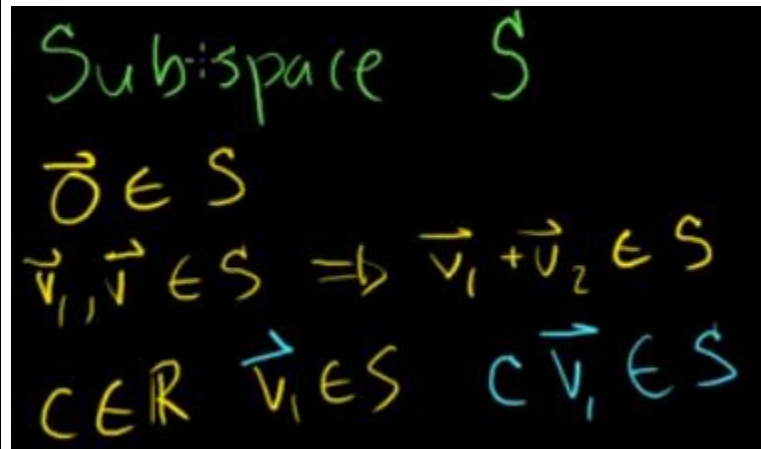
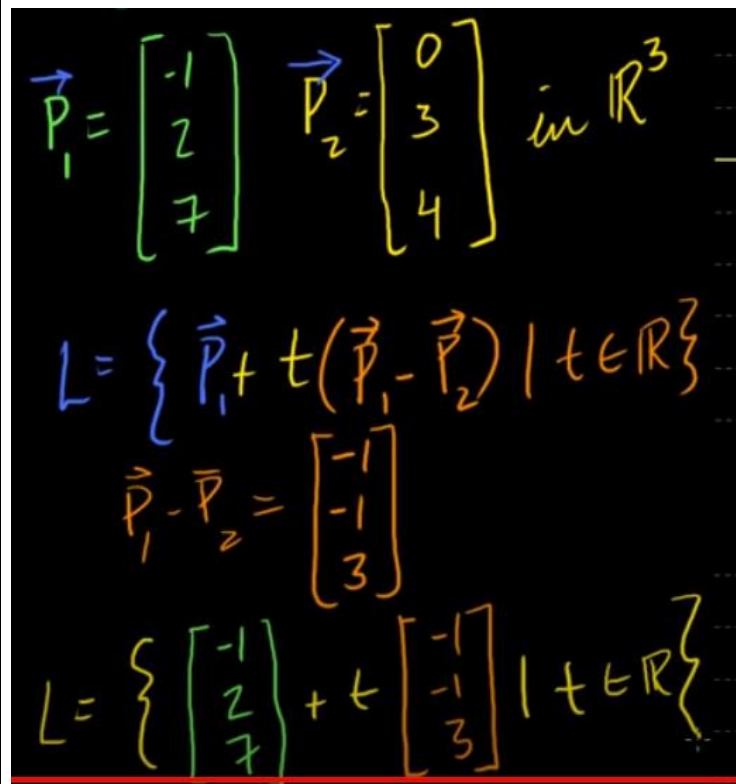


Properties of a subspace



Subspace S
 $\vec{0} \in S$
 $\vec{v}_1, \vec{v}_2 \in S \Rightarrow \vec{v}_1 + \vec{v}_2 \in S$
 $c \in \mathbb{R} \quad \vec{v}_1 \in S \quad c\vec{v}_1 \in S$

Equation of a line that passes through P1 and P2 in \mathbb{R}^3



$\vec{P}_1 = \begin{bmatrix} -1 \\ 2 \\ 7 \end{bmatrix} \quad \vec{P}_2 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} \text{ in } \mathbb{R}^3$
 $L = \left\{ \vec{P}_1 + t(\vec{P}_1 - \vec{P}_2) \mid t \in \mathbb{R} \right\}$
 $\vec{P}_1 - \vec{P}_2 = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$
 $L = \left\{ \begin{bmatrix} -1 \\ 2 \\ 7 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} \mid t \in \mathbb{R} \right\}$

This implies a line in \mathbb{R}^3 can be represented using the following 3 equations.

$$\begin{aligned}x &= -1 + -1t \\y &= 2 + -1t \\z &= 7 + 3t\end{aligned}$$

<https://www.youtube.com/watch?v=hWhs2clj7Cw>

Square of the length of a vector is dot product of the vector with itself

$$\begin{aligned}\|\vec{a}\| &= \sqrt{a_1^2 + a_2^2 + \dots + a_n^2} & \|\vec{a}\| &= \sqrt{\vec{a} \cdot \vec{a}} \\ \vec{a} \cdot \vec{a} &= \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = a_1^2 + a_2^2 + \dots + a_n^2 & \boxed{\|\vec{a}\|^2 = \vec{a} \cdot \vec{a}}\end{aligned}$$

https://www.youtube.com/watch?v=WNulhXo39_k

Cauchy Swarz inequality

$$\|\vec{y}\| \|\vec{x}\| \geq |\vec{x} \cdot \vec{y}|$$

If $\vec{x} = c\vec{y}$, then this turns to an equality.

Watch the entire proof at https://www.youtube.com/watch?v=r2PogGDI8_U

Triangle inequality

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

If $\vec{x} = c\vec{y}$, then this turns to an equality.

Watch the entire proof at <https://www.youtube.com/watch?v=PsNidCBr5II>

Angle between two vectors of any dimension.

$$(\vec{a} \cdot \vec{b}) = \|\vec{a}\| \|\vec{b}\| \cos \theta$$

This implies if two vectors are perpendicular/orthogonal, dot product of the two vectors is 0.

Watch the entire proof at https://www.youtube.com/watch?v=5AWob_z74Ks

Given an R3 plane containing the point (x_0, y_0, z_0) and normal to the plane is represented by (n_1, n_2, n_3) , equation of the plane is given as:

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$$

Watch the entire proof at <https://www.youtube.com/watch?v=UJxgcVaNTqY>

Cross-product of two vectors in R3 is defined as

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \vec{a} \times \vec{b} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

The cross-product is orthogonal to both vectors a and b .

Watch the entire proof at <https://www.youtube.com/watch?v=pJzmiywagfY>

Properties of dot product and cross product

$$\begin{aligned} \vec{a} \cdot \vec{b} &= \|\vec{a}\| \|\vec{b}\| \cos \theta \\ \|\vec{a} \times \vec{b}\| &= \|\vec{a}\| \|\vec{b}\| \sin \theta \end{aligned}$$

Watch the entire proof at <https://www.youtube.com/watch?v=7MKA2QIKvHc>

In the reduced row echelon form,

- if the number of pivot entries is equal to the number of columns, then there's a unique solution.
- If there are free (independent) variables, then there are infinite solutions.
- If you get $0 = \langle a \rangle$, then there's no solution.

Watch the video here <https://www.youtube.com/watch?v=JVDrlTdzxiI>

Matrix multiplication Ax can be interpreted as a linear combination of the column vectors of A , using the weights in x .

Handwritten notes illustrating matrix multiplication as a linear combination of column vectors:

$$A = \begin{bmatrix} 3 & 1 & 0 & 3 \\ 2 & 4 & 7 & 0 \\ -1 & 2 & 3 & 4 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad A\vec{x} = \begin{bmatrix} 3x_1 + 1x_2 + 0x_3 + 3x_4 \\ 2x_1 + 4x_2 + \dots \\ -1x_1 + 2x_2 + \dots \end{bmatrix}$$

Below the matrix, the columns are labeled $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$.

$$A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{v}_4]$$

$$A\vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{v}_4$$

An arrow points to the expression with the text: "Linear combination of column vectors of A"

Or as dot product of transpose of the row vectors in A with x .

Handwritten notes illustrating matrix multiplication as a dot product of row vectors:

$$\begin{bmatrix} -3 & 0 & 3 & 2 \\ 1 & 7 & -1 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \cdot 2 + 0 \cdot (-3) + 3 \cdot 4 + 2 \cdot (-1) \\ 1 \cdot 2 + 7 \cdot (-3) + (-1) \cdot 4 + 9 \cdot (-1) \end{bmatrix}$$

$$= \begin{bmatrix} -6 + 0 + 12 - 2 \\ 2 - 21 - 4 - 9 \end{bmatrix} = \begin{bmatrix} 4 \\ -32 \end{bmatrix}$$

Row vectors are defined as:

$$\vec{a}_1 = \begin{bmatrix} -3 \\ 0 \\ 3 \\ 2 \end{bmatrix} \quad \vec{a}_2 = \begin{bmatrix} 1 \\ 7 \\ -1 \\ 9 \end{bmatrix}$$

The final expression is shown as:

$$\begin{bmatrix} \vec{a}_1^T \\ \vec{a}_2^T \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{a}_1 \cdot \vec{x} \\ \vec{a}_2 \cdot \vec{x} \end{bmatrix}$$

Watch here at <https://www.youtube.com/watch?v=7Mo4S2wyMg4>

Null space is defined as that subspace of the vector space A , such that $Ax = 0$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad N(A) = \{ \vec{x} \in \mathbb{R}^4 \mid A\vec{x} = \vec{0} \}$$

$\vec{x} \in \mathbb{R}^4$

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 + 2x_2 + 3x_3 + 4x_4 &= 0 \\ 4x_1 + 3x_2 + 2x_3 + x_4 &= 0 \end{aligned}$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 & 0 \\ 4 & 3 & 2 & 1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow \begin{aligned} x_1 - x_3 - 2x_4 &= 0 & x_1 &= x_3 + 2x_4 \\ x_2 + 2x_3 + 3x_4 &= 0 & x_2 &= -2x_3 - 3x_4 \end{aligned}$$

$$A\vec{x} = \vec{0}$$

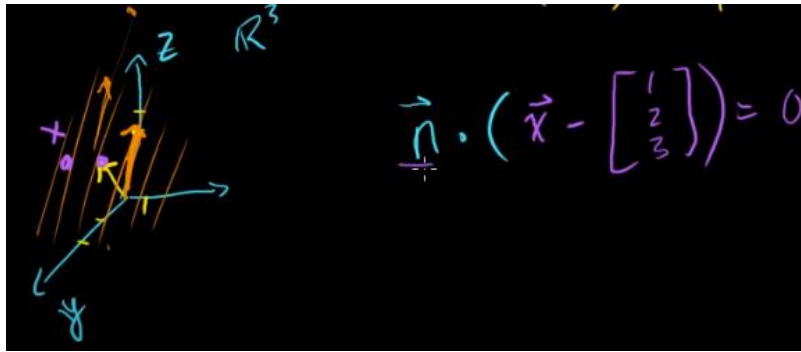
$$N(A) = \text{span} \left(\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

Null space of a vector space A is zero matrix, if and only if A is linearly independent, in which case it has zero free variables in RREF.

Two different ways to find the equation of an R^3 plane.

1. Using the normal vector, and two vectors in the plane like this:



where n is the normal vector. To

get n , find the cross product of the given vectors.

2. Using the generic vector $[x, y, z]$ and applying rref.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{bmatrix} \begin{matrix} x \\ y \\ z \end{matrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & x \\ 0 & 1 & -2 & -1 & 2x-y \\ 0 & 1 & -2 & -1 & z-3x \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & -2 & -1 & 2x-y \\ 0 & 0 & 0 & 0 & 2x-y-z+3x \end{bmatrix}$$

$$2x-y-z+3x=0 \Rightarrow 5x-y-z=0$$

Watch the entire video at <https://www.youtube.com/watch?v=EGNIXtjYABw>

Any arbitrary linear transformation of a vector can be represented as a product of a matrix where each column is the transformation of the basis vectors, and the given vector.

$$\vec{x} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + \dots + x_n \vec{e}_n$$

$$T(\vec{x}) = T(x_1 \vec{e}_1 + x_2 \vec{e}_2 + \dots + x_n \vec{e}_n)$$

$$= T(x_1 \vec{e}_1) + T(x_2 \vec{e}_2) + \dots + T(x_n \vec{e}_n)$$

$$= x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + \dots + x_n T(\vec{e}_n)$$

$$T(\vec{x}) = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \dots & T(\vec{e}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

all L.T. matrix vector product

Watch the entire working at <https://www.youtube.com/watch?v=PErhLkQcpZ8>

Linear transformation of a shape is demoed at <https://www.youtube.com/watch?v=MIAmN5kgp3k>

Sum of linear transformation S and T (each represented by matrices A and B). can be represented by A + B.

$$\begin{aligned}(S+T)(\vec{x}) &= S(\vec{x}) + T(\vec{x}) \\ &= A\vec{x} + B\vec{x} \\ &= (A+B)\vec{x}\end{aligned}$$

Similarly, scalar multiple of the transformation S (represented by A) can be represented as scalar times A.

Watch the full proof at <https://www.youtube.com/watch?v=wHuY97vss18>

Rotating a vector in \mathbb{R}^2 is a linear transformation. Assuming the angle of rotation is θ , the transformation matrix is

$$\begin{aligned}\text{Rot}_\theta: \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ \text{Rot}_\theta(\vec{x}) &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\end{aligned}$$

Watch the full proof at <https://www.youtube.com/watch?v=IPWflq5Dzql>

Projection of a vector x on another vector L can be derived as follows.

$\text{Proj}_L(\vec{x}) = \text{some vector in } L$
 where $\vec{x} - \text{Proj}_L(\vec{x})$
 is orthogonal to L
 $= c\vec{v}$
 $(\vec{x} - c\vec{v}) \cdot \vec{v} = 0$
 $\vec{x} \cdot \vec{v} - c\vec{v} \cdot \vec{v} = 0$
 $\vec{x} \cdot \vec{v} = c\vec{v} \cdot \vec{v}$

NOTE: L can be represented as a scalar multiple of v .

Hence, projection is $\text{Proj}_L(\vec{x}) = c\vec{v} = \left(\frac{\vec{x} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \right) \vec{v}$

Watch details of the proof at <https://www.youtube.com/watch?v=27vT-NWuw0M>

Projection of a vector x onto another vector v can be rewritten as $\text{Proj}_L(\vec{x}) = (\vec{x} \cdot \hat{u}) \hat{u}$, given u is the unit vector in the direction of v . It can also be proved that this projection is a linear transformation, represented by the matrix below.

$A = \left[\begin{array}{cc} ([u_1] \cdot [u_1]) [u_1] & ([u_1] \cdot [u_2]) [u_1] \\ ([u_2] \cdot [u_1]) [u_2] & ([u_2] \cdot [u_2]) [u_2] \end{array} \right]$

that simplifies to

$A = \begin{bmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{bmatrix}$

Watch the full proof at <https://www.youtube.com/watch?v=JK-8XNloAkl>

Composition of two linear transformations can be represented by the matrix obtained by multiplying the individual matrices that represent each transformation. Thus, if B represents the T and A represents S, then the composition of the two transformations is $S \circ T(x) = AB$.

Watch the full proof at <https://www.youtube.com/watch?v=BugcKpe5ZQs>

If f is invertible, then $f(x) = y$ has a unique solution for x. This implies that there's only one x-value that produces the specified y-value.

Watch the full explanation at <https://www.youtube.com/watch?v=7GEUgRcnfVE>

A function f is invertible, only if it's both surjective and injective.

Watch the full explanation at <https://www.youtube.com/watch?v=QIU1daMN8fw>

A linear transformation from R^n to R^m is onto (surjective) only if the column space of the transformation matrix A equals the co-domain. In other words, $\text{rref}(A)$ has a pivot entry for every row in A. This means, the given transformation is surjective only if rank of the transformation matrix is m.

Watch the full explanation at <https://www.youtube.com/watch?v=eR8vEdJTvd0>

For $Ax = b$, the solution set comprises of a particular vector, combined with a null-space.

$$A\vec{x} = \vec{b}$$

$$[A | \vec{b}]$$

$$\downarrow$$

$$[\text{rref}(A) | \vec{b}']$$

$$\vec{x} = \underbrace{\vec{b}'}_{\vec{x}_p} + \underbrace{a\vec{n}_1 + b\vec{n}_2 + \dots + c\vec{n}_k}_{\vec{x}_n}$$

A linear transformation is one-to-one only if the null space is empty. In this case, column vectors of matrix A are linearly independent.

Watch the full explanation at <https://www.youtube.com/watch?v=M3FuL9qKTBs>

For a linear transformation to be invertible, dimensions of the input and output space must be same. This means, the transformation matrix A must be a square matrix. Also implies that the $\text{rref}(A)$ is the identity matrix.

Watch more explanation at <https://www.youtube.com/watch?v=Yz2OosyMTmY>