AQ1.7

2) Ravi Anand went to buy fruits for himself from a shop. There was a basket of fruits containing 10 apples and 20 oranges, out of which 3 apples and 5 oranges are rotten. If he chooses two fruits randomly, what is the probability that either both are oranges or both are not rotten? ***1 point***

- ○ $\frac{316}{485}$
- ◉ $\frac{316}{435}$
- ○ $\frac{217}{485}$
- ○ $\frac{158}{435}$

AQ1.8

Let A = both the chosen fruits are oranges and B= both are non rotten. Then, find P(AUB) which is P(A)+P(B)-P(A & B) . P(A)= 20/30*19
P(B) = 22/30 * 21/29
P(A &B) = 15/30 * 14/29.

1) The hats of 5 persons are identical and they get mixed up. Each person picks a hat at random. What is the probability that none of them will get their own hat? ***1 point***

- ○ $\frac{22}{120}$
- ◉ $\frac{44}{120}$
- ○ $\frac{66}{120}$
- ○ $\frac{11}{120}$

AQ1.10

1) During an IITM online exam, Shalini answered only 2 questions and her answers can either be correct or incorrect with equal probability. The correctness of the two answers are independent. What is the probability that both of her answers are correct, given that at least one is correct? ***1 point***

- ○ $\frac{2}{3}$
- ◉ $\frac{1}{3}$
- ○ $\frac{1}{2}$
- ○ $\frac{1}{4}$

5) Each set of a 3-set badminton match between Himanshu and Hari is won with equal probability by either player. What is the conditional probability that Himanshu wins more sets than Hari given that Himanshu wins the first set? (Answer the question cor- rect to two decimal points.)

0.75

AQ1.15

3) In an examination conducted by IITM on the topic Statistics, there are 5 multiple select questions (one or more options can be correct) and each question has four options. Aman decided to mark the answer at random. Find the probability that he gets exactly two questions correct (assume that the answer of each question is independent of the other questions). ***1 point***

- ○ $\frac{5!}{2! \times 3!}(\frac{1}{15})^2 \times (\frac{14}{15})^3$
- ○ $\frac{5!}{2! \times 3!}(\frac{14}{15})^2 \times (\frac{1}{15})^3$
- ○ $\frac{5!}{2! \times 3!}(\frac{1}{5})^2 \times (\frac{4}{5})^3$
- ○ $\frac{5!}{2! \times 3!}(\frac{4}{5})^2 \times (\frac{1}{5})^3$

Hint: one or more options removes the possibility of all options unselected, so number of options are (16-1)=15.

Practice1

9. A person has bought a bed from an online furniture store. The seller delivers the disassembled bed parts along with some screws to assemble it. The probability of a screw being defective is 0.1 independent of all other screws. To compensate for the manufacturing error, the seller sends two extra screws in the package where the bed needs exactly 8 screws to assemble. What is the probability that the buyer will be able to assemble the bed? (Enter the answer correct to 4 decimal accuracy)

Graded1

1. The probability that an electrical machine will work more than 5 years but less than 8 years is 0.6 and the probability that it will work at least 8 years is 0.1. What is the probability that the machine will work for more than 5 years? [1 mark]

5. Two friends Ravi and Sonali are playing a game in which they are hitting a target in rounds. In each round, both hit the target independent of each other with a probability of 0.5. The first one who hits the target three times wins the game. What is the probability that in the fifth round Sonali wins the game? [2 marks]

Hint: Sonali not achieving target doesn't mean Ravi achieves target.

7. In a town, 60% of the residents are eligible for voting in an election but only 80 % of the eligible residents voted in the election. A person is randomly selected from the town. What is the conditional probability that the person is eligible for the voting given that he or she did not vote? [2 mark]

Hint: If person is not eligible, he cannot vote.

9) Three different tasks were assigned to three persons A, B, and C. Previous records show that A, B, and C will complete their tasks independent of each other with probabilities of $\frac{1}{2}, \frac{2}{3}$ and $\frac{3}{4}$ respectively. If it is known that exactly two of them have completed their tasks, then what is the conditional probability that A has not completed his task? ***3 points***

Hint: While finding probability of 2 completing task, multiply probability of the third person NOT completing also.

10. There are twenty boxes out of which exactly fifteen contains gifts and five are empty. Five boxes are removed randomly. Now, a person selects one box from the remaining boxes, then what is the probability that the person selects the empty box? [3 marks] (Hint: Consider all the cases of removing empty boxes and apply the law of total probability)

Week2 Tutorial 3

4) Let X ~ Geometric(p) and

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is an even number} \\ \frac{x+1}{2} & \text{if } x \text{ is an odd number} \end{cases}$$

Find the probability that $f(X) = k$
where k is in the range of $f(X)$.

○ $(1-p)^{2(k-1)}p(2-p)$

○ $(1-p)^{k-1}p(2-p)$

○ $(1-p)^{2k-1}p(2-p)$

○ $(1-p)^{2k-1}p(1-p)$

P(f(X)=1) = P(X= 1) or P(X= 2)
= p + (1-p)p (Since, X~ Geometric(p))
= p(2-p)

P(f(X)=2) = P(X= 3) or P(X= 4)
= $(1-p)^2 p + (1-p)3p$
=$(1-p)^2 p(2-p)$

P(f(X)=3) = P(X= 5) or P(X= 6)
=$(1-p)^4 p + (1-p)^5 p = (1-p)^4 p(2-p)$ and so on

Now see the pattern (you can find for more f(x)) and generalize the formula.

Graded2

7) A shopkeeper sells mobile phones. The demand for mobile phone follows a Poisson distribution with mean 4.6 per week. The shopkeeper has 5 mobile phones in his shop at the beginning of a week. Find the probability that this will not be enough to satisfy the demand for mobile phones in that week. Enter your answer correct up to two decimals accuracy.

0.31

Hint: Find probability that the demand exceeds 5, P(X > 5) = 1 – P(X<=5)

10) Suppose the probability that any given person will independently believe a tale about the existence of a parallel universe is 0.6. What is the probability that the eighth person to hear this tale about existence of a parallel universe is the fifth one to believe it? ***1 point***

Hint: Negative binomial. N = 8, p = 0.6, r = 5. Apply this to $N-1_{C_{r-1}} p^r (1-p)^{N-r}$

Week3-Tutorial1

3) An urn contains n tickets numbered from 1 to n. Two tickets are drawn (without replacement). Let X denote the smaller, Y the larger of the two numbers so obtained. Describe the joint distribution of X and Y.

X denotes the smaller of two numbers.Hence, range of X will be {1, 2, 3, ... , n − 1}.
Y denotes the larger of two numbers. Hence, range of Y will be {2, 3, 4, ... , n}.
Let f_{XY} be the joint probability mass function of X and Y .

If Y can take values between 1 and 10 (say), X can take 1 less value. Both are uniform distributions. Thus, P(Y) = 1/n and P(X) = 1/n-1. These are independent and hence probability is 1/n*(n-1). Since the X or Y can occur first, the result must be multiplied by 2. Thus, final answer is

$$f_{XY}(x,y) = \begin{cases} 0 & \text{if, } x \geq y \\ \frac{2}{n(n-1)} & \text{if, } x < y \end{cases}$$

Practice3

5) Let X and Y be two independent random variables with PMFs

$$f_X(k) = f_Y(k) = \begin{cases} \frac{1}{6} & \text{for } k = 1,2,3,4,5,6. \\ 0 & \text{otherwise} \end{cases}$$

Define $Z = X - Y$. Find the value of $f_Z(3)$

In such problems, represent distribution of z in terms of X and Y.

In this case, Y = X – Z. Thus, fz(3) = fxy(x=4, y=1) + fxy(x=5, y=2) + fxy(x=6, y=3)

Week4-Tutorial2

12) Suppose the random variable Nrepresenting the number of customers who walk into a boutique on any particular day follows the Poisson distribution with parameter λ equal to 12. Assume that each customer who visits the boutique will make a purchase with probability 0.8 independent of other customers and independent of the value of N. Let X denote the number of customers who make a purchase in a day. Find $E(X)$.

N is Poisson distribution Poisson(12). X|N = Binomial(n, 0.8). Thus, X is Poisson(0.8*12) = Poisson(9.6). Expectation of X is hence 9.6 (equals λ)

This is because Poisson superimposed with Binomial gives a Poisson with lambda multiplied by the Binomial probability.

Practice4

3) The number of spam messages (X) sent to a server in a day has Poisson distribution with parameter $\lambda = 21$. Each spam message independently has a probability of $p = \frac{1}{3}$ of not being detected by the spam filter. Let Y denote the number of spam messages detected by the filter in a day. Calculate the expected value of $X + Y$.

X is Poisson(21)

Y is Poisson(21 * 2/3) = Poisson(14) {Refer to the above problem.}

E[X + Y] = E[X] + E[Y] = 21 + 14 = 35

Graded4

6) Find a bound on the probability that on a particular day, number of reservations made will lie in between 6 and 14 ***2 points*** using Chebyshev's inequality.

$$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$$

Here kσ = 4.

So, RHS is 1-1/8 = 7/8

AQ5.5

Consider the following probability density function f_X of a random variable X

$$f_X(x) = \begin{cases} kx & 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

8) Find the value of $F(0.5)$ where F is the CDF of X. (Write your answer correct to two decimal places.)

0.25

First, integrate kx from 0 to 1 and equate to 1, to obtain the value of k to be 2. Thus,

Now, F(0.5) = P(X < 0.5) = integral of 2x from 0 to 0.5. This results in 0.25.

AQ5.7

1) Let X be a continuous random variable with the following probability density function:

$$f_X(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & \text{Otherwise} \end{cases}$$

Find the probability distribution function of $Y = X^2$.

Method 1

$f = 3x^2$

$g = y = x^2 \Rightarrow +\sqrt{y} = x = g^{-1}$

Now $g' = 2x$.

So $g'(g^{-1}) = 2(\pm\sqrt{y})$

Since we're dealing with $+x$ axis

Denominator $g'(g^{-1}) = 2\sqrt{y}$

Numerator; $f(g^{-1}) = 3(\cancel{\pm}\sqrt{y})^2$

$= 3y$.

So PDF $= \dfrac{3y}{2\sqrt{y}} = \dfrac{3\sqrt{y}}{2}$

Here, we applied PDF $= \dfrac{1}{g'(g^{-1})} f(g^{-1})$

Method 2

CDF $= \int 3x^2$ between limits obtained from given function $Y<(y=x^2)$

$Y<(y=x^2) \Rightarrow x^2<y \Rightarrow x<\sqrt{y}$ (ignore $-\sqrt{y}$)

CDF $= \int_0^{\sqrt{y}} 3x^2 = \dfrac{3x^3}{3}\Big|_0^{\sqrt{y}} = \cancel{3}\, y^{3/2}$

Now derivative of above CDF

$\dfrac{3}{2} y^{1/2} = \dfrac{3\sqrt{y}}{2}$

6) Let $X \sim \text{Normal}(\mu, \sigma^2)$. What will be the distribution of $aX + b$ where a and b are constants?

$$X \sim \text{Normal}(b + a\mu, a^2\sigma^2)$$

Week5-Tutorial2

6) Let $X \sim$Uniform $[0, 20]$. Find the CDF of $Y = \max\{4, \min\{8, X\}\}$.

Let

$$g(x) = \max\{4, min\{8, X\}\}$$

$$g(x) = \begin{cases} \max\{4, x\} & \text{for } 0 < x < 8 \\ \max\{4, 8\} & \text{for } 8 < x < 20 \end{cases}$$

$$\Rightarrow g(x) = \begin{cases} 4 & \text{for } 0 < x < 4 \\ x & \text{for } 4 < x < 8 \\ 8 & \text{for } 8 < x < 20 \end{cases}$$

$\text{Support}(Y) = [4, 8]$

For $-\infty < y < 4$, $F_Y(y) = 0$

For $y = 4$, $F_Y(y) = P(Y \leq y) = P(0 < x < 4) = \frac{4}{20}$

For $4 < y < 8$, $F_Y(y) = P(Y \leq y) = \frac{4}{20} + \frac{y}{20} - \frac{4}{20} = \frac{y}{20}$

For $y \geq 8$, $F_Y(y) = 1$

$$F_Y(y) = \begin{cases} 0 & \text{for } -\infty < y < 4 \\ \frac{y}{20} & \text{for } 4 \leq y < 8 \\ 1 & \text{for } y \geq 8 \end{cases}$$

4) If $X \sim \text{Normal}(10, 25)$, what is the value of $E[2X^2]$?

Use formula Variance = E[X^2] - (Mean)^2

Variance and Mean is 25 and 10 respectively. Find E[X^2] and multiply by 2.

Graded5

7) The number of days in advance by which airline tickets are purchased by travelers is exponentially distributed with an average of 28 days. If there is an 80% chance that a traveler will purchase tickets fewer than d days in advance, then what is the value of d?

Write your answer to the nearest integer.

45

Here, mean is 28 days. This implies $1/\lambda$ = 28 days. Now, use the exponential formula like below

$\lambda = \frac{1}{28}$

PDF of exponential dist is $\lambda e^{-\lambda x}$.

Thus probability is $\int \frac{1}{28} \cdot e^{-\frac{x}{28}}$

Integrate between 0 and d you should get 0.8.

$$\int_0^d \frac{1}{28} e^{-\frac{x}{28}} = 0.8 \Rightarrow -e^{-\frac{x}{28}}\Big|_0^d = 0.8 \Rightarrow$$

$$1 - e^{-\frac{x}{28}} = 0.8 \Rightarrow -\frac{x}{28} = -\ln 2 \Rightarrow x = 45$$

8) A firm produces machines with a lifespan, whose distribution has a mean of 200 months and standard deviation of 50 months. The firm wishes to introduce a warranty scheme in which it would like to replace all the dysfunctional machines with new ones within warranty period. But they do not wish to do so for more than 11.9% of the machines they produce. If the lifespan of the machine is assumed to follow a normal distribution, how long a guarantee period should be offered?

$N(200, 50)$

$P(X < p) = 0.119$.

$\frac{X-200}{50} < \frac{p-200}{50} = 0.119$

$\frac{X-200}{50} = -1.18 \cancel{\times 50 + 2}$

$X = -1.18 \times 50 + 200 = \underline{141}$

Week6

Let X and Y be continuous random variables with joint density

$$f_{XY}(x,y) = \begin{cases} 6x^2y & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

11) Find $P(X \geq Y)$.

$$f_{XY}(x,y) = \begin{cases} 6x^2y \;\; 0 < x < 1; 0 < y < 1 \\ 0 \;\; \text{otherwise} \end{cases}$$

$P(X < Y) = \int_0^1 \int_0^y 6x^2y \; dx \; dy = \int_0^1 2x^3y]_0^y \; dy = \int_0^1 2y^4 dy = \frac{2}{5} - -(A)$

Thus,

$P(X \geq Y) = 1 - \frac{2}{5} = \frac{3}{5} = 0.6$

I proceeded as in step (A) so as to make the integration easier, you can also do $P(X \geq Y)$ by doing the integration $\int_0^1 \int_y^1 6x^2y \; dx \; dy$

Week6 – Tutorial3

1) The joint pdf of two random variables X and Y is given by

$$f_{XY}(x,y) = \begin{cases} \frac{3}{4} + xy & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the value of $P(Y > \frac{1}{2} | X = \frac{1}{4})$. Write your answer correct to two decimal places.

Such problems can be solved in 3 steps.

1. Fy|x = fxy/fx. Compute fx (marginal X) by integrating over Y (in this case 0-1). Then divide the specified fxy (in this case ¾+xy) by fx.
2. Now, substitute x with ¼ in the resulting expression.
3. Now, find P(Y > ½|X=¼) by integrating above expression between ½ and 1

Graded6

Let $Y = XZ + X$, where $X \sim \text{Uniform}\{1,2,3\}$ and $Z \sim \text{Normal}(1,4)$ are independent. Find the value of $f_{X|Y=2}(2)$.

Given that $X \sim \text{Uniform}\{1,2,3\}$ and $Z \sim \text{Normal}(1,4)$ are independent.
$Y = XZ + X$

Since Var(2Z) = 4Var(Z) and Var(3Z) = 9Var(Z), we can write

$Y|X = 1 = Z + 1 \sim \text{Normal}(2,4)$
$Y|X = 2 = 2Z + 2 \sim \text{Normal}(4,16)$
$Y|X = 3 = 3Z + 3 \sim \text{Normal}(6,36)$

Therefore,
$f_{Y|X=1}(y) = \frac{1}{2\sqrt{2\pi}} \exp\left(\frac{-(y-2)^2}{8}\right)$
$f_{Y|X=2}(y) = \frac{1}{4\sqrt{2\pi}} \exp\left(\frac{-(y-4)^2}{32}\right)$
$f_{Y|X=3}(y) = \frac{1}{6\sqrt{2\pi}} \exp\left(\frac{-(y-6)^2}{72}\right)$

$$f_{X|Y=2}(2) = \frac{f_{Y|X=2}(2).f_X(2)}{f_{Y|X=2}(2).f_X(2) + f_{Y|X=1}(2).f_X(1) + f_{Y|X=3}(2).f_X(3)}$$

AQ8.4

4) Let $X_1, X_2, \ldots, X_n \sim$ i.i.d. Normal(μ, σ^2). An estimator $\hat{\sigma}^2$ for σ^2 is given by

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \overline{X}^2$$

Find the bias of $\hat{\sigma}^2$.

$$E(\hat{\sigma}^2) = E\,\frac{1}{n}\sum_{i=1} X_i^2 - \overline{X}^2$$

$$= \frac{1}{n}\left(n\,E X_i^2 - n\,E\overline{X}^2\right)$$

$$= \sigma^2 + \mu^2 - \left[\frac{\sigma^2}{n} + \mu^2\right]$$

$$= \sigma^2 - \frac{\sigma^2}{n}$$

$$\text{Bias} = \sigma^2 - \frac{\sigma^2}{n} - \sigma^2$$

$$= -\frac{\sigma^2}{n}.$$

Solve with us-8

4. Suppose that we want to estimate the true average number of eggs a queen bee lays with 95% confidence. The margin of error we are willing to accept is 0.3. Suppose we also know that standard deviation is 9. What sample size should we use?

$$P(|\hat{\mu} - \mu| \leq 0.3) = 0.95$$

$$\Rightarrow P\left(|\frac{\hat{\mu} - \mu}{\sigma/\sqrt{n}}| \leq \frac{0.3}{\sigma/\sqrt{n}}\right) = 0.95$$

$$\Rightarrow P\left(|Z| \leq \frac{0.3}{\sigma/\sqrt{n}}\right) = 0.95$$

Now, $2F_Z(\frac{0.3}{\sigma/\sqrt{n}}) - 1 = 0.95$. This implies $F_Z(\frac{0.3}{\sigma/\sqrt{n}}) = 1.95/2 = 0.975$ and hence,

$$\frac{0.3}{\sigma/\sqrt{n}} = 1.96$$

$$\Rightarrow \sqrt{n} = 9 \times \frac{1.96}{0.3}$$

$$\Rightarrow n = 3457.44$$

Practice8

11. The distribution of the diameter of screws produced by a certain machine is normally distributed with μ and σ unknown. We observe a random sample $9.8, 10.2, 10.4, 9.8, 10.0, 10.2$ and 9.6 (in cm).
Find a 95% confidence interval for the mean diameter of screws.
Hint: Use $P(-2.447 < t_6 < 2.447) = 0.95$ and S(sample standard deviation) $= 0.283$.

Given that $S = 0.283$, $n = 7$, $\beta = 0.95$

$$\text{Now, } \bar{X} = \frac{9.8 + 10.2 + 10.4 + 9.8 + 10.0 + 10.2 + 9.6}{7} = 10$$

Using t-distribution, $\dfrac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$.

$$\frac{\alpha}{S/\sqrt{n}} = 2.447$$

$$\alpha = 2.447 \times \frac{0.283}{\sqrt{7}}$$

$$= 0.26$$

$P(|\hat{\mu} - \mu| < 0.26) = 0.95$

So, 95% confidence interval is $[10 - 0.26, 10 + 0.26] = [9.74, 10.26]$.

AQ9.5 Q.3

To estimate the fraction of people (p) who support the construction of a shopping mall in the city, a local newspaper conducted a poll of the city residents. Out of the 68 people in the sample, 21 supported the construction.

3) Suppose one believe that her prior mean is 0.6 and prior standard deviation is 0.15. Use a Beta(α, β) prior that matches her prior belief and find the posterior mean. Write your answer correct to two decimal places.

$$\frac{\alpha}{\alpha+\beta} = 0.6 \qquad \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = (0.15)^2$$

$$\boxed{\frac{\alpha}{(\alpha+\beta)^2} = \frac{0.6}{\alpha}} \qquad \frac{(0.6)^2\beta}{\alpha} \times \frac{0.6}{(\alpha+0.6)} = (0.15)^2$$

$$\boxed{\frac{\alpha}{0.6} + 1 = \alpha+\beta+1} \qquad (0.6)^2 \frac{\beta \cdot 0.4\alpha \cancel{(0.6)}}{\cancel{0.6}(\alpha+0.6)} = (0.15)^2$$

$$\frac{\alpha+0.6}{0.6} = \alpha+\beta+1 \qquad \alpha+0.6 = \frac{(0.6)^2(0.4)}{(0.15)^2}$$

$$\beta = \frac{0.4\alpha}{0.6} \qquad \alpha+0.6 = 6.4$$

$$\boxed{\alpha = 5.8}$$

$$\beta = \frac{5.8 \times 0.4}{0.6}$$

$$\boxed{\beta = 3.86}$$

$$\propto p^{21}(1-p)^{47} \, p^{5.8-1}(1-p)^{3.86-1}$$

$$\propto p^{26.8-1}(1-p)^{50.8-1}$$

Beta$(26.8, 50.86)$

posterior mean $= \dfrac{26.8}{50.86+26.8} = \underline{\underline{0.345}}$

AQ9.5 Q.4

4) Let p represent the proportion of defective items that are coming out of the production line. Suppose that 10 items are sampled from a production line and 3 are found to be defective. Use Beta$(0.5, 0.5)$ prior to estimate the unknown proportion p of defective items. Write your answer correct to three decimal places.

Likelihood $= p\,(1-p)$

Prior $= p^{0.5-1}(1-p)^{0.5-1}$

Posterior $\propto$ Likelihood $\times$ prior

$\propto p^{3}(1-p)^{7} \cdot p^{-0.5}(1-p)^{-0.5}$

$\propto p^{2.5}(1-p)^{6.5}$

This is Beta$(3.5, 7.5)$

Hence expected value is

$\frac{3.5}{3.5+7.5} = \frac{3.5}{11} = \underline{\underline{0.318}}$

AQ9.5 Q.5

5) The number of items sold by an online store in an hour has the Poisson(λ) distribution. The number of items sold hourly over the next eight hours are: $3, 2, 0, 8, 2, 4, 6, 1$. Find the Bayesian estimator (posterior mean) of λ using Gamma$(2, 3)$ prior. Write your answer correct to two decimal places.

Posterior = Likelihood × Prior.

Prior $\propto \lambda^1 . e^{-3\lambda}$ $\because \alpha = 2, \beta = 3$

Likelihood $\propto e^{-8\lambda} . \lambda^{26}$

$(e^{-\lambda} . \lambda^{x_1} . e^{-\lambda} \lambda^{x_2} . e^{-\lambda} x^3 \cdots e^{-\lambda} \lambda^n)$

8 times

Thus posterior $= e^{-8\lambda} . \lambda^{26} \lambda e^{-3\lambda}$

$= e^{-11\lambda} . \lambda^{27}$

This is gamma(28, 11)

Expected ~~mean~~ value of gamma

is $\frac{\alpha}{\beta} = \frac{28}{11} = 2.545$

Practice9

Suppose that the number of customers arriving in a restaurant in a one day time period follows the Poisson distribution with unknown parameter λ. Previous records suggest that the prior probabilities of λ are $P(\lambda = 10) = 0.4$ and $P(\lambda = 8) = 0.6$. If on a particular day 15 people arrive at the restaurant, find the posterior mode of λ.

$$\begin{aligned} P(X = 15) &= P(X = 15 \mid \lambda = 10) P(\lambda = 10) + P(X = 15 \mid \lambda = 8) P(\lambda = 8) \\ &= \frac{e^{-10} 10^{15}}{15!} \times 0.4 + \frac{e^{-8} 8^{15}}{15!} \times 0.6 \end{aligned}$$

Now,

$$\begin{aligned} P(\lambda = 10 \mid X = 15) &= P(X = 15 \mid \lambda = 10) P(\lambda = 10) / P(X = 15) \\ &= \frac{e^{-10} 10^{15} \times 0.4}{e^{-10} 10^{15} \times 0.4 + e^{-8} 8^{15} \times 0.6} \end{aligned}$$

And

$$P(\lambda = 8 \mid X = 15) = P(X = 15 \mid \lambda = 8)P(\lambda = 8)/P(X = 15)$$
$$= \frac{e^{-8}8^{15} \times 0.6}{e^{-10}10^{15} \times 0.4 + e^{-8}8^{15} \times 0.6}$$

$$\Rightarrow \frac{P(\lambda = 10 \mid X = 15)}{P(\lambda = 8 \mid X = 15)} = \frac{e^{-10}10^{15} \times 0.4}{e^{-8}8^{15} \times 0.6}$$
$$= 2.56$$
$$\Rightarrow P(\lambda = 10 \mid X = 15) > P(\lambda = 8 \mid X = 15)$$

Hence, the posterior mode of λ is 10.

AQ10.6

You have a coin and you would like to check whether it is fair or biased. Let p be the probability of heads, $p = P(H)$. Let the null and alternative hypothesis be $H_0 : p = 0.5$ and $H_A : p > 0.5$. You toss the coin 100 times and observe 55 heads.

1) Can you reject H_0 at significance level $\alpha = 0.05$? ***1 point***

$H_0 : p = 0.5$ against $H_1 : p > 0.5$ The coin was tossed 100 times and 55 heads were observed.

Now, let $\overline{X} = \dfrac{X_1 + X_2 + \ldots X_n}{100}$, where $X_i = 1$, if heads occur, 0 otherwise

Thus, $\overline{X}$ is a random variable such that $E(\overline{X}) = p\,, Var(\overline{X}) = \dfrac{p(1-p)}{100}$ (since each $X_i \sim Ber(p)$)

Thus, $\alpha = P(\overline{X} > C|H_0) = P(\overline{X} - 0.5 > C - 0.5) = P(\dfrac{\overline{X} - 0.5}{0.5/10} > \dfrac{C - 0.5}{0.5/10}) =$
$P(Z > \dfrac{C - 0.5}{0.5/10}) = P(Z > 20(C - 0.5))$

Thus, $\alpha = 1 - P(Z \leq 20(C - 0.5))$

Hence, $P - \text{Value}$ for the given random sampling can be obtained by substituting $\overline{X}_{obs} = 0.55$ in the above equation in place of C.

Thus, $P - \text{Value} = 1 - F_Z(1) \approx 0.1586$, which is fairly large(greater that α) Hence, we cannot reject H_0 at level α.

Others

9) Let $X \sim \text{Binomial}(3, 0.75)$ and $Y \sim \text{Binomial}(5, 0.25)$ be two independent random variables. Find the value of $P(\max\{X, Y\} = 1)$. Write your answer correct to three decimal places.

0.089

$$P(\text{Max}(x,y)=1)$$

$$= P(X=1,Y=1) + P(X=0, Y=1) + P(X=1,Y=0)$$

$$= P(X=1)P(Y=1) + P(X=0)P(Y=1) + P(X=1)P(Y=0)$$

$$= {}^3C_1\left(\tfrac{3}{4}\right)^1\left(\tfrac{1}{4}\right)^2 \times {}^5C_1\left(\tfrac{1}{4}\right)^1\left(\tfrac{3}{4}\right)^4 + \left(\tfrac{1}{4}\right)^3 \times {}^5C_1\left(\tfrac{1}{4}\right)^1\left(\tfrac{3}{4}\right)^4 + {}^3C_1\left(\tfrac{3}{4}\right)^1\left(\tfrac{1}{4}\right)^2 \times \left(\tfrac{3}{4}\right)^5$$

$$= 15\left(\tfrac{3}{4}\right)^5\left(\tfrac{1}{4}\right)^3 + 5\times\left(\tfrac{1}{4}\right)^4\left(\tfrac{3}{4}\right)^4 + 3\times\left(\tfrac{1}{4}\right)^2\left(\tfrac{3}{4}\right)^6$$

$$= \frac{3645+405+2187}{65536} = \frac{6237}{65536} \approx 0.095$$

https://discourse.onlinedegree.iitm.ac.in/t/quiz-1-june-20-question-9/10370/4?u=anand

10) Let X be a discrete random variable with the following probability mass function

$$P(X = k) = \begin{cases} 0.4 & \text{if } k = 1 \\ 0.15 & \text{if } k = 2 \\ 0.35 & \text{if } k = 3 \\ t & \text{if } k = 4 \end{cases}$$

Find the value of $P(X = 4|X > 2)$. Write your answer correct to two decimal places.

$$P(X=k)=\begin{cases}0.4 \text{ if } k=1\\ 0.15 \text{ if } k=2\\ 0.35 \text{ if } k=3\\ t \text{ if } k=4\end{cases}$$

Then, $0.4+0.15+0.35+t=1 \Rightarrow t=0.1$

Thus, $P(X=4|X>2)=\frac{P(X=4\cap X>2)}{P(X>2)}=\frac{P(X=4)}{P(X=3)+P(X=4)}=\frac{0.1}{0.35+0.1}=\frac{0.1}{0.45}\approx 0.22$

8) For a certain lottery, a three-digit number is randomly selected (from 000 to 999). If a ticket matches the number exactly, it is worth Rs. 1000. If the ticket matches exactly two of the three digits, it is worth Rs. 200. Otherwise it is worth nothing. Let X be the worth of the ticket. Calculate $P(X=200)$. Write your answer correct to three decimal places.

Let your ticket number is xyz $(x, y, z$ are digits). then worth of this will be 200 if two of the digits (x, y, z)(in their places as well) match the selected number.
So, the probability that two digits match will be $3\times\frac{9}{10}\times\frac{1}{10}\times\frac{1}{10}$

This can also be solved as a binomial problem. 3C2*(1/10)^2*(9/10)

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Let $X_1, X_2, \cdots, X_{20} \sim i.i.d.$ Geometric$(\frac{1}{2})$. Define $Z_1=\min(X_1, X_2, \cdots, X_{20})$ and $Z_2=\max(X_1, X_2, \cdots, X_{20})$.

15) Calculate $F_{Z_1}(2)$. ***1 point***

16) Calculate $F_{Z_2}(2)$.

$X_1, X_2, \ldots X_{20}$~$i.i.d$ Geo$(\frac{1}{2})$

$Z_1 = \min\{X_1, X_2, \ldots X_{20}\}$
$F_{Z_1}(2) = P(Z_1 \leq 2) = P(\min\{X_1, X_2, \ldots X_{20}\} \leq 2)$ but if you go by this, it will be a bit harder to tackle the problem.

So, proceed like this:

$P(\min\{X_1, X_2, \ldots X_{20}\} > 2) = P(X_1 > 2) \ldots P(X_{20} > 2) = [P(X > 2)]^{20}$
$= [(1 - P(X \leq 2))]^{20} = (1 - \frac{1}{2} - \frac{1}{4})^{20} = (\frac{1}{4})^{20}$

Thus, $P(\min\{X_1, X_2, \ldots X_{20}\} \leq 2) = 1 - (\frac{1}{4})^{20}$

https://discourse.onlinedegree.iitm.ac.in/t/refresher-week-q-15-and-q16/9923/4?u=anand