Multivariate Normal dist$^{\underline{n}}$

$\underline{X} = (X_1 \; X_2 \; \cdots\cdots \; X_n)^T$ is vector which is an ~~d~~ r.v. following Normal

$$f_{\underline{X}}(\underline{x}) = \frac{1}{(2\pi)^{n/2}\sqrt{|\Sigma|}} \, e^{-\frac{1}{2}(\underline{x}-\underline{\mu})^T \Sigma^{-1}(\underline{x}-\underline{\mu})}$$

$$\underline{x} \in \mathbb{R}^N$$

Here, $E(\underline{X}) = \underline{\mu}$

and Σ is a variance covariance matrix of elements of $\underline{X}$

$$\Sigma = \begin{pmatrix} V(X_1) & cov(X_1 X_2) & \cdots\cdots & Cov(X_1 X_n) \\ & V(X_2) & \cdots\cdots & cov(X_2 X_n) \\ & & & \\ & & & V(X_n) \end{pmatrix}$$

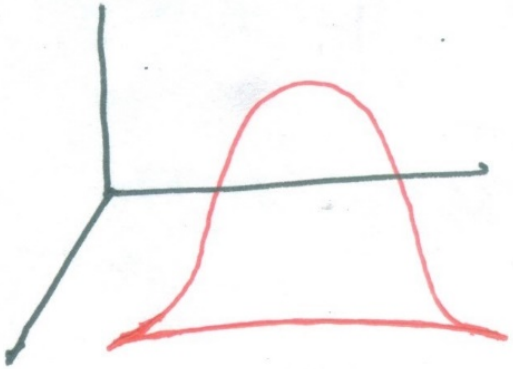
Example $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$f_{\underline{X}}(\underline{x}) = \frac{1}{(2\pi)\sqrt{|\Sigma|}} e^{-\frac{1}{2}(\underline{x}-\underline{\mu})^T \Sigma^{-1} (\underline{x}-\underline{\mu})}$$

Exponent $$\left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}\right]^T \begin{pmatrix} V(x_1) & cov(x_1, x_2) \\ & V(x_2) \end{pmatrix}^{-1} \left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}\right]$$

$$\left[(x_1 - \mu_1) \quad (x_2 - \mu_2)\right] \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}^{-1} \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix}$$

$$\underline{X} \sim N_n(\underline{\mu}, \Sigma)$$



Results :

1. $\underline{X} \sim N_n(\underline{\mu}, \Sigma)$

Then for any $\underline{l}$, $\underline{l}'\underline{X} \sim N_1(\underline{l}'\underline{\mu}, \underline{l}'\Sigma\underline{l})$

$$E(\underline{l}'\underline{x}) = \underline{l}'E(\underline{x}) = \underline{l}'\underline{\mu}$$

$$V(\underline{l}'x) = \underline{l}'V(x)\underline{l} = \underline{l}'\Sigma\underline{l}$$

→ For a multivariate Normal, each component separately is univariate Normal.

$$\underline{l} = (0\ 0 \cdots 0\ 1\ 0 \cdots 0)$$

So, If $\exists$ $\underline{\mu}$ and Σ such that for any $\underline{l}$, $\underline{l}'\underline{X} \sim N(\underline{l}'\underline{\mu}, \underline{l}'\Sigma\underline{l})$ then $\underline{X} \sim N_n(\underline{\mu}, \Sigma)$.

Variance covariance matrix

If $\Sigma = \text{diag}(\sigma_1 \quad \sigma_2 \; \cdots\cdots \; \sigma_n)$

then • $x_1, x_2, \cdots\cdots, x_n$ are independent

• $\text{cov}(x_i, x_j) = 0$

• $x_1, x_2 \cdots\cdots x_n$ are n independently Normally distributed r.v.s having means $\mu_1, \mu_2 \cdots\cdots \mu_n$ and variance 1, then

$$\underline{X} = (x_1 \cdots\cdots x_n)^T \sim N_n(\underline{\mu}, I)$$

Again if $\underline{\mu} = \underline{0}$

then $\underline{X} \sim N_n(\underline{0}, I)$

So, $\left.\begin{array}{l} x_1 \sim N(0,1) \\ x_2 \sim N(0,1) \\ \vdots \\ x_n \sim N(0,1) \end{array}\right\}$ independent then $\underline{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \sim N_n(\underline{0}, I)$

Let $\underline{U} = (x_1\ x_2 \cdots x_r)^T$

and $\underline{V} = (x_{r+1}\ x_{r+2} \cdots x_n)^T$

Thus, $\underline{X} = \begin{pmatrix} \underline{U} \\ \underline{V} \end{pmatrix}$

$$\underline{X} \sim N_n(\underline{\mu}, \Sigma)$$

Then $\underline{U} \sim N_r(\underline{\mu}_1, \Sigma_1)$

and $\underline{V} \sim N_{n-r}(\underline{\mu}_2, \Sigma_2)$

Here, $\underline{\mu}_1 = (\mu_1\ \mu_2 \cdots \mu_r)^T$

and $\underline{\mu}_2 = (\mu_{r+1}\ \mu_{r+2} \cdots \mu_n)^T$ Thus, $\underline{\mu} = \begin{pmatrix} \underline{\mu}_1 \\ \underline{\mu}_2 \end{pmatrix}$

$$
\Sigma = \left(\begin{array}{ccccc|ccc}
\sigma_1 & \sigma_{12} & \sigma_{13} & \cdots & \sigma_{1r} & \sigma_{1\,r+1} & \cdots & \sigma_{1n} \\
 & \sigma_2 & \sigma_{23} & \cdots & \sigma_{2r} & \sigma_{2\,r+1} & \cdots & \sigma_{2n} \\
 & & & & \vdots & & & \\
 & & & & \sigma_r & \sigma_{r\,r+1} & \cdots & \sigma_{rn} \\
\hline
 & & & & & \sigma_{r+1} & \cdots & \sigma_{r+1\,n} \\
 & & & & & & & \vdots \\
 & & & & & & & \sigma_n
\end{array}\right)
$$

$$
\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}
$$

If $\underline{U}, \underline{V}$ are independent then $\Sigma_{12} = 0$

Let $C_{m\times n}$ matrix

Define $\underline{Y}_{m\times 1} = \underset{\sim}{C}_{m\times n} \underline{X}_{n\times 1}$

Then $E(\underline{Y}) = C_{m\times n} E(\underline{X})$

$= C\underline{\mu}$

and $V(\underline{Y}) = C \Sigma C^T$

$\Rightarrow \underline{Y} \sim N_m (C\underline{\mu}, C\Sigma C^T)$

Consider a linear fn. $\underline{\ell}^T \underline{Y}$

$\underline{\ell}^T \underline{Y} \sim N_1 (\underline{\ell}' C \underline{\mu}, \underline{\ell}' C \Sigma C' \ell)$

[$\underline{\ell}' \underset{\sim}{C} \underline{X}$ is also a linear fn. defined on $\underline{X}$

$\underline{p}' \underline{X} \sim N_1 (\underline{p}' \underline{\mu}, \underline{p}^T \Sigma \underline{p})$ or $N_1 (\underline{\ell}^T C \underline{\mu}, \ell^T C \Sigma C^T \ell)$ $\boxed{p = C^T \underline{\ell}}$

$$\left.\begin{array}{l} \underline{U}_1 \sim N_n(\underline{\mu}_1, \Sigma_1) \\ \underline{U}_2 \sim N_n(\underline{\mu}_2, \Sigma_2) \end{array}\right\rangle \text{indep}$$

Then, $\underline{U}_1 + \underline{U}_2 = \underline{U} \sim N_n(\underline{\mu}, \Sigma)$

$$\underline{\mu} = (\underline{\mu}_1 + \underline{\mu}_2) \quad \Sigma = \cancel{\begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}} \; \Sigma_1 + \Sigma_2$$

Similarly

$$\left.\begin{array}{l} X_1 \sim N_1(\mu_1, \sigma_1^2) \\ X_2 \sim N_1(\mu_2, \sigma_2^2) \end{array}\right\rangle \text{indep.}$$

$$Y = X_1 + X_2 \sim N_1(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

Mixture of Normal ?

Important Result (CLT)

$X_1, X_2, \ldots\ldots, X_n$ are iid r.v.s with $E(x_i) = \mu$
and $V(x_i) = \sigma 2$

Define $S_n = \dfrac{\sum_i x_i}{n}$

$$\frac{S_n - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$ for large value of n..

Example $X_i \sim \text{Rec}(0,1)$

$$E(x_i) = \frac{1}{2} \text{ and } V(x_i) = \frac{1}{12}$$

$$S_n = \frac{1}{n}\sum_{i=1}^{n} x_i \quad , \quad S_n - \mu = \frac{1}{n}\sum_{i=1}^{n} x_i - \frac{1}{2}$$

$$= \frac{1}{n}\left(\sum_{i=1}^{n} x_i - \frac{n}{2}\right)$$

$$\frac{\sqrt{n}(S_n - \mu)}{\sigma} = \sqrt{12}\sqrt{n}\,\frac{1}{n}\left(\sum_{i=1}^{n} x_i - \frac{n}{2}\right)$$

$$= \frac{\sqrt{12}}{\sqrt{n}}\left(\sum_{i=1}^{n} x_i - \frac{n}{2}\right)$$

$$= \sqrt{\frac{12}{n}}\left(\sum_{i=1}^{n} x_i - \frac{n}{2}\right)$$

<u>For $= n = 12$</u>

$$\left(\sum_{i=1}^{12} x_i - 6\right) \sim N(0,1)$$

<u>Note:</u> 1. Draw 12 random samples from Uni $(0,1)$ [≡ random numbers [0,1]]

2. (Sum of random No.s $- 6$) $\sim N(0,1)$

3. 2 provides one sample of standard Gaussian

$$X_1 \sim \text{Uni}\,(0,1), \quad X_2 \sim \text{Uni}\,(0,1) \quad \text{indep} \quad \left[\equiv 2 \text{ random numbers } [0,1] \right]$$

$$U_1 = \sqrt{-2\ln X_1}\,\cos 2\pi X_2, \quad U_2 = \sqrt{-2\ln X_2}\,\sin 2\pi X_2 \quad \text{indep } N(0,1)$$

Note: 2 random numbers from $[0,1] \longrightarrow$ 2 random samples from $N(0,1)$.