Trigonometry

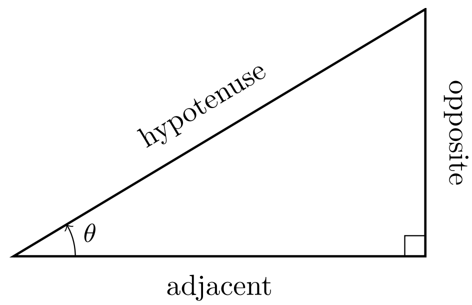
Tarang Janawalkar

Contents

1 Definitions

1.1 Trigonometric Functions

The trigonometric definitions below are derived from the following right triangle construction.



$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \csc(\theta) = \frac{\text{hypotenuse}}{\text{opposite}}$$
$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \sec(\theta) = \frac{\text{hypotenuse}}{\text{adjacent}}$$
$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} \qquad \cot(\theta) = \frac{\text{adjacent}}{\text{opposite}}$$

1.2 Inverse Trigonometric Functions

$$y = \arccos(x) \iff x = \cos(y) \qquad y = \operatorname{arccsc}(x) \iff x = \csc(y)$$
$$y = \arcsin(x) \iff x = \sin(y) \qquad y = \operatorname{arcsec}(x) \iff x = \sec(y)$$
$$y = \arctan(x) \iff x = \tan(y) \qquad y = \operatorname{arccot}(x) \iff x = \cot(y)$$

1.3 Properties as a Real Function

Let $n \in \mathbb{Z}$ be a constant.

Function	**Period**	**Parity**	**Domain**	**Range**
$\sin(x)$	2π	odd	$\mathbb{R}$	$[-1, 1]$
$\cos(x)$	2π	even	$\mathbb{R}$	$[-1, 1]$
$\tan(x)$	π	odd	$\mathbb{R}\backslash\left\{\left(n+\frac{1}{2}\right)\pi\right\}$	$\mathbb{R}$
$\cot(x)$	π	odd	$\mathbb{R}\backslash\left\{n\pi\right\}$	$\mathbb{R}$
$\sec(x)$	2π	even	$\mathbb{R}\backslash\left\{\left(n+\frac{1}{2}\right)\pi\right\}$	$(-\infty, -1] \cup [1, \infty)$
$\csc(x)$	2π	odd	$\mathbb{R}\backslash\left\{n\pi\right\}$	$(-\infty, -1] \cup [1, \infty)$

Function	**Parity**	**Domain**	**Range**
$\arcsin(x)$	odd	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\arccos(x)$	–	$[-1, 1]$	$[0, \pi]$
$\arctan(x)$	odd	$\mathbb{R}$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\text{arccot}(x)$	odd	$\mathbb{R}$	$[0, \pi]$
$\text{arcsec}(x)$	–	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] \setminus \{\frac{\pi}{2}\}$
$\text{arccsc}(x)$	odd	$(-\infty, -1] \cup [1, \infty)$	$[-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$

1.4 Symmetry

$$\sin(-x) = -\sin(x) \qquad \csc(-x) = -\csc(x)$$
$$\cos(-x) = \cos(x) \qquad \sec(-x) = \sec(x)$$
$$\tan(-x) = -\tan(x) \qquad \cot(-x) = -\cot(x)$$

1.5 Periodicity

Let $n \in \mathbb{Z}$ be a constant.

$$\sin(x + 2\pi n) = \sin(x) \qquad \csc(x + 2\pi n) = \csc(x)$$
$$\cos(x + 2\pi n) = \cos(x) \qquad \sec(x + 2\pi n) = \sec(x)$$
$$\tan(x + \pi n) = \tan(x) \qquad \cot(x + \pi n) = \cot(x)$$

2 Trigonometric Identities

2.1 Pythagorean Identities

$$\sin^2(x) + \cos^2(x) = 1$$

Dividing by either the sine or cosine function gives:

$$\tan^2(x) + 1 = \sec^2(x)$$
$$1 + \cot^2(x) = \csc^2(x)$$

2.2 Angle Sum Identities

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y) \qquad \csc(x \pm y) = \frac{1}{\sin(x)\cos(y) \pm \cos(x)\sin(y)}$$
$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y) \qquad \sec(x \pm y) = \frac{1}{\cos(x)\cos(y) \mp \sin(x)\sin(y)}$$
$$\tan(x \pm y) = \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)} \qquad \cot(x \pm y) = \frac{\cot(x)\cot(y) \mp 1}{\cot(x) \pm \cot(y)}$$

2.3 Double-Angle Identities

$$\sin(2x) = 2\sin(x)\cos(x) \qquad \csc(2x) = \frac{\sec(x)\csc(x)}{2}$$

$$\cos(2x) = \cos^2(x) - \sin^2(x) \qquad \sec(2x) = \frac{\sec^2(x)\csc^2(x)}{\csc^2(x) - \sec^2(x)}$$

$$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)} \qquad \cot(2x) = \frac{\cot^2(x) - 1}{2\cot(x)}$$

2.4 Power Reducing Identities

$$\sin^2(x) = \frac{1 - \cos(2x)}{2} \qquad \csc^2(x) = \frac{2}{1 - \cos(2x)}$$

$$\cos^2(x) = \frac{1 + \cos(2x)}{2} \qquad \sec^2(x) = \frac{2}{1 + \cos(2x)}$$

$$\tan^2(x) = \frac{1 - \cos(2x)}{1 + \cos(2x)} \qquad \cot^2(x) = \frac{1 + \cos(2x)}{1 - \cos(2x)}$$

2.5 Half Angle Identities

$$\sin\left(\frac{x}{2}\right) = (-1)^{\lfloor \frac{x}{2\pi} \rfloor}\sqrt{\frac{1 - \cos(x)}{2}} \qquad \csc\left(\frac{x}{2}\right) = (-1)^{\lfloor \frac{x}{2\pi} \rfloor}\sqrt{\frac{2\sec(x)}{\sec(x) - 1}}$$

$$\cos\left(\frac{x}{2}\right) = (-1)^{\lfloor \frac{x+\pi}{2\pi} \rfloor}\sqrt{\frac{1 + \cos(x)}{2}} \qquad \sec\left(\frac{x}{2}\right) = (-1)^{\lfloor \frac{x+\pi}{2\pi} \rfloor}\sqrt{\frac{2\sec(x)}{\sec(x) + 1}}$$

$$\tan\left(\frac{x}{2}\right) = \frac{1 - \cos(x)}{\sin(x)} \qquad \cot\left(\frac{x}{2}\right) = \frac{\sin(x)}{1 - \cos(x)}$$

2.6 Werner Identities

$$\begin{aligned}
2\sin(x)\sin(y) &= \cos(x - y) - \cos(x + y) \\
2\cos(x)\cos(y) &= \cos(x - y) + \cos(x + y) \\
2\sin(x)\cos(y) &= \sin(x - y) + \sin(x + y) \\
-2\cos(x)\sin(y) &= \sin(x - y) - \sin(x + y)
\end{aligned}$$

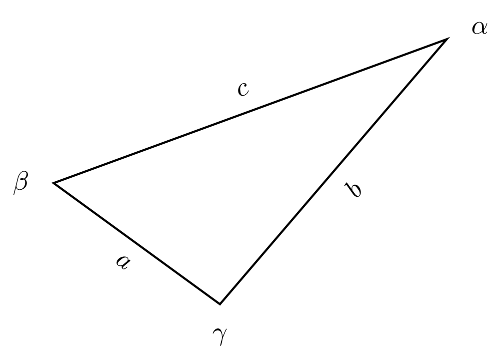
2.7 Prosthaphaeresis Identities

$$\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\sin(x) - \sin(y) = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\cos(x) - \cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

2.8 Inverse Reciprocal Identities

$$\arcsin\left(\frac{1}{x}\right) = \operatorname{arccsc}(x) \qquad \operatorname{arccsc}\left(\frac{1}{x}\right) = \arcsin(x)$$
$$\arccos\left(\frac{1}{x}\right) = \operatorname{arcsec}(x) \qquad \operatorname{arcsec}\left(\frac{1}{x}\right) = \arccos(x)$$
$$\arctan\left(\frac{1}{x}\right) = \operatorname{arccot}(x) \qquad \operatorname{arccot}\left(\frac{1}{x}\right) = \arctan(x)$$

3 Geometric Identities



3.1 Area of a Triangle

$$A = \frac{1}{2}ab\sin(\gamma)$$

3.2 Sine Rule

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$$

3.3 Cosine Rule

$$a^2 = b^2 + c^2 - 2bc\cos(\alpha)$$

3.4 Tangent Rule

$$\frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{\alpha+\beta}{2}\right)} = \frac{a-b}{a+b}$$

3.5 Mollweide's Identity

$$\frac{b-c}{a} = \frac{\sin\left(\frac{\beta-\gamma}{2}\right)}{\cos\left(\frac{\alpha}{2}\right)}$$

3.6 Newton's Identity

$$\frac{b+c}{a} = \frac{\cos\left(\frac{\beta-\gamma}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$$

4 The Unit Circle

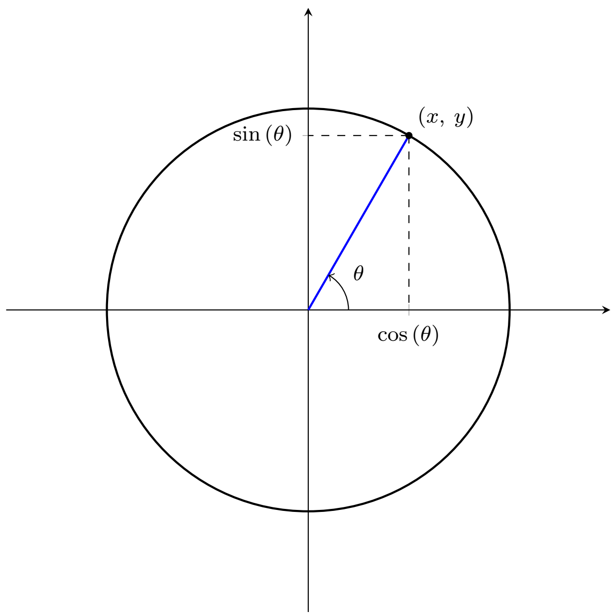
The unit circle C is defined as the set of all points $(x,\ y)$ which satisfy the equation

$$x^2 + y^2 = 1,$$

or, mathematically,

$$C = \left\{(x,\ y) \mid x^2 + y^2 = 1\right\}.$$

A graph of this function is shown below.

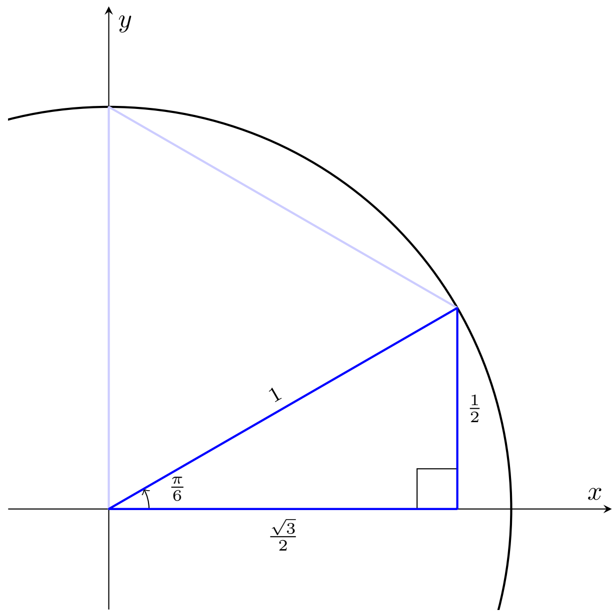


A point that lies on this circle can be described by the angle it forms with the positive x-axis. This point has the components:

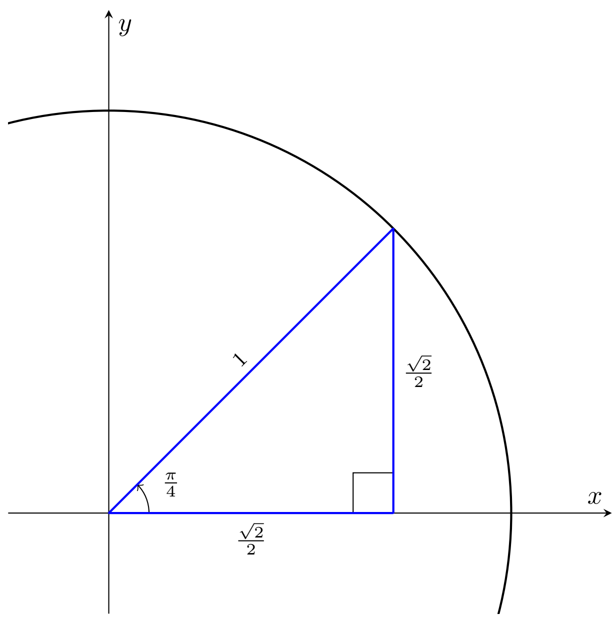
$$(x,\ y) = (\cos(\theta),\ \sin(\theta))\,.$$

4.1 Special Triangles

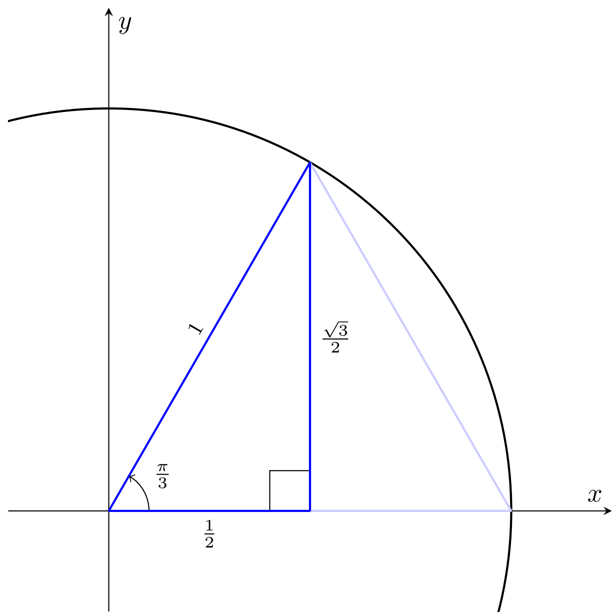
The following right triangle constructions provide coordinates of points that form an angle of $\pi/6$, $\pi/4$, or $\pi/3$ from the positive x-axis.



y
1
$\frac{1}{2}$
$\frac{\pi}{6}$
x
$\frac{\sqrt{3}}{2}$



y
1
$\frac{\sqrt{2}}{2}$
$\frac{\pi}{4}$
x
$\frac{\sqrt{2}}{2}$



This is summarised in the table below.

Angle from Positive x-axis	**Horizontal Component**	**Vertical Component**
$\frac{\pi}{6} = 30°$	$\frac{\sqrt{3}}{2} \approx 0.866\,025$	$\frac{1}{2} = 0.5$
$\frac{\pi}{4} = 45°$	$\frac{\sqrt{2}}{2} \approx 0.707\,107$	$\frac{\sqrt{2}}{2} \approx 0.707\,107$
$\frac{\pi}{6} = 30°$	$\frac{1}{2} = 0.5$	$\frac{\sqrt{3}}{2} \approx 0.866\,025$

4.2 2-Argument Arctangent

Consider a point $(x,\ y)$ that lies on the Cartesian plane. The angle measure from the positive x-axis and the ray from the origin to the point $(x,\ y)$ is defined by the 2-argument arctangent function:

$$\operatorname{atan2}(y,\ x) = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x > 0, \\ \arctan\left(\frac{y}{x}\right) + \pi & \text{if } x < 0 \text{ and } y \geqslant 0, \\ \arctan\left(\frac{y}{x}\right) - \pi & \text{if } x < 0 \text{ and } y < 0, \\ +\frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0, \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0, \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

This definition allows an angle measure to be obtained in any quadrant of the Cartesian plane, on the interval $(-\pi, \pi]$. When x and y describe the real and imaginary parts of a complex number

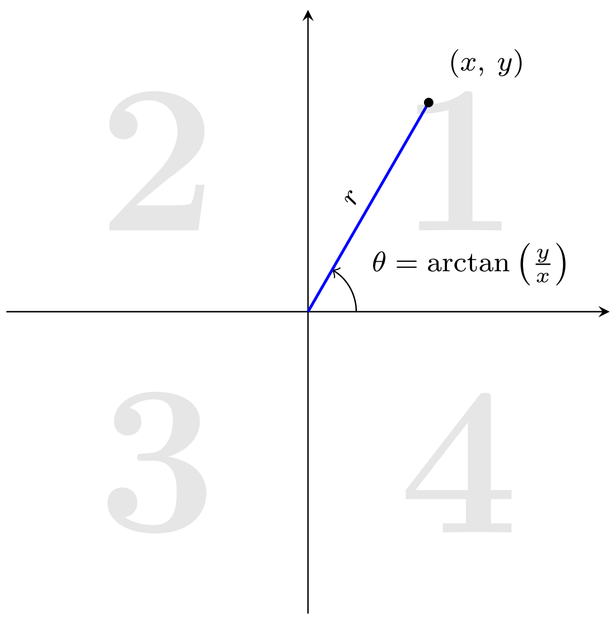
$z = x + yi$, the 2-argument arctangent is precisely the principal argument of z:

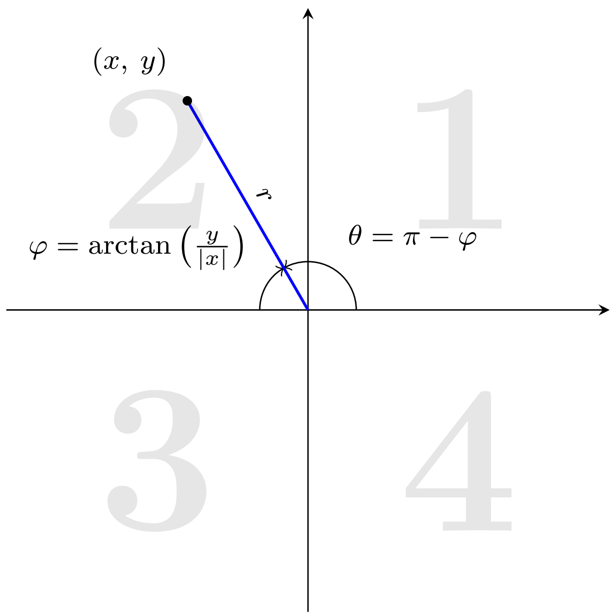
$$\mathrm{Arg}\,(z) = \mathrm{atan2}\,(y,\ x)\,,$$

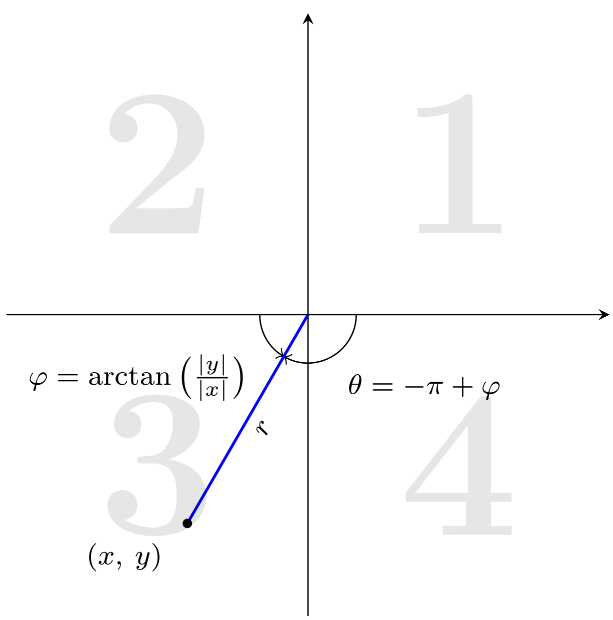
defined on the same interval $(-\pi, \pi]$. Note that the set of all arguments of a complex number is defined by the general argument:

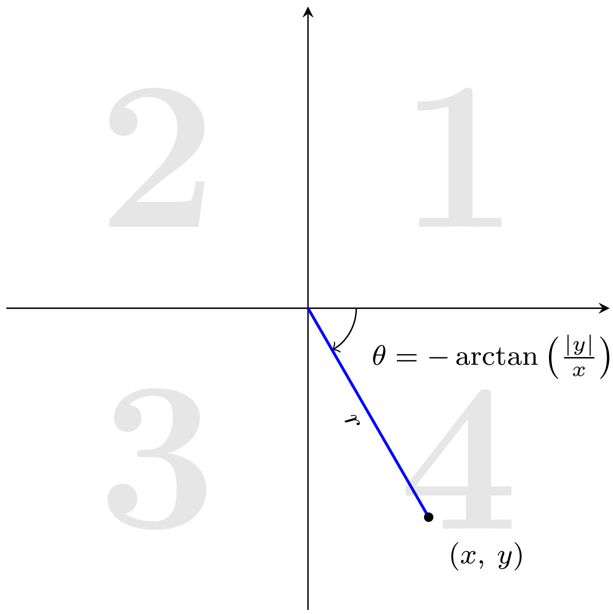
$$\arg\,(z) = \{\mathrm{Arg}\,(z) + 2\pi n \,|\, n \in \mathbb{Z}\}\,.$$

These angles can be geometrically determined by considering the quadrant in which the point lies.




$(x,\ y)$
2
1
r
$\varphi = \arctan\left(\frac{y}{|x|}\right)$
$\theta = \pi - \varphi$
3
4


2
1
$\varphi = \arctan\left(\frac{|y|}{|x|}\right)$
$\theta = -\pi + \varphi$
r
3
4
$(x,\ y)$



This is summarised below.

Quadrant	**Sign** x	**Sign** y	**Angle Measure** θ
First	$+$	$+$	$\arctan\left(\frac{y}{x}\right)$
Second	$-$	$+$	$\pi - \arctan\left(\frac{y}{\lvert x\rvert}\right)$
Third	$-$	$-$	$-\pi + \arctan\left(\frac{\lvert y\rvert}{\lvert x\rvert}\right)$
Fourth	$+$	$-$	$-\arctan\left(\frac{\lvert y\rvert}{x}\right)$