9) Let $\{a_n\}$ be a sequence defined by $a_n = n^{\frac{1}{n}}$, has limit 1. Which of the following option(s) is(are) true?

[Hint: Use v),vi) and viii) to find out the limits of the sequences given in the options.]

- [x] $\lim\limits_{n\to\infty} \ln(n^{\frac{1}{n}}) = 0$
- [x] $\lim\limits_{n\to\infty} \ln(1+n^{\frac{1}{n}}) = 0$
- [ ] $\lim\limits_{n\to\infty} \dfrac{\ln 2 - \frac{1}{n}}{\ln(1+n^{\frac{1}{n}})} = 1$
- [x] $\lim\limits_{n\to\infty} (4n^{\frac{3}{n}} - 1) = 3$

$\lim_{n\to\infty} \ln(1+n^{\frac{1}{n}}) = \ln(1+\lim_{n\to\infty} n^{\frac{1}{n}}) = \ln(1+1) = \ln(2)$

Here, limits can be brought inside the function as log is a continuous function.

$\lim_{n\to\infty} \frac{\ln 2 - \frac{1}{n}}{\ln(1+n^{\frac{1}{n}})} = \frac{\ln 2}{\ln 2} = 1$, the denominator can be calculated from the above and

$\lim_{n\to\infty} \ln 2 - \frac{1}{n} = \ln 2$

$\lim_{n\to\infty} (4n^{\frac{3}{n}} - 1) = 4\lim_{n\to\infty} (n^{\frac{1}{n}})^3 - 1 = 4 - 1 = 3$

Graded2

If the functions $f(t)$ and $g(t)$ denoting the profits of Company A and Company B, respectively, are known to be continuous at $t = 3$, then what will be the values of n and m?

9) Assuming g to be continuous at $t=3$, choose the correct option from the following.

○ $\lim\limits_{t \to 3-} \frac{g(t)-g(3)}{t-3} = 18$ and $\lim\limits_{t \to 3+} \frac{g(t)-g(3)}{t-3} = 18$ hence g is differentiable at $t=3$

○ $\lim\limits_{t \to 3-} \frac{g(t)-g(3)}{t-3} = 9$ and $\lim\limits_{t \to 3+} \frac{g(t)-g(3)}{t-3} = 9$, hence g is differentiable at $t=3$.

○ $\lim\limits_{t \to 3-} \frac{g(t)-g(3)}{t-3} = 18$ and $\lim\limits_{t \to 3+} \frac{g(t)-g(3)}{t-3} = 9$, hence g is not differentiable at $t=3$.

○ $\lim\limits_{t \to 3-} \frac{g(t)-g(3)}{t-3} = 9$ and $\lim\limits_{t \to 3+} \frac{g(t)-g(3)}{t-3} = 18$, hence g is not differentiable at $t=3$.

Doubt: Not sure how this expression evaluates differentiability.

7. Let f and g be two functions which are differentiable at each $x \in \mathbb{R}$. Suppose that, $f(x) = g(x^2+5x)$, and $f'(0) = 10$. Find the value of $g'(0)$. [Answer: 2]

Solution:

Given $f(x) = g(x^2+5x) \implies f'(x) = g'(x^2+5x)(2x+5)$

So $f'(0) = 5g'(0) \implies g'(0) = \frac{10}{5} = 2$

AQ3.4

6) Let $f(x) = 3x+1$ then find the value of the integral $\int_0^2 f(x)\,dx$ using limit of Riemann sums as $n \to \infty$, for the given partition $P = \{0 = x_0, x_1 = \frac{2}{n}, \ldots, x_i = \frac{2 \times i}{n}, \ldots, x_n = 2\}, i = 1, 2, \ldots, n$ and $x_i^* \in [x_{i-1}, x_i]$, where $x_i^* = \frac{2 \times i}{n}$.

4) Find the value of given definite integral $\int_{-2021}^{2021} (x^{2021}.\cos 2021x + \sin 2021x)\,dx$.

No need to simplify the integral. Note that $sin(2021x)$ is an odd function. The area under the graph of an odd function is 0 in the interval $[-2021, 2021]$. Similarly, can you find the nature of the composite function in the first term and compute the area (integral)?

$$\Rightarrow \quad f(x) = 3x+1$$

$$\Rightarrow \quad \left[x_i^* = \frac{2i}{n}\right] \; ; \quad \Delta x_i = \frac{2i}{n} - \frac{2(i-1)}{n}$$

$$\Rightarrow \quad \left[\Delta x_i = \frac{2}{n}\right]$$

Now, $\displaystyle\int_0^2 f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[3\left(\frac{2i}{n}\right)+1\right] \times \frac{2}{n} .$$

$$= \lim_{n\to\infty} \left[\sum \frac{12i}{n^2} + \sum \frac{2}{n}\right]$$

$$= \lim_{n\to\infty} \left[\frac{12}{n^2}\sum_{i=1}^{n} i + \frac{2}{n}\sum_{i=1}^{n} 1\right]$$

$$= \lim_{n\to\infty} \left[\frac{12}{n^2} \times \frac{n(n+1)}{2} + \frac{2}{n} \times n\right]$$

$$= \lim_{n\to\infty} \left[6\left(1+\frac{1}{n}\right) + 2\right]$$

$$= \lim_{n\to\infty} 8 + \lim_{n\to\infty} \frac{1}{n} \qquad \left\{\lim_{n\to\infty} \frac{1}{n} = 0\right\}$$

$$\boxed{\int_0^2 f(x)\,dx = 8}$$

Practice3

3) Suppose $\displaystyle\int x\ln(1+x)dx = f(x)\ln(x+1) - \frac{x^2}{4} + Ax + B$, where B is the constant of integration. Which of the following are correct?

Solution: By using integration by parts:

$$
\begin{aligned}
\int x \ln(1+x)\ dx &= \ln(1+x) \int x\ dx - \int \left\{ \frac{d(\ln(1+x))}{dx} \int x\ dx \right\} dx \\
&= \frac{x^2 \ln(1+x)}{2} - \int \frac{x^2}{2(1+x)}\ dx \\
&= \frac{x^2 \ln(1+x)}{2} - \int \frac{x^2+x-x}{2(1+x)}\ dx \\
&= \frac{x^2 \ln(1+x)}{2} - \int \frac{x^2+x}{2(1+x)}\ dx + \int \frac{x}{2(1+x)}\ dx \\
&= \frac{x^2 \ln(1+x)}{2} - \frac{x^2}{4} + \int \frac{x+1-1}{2(1+x)}\ dx \\
&= \frac{x^2 \ln(1+x)}{2} - \frac{x^2}{4} + \int \frac{x+1}{2(1+x)}\ dx - \int \frac{1}{2(1+x)}\ dx \\
&= \frac{x^2 \ln(1+x)}{2} - \frac{x^2}{4} + \frac{x}{2} - \frac{1}{2} \ln(1+x) + B \\
&= \frac{x^2-1}{2} \ln(1+x) - \frac{x^2}{4} + \frac{x}{2} + B
\end{aligned}
$$

If we equate coefficients then $f(x) = \dfrac{x^2-1}{4}$, and $A = \dfrac{1}{2}$.

https://www.youtube.com/watch?v=fTCDNe4GDHI

Consider a basis $S=\{v_1, v_2, v_3\}$ for R^3

$v_1=(1,1,1)$, $v_2=(1,1,0)$, $v_3=(1,0,0)$

Let $T: R^3 \to R^2$ be the linear transformation for which

$T(v_1)=(1,0)$, $T(v_2)=(2,-1)$, $T(v_3)=(4,3)$

Find a formula for $\underline{T(x_1,x_2,x_3)}$ and then use that formula to compute $\underline{T(2,-3,5)}$

Sol $T: V \to W$, V- finite dimensional vector space

$S=\{v_1, v_2, \dots, v_n\}$, for every $v \in V$, $T(v)=c_1T(v_1)+c_2T(v_2)+\cdots+c_nT(v_n)$

where $c_1, c_2, \dots, c_n$ are coefficients used to express v using $v_1, v_2, \dots, v_n$

$v=c_1(1,1,1)+c_2(1,1,0)+c_3(1,0,0)$

$v=(x_1,x_2,x_3)$

$c_1+c_2+c_3=x_1$

$c_1+c_2=x_2$

$c_1=x_3$

$$\boxed{\begin{array}{l} c_1=x_3 \\ c_2=x_2-x_3 \\ c_3=x_1-x_2 \end{array}}$$

$$\begin{aligned} T(x_1,x_2,x_3) &= c_1T(v_1)+c_2T(v_2)+c_3T(v_3) \\ &= c_1(1,0)+c_2(2,-1)+c_3(4,3) \\ &= x_3(1,0)+(x_2-x_3)(2,-1)+(x_1-x_2)(4,3) \\ &= (x_3+2x_2-2x_3+4x_1-4x_2, \\ &\quad -x_2+x_3+3x_1-3x_2) \end{aligned}$$

$$\Rightarrow T(x_1,x_2,x_3)=(4x_1-2x_2-x_3, 3x_1-4x_2+x_3)$$

AQ8.1

Let $W=\{(x,y,z) \mid x=2y+z\}$ be a subspace of $\mathbb{R}^3$. Let $\beta=\{(2,1,0),(1,0,1)\}$ be a basis of W. Let $T: W \to \mathbb{R}^2$ be a linear transformation such that $T(2,1,0)=(1,0)$ and $T(1,0,1)=(0,1)$. Answer the questions 3, 4 and 5 using the given information.

5) What will be the matrix representation of T with respect to the basis β for W and $\gamma=\{(1,1),(1,-1)\}$ for $\mathbb{R}^2$?

(8) $\{(2,1,0),(1,0,1)\}$ is a basis of W.

$T(2,1,0)=(1,0)$ & $T(1,0,1)=(0,1)$

Let $(x,y,z)\in W$. Then

$(x,y,z)=c_1(2,1,0)+c_2(1,0,1)$ — ①

From ①; we get $c_1=y$, $c_2=z$.

$\therefore (x,y,z)=y(2,1,0)+z(1,0,1)$

Thus; $T(x,y,z)=yT(2,1,0)+zT(1,0,1)$

$=y(1,0)+z(0,1)$

$T(x,y,z)=(y,z)$

Now; We have basis $\{(1,1),(1,-1)\}$ for $\mathbb{R}^2$

$T(2,1,0)=(1,0)=\frac{1}{2}(1,1)+\frac{1}{2}(1,-1)$

$T(1,0,1)=(0,1)=\frac{1}{2}(1,1)-\frac{1}{2}(1,-1)$

Thus; $m(T)=\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$

$$\theta = cos^{-1}\left(\frac{u \cdot v}{\sqrt{(v \cdot v) \times (u \cdot u)}}\right)$$

Let us compute the angle θ between $(1,0,0)$ and $(1,0,1)$.

$(1,0,0)\cdot(1,0,1) = 1, (1,0,1)\cdot(1,0,1) = 2, (1,0,0)\cdot(1,0,0) = 1.$

Hence, $\theta = cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$ radians or 45°.

Similarly, the angle between between $(1,0,0)$ and $(1,1,1)$ is $cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

Representing vector as a linear combination

$\left\{\frac{1}{\sqrt{10}}(1,3), \frac{1}{\sqrt{10}}(-3,1)\right\}$ is an orthonormal basis of $\mathbb{R}^2$. Write $(2,5)$ as a linear combination in terms of these basis vectors.

$$(2,5) = c_1 \frac{1}{\sqrt{10}}(1,3) + c_2 \frac{1}{\sqrt{10}}(-3,1)$$

$$c_1 = \left\langle (2,5), \frac{1}{\sqrt{10}}(1,3) \right\rangle = \frac{1}{\sqrt{10}}(2\times 1 + 5\times 3) = \frac{1}{\sqrt{10}} 17.$$

$$c_2 = \left\langle (2,5), \frac{1}{\sqrt{10}}(-3,1) \right\rangle = \frac{1}{\sqrt{10}}(2\times(-3) + 5\times 1) = \frac{-1}{\sqrt{10}}.$$

$$(2,5) = \frac{17}{\sqrt{10}} v_1 + \frac{-1}{\sqrt{10}} v_2 = \frac{17}{\sqrt{10}} \times \frac{1}{\sqrt{10}}(1,3) - \frac{1}{\sqrt{10}} \times \frac{1}{\sqrt{10}}(-3,1)$$

$$= \frac{17}{10}(1,3) - \frac{1}{10}(-3,1).$$

Finding the projection of a vector into a subspace

$V = \mathbb{R}^2$, $W = \langle(3,1)\rangle$, $v = (1,3)$. Then $proj_W(v) = (1.8, 0.6)$.

$\frac{1}{\sqrt{10}}(3,1)$ is an o.n. basis for W.

$proj_W(v) = \langle v, \frac{1}{\sqrt{10}}(3,1)\rangle \frac{1}{\sqrt{10}}(3,1) = \langle (1,3),(3,1)\rangle \frac{1}{10}(3,1)$

$= \frac{(1\times 3 + 3\times 1)}{10}(3,1) = \frac{6}{10}(3,1)$

$= (1.8, 0.6)$.

$V = \mathbb{R}^3$, $W = \langle(1,0,0),(0,1,0)\rangle$, $v = (2,3,5)$.
Then $proj_W(v) = (2,3,0)$.

o.n. basis $\langle (1,0,0), (0,1,0)\rangle$.

$proj_W(v) = \langle (2,3,5),(1,0,0)\rangle (1,0,0) + \langle (2,3,5),(0,1,0)\rangle (0,1,0)$

$= 2(1,0,0) + 3(0,1,0) = (2,3,0)$.

Let W be the 2-dimensional subspace of $V = \mathbb{R}^3$ spanned by the orthogonal vectors $v_1 = (1,2,1)$ and $v_2 = (1,-1,1)$. What is the projection of $v = (-2,2,2)$ on W?

$$proj_{v_1} v = \frac{\langle v, v_1\rangle}{\langle v_1, v_1\rangle} v_1 = \frac{4}{6}(1,2,1) = \frac{2}{3}(1,2,1).$$

$$proj_{v_2} v = \frac{\langle v, v_2\rangle}{\langle v_2, v_2\rangle} v_2 = -\frac{2}{3}(1,-1,1).$$

Hence
$$\begin{aligned} proj_W(v) &= proj_{v_1}(v) + proj_{v_2}(v) \\ &= \frac{2}{3}(1,2,1) - \frac{2}{3}(1,-1,1) \\ &= (0,2,0). \end{aligned}$$

Practice10

An earthworm is crawling about on a sunny day. We draw coordinates axes on the ground where it is crawling and find that the body temperature of the earthworm at a point (x, y) is given by the function:

$$T(x, y) = 2x^2 + 3xy + y^2$$

To arrive at the answer, take partial derivative of T(x, y) on x at (1,1), multiply with x component of the unit vector in direction (2,3); take partial derivative of T(x, y) on y at (1,1), multiply with y component of the unit vector in direction (2,3); add them together.

$$(7, 5) \cdot \frac{1}{\sqrt{13}} (2, 3) = \frac{29}{\sqrt{13}}$$

Thus,

If the earthworm moves parallel to the straight line joining the origin and the point $(2, 3)$, then rate of change of its body temperature at the point $(1, 1)$ is given by $\frac{29}{\sqrt{13}}$.