Notes

- For a function to be considered sequence, its domain must be in natural numbers.
- If the limit of a function from the left side is different than the right side, then limit doesn't exist
- If the left-hand limit and right-hand limit of a function exist and are equal, then the limit exists and is equal to either.
- Inverse of a non-differentiable function is also not differentiable
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Some limits

- $\lim_{x \to 0} \frac{sin(x)}{x} = 1$
- $\lim_{x \to 0} \frac{log(1+x)}{x} = 1$
- $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

3. Let $x \in \mathbb{R}$. Then the sequence $\left\{ \sum_{k=0}^{n} \frac{x^k}{k!} \right\}$ is convergent and converges to e^x.

4. Let $x \in \mathbb{R}$. Then $\left\{ \left(1 + \frac{x}{n}\right)^n \right\}$ converges to e^x.

5. The sequence $\left\{ n \left(\frac{\sqrt{2\pi n}}{n!} \right)^{\frac{1}{n}} \right\}$ converges to e.

6. The sequence $\left\{ \frac{n}{\sqrt[n]{n!}} \right\}$ converges to e.

Useful rules regarding convergence of sequences

1. If $a_n \to a$, then every subsequence of $\{a_n\}$ also converges to a.
2. If $a_n \to a$ and $b_n \to b$, then $a_n + b_n \to a + b$.
3. If $a_n \to a$ and $c \in \mathbb{R}$, then $ca_n \to ca$.
4. If $a_n \to a$ and $b_n \to b$, then $a_n - b_n \to a - b$.
5. If $a_n \to a$ and $b_n \to b$, then $a_n b_n \to ab$.
6. If $a_n \to a$ and f is a polynomial function in one variable, then $f(a_n) \to f(a)$.
7. If $a_n \to a$ and $b_n \to b$ and $b \neq 0$, then $\dfrac{a_n}{b_n} \to \dfrac{a}{b}$.
8. If $a_n \to a$ and $c \in \mathbb{R}$, then $c^{a_n} \to c^a$.
9. If $a_n \to a$ and $c \in \mathbb{R}$ such that $a_n > 0 \forall n$ and $a, c > 0$, then $log_c(a_n) \to log_c(a)$.
10. The sandwich principle : If $a_n \to a$ and $b_n \to a$ and $\{c_n\}$ is a sequence such that $a_n \leq c_n \leq b_n$, then $c_n \to a$.

Continuity

Defintion : f is continuous at a if the limit of f at a exists and $\lim_{x \to a} f(x) = f(a)$. f is continuous at a is equivalent to $f(a_n) \to f(a)$ whenever $a_n \to a$.

Equation of the tangent (Linear approximation)

Let f be a function differentiable at the point a. Then the tangent to f at a exists and is given by

$$y = f'(a)(x - a) + f(a).$$

Examples follow:

$f(x) = 5x^3 - 17x^2 + \pi x - 0.5 \quad ; \quad a = 0.$

$f'(x) = 15x^2 - 34x + \pi$. Eqn. of tangent to f at 0 is

$f'(0) = \pi$. $\therefore$ $y = \pi(x-0) + f(0) = \pi x - 0.5$.

$f(x) = cos(x) \quad ; \quad a = \frac{\pi}{3}$

$f'(x) = -\sin(x)$

$f'(\pi/3) = -\sin(\pi/3) = -\sqrt{3}/2$ $\quad \therefore \; y = -\frac{\sqrt{3}}{2}(x - \pi/3) + \cos(\pi/3)$

$= -\frac{\sqrt{3}}{2}(x - \pi/3) + \frac{1}{2}$.

$f(x) = xtan(x) \quad ; \quad a = \frac{\pi}{4}$

$f'(x) = 1 \times \tan(x) + x \times \sec^2(x)$

$= \tan x + x \sec^2(x)$ $\quad \therefore \; y = (1 + \frac{\pi}{2})(x - \pi/4) + \frac{\pi}{4}$

$f'(\pi/4) = \tan(\pi/4) + \frac{\pi}{4}\sec^2(\pi/4)$

$= 1 + \frac{\pi}{4} \times 2 = 1 + \frac{\pi}{2}$.

Critical points

A point a is called a critical point of a function f(x) if either f is not differentiable at a or f'(a)=0

Every turning point is a critical point, but converse is not true. Critical points include saddle points also. Include local extrema of a function f on a closed interval I = [a, b]. Thus, to find the maximum and minimum, we find the critical points and the boundary points and check the value of f on all of them.

$f(x) = x^3 - 12x$

$f'(x) = 3x^2 - 12$ Setting it to 0, $3x^2 - 12 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$.

we obtain Critical points: ± 2. $f''(x) = 6x$.

$f''(2) = 12 > 0$ $\quad f''(-2) = -12 < 0$

$\therefore$ 2 is a local minimum & -2 is a local maximum

Reimann sum

The Riemann sum of f w.r.t. the above data is defined as

$$S(P) = \sum_{i=1}^{n} f(x_i^*)\Delta x_i.$$

Fundamental theorem of calculus

Suppose f is continuous on the domain D which includes the interval $[a, b]$. Then an anti-derivative for f on (a, b) is given by

$$F(x) = \int_a^x f(t)dt.$$

Conversely, if f is continuous on the domain D which includes the interval $[a, b]$ and F is the (indefinite) integral of f, then the (definite) integral from a to b of f can be computed by

$$\int_a^b f(x)dx = F(b) - F(a).$$

$\frac{d}{dx}$ a^x

$$\ln(a)a^x$$

Some common integrals

$\int 1\ dx = x + c$

$\int a\ dx = ax + c$

$\int x^n\ dx = \frac{x^{n+1}}{n+1} + c$, for $n \neq -1$

$\int sin\ x\ dx = -cos\ x + c$

$\int cos\ x\ dx = sin\ x + c$

$\int sec^2\ x\ dx = tan\ x + c$

$\int cosec^2\ x\ dx = -cot\ x + c$

$\int sec\ x\ tan\ x\ dx = sec\ x + c$

$\int cosec\ x\ cot\ x\ dx = -cosec\ x + c$

$\int \frac{1}{x}\ dx = ln|x| + c$

$\int e^x\ dx = e^x + c$

$\int a^x\ dx = \frac{a^x}{lna} + c$, for $a > 0$, and $a \neq 1$

$\int \frac{1}{\sqrt{1-x^2}}\ dx = sin^{-1}\ x + c$

$\int \frac{1}{1+x^2}\ dx = tan^{-1}\ x + c$

$\int \frac{1}{|x|\sqrt{x^2-1}}\ dx = sec^{-1}\ x + c$