☑

If A and B are square matrices of order 3 and $A + B = 0$, then $B = -A$.

☑

If A is a scalar matrix of order 3, B is a non-zero square matrix of order 3 and $AB = 0$, then $A = 0$.

☑ Every system of linear equations has either a unique solution, no solution or infinitely many solutions.

☑

If $Ax = b$ is a system of linear equations which has a solution, then the system of linear equations $cAx = b$, where $c \neq 0$, will also have a solution.

☑

If $Ax = b$ is a system of linear equations which has a solution, then $\frac{1}{c}Ax = b$, where $c \neq 0$, will also have a solution.

In the case of a square matrix, linearly independent columns do imply every Ax=b has a unique solution.

7) Let x_1 and x_2 be solutions of the system of linear equations $Ax = b$.

Then,

$x_1 + x_2$ is a solution of the system of linear equations $Ax = 2b$.

$x_1 - x_2$ is a solution of the system of linear equations $Ax = 0$.

8) Let v be a solution of the systems of linear equations $A_1 x = b$ and $A_2 x = b$. Which of the following options are correct ?

☐

v is a solution of the system of linear equations $(A_1 + A_2)x = 2b$.

☐

v is a solution of the system of linear equations $(A_1 - A_2)x = 0$.

Scalar multiplication by c = multiplication by scalar matrix cI.

$$\begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} c & 2c \\ 3c & 4c \\ 5c & 6c \end{bmatrix} = c \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

There are 3 possibilities for the solutions to a linear system of equations:

1) The system has a single unique solution. Geometrically, the representations intersect (lines in R2 and planes in R3)
2) The system has infinitely many solutions. Two or more equations could be scalar multipliers of each other. Geometrically, the representations are same/coincide (lines in R2 and planes in R3)
3) The system has no solution. Two or more equations could be contradicting to each other. Geometrically, the representations are parallel (lines in R2 and planes in R3)

- det(I) = 1

- det(AB) = det(A) * det(B)

- det(A) det(A^{-1}) = I. Here det(A^{-1}) is called inverse of the matrix A and is given by 1/det(A)

Type1: If you switch position of two rows (or columns) in a matrix, determinant changes its sign.

Type2: If you add multiple of one row (or a column) to another, determinant is unaffected.

Type3: If any of the rows (or columns) of a matrix is multiplied by a scalar, determinant is also multiplied by the scalar. This implies that if every row (or column) in an m * m matrix is multiplied by the scalar, the determinant is multiplied by (scalar)^m

- Determinant of a matrix with a row (or column) comprising of all zeros is 0.

- Determinant of a matrix in which one row (or column) is a linear combination of other rows (resp. columns) is 0.

- Determinant of an triangular (upper/lower) matrix is products of its diagonal elements.

- Determinant of transpose of a matrix is the same as that of the original matrix.

- Determinant of a product of matrices is the product of its determinants.

Let $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a triangle. The area of the triangle ABC is given by $\frac{1}{2}|det(D)|$ square units, where $D = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$

Suppose for a real 3 * 3 matrix A, there exists a real 3 * 3 matrix P such that D = PAP^{-1} is a real 3 * 3 diagonal matrix. In that case,

- det(A) must be equal to det(D)

- If D is an identity matrix of order 3, A must also be an identity matrix of order 3.

Cramer's rule

Consider the system of linear equations $Ax = b$ where

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 5 \\ 4 & 3 & 1 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

As described in the procedure, calculate $det(A) = -37$. Since it is non-zero, we can apply Cramer's rule. Follow the next steps in the procedure :

$$A_{x_1} = \begin{bmatrix} 0 & 0 & 3 \\ 2 & 2 & 5 \\ 1 & 3 & 1 \end{bmatrix} \qquad A_{x_2} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 5 \\ 4 & 1 & 1 \end{bmatrix} \qquad A_{x_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 4 & 3 & 1 \end{bmatrix}$$

$$det(A_{x_1}) = 12 \qquad det(A_{x_2}) = -27 \qquad det(A_{x_3}) = 4.$$

$$x_1 = \frac{det(A_{x_1})}{det(A)} = -\frac{12}{37}$$

$$x_2 = \frac{det(A_{x_2})}{det(A)} = \frac{27}{37}$$

$$x_3 = \frac{det(A_{x_3})}{det(A)} = \frac{4}{37}$$

Inverse of a matrix exists, only if its determinant is non-zero. This is because, by definition, $AA^{-1}=I$, which implies $\det(A^{-1}) = 1/\det(A)$.

Finding the inverse of a matrix

- Find the determinant of the matrix. Call this *d*
- Find the adjugate of the matrix (transpose of the cofactor matrix). Call this *Adj*
- Inverse = *Adj / d*

A homogeneous system of linear equations with n equations in n unknowns :

- has a unique solution 0 if its coefficient matrix is invertible, i.e. its determinant is non-zero.
- has an infinite number of solutions if its coefficient matrix is not invertible i.e. its determinant is 0.

If v is a solution of the system of linear equations $Ax = b$, then $\frac{1}{c}v$ is a solution of system of linear equations $cAx = b$, where $c \neq 0$.

Let $Ax = b$ be a system of linear equations. If A is invertible, then $adj(A)x = b$ also has a solution.

Adjoint of Identity matrix (or zero matrix) is same as the original matrix.

If A is an upper triangular 3×3 matrix, then the adjoint matrix of A is also an upper triangular matrix.

If A is an invertible upper triangular 3×3 matrix, then the inverse matrix of A is also an upper triangular matrix.

$adj(AB) = adj(B)adj(A)$.

$adj(A + B) = adj(A) + adj(B)$.

$adj(A^T) = adj(A)^T$.

$adj(A^{-1}) = adj(A)^{-1}$.

☐ The reduced row echelon form of a diagonal matrix must be the identity matrix.

☐ The reduced row echelon form of a scalar matrix must be the identity matrix.

For homogeneous system of linear equations, there are only two possibilities:

1. 0 is the unique solution. This is called trivial solution.
2. There are infinitely many solutions other than 0.

In a homogeneous system of equations, if there are more variables than equations, then it is guaranteed to have nontrivial solutions.

Vector spaces

i) $v_1 + v_2 = v_2 + v_1$ for all $v_1, v_2 \in V$

ii) $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$ for all $v_1, v_2, v_3 \in V$

iii) There exists an element in V denoted by 0 such that $v + 0 = v$ for all $v \in V$

iv) For each element $v \in V$ there exists an element $v' \in V$ such that $v + v' = 0$

v) For each element $v \in V$, $1v = v$

vi) For each pair of elements $a, b \in \mathbb{R}$ and each element $v \in V$, $(ab)v = a(bv)$

vii) For each element $a \in \mathbb{R}$ and each pair of elements v_1 and v_2, $a(v_1 + v_2) = av_1 + av_2$

viii) For each pair of elements $a, b \in \mathbb{R}$ and each element $v \in V$, $(a + b)v = av + bv$

How to check if a given vector space (subspace) is valid or not?

If all vectors in the space are closed under addition or scalar multiplication, then the given vector space is valid.

$$x + y = z + y \implies x = z \; \forall x, y, z \in V.$$

$$ax = bx, \; \forall \, x \in V \implies a = b, \text{ where } a, b \in \mathbb{R}.$$

$$ax = ay, \; \forall \, a \in \mathbb{R} \implies x = y, \text{ where } x, y \in V.$$

Let V be a plane plane parallel to the XY-plane. Any plane parallel to XY-plane is given by $z = c$. We define addition of two vectors $v_1 = (x_1, y_1, c)$ and $v_2 = (x_2, y_2, c)$ on V as follows: First project v_1 and v_2 on the XY-plane (we will get the vectors $(x_1, y_1, 0)$ and $(x_2, y_2, 0)$ by projection on the XY-plane) and then calculate the addition of the vectors we obtained by the projection on the XY-plane (we will obtain $(x_1 + x_2, y_1 + y_2, 0)$). Then project the obtained vector back to the plane V (we will obtain the vector $(x_1 + x_2, y_1 + y_2, c)$).

When v1 is projected to XY plane, we get (1,2,0). When v2 is projected to XY plane, we get (0.3,0)

When v1 is added to v2, we get (1, 5, 2)

If a set is linearly dependent, then so is every superset of it.

Set of vectors containing the 0 vector is always dependent, since it can produce non-trivial solutions for linear combinations amongst them.

Hence, any set of r vectors in $\mathbb{R}^n$ with $r > n$ are linearly dependent.

2) If S_1 is a maximal linear independent set and S_2 is a minimal spanning set of a vector space V, then which of the following option(s) is (are) true?

Then,

For any $v \in S_1$, $S_1 \setminus \{v\}$ is a linearly independent.

For any $v \in V \setminus S_1$, $S_1 \cup \{v\}$ is a linearly dependent.

For any $v \in V$, $S_2 \cup \{v\}$ is a spanning set of V.

Rank is the number of elements in the basis of a vector. It's the number of non-zero rows, when vectors are represented as rows. It's the number of non-pivot elements, when vectors are represented as columns

Nullity is the number of independent elements when the vectors are represented as a matrix.

Nullity + Rank = number of columns.

- [x] If $A_{2\times3}$ is a non zero matrix, then nullity of the matrix ≤ 2.
- [x] If $A_{3\times2}$ is a non zero matrix, then nullity of the matrix ≤ 1.
- [x] Let A and B be two square matrices of order 3, if nullity of matrix AB is 0, then nullity of matrix A is also zero.
- [x] Let A and B be two square matrices of order 3, if nullity of matrix AB is 0, then nullity of matrix B is also zero.

4) Let S denote the set of solutions of the homogeneous system of linear equations $Ax = 0$.

Then,

- [x] If x_1 and x_2 are in S, then $x_1 - x_2$ is in S.
- [x] If x is in S, then cx is in S, for any $c \in \mathbb{R}$.
- [x] If x_1 and x_2 are in S, then $\alpha x_1 + \beta x_2$ is in S, for any $\alpha, \beta \in \mathbb{R}$.
- [x] If x is in S, then $A^n x = 0$ for any $n \in \mathbb{N} \setminus \{0\}$.

2) Which of the following is true for a homogeneous system of linear equations $Ax = 0$?

Then,

- [x] It can have infinitely many solutions.
- [x] It can have a unique solution.
- [x] Whenever it has a non-trivial(non zero) solution, it must have infinitely many solutions

3) Which of the following options are correct for a square matrix A of order $n \times n$, where n is any natural number?

Then,

- [x] If the determinant is non-zero, then the nullity of A must be 0.

- [x] If the nullity of A is non-zero, then the determinant of A must be 0.

- [x] If nullity of a 3×3 matrix is c for some natural number c, $0 \leq c \leq 3$, then the nullity of $-A$ will also be c.
- [x] Nullity of the zero matrix of order $n \times n$, is n.
- [x] There exist square matrices A and B of order $n \times n$, such that nullity of both A and B is 0, but the nullity of $A + B$ is n.

6) Let $Ax = 0$ be a homogeneous system of linear equations which has infinitely many solutions, where A is an $m \times n$ matrix (where, $m > 1, n > 1$). Which of the following statements are possible?

- [x] $rank(A) = m$ and $m < n$.
- [x] $nullity(A) = n$.
- [x] $nullity(A) \neq 0$.

Following is not possible:

- [ ] $rank(A) = m$ and $m > n$.
- [ ] $rank(A) = m$ and $m = n$.

- [ ] The null-space of a non-zero scalar matrix is always the zero vector space.
- [x] The nullity of a non-zero scalar matrix is always 0.
- [x] The rank of a non-zero scalar matrix is always same as the order of the matrix.

Linear Transformation

A function $f : V \to W$ between two vector spaces V and W is said to be a linear transformation if for any two vectors v_1 and v_2 in the vector space V and for any $c \in \mathbb{R}$ (scalar) the following conditions hold :

- $f(v_1 + v_2) = f(v_1) + f(v_2)$
- $f(cv_1) = cf(v_1)$

In a linear transformation, output vectors can be represented as a linear combination of input vectors. The coefficients of this linear combination (c1, c2, .., cn) can be used to compute transformation of a vector from one space to another.

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If the Identity matrix of order 2 is the matrix representation of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, with respect to the standard ordered basis for both the domain and co-domain, then it is also the matrix representation of T with respect to any other basis β for both the domain and co-domain.

☑ Let T be an isomorphism between two vector spaces, and A be the matrix representation of T with respect to some basis. Then the nullity of A is 0.

☑

If $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is a matrix representation of $T : \mathbb{R}^2 \to \mathbb{R}^2$ with respect to the standard ordered basis of $\mathbb{R}^2$, for both the domain and co-domain, then the matrix representation of $T \circ T$ with respect to the same bases will also be A.

In a linear transformation problem,

Beta vectors is the set of vectors in the input vector space.

Gamma vectors is the set of vectors in the output vector space.

Transformation of each Beta vector is a linear combination of Gamma vectors. The resulting coefficients for one column of the transformation matrix.

Consider a linear transformation $T : V \to W$, two ordered bases β_1 and β_2 for V, and two ordered bases γ_1 and γ_2 for W.

Let A be the matrix corresponding to T with respect to the bases β_1 and γ_1 and B be the matrix corresponding to T with respect to the bases β_2 and γ_2.

Then A is equivalent to B !

If A = QBP, then A and B are called equivalent. If A = $P^{-1}BP$, then A and B are called similar matrices.

If A is equivalent (or similar) to B, B is also equivalent to A.

A equivalent (or similar) to B ***and*** B equivalent (or similar) to C ***implies*** A equivalent (or similar) to C.

Ranks of equivalent (or similar) matrices are equal.

Determinants of equivalent (or similar) matrices are equal.

Equivalent matrices can be transformed between each other using elementary row/column operations.