Let u and v be two vectors in $\mathbb{R}^2$. Then we can compute the angle θ between the vectors u and v using the dot products as :

$$cos(\theta) = \frac{u \cdot v}{\sqrt{(v \cdot v) \times (u \cdot u)}} \quad i.e. \quad \theta = cos^{-1}\left(\frac{u \cdot v}{\sqrt{(v \cdot v) \times (u \cdot u)}}\right)$$

More generally, the length of the vector $(x, y, z) \in \mathbb{R}^3$ is $\sqrt{x^2 + y^2 + z^2} = \sqrt{(x, y, z) \cdot (x, y, z)}$.

Consider 3 vectors a, b, c in $\mathbb{R}^3$ and a scalar λ in $\mathbb{R}$. Then,

- ☑ $\lambda(a \cdot b) = (\lambda a) \cdot b$
- ☑ $\lambda(a \cdot b) = (\lambda b) \cdot a$
- ☑ $(a + c) \cdot b = a \cdot b + c \cdot b$
- ☑ $a \cdot a = 0, b \cdot b = 0$ if and only if a, b are null vectors.

An inner product on a vector space V is a function $\langle \, . \, , \, . \, \rangle : V \times V \to \mathbb{R}$ satisfying the following :

- $\langle v, v \rangle > 0$ for all $v \in V \setminus \{0\}$; $\langle v, v \rangle = 0$ if and only if $v = 0$.
- $\langle v_1 + v_2, v_3 \rangle = \langle v_1, v_3 \rangle + \langle v_2, v_3 \rangle$
- $\langle v_1, v_2 \rangle = \langle v_2, v_1 \rangle$
- $\langle cv_1, v_2 \rangle = c\langle v_1, v_2 \rangle = \langle v_1, cv_2 \rangle$.

A norm on a vector space V is a function satisfying the following conditions:

- $||x + y|| \leq ||x|| + ||y||$, for all $x, y \in V$

  $v + w$ $\quad v \quad w$

- $||cx|| = |c|||x||$ for all $c \in \mathbb{R}$ and for all $x \in V$

  $cv \quad |c|\,||v||$

- $||x|| \geq 0$ for all $x \in V$; $||x|| = 0$ if and only if $x = 0$

  $||v|| \quad v \quad ||v|| = 0$ iff $v = 0$.

Two vectors u and v of an inner product space V are said to be orthogonal if $\langle u, v \rangle = 0$.

Let $\{v_1, v_2, \ldots, v_k\}$ be an orthogonal set of vectors in the inner product space V.

Then $\{v_1, v_2 \ldots, v_k\}$ is a linearly independent set of vectors.

Let V be an inner product space. A basis consisting of mutually orthogonal vectors is called an orthogonal basis.

Examples of orthogonal bases :

1. the standard basis.
2. $\{(4, 3, -2), (-3, 2, -3), (-5, 18, 17)\} \subseteq \mathbb{R}^3$.
3. consider $\mathbb{R}^2$ with the inner product
   $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1 y_1 - (x_1 y_2 + x_2 y_1) + 2 x_2 y_2$.
   Then $\{(1, 1), (1, 0)\}$ is an orthogonal basis.

An orthonormal set of vectors of an inner product space V is an orthogonal set of vectors such that the norm of each vector of the set is 1.

Let V be an inner product space. If $\Gamma = \{v_1, v_2, \ldots, v_k\}$ is an orthogonal set of vectors, then we can obtain an orthonormal set of vectors β from Γ by

$$\beta = \left\{ \frac{v_1}{||v_1||}, \frac{v_2}{||v_2||}, \ldots, \frac{v_k}{||v_k||} \right\} \quad .$$

An orthonormal basis is an orthonormal set of vectors which forms a basis.

Equivalently : An orthonormal basis is an orthogonal basis where the norm of each vector is 1.

Let A be the coefficient matrix of the given system of linear equations. Let a matrix B contain the column vectors of A, which are normalized by their respective norms, as its columns (i.e. first column vector of A normalized by its norm is the first column of B). Which of the following statements are true ?

- [x] The determinant of BB^T is 1.
- [x] BB^T is an identity matrix.
- [x] BB^T is a scalar matrix.
- [x] BB^T is a diagonal matrix.

Gram-Schmidt process to find orthonormal basis from a set of basis vectors

Consider the basis $\beta = \{(1,2,2), (-1,0,2), (0,0,1)\}$ for $\mathbb{R}^3$. Can we use this to obtain an orthonormal basis for $\mathbb{R}^3$?

Let $v_1 = (1,2,2)$. We want a vector which is orthogonal to v_1, i.e. a vector in $\langle v_1 \rangle^{\perp}$, so we use the projection P_{v_1} to v_1.

$$
\begin{aligned}
\text{Define } v_2 &= (-1,0,2) - P_{v_1}((-1,0,2)) = \\
&= (-1,0,2) - \frac{\langle (-1,0,2),(1,2,2)\rangle}{\langle (1,2,2),(1,2,2)\rangle}(1,2,2) \\
&= \left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right) .
\end{aligned}
$$

$$
\begin{aligned}
\text{Define } v_3 &= (0,0,1) - P_{v_1}((0,0,1)) - P_{v_2}((0,0,1)) \\
&= (0,0,1) - \frac{\langle (0,0,1)(1,2,2)\rangle}{\langle (1,2,2),(1,2,2)\rangle}(1,2,2) \\
&\quad - \frac{\langle (0,0,1), \left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right)\rangle}{\langle \left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right), \left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right)\rangle}\left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right) \\
&= \left(\frac{2}{9}, -\frac{2}{9}, \frac{1}{9}\right) .
\end{aligned}
$$

Check $\langle v_1, v_2 \rangle = \langle v_1, v_3 \rangle = \langle v_2, v_3 \rangle = 0$

Thus $\{v_1, v_2, v_3\}$ is an orthogonal basis and dividing each vector by its norm yields an orthonormal basis

$$
\left\{ \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right), \left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right), \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right) \right\}.
$$

3) Suppose W_1 and W_2 are subspaces of a vector space V. Let P_{W_1} and P_{W_2} denote the projection from V to W_1 a
V to W_2 respectively. Which of the following statements is true?

Then,

◯ If $P_{W_1} + P_{W_2}$ is a projection from V to $W_1 + W_2$, then $P_{W_1} \circ P_{W_2} + P_{W_2} \circ P_{W_1} = 0$.

Week10

To prove continuity/discontinuity of a curve in two variables x and y at a specified point, consider y = mx. After substituting, if you get an expression in m, then it means the curve isn't continuous since the limit value at the specified point depends on the value of m.