberg factorisation: $\mathbf{AQ} = \mathbf{QH}$ for $\mathbf{Q}, \mathbf{H} \in \mathbb{R}^{n\times n}$. Reduced factorisation: $\mathbf{AQ}_m = \mathbf{Q}_{m+1}\bar{\mathbf{H}}_m$ for $\mathbf{Q}_m \in \mathbb{R}^{n\times m}$, $\bar{\mathbf{H}}_m \in \mathbb{R}^{m+1\times m}$.

$$\mathbf{q}_{j+1} = (\mathbf{Aq}_j - h_{1j}\mathbf{q}_1 - h_{2j}\mathbf{q}_2 - \cdots - h_{jj}\mathbf{q}_j)/h_{j+1,j} = \frac{1}{h_{j+1,j}}\left(\mathbf{Aq}_j - \sum_{i=1}^{j} h_{ij}\mathbf{q}_i\right)$$

Arnoldi's method apply Gram-Schmidt process to $\mathcal{K}_m(\mathbf{A},\, \mathbf{b})$.

$$\mathbf{q}_{j+1} = \frac{1}{\|\mathbf{v}_{j+1}\|}\left(\mathbf{Aq}_j - \sum_{i=1}^{j}(\mathbf{q}_i^\top \mathbf{Aq}_j)\, \mathbf{q}_i\right), \quad \mathbf{Aq}_j = \sum_{i=1}^{j}(\mathbf{q}_i^\top \mathbf{Aq}_j)\, \mathbf{q}_i + \|\mathbf{v}_{j+1}\|\mathbf{q}_{j+1}$$

$$\bar{\mathbf{H}}_1 = \begin{bmatrix} \mathbf{q}_1^\top \mathbf{Aq}_1 \\ \|\mathbf{v}_2\| \end{bmatrix}, \quad \bar{\mathbf{H}}_2 = \begin{bmatrix} \mathbf{q}_1^\top \mathbf{Aq}_1 & \mathbf{q}_1^\top \mathbf{Aq}_2 \\ \|\mathbf{v}_2\| & \mathbf{q}_2^\top \mathbf{Aq}_2 \\ & \|\mathbf{v}_3\| \end{bmatrix}, \quad \ldots, \quad \bar{\mathbf{H}}_m = \begin{bmatrix} \mathbf{H}_m \\ h_{m+1,m}\mathbf{e}_m^\top \end{bmatrix}$$

Left-multiply $\bar{\mathbf{H}}_m$ by $\mathbf{Q}_{m+1}$ to show $\mathbf{Q}_{m+1}\bar{\mathbf{H}}_m = \mathbf{Q}_m\mathbf{H}_m + h_{m+1,m}\mathbf{q}_{m+1}\mathbf{e}_m^\top$, where $\mathbf{H}_m \in \mathbb{R}^{m\times m}$. Left-multiply this result by $\mathbf{Q}_m^\top$ to show $\mathbf{Q}_m^\top \mathbf{AQ}_m = \mathbf{H}_m$.

Sparse Linear Systems

- Assume $\mathbf{x}^{(m)} \in \mathbf{x}^{(0)} + \mathcal{K}_m(\mathbf{A},\, \mathbf{r}^{(0)})$, for initial residual $\mathbf{r}^{(0)} = \mathbf{b} - \mathbf{Ax}^{(0)} = \beta\mathbf{q}_1$ where $\mathbf{q}_1$ is taken from the Gram-Schmidt process and $\beta = \|\mathbf{r}^{(0)}\|$.
- Solution form: $\mathbf{x}^{(m)} = \mathbf{x}^{(0)} + \mathbf{Q}_m\mathbf{y}_m$, solve $\mathbf{y}_m$ using $\mathbf{r}^{(m)} = \mathbf{b} - \mathbf{Ax}^{(m)} \perp \mathcal{W}_m$.
- FOM: $\mathcal{W}_m = \mathcal{K}_m(\mathbf{A},\, \mathbf{r}^{(0)})$.
- GMRES: $\mathcal{W}_m = \mathbf{A}\mathcal{K}_m(\mathbf{A},\, \mathbf{r}^{(0)})$.

Left preconditioning ($\mathbf{M}^{-1}\mathbf{Ax} = \mathbf{M}^{-1}\mathbf{b}$ where $\tilde{\mathbf{r}}^{(0)} = \mathbf{M}^{-1}\mathbf{r}^{(0)}$):

- Assume $\mathbf{x}^{(m)} \in \mathbf{x}^{(0)} + \mathcal{K}_m(\mathbf{M}^{-1}\mathbf{A},\, \tilde{\mathbf{r}}^{(0)})$, with $\mathbf{q}_1 = \tilde{\mathbf{r}}^{(0)}/\beta$ for $\beta = \|\tilde{\mathbf{r}}^{(0)}\|$.
- Arnoldi decomposition: $(\mathbf{M}^{-1}\mathbf{A})\,\mathbf{Q}_m = \mathbf{Q}_{m+1}\bar{\mathbf{H}}_m$.
- Solution form: $\mathbf{x}^{(m)} = \mathbf{x}^{(0)} + \mathbf{Q}_m\mathbf{y}_m$, solve $\mathbf{y}_m$ using $\tilde{\mathbf{r}}^{(m)} = \mathbf{M}^{-1}\mathbf{r}^{(m)} \perp \mathcal{W}_m$.
- FOM: $\mathcal{W}_m = \mathcal{K}_m(\mathbf{M}^{-1}\mathbf{A},\, \tilde{\mathbf{r}}^{(0)})$.
- GMRES: $\mathcal{W}_m = (\mathbf{M}^{-1}\mathbf{A})\,\mathcal{K}_m(\mathbf{M}^{-1}\mathbf{A},\, \tilde{\mathbf{r}}^{(0)})$.

Right preconditioning ($\mathbf{AM}^{-1}\tilde{\mathbf{x}} = \mathbf{b}$ where $\mathbf{M}\tilde{\mathbf{x}} = \mathbf{x}$):

- Assume $\mathbf{x}^{(m)} \in \mathbf{x}^{(0)} + \mathbf{M}^{-1}\mathcal{K}_m(\mathbf{AM}^{-1},\, \mathbf{r}^{(0)})$.
- Arnoldi decomposition: $(\mathbf{AM}^{-1})\,\mathbf{Q}_m = \mathbf{Q}_{m+1}\bar{\mathbf{H}}_m$.
- Solution form: $\mathbf{x}^{(m)} = \mathbf{x}^{(0)} + \mathbf{M}^{-1}\mathbf{Q}_m\mathbf{y}_m$, solve $\mathbf{y}_m$ using $\mathbf{r}^{(m)} \perp \mathcal{W}_m$.
- FOM: $\mathcal{W}_m = \mathcal{K}_m(\mathbf{AM}^{-1},\, \mathbf{r}^{(0)})$.
- GMRES: $\mathcal{W}_m = (\mathbf{AM}^{-1})\,\mathcal{K}_m(\mathbf{AM}^{-1},\, \mathbf{r}^{(0)})$.

No/right preconditioning GMRES solution minimises $\|\mathbf{r}^{(m)}\|$ over the affine space:

$$\mathbf{y}_m = \arg\min_{\mathbf{y}}\|\beta\mathbf{e}_1 - \bar{\mathbf{H}}_m\mathbf{y}\| \implies \mathbf{y}_m = \beta\bar{\mathbf{H}}_m^\dagger\mathbf{e}_1$$

Finite Volume Method

$$\underbrace{\frac{\partial u}{\partial t}}_{\text{unsteady}} + \underbrace{\boldsymbol{\nabla}\cdot(\mathbf{v}u)}_{\text{advection}} = \underbrace{\boldsymbol{\nabla}\cdot(\mathbf{D}\boldsymbol{\nabla}u)}_{\text{diffusion}} + \underbrace{R}_{\text{reaction}}$$

$$\partial_t u = -\boldsymbol{\nabla}\cdot\mathbf{q} + R,\ \mathbf{q} = \mathbf{v}u - \mathbf{D}\boldsymbol{\nabla}u$$

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} \approx \frac{\mathrm{d}\bar{u}_i}{\mathrm{d}t} = \frac{1}{V_i}\left(q_{w_i} - q_{e_i}\right) + \bar{R}_i$$

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{Au} + \mathbf{b}(t) \ \text{ or } \mathbf{G}(t,\ \mathbf{u}(t))$$

Newton Methods ($\mathbf{u}^{(n+1)} := \mathbf{x}^{(k)}$)

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{G}(\mathbf{u}) \implies \mathbf{F}\left(\mathbf{u}^{(n+1)}\right) = \mathbf{0}$$

$$\mathbf{F}\left(\mathbf{x}^{(k)}\right) + \mathbf{J}\left(\mathbf{x}^{(k)}\right)\left(\mathbf{x} - \mathbf{x}^{(k)}\right) = \mathbf{0}$$

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \mathbf{J}\left(\mathbf{x}^{(k)}\right)^{-1}\mathbf{F}\left(\mathbf{x}^{(k)}\right)$$

Newton (quad.): $m = 1$
Shamanskii (suplin.): m
Chord (lin.): $m = \infty$

Convergence Theory

$$\lim_{k\to\infty}\left\|\mathbf{x}^{(k)} - \mathbf{x}^*\right\| = 0 \implies \left\{\mathbf{x}^{(k)}\right\}_{k=0}^{\infty} \to \mathbf{x}^*$$

Cauchy if for all $\epsilon > 0$

$$\exists M : \left\|\mathbf{x}^{(k)} - \mathbf{x}^{(m)}\right\| < \epsilon : \forall k, m > M$$

Sequence converges ($\exists K > 0$):
Quad. $\left\|\mathbf{x}^{(k+1)} - \mathbf{x}^*\right\| \leqslant K\left\|\mathbf{x}^{(k)} - \mathbf{x}^*\right\|^2$
Suplin. $\left\|\mathbf{x}^{(k+1)} - \mathbf{x}^*\right\| \leqslant K\left\|\mathbf{x}^{(k)} - \mathbf{x}^*\right\|^\alpha$
Linear $\left\|\mathbf{x}^{(k+1)} - \mathbf{x}^*\right\| \leqslant \alpha\left\|\mathbf{x}^{(k)} - \mathbf{x}^*\right\|$
order $\alpha \in (0, 1)$, factor $\sigma \in (0, 1)$, $k \gg 1$.
Lipschitz continuous $\mathbf{J} \in \mathrm{Lip}_\gamma(D)$ if

$$\|\mathbf{J}(\mathbf{x}) - \mathbf{J}(\mathbf{y})\| \leqslant \gamma\|\mathbf{x} - \mathbf{y}\| : \forall \mathbf{x}, \mathbf{y} \in D$$

Useful Identities

$$\|\mathbf{x} + \mathbf{y}\| \leqslant \|\mathbf{x}\| + \|\mathbf{y}\|,\ \|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$$

$$\|\mathbf{A} + \mathbf{B}\| \leqslant \|\mathbf{A}\| + \|\mathbf{B}\|,\ \|\mathbf{Ax}\| \leqslant \|\mathbf{A}\|\|\mathbf{x}\|$$

$$\mathbf{x} = \int_0^1 \mathbf{x}\, \mathrm{d}t$$

$$\left\|\int_a^b \mathbf{F}(t)\, \mathrm{d}t\right\| \leqslant \int_a^b \|\mathbf{F}(t)\|\, \mathrm{d}t.$$

$$\mathbf{F}(\mathbf{x} + \mathbf{h}) - \mathbf{F}(\mathbf{x}) = \int_0^1 \mathbf{J}(\mathbf{x} + t\mathbf{h})\, \mathbf{h}\, \mathrm{d}t$$

$$\int_a^b \frac{\partial}{\partial t}\mathbf{F}(t)\, \mathrm{d}t = \mathbf{F}(b) - \mathbf{F}(a)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^b f(x,\, t)\, \mathrm{d}t\right) = \int_a^b \frac{\partial f(x,\, t)}{\partial x}\, \mathrm{d}t$$

Inexact Newton Method (≈quad.)

$$\tilde{\mathbf{J}}(\mathbf{x})\, \mathbf{e}_j = \left(\mathbf{F}(\mathbf{x} + \varepsilon\mathbf{e}_j) - \mathbf{F}(\mathbf{x})\right)/\varepsilon$$

FOM Solution and Residual Norm Derivations

No preconditioning

$$\begin{aligned}
\mathbf{Q}_m^\top \mathbf{r}^{(m)} &= \mathbf{0} \\
\mathbf{Q}_m^\top \left(\mathbf{b} - \mathbf{A}\mathbf{x}^{(m)}\right) &= \mathbf{0} \\
\mathbf{Q}_m^\top \left(\mathbf{b} - \mathbf{A}\mathbf{x}^{(0)} - \mathbf{A}\mathbf{Q}_m\mathbf{y}_m\right) &= \mathbf{0} \\
\mathbf{Q}_m^\top \left(\mathbf{r}^{(0)} - \mathbf{A}\mathbf{Q}_m\mathbf{y}_m\right) &= \mathbf{0} \\
\mathbf{Q}_m^\top \mathbf{r}^{(0)} - \left(\mathbf{Q}_m^\top \mathbf{A}\mathbf{Q}_m\right)\mathbf{y}_m &= \mathbf{0} \\
\beta\mathbf{Q}_m^\top \mathbf{q}_1 - \mathbf{H}_m\mathbf{y}_m &= \mathbf{0} \\
\mathbf{H}_m\mathbf{y}_m &= \beta\mathbf{e}_1
\end{aligned}$$

$$\begin{aligned}
\left\|\mathbf{r}^{(m)}\right\| &= \left\|\mathbf{b} - \mathbf{A}\mathbf{x}^{(m)}\right\| \\
&= \left\|\mathbf{b} - \mathbf{A}\mathbf{x}^{(0)} - \mathbf{A}\mathbf{Q}_m\mathbf{y}_m\right\| \\
&= \left\|\mathbf{r}^{(0)} - \mathbf{Q}_{m+1}\bar{\mathbf{H}}_m\mathbf{y}_m\right\| \\
&= \left\|\mathbf{Q}_m\mathbf{Q}_m^\top\mathbf{r}^{(0)} - \mathbf{Q}_{m+1}\bar{\mathbf{H}}_m\mathbf{y}_m\right\| \\
&= \left\|\beta\mathbf{Q}_m\mathbf{Q}_m^\top\mathbf{q}_1 - \left(\mathbf{Q}_m\mathbf{H}_m + h_{m+1,m}\mathbf{q}_{m+1}\mathbf{e}_m^\top\right)\mathbf{y}_m\right\| \\
&= \left\|\beta\mathbf{Q}_m\mathbf{e}_1 - \mathbf{Q}_m\mathbf{H}_m\mathbf{y}_m - h_{m+1,m}\mathbf{q}_{m+1}\mathbf{e}_m^\top\mathbf{y}_m\right\| \\
&= \left\|\beta\mathbf{Q}_m\mathbf{e}_1 - \beta\mathbf{Q}_m\mathbf{H}_m\mathbf{H}_m^{-1}\mathbf{e}_1 - h_{m+1,m}\mathbf{q}_{m+1}\mathbf{e}_m^\top\mathbf{y}_m\right\| \\
&= h_{m+1,m}\left\|\mathbf{q}_{m+1}\mathbf{e}_m^\top\mathbf{y}_m\right\| = h_{m+1,m}\left|\mathbf{e}_m^\top\mathbf{y}_m\right|
\end{aligned}$$

Left preconditioning

Same solution $\mathbf{y}_m$, with appropriate substitutions.

$$\begin{aligned}
\left\|\mathbf{r}^{(m)}\right\| &= \left\|\mathbf{M}\tilde{\mathbf{r}}^{(m)}\right\| \\
&= \left\|\mathbf{M}\mathbf{M}^{-1}\left(\mathbf{b} - \mathbf{A}\mathbf{x}^{(0)} - \mathbf{A}\mathbf{Q}_m\mathbf{y}_m\right)\right\| \\
&= \left\|\mathbf{M}\left(\mathbf{M}^{-1}\mathbf{r}^{(0)} - \left(\mathbf{M}^{-1}\mathbf{A}\right)\mathbf{Q}_m\mathbf{y}_m\right)\right\| \\
&= \left\|\mathbf{M}\left(\tilde{\mathbf{r}}^{(0)} - \mathbf{Q}_{m+1}\bar{\mathbf{H}}_m\mathbf{y}_m\right)\right\| \\
&= h_{m+1,m}\left\|\mathbf{M}\mathbf{q}_{m+1}\mathbf{e}_m^\top\mathbf{y}_m\right\|
\end{aligned}$$

Right preconditioning

Same solution $\mathbf{y}_m$ and residual norm, with appropriate substitutions.

GMRES Solution and Residual Norm Derivations

No preconditioning

$$\begin{aligned}
\left(\mathbf{A}\mathbf{Q}_m\right)^\top \mathbf{r}^{(m)} &= \mathbf{0} \\
\left(\mathbf{A}\mathbf{Q}_m\right)^\top \left(\mathbf{b} - \mathbf{A}\mathbf{x}^{(m)}\right) &= \mathbf{0} \\
\left(\mathbf{A}\mathbf{Q}_m\right)^\top \left(\mathbf{b} - \mathbf{A}\mathbf{x}^{(0)} - \mathbf{A}\mathbf{Q}_m\mathbf{y}_m\right) &= \mathbf{0} \\
\left(\mathbf{Q}_{m+1}\bar{\mathbf{H}}_m\right)^\top \left(\mathbf{r}^{(0)} - \mathbf{Q}_{m+1}\bar{\mathbf{H}}_m\mathbf{y}_m\right) &= \mathbf{0} \\
\bar{\mathbf{H}}_m^\top\mathbf{Q}_{m+1}^\top \left(\beta\mathbf{q}_1 - \mathbf{Q}_{m+1}\bar{\mathbf{H}}_m\mathbf{y}_m\right) &= \mathbf{0} \\
\beta\bar{\mathbf{H}}_m^\top\mathbf{Q}_{m+1}^\top\mathbf{q}_1 - \bar{\mathbf{H}}_m^\top\bar{\mathbf{H}}_m\mathbf{y}_m &= \mathbf{0} \\
\bar{\mathbf{H}}_m^\top\bar{\mathbf{H}}_m\mathbf{y}_m &= \beta\bar{\mathbf{H}}_m^\top\mathbf{e}_1
\end{aligned}$$

$$\begin{aligned}
\left\|\mathbf{r}^{(m)}\right\| &= \left\|\mathbf{b} - \mathbf{A}\mathbf{x}^{(m)}\right\| \\
&= \left\|\mathbf{b} - \mathbf{A}\mathbf{x}^{(0)} - \mathbf{A}\mathbf{Q}_m\mathbf{y}_m\right\| \\
&= \left\|\mathbf{r}^{(0)} - \mathbf{Q}_{m+1}\bar{\mathbf{H}}_m\mathbf{y}_m\right\| \\
&= \left\|\mathbf{Q}_{m+1}\mathbf{Q}_{m+1}^\top\mathbf{r}^{(0)} - \mathbf{Q}_{m+1}\bar{\mathbf{H}}_m\mathbf{y}_m\right\| \\
&= \left\|\beta\mathbf{Q}_{m+1}\mathbf{Q}_{m+1}^\top\mathbf{q}_1 - \mathbf{Q}_{m+1}\bar{\mathbf{H}}_m\mathbf{y}_m\right\| \\
&= \left\|\beta\mathbf{Q}_{m+1}\mathbf{e}_1 - \mathbf{Q}_{m+1}\bar{\mathbf{H}}_m\mathbf{y}_m\right\| \\
&= \left\|\mathbf{Q}_{m+1}\left(\beta\mathbf{e}_1 - \bar{\mathbf{H}}_m\mathbf{y}_m\right)\right\| = \left\|\beta\mathbf{e}_1 - \bar{\mathbf{H}}_m\mathbf{y}_m\right\|
\end{aligned}$$

Left preconditioning

Same solution $\mathbf{y}_m$, with appropriate substitutions.

$$\begin{aligned}
\left\|\mathbf{r}^{(m)}\right\| &= \left\|\mathbf{M}\tilde{\mathbf{r}}^{(m)}\right\| \\
&= \left\|\mathbf{M}\mathbf{M}^{-1}\left(\mathbf{b} - \mathbf{A}\left(\mathbf{x}^{(0)} + \mathbf{Q}_m\mathbf{y}_m\right)\right)\right\| \\
&= \left\|\mathbf{M}\mathbf{M}^{-1}\left(\mathbf{b} - \mathbf{A}\mathbf{x}^{(0)} - \mathbf{A}\mathbf{Q}_m\mathbf{y}_m\right)\right\| \\
&= \left\|\mathbf{M}\left(\mathbf{M}^{-1}\mathbf{r}^{(0)} - \left(\mathbf{M}^{-1}\mathbf{A}\right)\mathbf{Q}_m\mathbf{y}_m\right)\right\| \\
&= \left\|\mathbf{M}\left(\tilde{\mathbf{r}}^{(0)} - \mathbf{Q}_{m+1}\bar{\mathbf{H}}_m\mathbf{y}_m\right)\right\| \\
&= \left\|\mathbf{M}\mathbf{Q}_{m+1}\left(\beta\mathbf{e}_1 - \bar{\mathbf{H}}_m\mathbf{y}_m\right)\right\|
\end{aligned}$$

Right preconditioning

Same solution $\mathbf{y}_m$ and residual norm, with appropriate substitutions.