NLO QED Calculation of Pair Production Process at Lepton Colliders

B.Tech project submitted by

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Declaration

This is to declare that the project titled **NLO QED calculation of pair production process at lepton colliders** submitted by me to the Department of Physics, Indian Institute of Technology, Guwahati, for the partial fulfillment of the requirement for the degree of **Tarannum** is a bonafide work carried out by me under the supervision of **Dr. M. C. Kumar**.

The content of this work, in full or in parts, has not been submitted elsewhere for the award of any degree or diploma. I also declare that this report is based on my personal research and I have provided all the references and resources used in its preparation.

Name: Tarannum
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Certificate

This is to certify that the work contained in this project titled **NLO QED calculation of pair production process at lepton colliders** is a bonafide work of Tarannum (Roll. 200121058), carried out in the Department of Physics, IIT Guwahati under my supervision and has not been submitted elsewhere for a degree.

Signature of the supervisor:

Acknowledgment

I am profoundly thankful to my project supervisor, **Dr. M. C. Kumar**, for their invaluable guidance, continuous support, and expert advice throughout the entire research process. Their insights and encouragement have significantly enriched the quality of this project.
I extend my thanks to the Physics Department, IIT Guwahati for providing access to resources, facilities, and a conducive academic environment. The scholarly atmosphere has been instrumental in shaping the outcome of this project.

Contents

1 Abstract

We present theoretical analysis of e^+e^- annihilation into $\mu^+\mu^-$ at different energy scales taking into account full next-to-leading order.

This project focuses on conducting a comprehensive Next-to-Leading Order (NLO) Quantum Electrodynamics (QED) calculation for the pair production process at lepton colliders. The study centers on precise theoretical predictions for lepton pair production, such as electron-positron (e^+e^-) or muon-antimuon ($\mu^+\mu^-$), incorporating radiative corrections up to the NLO level. Emphasizing the importance of quantum corrections beyond tree-level accuracy, the project aims to refine predictions for scattering amplitudes, cross-sections, and associated observables at lepton colliders.

Here we enhance our understanding of lepton pair production at lepton colliders, providing more accurate predictions that align with experimental data.

2 Introduction

Lepton colliders, such as the Large Electron-Positron Collider (LEP) and potential future facilities, play a pivotal role in advancing our understanding of fundamental particle interactions. Among the processes scrutinized at these colliders, lepton pair production, specifically electron-positron (e^+e^-) and muon-antimuon ($\mu^+\mu^-$) production, stands as a cornerstone process. Theoretical descriptions of such interactions demand precision beyond the conventional tree-level calculations. Quantum corrections at the Next-to-Leading Order (NLO) in QED become essential to refine predictions and align theoretical models with experimental observations.

In processes involving the production of particle-antiparticle pairs at lepton colliders, quantum electrodynamics (QED) plays a central role. At leading order (LO), these processes are described by tree-level Feynman diagrams. However, to achieve higher precision in theoretical predictions that match experimental data, it is essential to include higher-order corrections.

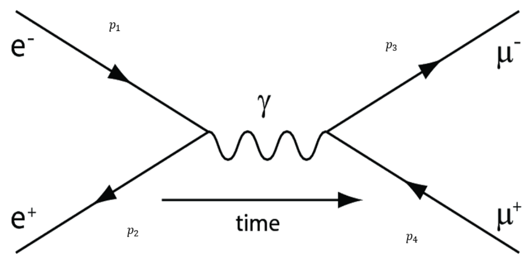
Next-to-leading-order (NLO) calculations in QED provide crucial corrections beyond LO, accounting for virtual and real photon emissions, as well as interference effects between different Feynman diagrams. These corrections are essential for accurately predicting observables such as cross sections, differential distributions, and event shapes.

In this pursuit, this project embarks on NLO QED calculation for lepton pair production, addressing the intricacies associated with collinear and soft photon emissions. The outcomes of this research are poised to advance our comprehension of fundamental particle processes, enhance the precision of theoretical predictions, and contribute to the ongoing synergy between theory and experiment in the realm of particle physics.

3 Process: $e^- + e^+ \longrightarrow \mu^- + \mu^+$

3.1 Unpolarized Cross-section

Amplitude for the process using Feynman rule:



$$= \bar{v}^{s'}(p')\big(-ie\gamma^{\mu}\big)u^{s}(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}^{r}(k)\big(-ie\gamma^{\nu}\big)v^{r'}(k').$$

Squared matrix element:

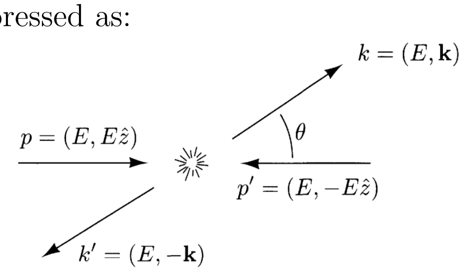
$$|\mathcal{M}|^2 = \frac{e^4}{q^4}\Big(\bar{v}(p')\gamma^{\mu}u(p)\bar{u}(p)\gamma^{\nu}v(p')\Big)\Big(\bar{u}(k)\gamma_{\mu}v(k')\bar{v}(k')\gamma_{\nu}u(k)\Big).$$

Simplifying we get final result:

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{8e^4}{q^4}\Big[(p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k) + m_{\mu}^2(p\cdot p')\Big].$$

Here, including the electron mass in the calculation is feasible but inconsequential. This is because the ratio $m_e/m_\mu \approx 1/200$ is significantly smaller than the fractional error introduced by neglecting higher-order terms in the perturbation series.

To derive a more detailed equation, it is necessary to focus on a specific frame of reference and represent the vectors p, p', k, k', and q in relation to the fundamental kinematic variables—energies and angles—in that particular frame. The selection of the frame typically depends on the experimental conditions. In this text, we generally opt for the most straightforward approach, assessing cross sections in the center-of-mass frame. For this particular selection, the initial and final 4-momenta for the process e+e- → mu-mu+ can be expressed as:



$$|\mathbf{k}| = \sqrt{E^2 - m_\mu^2}$$

$$\mathbf{k}\cdot\hat{z} = |\mathbf{k}|\cos\theta$$

To compute the squared matrix element we need

$$q^2 = (p+p')^2 = 4E^2; \qquad p\cdot p' = 2E^2;$$

$$p\cdot k = p'\cdot k' = E^2 - E|\mathbf{k}|\cos\theta; \qquad p\cdot k' = p'\cdot k = E^2 + E|\mathbf{k}|\cos\theta.$$

We can rewrite equation as:

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{8e^4}{16E^4}\Big[E^2(E-|\mathbf{k}|\cos\theta)^2 + E^2(E+|\mathbf{k}|\cos\theta)^2 + 2m_\mu^2 E^2\Big]$$
$$= e^4\Bigg[\left(1+\frac{m_\mu^2}{E^2}\right)+\left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\Bigg].$$

Plugging this expression into the cross-section formula, $|v_A - v_B| = 2$ and $E_A = E_B = E_{cm}/2$:

$$\frac{d\sigma}{d\Omega} = \frac{1}{2E_{\text{cm}}^2}\frac{|\mathbf{k}|}{16\pi^2 E_{\text{cm}}}\cdot\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2$$
$$= \frac{\alpha^2}{4E_{\text{cm}}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\Bigg[\left(1+\frac{m_\mu^2}{E^2}\right)+\left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\Bigg].$$

Integrating over dQ, we find the total cross section:

$$\sigma_{\text{total}} = \frac{4\pi\alpha^2}{3E_{\text{cm}}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left(1+\frac{1}{2}\frac{m_\mu^2}{E^2}\right).$$

In the high-energy limit where $E \gg m_\mu$,

$$\frac{d\sigma}{d\Omega} \underset{E\gg m_\mu}{\longrightarrow} \frac{\alpha^2}{4E_{\text{cm}}^2}\left(1+\cos^2\theta\right);$$
$$\sigma_{\text{total}} \quad \underset{E\gg m_\mu}{\longrightarrow} \frac{4\pi\alpha^2}{3E_{\text{cm}}^2}\left(1-\frac{3}{8}\left(\frac{m_\mu}{E}\right)^4-\cdots\right).$$

4 Radiative Corrections

Up to this point, our focus has been exclusively on tree-level processes, referring to diagrams devoid of loops. However, these processes inevitably incorporate higher-order contributions, termed radiative corrections, originating from diagrams that feature loops. In Quantum Electrodynamics (QED), bremsstrahlung, which involves the emission of additional final-state photons during a reaction, is another significant source of radiative corrections. In the upcoming exploration, we will delve into both categories of radiative corrections and discover the inconsistency of incorporating one without also considering the other.

Radiative corrections are quantum corrections that arise from the exchange of virtual particles, particularly photons, in loop diagrams. They play a fundamental role in refining theoretical predictions, ensuring the consistency of quantum field theory with experimental results, and accounting for quantum fluctuations in particle interactions.

Photon Emission and Absorption: In quantum field theory, particles can emit and absorb virtual photons as they interact with the electromagnetic field. These virtual photons correspond to loops in Feynman diagrams and contribute to the radiative corrections.

4.1 Reasons for Radiative Corrections:

Quantum Fluctuations: In the quantum realm, particles are subject to fluctuations and uncertainties. Virtual particles, such as virtual photons, represent quantum fluctuations in the electromagnetic field.

Higher-Order Processes: Radiative corrections are associated with higher-order processes beyond the leading-order interactions described by tree-level diagrams. These corrections become increasingly important at higher energies and precision levels.

Quantum Field Theory Precision: Radiative corrections are crucial for achieving precision in quantum field theory predictions. They refine theoretical predictions to match experimental measurements more accurately.

Experimental Consequences: Radiative corrections often have observable consequences, influencing measurable quantities such as cross-sections and decay rates. Accurate theoretical predictions, including radiative corrections, are essential for interpreting experimental data and testing the validity of the underlying theory.

5 Perturbative Expansion

The QED Lagrangian is given by:

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

The scattering amplitude M for $e^- + e^+ \longrightarrow \mu^- + \mu^+$ in perturbative expansion:

$$M = M_0 + eM_1 + e^2 M_2 + \dots$$

where:

$$\begin{aligned} &M_0 \text{ is the leading-order term,} \\ &M_1 \text{ is the first-order correction,} \\ &M_2 \text{ is the second-order correction,} \\ &\quad \dots . \end{aligned}$$

In Quantum Electrodynamics (QED), the primary contribution to a cross section or decay rate typically stems from the Feynman diagram with the least number of interaction vertices, referred to as the lowest-order (LO) diagram. For the annihilation process $e^+e^- \to \mu^+\mu^-$, a singular lowest-order QED diagram is present, depicted in Figure below. Within this diagram, two QED interaction vertices exist, each contributing a factor of $ie\gamma^\mu$ to the matrix element. Consequently, irrespective of other considerations, the squared matrix element $|M|^2$ will be proportional to e^4 or, equivalently, $|M|^2 \propto \alpha^2$, where α denotes the dimensionless fine-structure constant $\alpha = \frac{e^2}{4\pi}$. In a general context, each QED vertex introduces a factor of α into the expressions for cross sections and decay rates.

Apart from the lowest-order diagram illustrated in Figure, there exists an infinite array of higher-order diagrams leading to the same final state. For instance, three of the next-to-leading-order (NLO) diagrams for $e^+e^- \to \mu^+\mu^-$, each featuring four interaction vertices, are depicted in Figure. Considered in isolation, the squared matrix element for each of these diagrams incorporates a factor of α for each of the four QED vertices, resulting in $|M|^2 \propto \alpha^4$. However, in quantum mechanics, individual Feynman diagrams for a specific process cannot be considered independently; the total amplitude M_{fi} for a given process is the sum of all individual amplitudes yielding the same final state. In the case of $e^+e^- \to \mu^+\mu^-$, this sum can be expressed as:
M_{fi}=$M_{\mathrm{LO}} + \sum_j M_{1,j} + \dots$
where M_{LO} represents the lowest-order term, and the summation runs over j, accounting for additional terms.

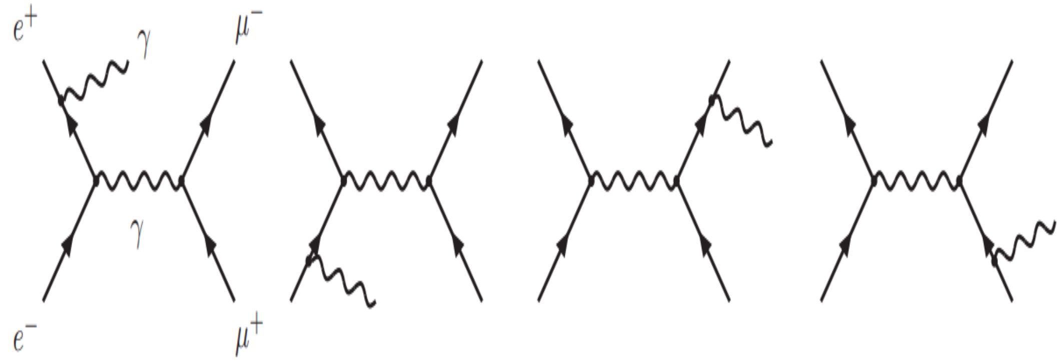
The explicit dependence of each term in the equation on α can be illustrated by expressing it as
$M_{fi} = \alpha M_{\mathrm{LO}} + \alpha^2 \sum_j M_{1,j} + \dots,$
where the terms are explicitly scaled by powers of the fine-structure constant α.
Now matrix element squared given by:

$$|\mathcal{M}_{fi}|^2 = \left(\alpha M_{\mathrm{LO}} + \alpha^2 \sum_j M_{1,\mathrm{j}} + \cdots\right)\left(\alpha M^*_{\mathrm{LO}} + \alpha^2 \sum_k M^*_{1,\mathrm{k}} + \cdots\right)$$
$$= \alpha^2 |M_{\mathrm{LO}}|^2 + \alpha^3 \sum_j \left(M_{\mathrm{LO}} M^*_{1,j} + M^*_{\mathrm{LO}} M_{1,j}\right) + \alpha^4 \sum_{jk} M_{1,j} M^*_{1,\mathrm{k}} + \cdots .$$

In general, individual amplitudes are complex, and contributions from different diagrams can interfere either positively or negatively. Equation presents the QED perturbation expansion in terms of powers of α. In QED, the dimensionless coupling constant $\alpha \approx 1/137$ is small enough that this series converges rapidly, with dominance by the lowest-order (LO) term. The interference terms between the lowest-order diagram and the next-to-leading-order (NLO) diagrams, such as $(M_{\mathrm{LO}} M^*_{1,j} + M^*_{\mathrm{LO}} M_{1,j})$, are suppressed by a factor of $\alpha \approx 1/137$ relative to the lowest-order term. Consequently, if all higher-order terms are neglected, QED calculations can be reasonably expected to be accurate to approximately O(1%).

6 Tree - Level corrections

Radiative corrections to the process $e^+e^- \to \mu^+\mu^-\gamma$ involve both virtual corrections and real photon emission. These corrections are part of the next-to-leading order (NLO) contributions to the scattering amplitude.

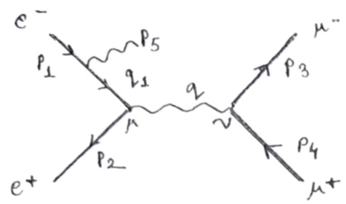


The tree diagrams for $e^+e^- \to \mu^+\mu^-\gamma$

Real photon emission involves the actual creation of a photon during a particle interaction. This photon is a real, on-shell particle that can be detected experimentally.
Real photon emission is represented by specific Feynman diagrams where a photon line is connected to the internal lines of the interaction. These diagrams contribute to next-to-leading order (NLO) and higher-order terms in perturbation theory.
Real photon emission introduces challenges related to collinear and infrared divergences in the calculations. Collinear divergences arise when the emitted photon becomes collinear with a charged particle, and infrared divergences occur in the limit of low photon energies.
Real photon emission is complementary to virtual corrections. While virtual corrections involve internal loops and virtual particles, real photon emission involves the actual creation of a photon in the final state.

6.1 Calculation of Feynman Amplitude:

(i)



$$M_1 = \vec{u}(p_3)\,[-ie\gamma_\mu]\; v(p_4)\left[\frac{-ig^{\mu\nu}}{q^2}\right]\vec{v}(p_2)\,[-ie\gamma_\mu]\;\left[\frac{i(\not{q}_1 + m_e)}{q_1^2 - m_e^2}\right]$$

$$\times [-ie\gamma_\rho]\;\; u(p_1)\;\varepsilon^{*\rho}(p_5)$$

$$= \frac{ie^3}{q^2\,[q_1^2 - m_e^2]^2}\;\left[\vec{u}(p_3)\gamma^\mu v(p_4)\vec{v}(p_2)\gamma_\mu(\not{q}+m_e)\gamma_\rho\; u(p_1)\right]$$

$$M_1^\dagger = \frac{-ie^3}{q^2[q_1^2 - m_e]^2}\;\left[\vec{u}(p_1)\gamma_{\rho'}(\not{q}+m_e)\,\gamma_{\mu'}\, v(p_2)\vec{v}(p_4)\gamma^{\mu'} u(p_3)\right]\,\varepsilon^{\rho'}(p_5)$$

Here, denominator we have $q_1^2 - m_e^2$

we know, $q_1 = p_1 - p_5$

$\Rightarrow q_1^2 = p_1^2 + p_5^2 - 2p_1 \cdot p_5$

$\Rightarrow q_1^2 - m_e^2 = m_e^2 + m_\gamma^2 - 2p_1 \cdot p_5 - m_e^2 \quad = -2p_1 p_5$

For e^-: $p_1 = (E_1, \vec{p}_1) = [E_1, 0, 0, P]$

photon: $p_5 = [E_5, E_5 \sin\theta\cos\phi, E_5 \sin\phi\sin\theta, E_5\cos\theta]$

$p_1 \cdot p_5 = E_1 \cdot E_5 - \mathbb{P} \cdot E_5 \cos\theta$

$= E_5\,(E_1 - \mathbb{P}\cos\theta)$

$E_5 \to 0 \quad \Rightarrow$ soft - singularity

Denominator term :

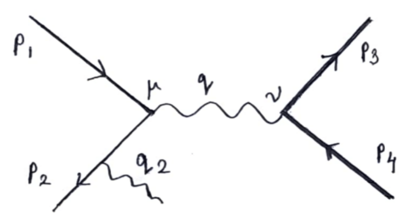
$$\frac{1}{q_1^2 - m_e^2} = \frac{-1}{2E_5\left[E_1 - |\vec{p}_1|\cos\theta\right]}$$

In High - Energy limits : $\quad m \to 0$

$\Rightarrow$ if $\theta \to 0$ denominator vanishes .

$\Rightarrow$ collinear singularity .

(ii)

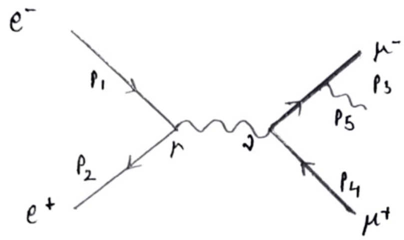


$q_2 = p_2 - p_5$

$$M_2 = \vec{u}(p_3)\left[-ie\gamma_\nu\right]\vec{v}(p_4)\left[\frac{-ig^{\mu\nu}}{q^2}\right]\varepsilon^{*\rho}(p_5)\left[\frac{i(\not{q}_2 + m_e)}{q_2^2 - m_e^2}\right]$$

$$\left[-ie\gamma_\rho\right]\vec{v}(p_2)\left[-ie\gamma_\mu\right]\vec{u}(p_1)$$

(iii)



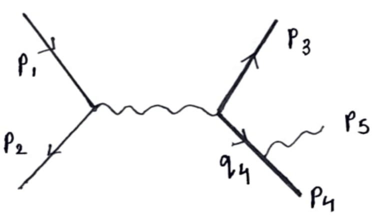
$q_3 = p_3 - p_5$

M_3 :

$$\vec{u}(p_3)\,\varepsilon^{*\rho}(p_5)\left[\frac{i(\not{q}_3+m_e)}{q_3^2-m_e^2}\right][-ie\gamma_\rho]\,[-ie\gamma_\nu]\,v(p_4)$$

$$\times\left[\frac{-ig^{\mu\nu}}{q^2}\right]\vec{v}(p_2)\,[-ie\gamma_\mu]\,u(p_1)$$

(iv)



M_4 :

$$\vec{u}(p_3)\,[-ie\gamma_\nu]\,\varepsilon^{*\rho}(p_5)\left[\frac{i(\not{q}_4+m_e)}{q_4^2-m_e^2}\right][-ie\gamma_\rho]\,v(p_4)$$

$$\times\left[\frac{-ig^{\mu\nu}}{q^2}\right]\vec{v}(p_2)\,[-ie\gamma_\mu]\,u(p_1)$$

7 One-Loop Corrections

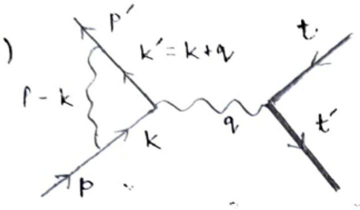
Virtual corrections involve loop diagrams where particles, such as electrons, positrons, or photons, exist only temporarily in intermediate states. These virtual particles are not directly observable but play a crucial role in quantum fluctuations and contribute to the overall quantum amplitudes.
Virtual corrections are represented by loop diagrams in Feynman diagrams, where internal lines represent virtual particles. These diagrams contribute to the quantum amplitudes beyond the tree-level processes and involve integrals over loop momenta.
Virtual corrections are represented by loop diagrams in Feynman diagrams, where internal lines represent virtual particles. These diagrams contribute to the quantum amplitudes beyond the tree-level processes and involve integrals over loop momenta.

Calculating virtual corrections involves complex mathematical expressions, including loop integrals.

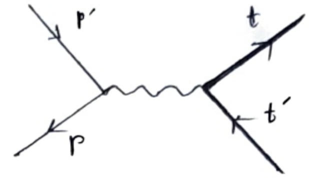
III) Virtual photon : One-loop

$e^- e^+ \longrightarrow \mu^- \mu^+$

(1)

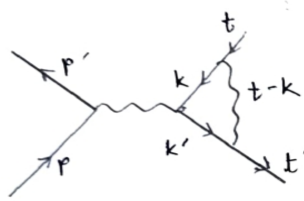


$\times$



$\sim \alpha^3$

(2)



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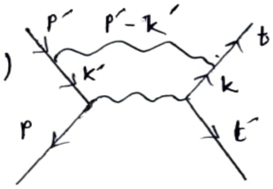

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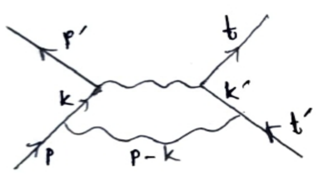

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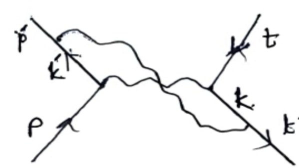

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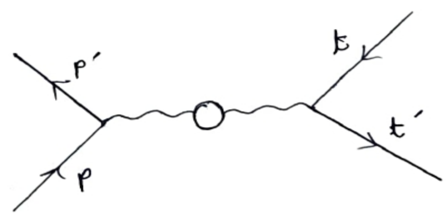

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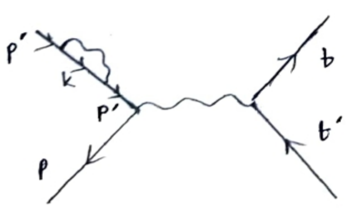

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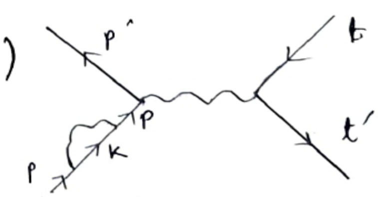

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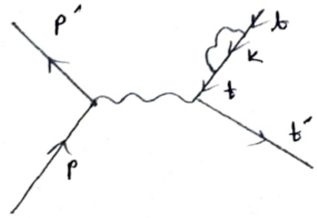

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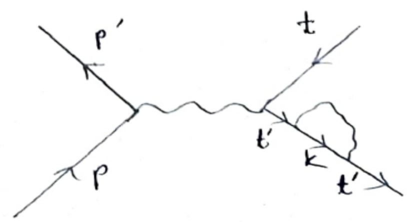


8 Soft Bremsstrahlung

Soft Bremsstrahlung arises as one type of radiative correction in the context of scattering processes involving charged particles. When charged particles scatter, they emit and absorb virtual photons due to their electromagnetic interactions. Soft Bremsstrahlung specifically refers to the emission of low-energy photons in this scattering process. Soft Bremsstrahlung poses a challenge because the emission of soft photons can lead to infrared divergences. These divergences arise because the probability of emitting a soft photon becomes infinite as the photon's energy approaches zero.

8.1 Computation

Considering process in which one photon is radiated during the scattering of an electron. If M_0 is the part of the amplitude that comes from electron's external field interaction then amplitude for the whole process is

$$\begin{aligned} i\mathcal{M} = -ie\bar{u}\left(p'\right) \Bigg(& \mathcal{M}_0\left(p', p-k\right) \frac{i(\not{p}-\not{k}+m)}{(p-k)^2-m^2} \gamma^\mu \epsilon_\mu^*(k) \\ & + \gamma^\mu \epsilon_\mu^*(k) \frac{i\left(\not{p}'+\not{k}+m\right)}{\left(p'+k\right)^2-m^2} \mathcal{M}_0\left(p'+k, p\right) \Bigg) u(p). \end{aligned}$$

For soft radiated photon : $|\mathbf{k}| \ll |\mathbf{p}'-\mathbf{p}|$. Then we can approximate

$$\mathcal{M}_0\left(p', p-k\right) \approx \mathcal{M}_0\left(p'+k, p\right) \approx \mathcal{M}_0\left(p', p\right),$$

ignoring $\not{k}$ in the numerators, simplifying first term:

$$\begin{aligned} (\not{p}+m)\gamma^\mu \epsilon_\mu^* u(p) &= \left[2p^\mu \epsilon_\mu^* + \gamma^\mu \epsilon_\mu^*(-\not{p}+m)\right] u(p) \\ &= 2p^\mu \epsilon_\mu^* u(p). \end{aligned}$$

In second term,

$$\bar{u}\left(p'\right) \gamma^\mu \epsilon_\mu^*\left(p'+m\right) = \bar{u}\left(p'\right) 2p'^\mu \epsilon_\mu^*.$$

denominators simplify to:

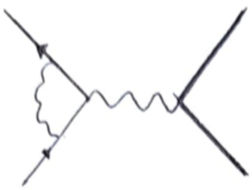
$$(p-k)^2-m^2=-2p\cdot k; \quad \left(p'+k\right)^2-m^2=2p'\cdot k.$$

Therefore the amplitude the soft-photon approximation becomes

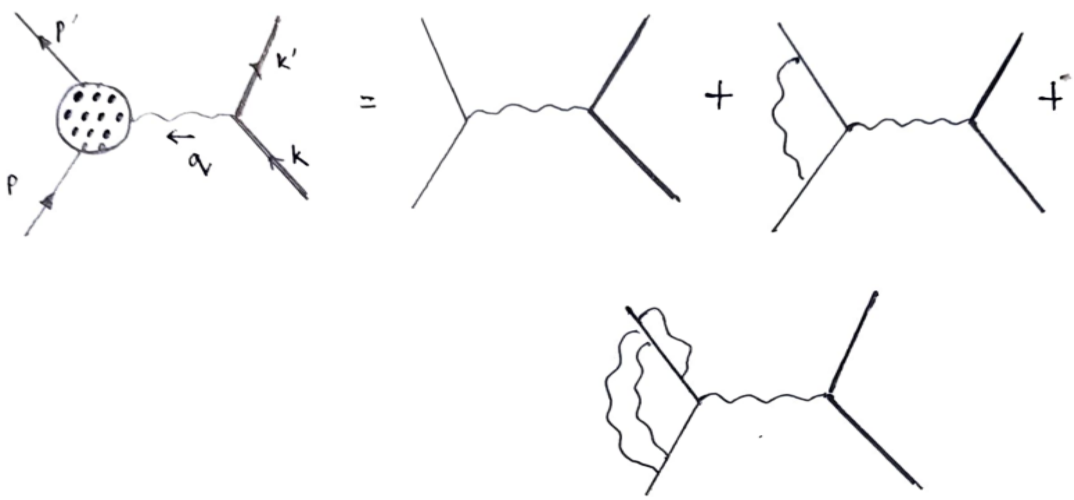
$$i\mathcal{M} = \bar{u}\left(p'\right)\left[\mathcal{M}_0\left(p', p\right)\right] u(p) \cdot \left[e\left(\frac{p'\cdot\epsilon^*}{p'\cdot k} - \frac{p\cdot\epsilon^*}{p\cdot k}\right)\right].$$

9 Formal Structure of Electron Vertex Function

The electron vertex function, also known as the electron-photon vertex or simply the vertex function, describes the interaction between an electron and a photon.



consider following class of diagram



where the gray circle indicates the sum of the lowest-order electron-photon vertex and all amputated loop corrections. We call this sum of vertex diagrams $-ie\Gamma^{\mu}(p',p)$. Then the amplitude for electron scattering from a heavy target is

$$i\mathcal{M} = ie^2\Big(\bar{u}(p')\,\Gamma^{\mu}(p',p)\,u(p)\Big)\frac{1}{q^2}\Big(\bar{u}(k')\gamma_{\mu}u(k)\Big).$$

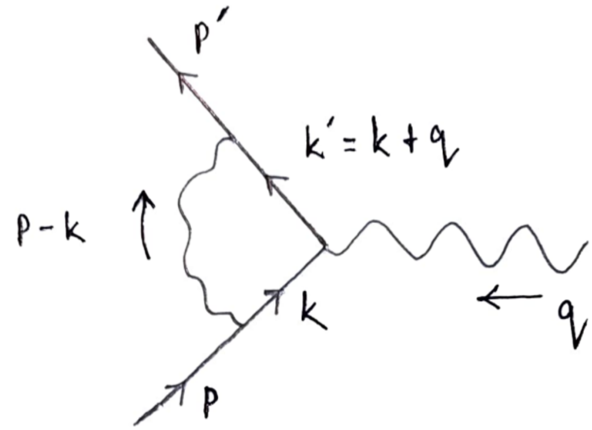
Where

$$\Gamma^{\mu}(p',p) = \gamma^{\mu}F_1(q^2) + \frac{i\sigma^{\mu\nu}q_{\nu}}{2m}F_2(q^2),$$

where F_1 and F_2 are unknown functions of q^2 called *form factors*. To lowest order, $F_1 = 1$ and $F_2 = 0$.
In principle, the form factors can be computed to any order in perturbation theory.

10 Evaluation of Electron Vertex Function

We will evaluate the one-loop contribution to the electron vertex function. Momenta as in diagram:



Using Feynman rules to evaluate in order α, that $\Gamma^{\mu} = \gamma^{\mu} + \delta\Gamma^{\mu}$, where

$$
\begin{aligned}
&\bar{u}\,(p')\,\delta\Gamma^{\mu}\,(p',p)\,u(p)\\
&= \int \frac{d^4k}{(2\pi)^4}\frac{-ig_{\nu\rho}}{(k-p)^2+i\epsilon}\bar{u}\,(p')\,(-ie\gamma^{\nu})\,\frac{i\,(\not{k}'+m)}{k'^2-m^2+i\epsilon}\gamma^{\mu}\frac{i(\not{k}+m)}{k^2-m^2+i\epsilon}\,(-ie\gamma^{\rho})\,u(p)\\
&= 2ie^2\int\frac{d^4k}{(2\pi)^4}\frac{\bar{u}\,(p')\,[\not{\gamma}^{\mu}\ \not{k}'+m^2\gamma^{\mu}-2m\,(k+k')^{\mu}]\,u(p)}{((k-p)^2+i\epsilon)\,(k^2-m^2+i\epsilon)\,(k^2-m^2+i\epsilon)}.
\end{aligned}
$$

We used the contraction identity $\gamma^{\nu}\gamma^{\mu}\gamma_{\nu} = -2\gamma^{\mu}$.
The $+i\epsilon$ terms in the denominators cannot be dropped; they are necessary for proper evaluation of the loop-momentum integral.

To calculate the integral we'll use the method of Feynman parameters.

10.1 Feynman Parametersiation

We use this method to squeeze the denominator into a single quadratic polynomial in k, raised to the third power.
Implementing the identity:

$$\frac{1}{AB} = \int_0^1 dx\frac{1}{[xA+(1-x)B]^2} = \int_0^1 dxdy\delta(x+y-1)\frac{1}{[xA+yB]^2}.$$

for example:

$$
\begin{aligned}
\frac{1}{(k-p)^2\,(k^2-m^2)} &= \int_0^1 dxdy\delta(x+y-1)\frac{1}{\left[x(k-p)^2+y\,(k^2-m^2)\right]^2}\\
&= \int_0^1 dxdy\delta(x+y-1)\frac{1}{\left[k^2-2xk\cdot p+xp^2-ym^2\right]^2}
\end{aligned}
$$

If we now let $\ell \equiv k - xp$, we see that the denominator depends only on ℓ^2. Integrating over d^4k would now be much easier, since $d^4k = d^4\ell$ and the integrand is spherically symmetric with respect to ℓ. The variables x and y that make this transformation

possible are called Feynman parameters.
The more general formula we will use is

$$\frac{1}{A_1 A_2 \cdots A_n} = \int_0^1 dx_1 \cdots dx_n \delta\left(\sum x_i - 1\right) \frac{(n-1)!}{\left[x_1 A_1 + x_2 A_2 + \cdots x_n A_n\right]^n}.$$

Applying formula to the denominator:

$$\frac{1}{\left((k-p)^2 + i\epsilon\right)\left(k'^2 - m^2 + i\epsilon\right)\left(k^2 - m^2 + i\epsilon\right)} = \int_0^1 dx dy dz \delta(x+y+z-1) \frac{2}{D^3}$$

where the new denominator D is

$$\begin{aligned} D &= x\left(k^2 - m^2\right) + y\left(k^2 - m^2\right) + z(k-p)^2 + (x+y+z)i\epsilon \\ &= k^2 + 2k \cdot (yq - zp) + yq^2 + zp^2 - (x+y)m^2 + i\epsilon. \end{aligned}$$

In the second line we have used $x+y+z=1$ and $k' = k+q$. Now shift k to complete the square:

$$\ell \equiv k + yq - zp.$$

After a bit of algebra we find that D simplifies to

$$D = \ell^2 - \Delta + i\epsilon,$$

where

$$\Delta \equiv -xyq^2 + (1-z)^2 m^2.$$

Since $q^2 < 0$ for a scattering process, Δ is positive; we can think of it as an effective mass term.

Now we are expressing the numerator of in terms of ℓ. This task is simplifed by noting that since D depends only on the magnitude of ℓ,

$$\begin{aligned} &\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\mu}{D^3} = 0 \\ &\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\mu \ell^\nu}{D^3} = \int \frac{d^4\ell}{(2\pi)^4} \frac{\frac{1}{4} g^{\mu\nu} \ell^2}{D^3}. \end{aligned}$$

The first identity follows from symmetry.
As for the second, note the integral vanishes by symmetry unless $\mu = \nu$.
Lorentz invariance therefore requires that we get something proportional to $g^{\mu\nu}$. To check the coefficient, contract each side with $g_{\mu\nu}$. Using these identities, we have

$$\begin{aligned} &\text{Numerator } = \bar{u}\left(p'\right)\left[\not{k}\gamma^\mu \not{k}' + m^2\gamma^\mu - 2m\left(k+k'\right)^\mu\right] u(p) \\ &\to \bar{u}\left(p'\right)\left[-\frac{1}{2}\gamma^\mu \ell^2 + (-y \not{q} + z \not{p})\gamma^\mu((1-y) \not{q} + z \not{p})\right. \\ &\left. + m^2\gamma^\mu - 2m\left((1-2y)q^\mu + 2zp^\mu\right)\right] u(p). \end{aligned}$$

$(k' = k + q.)$
We will now group everything into two terms, proportional to γ^μ and $i\sigma^{\mu\nu}q_\nu$.
Therefore we will accomplish expression of form:

$$\gamma^\mu \cdot A + (p^\mu + p^\mu) \cdot B + q^\mu \cdot C,$$

Attaining this form requires only the anticommutation relations (for example, $\not\gamma^\mu = 2p^\mu - \gamma^\mu \not p$) and the Dirac equation ($\not p u(p) = mu(p)$ and $\bar{u}(p') \not p' = \bar{u}(p') m$; this implies $\bar{u}(p') q u(p) = 0$). Also $x + y + z = 1$.
Therefore,

$$\begin{aligned} \text{Numerator } &= \bar{u}(p') [\gamma^\mu \cdot \left(-\frac{1}{2}\ell^2 + (1-x)(1-y)q^2 + \left(1 - 2z - z^2\right) m^2 \right) \\ &\quad + (p^\mu + p^\mu) \cdot mz(z-1) + q^\mu \cdot m(z-2)(x-y)] \, u(p). \end{aligned}$$

The coefficient of q^μ must vanish according to the Ward identity.

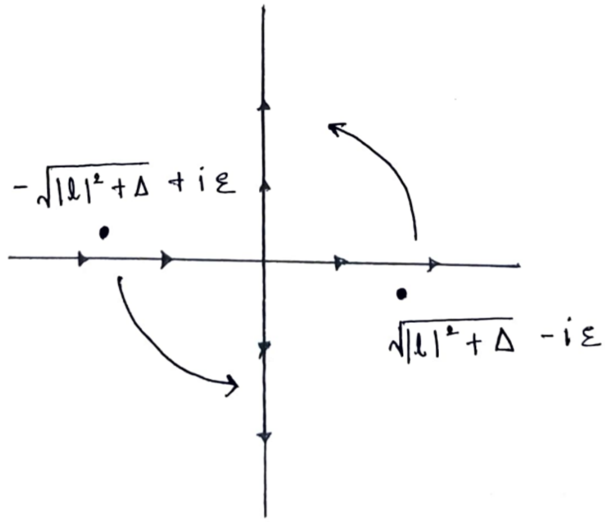
We now use the Gordon identity to eliminate $(p' + p)$ in favor of $i\sigma^{\mu\nu}q_\nu$. Our entire expression for the $\mathcal{O}(\alpha)$ contribution to the electron vertex then becomes

$$\begin{aligned} \bar{u}(p') \, \delta\Gamma^\mu(p', p) \, u(p) = 2ie^2 \int \frac{d^4\ell}{(2\pi)^4} \int_0^1 dxdydz\delta(x+y+z-1) \frac{2}{D^3} \\ \times \bar{u}(p') \left[\gamma^\mu \cdot \left(-\frac{1}{2}\ell^2 + (1-x)(1-y)q^2 + \left(1 - 4z + z^2\right) m^2 \right) \right. \\ \left. + \frac{i\sigma^{\mu\nu}q_\nu}{2m} \left(2m^2 z(1-z) \right) \right] u(p) \end{aligned}$$

where as before,

$$D = \ell^2 - \Delta + i\epsilon, \quad \Delta = -xyq^2 + (1-z)^2 m^2 > 0.$$

We can now perform the momentum integral. It is not difficult to evaluate the ℓ^0 integral as a contour integral, then do the spatial integrals in spherical coordinates. We will *Wick rotation*.
If it were not for the minus signs in the Minkowski metric, we could perform the entire four-dimensional integral in four-dimensional "spherical" coordinates. To remove the minus signs, consider the contour of integration in the ℓ^0-plane.



The contour of the ℓ^0 integration can be rotated as above.

The locations of the poles, and the fact that the integrand falls off sufficiently rapidly at large $[\ell^0]$, allow us to rotate the contour counterclockwise by 90°. We then define a *Euclidean* 4-momentum variable ℓ_E:

$$\ell^0 \equiv i\ell_E^0; \quad \ell = \ell_E.$$

Our rotated contour goes from $\ell_E^0 = -\infty$ to ∞. By simply changing variables to ℓ_E, we can now evaluate the integral in four-dimensional spherical coordinates.

We will first evaluate

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{[\ell^2-\Delta]^m} = \frac{i}{(-1)^m} \frac{1}{(2\pi)^4} \int d^4\ell_E \frac{1}{[\ell_E^2+\Delta]^m} = \frac{i(-1)^m}{(2\pi)^4} \int d\Omega_4 \int d\ell_E \frac{\ell_E^3}{[\ell_E^2+\Delta]^m}$$

(Here we need only the case $m = 3$, but the more general result will be useful for other loop calculations.)
The factor $\int d\Omega_4$ is the surface "area" of a four dimensional unit sphere, which happens to equal $2\pi^2$.(One way to compute this area is to use four-dimensional spherical coordinates,

$$x = (r \sin\omega \sin\theta \cos\phi,\ r \sin\omega \sin\theta \sin\phi,\ r \sin\omega \cos\theta,\ r \cos\omega).$$

The integration measure is then $d^4x = r^3 \sin^2\omega \sin\theta d\phi d\theta d\omega dr$.) From the rest of the integral we have

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{[\ell^2-\Delta]^m} = \frac{i(-1)^m}{(4\pi)^2} \frac{1}{(m-1)(m-2)} \frac{1}{\Delta^{m-2}}.$$

Similarly,

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^2}{[\ell^2-\Delta]^m} = \frac{i(-1)^{m-1}}{(4\pi)^2} \frac{2}{(m-1)(m-2)(m-3)} \frac{1}{\Delta^{m-3}}.$$

We will now introduce an artificial prescription to make our integral finite.
Going to the original expression and replace in the photon propagator

$$\frac{1}{(k-p)^2+i\epsilon} \longrightarrow \frac{1}{(k-p)^2+i\epsilon} - \frac{1}{(k-p)^2-\Lambda^2+i\epsilon},$$

where Λ is a very large mass.
The integrand is unaffected for small k, but cuts off smoothly when $k\Lambda$.
We can think of the second term as the propagator of a fictitious heavy photon, whose contribution is subtracted from that of the ordinary photon. In terms involving the heavy photon, the numerator algebra is unchanged and the denominator is altered by

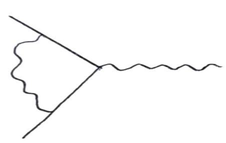
$$\Delta \longrightarrow \Delta_\Lambda = -xyq^2 + (1-z)^2m^2 + z\Lambda^2.$$

The integral is then replaced with a convergent integral, which can be Wick-rotated and evaluated:

$$\begin{aligned}\int \frac{d^4\ell}{(2\pi)^4}\left(\frac{\ell^2}{[\ell^2-\Delta]^3} - \frac{\ell^2}{[\ell^2-\Delta_\Lambda]^3}\right) &= \frac{i}{(4\pi)^2}\int_0^\infty d\ell_E^2\left(\frac{\ell_E^4}{[\ell_E^2+\Delta]^3} - \frac{\ell_E^4}{[\ell_E^2+\Delta_\Lambda]^3}\right)\\ &= \frac{i}{(4\pi)^2}\log\Big(\frac{\Delta_\Lambda}{\Delta}\Big).\end{aligned}$$

The convergent terms are modified by terms of order Λ^{-2},which we ignore.

This prescription for rendering Feynman integrals finite by introducing fictitious heavy particles is known as $Pauli-Villars\ regularisation$.
The fictitious photon has no physical significance, and this method is only one of many for defining the divergent integrals.
Using above formulae we obtain expression for the one-loop vertex correction:



$$\begin{aligned}&= \frac{\alpha}{2\pi}\int_0 dxdydz\delta(x+y+z-1)\\ &\times \bar{u}(p')\bigg(\gamma^\mu\Big[\log\frac{z\Lambda^2}{\Delta} + \frac{1}{\Delta}\Big((1-x)(1-y)q^2 + (1-4z+z^2)m^2\Big)\Big]\\ &+ \frac{i\sigma^{\mu\nu}q_\nu}{2m}\Big[\frac{1}{\Delta}2m^2z(1-z)\Big]\bigg)u(p).\end{aligned}$$

10.2 Interpretation

We notice that the divergence appears in the $F_1(q^2=0)$. We make the substitution

$$\delta F_1(q^2) \to \delta F_1(q^2) - \delta F_1(0)$$

There is also an infrared divergence in $F_1(q^2)$,coming from the $1/\Delta$ term. For example, at $q^2 = 0$ this term is

$$\int_0^1 dx\, dy\, dz\, \delta(x+y+z-1)\, \frac{1-4z+z^2}{\Delta(q^2=0)} = \int_0^1 dz \int_0^{1-z} dy\, \frac{-2+(1-z)(3-z)}{m^2(1-z)^2}$$
$$= \int^1 dz\, \frac{-2}{m^2(1-z)} + \text{finite terms.}$$

We can cure this problem by pretending that the photon has a small nonzero mass μ. Then in the denominator of the photon propagator, $(k-p)^2$ would become $(k-p)^2 - \mu^2$.

Now the form factors are:

$$F_1(q^2) = 1 + \frac{\alpha}{2\pi} \int_0^1 dx\, dy\, dz\, \delta(x+y+z-1)$$
$$\times \left[\log\left(\frac{m^2(1-z)^2}{m^2(1-z)^2 - q^2xy}\right) + \frac{m^2(1-4z+z^2) + q^2(1-x)(1-y)}{m^2(1-z)^2 - q^2xy + \mu^2 z} \right.$$
$$\left. - \frac{m^2(1-4z+z^2)}{m^2(1-z)^2 + \mu^2 z} \right] + \mathcal{O}(\alpha^2);$$

$$F_2(q^2) = \frac{\alpha}{2\pi} \int_0^1 dx\, dy\, dz\, \delta(x+y+z-1) \left[\frac{2m^2 z(1-z)}{m^2(1-z)^2 - q^2xy}\right] + \mathcal{O}(\alpha^2).$$

Neither the ultraviolet nor the infrared divergence affects $F_2(q^2)$. We can therefore evaluate unambiguously

$$F_2(q^2=0) = \frac{\alpha}{2\pi} \int_0^1 dx\, dy\, dz\, \delta(x+y+z-1)\, \frac{2m^2 z(1-z)}{m^2(1-z)^2}$$

Thus, we get a correction to the g-factor of the electron:

$$a_e \equiv \frac{g-2}{2} = \frac{\alpha}{2\pi} \approx .0011614.$$

11 The Electron Vertex Function: Infrared Divergence

To confront the infrared divergence in our result for $F_1(q^2)$ The dominant part, in the $\mu \to 0$limit, is

$$F_1(q^2) \approx \frac{\alpha}{2\pi}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\Bigg[\frac{m^2(1-4z+z^2)+q^2(1-x)(1-y)}{m^2(1-z)^2-q^2xy+\mu^2 z} - \frac{m^2(1-4z+z^2)}{m^2(1-z)^2+\mu^2 z}\Bigg]$$

Simplifying the expression by extracting and evaluating the divergent part of the integral. (we will retain only terms that diverge in the limit $\mu \to 0$)

The divergence occurs in the corner of Feynman-parameter space where $z \approx 1$ (and therefore $x \approx y \approx 0$). In this region we can set $z = 1$ and $x = y = 0$ in the numerators. We will set $z = 1$ in the μ^2 terms in the denominators. Using the delta function to evaluate the x-integral. we then have

$$F_1(q^2) = \frac{\alpha}{2\pi}\int_0^1 dz \int_0^{1-z} dy \left[\frac{-2m^2+q^2}{m^2(1-z)^2-q^2y(1-z-y)+\mu^2} - \frac{-2m^2}{m^2(1-z)^2+\mu^2}\right].$$

(The lower limit on the z-integral is unimportant.) Making the variable changes

$$y = (1-z)\xi, \quad w = (1-z),$$

this expression becomes

$$F_1(q^2) = 1 - \frac{\alpha}{2\pi} f_{\rm IR}(q^2)\log\Big(\frac{-q^2 \text{ or } m^2}{\mu^2}\Big) + \mathcal{O}(\alpha^2),$$

where the coefficient of the divergent logarithm is

$$f_{\rm IR}(q^2) = \int_0^1 \Big(\frac{m^2-q^2/2}{m^2-q^2\xi(1-\xi)}\Big)d\xi - 1.$$

We get for infrared divergence :

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm measured} \approx \left(\frac{d\sigma}{d\Omega}\right)_0 \left[1 - \frac{\alpha}{\pi}\log\left(\frac{-q^2}{m^2}\right)\log\left(\frac{-q^2}{E_\ell^2}\right) + \mathcal{O}(\alpha^2)\right].$$

This result is unambiguous and useful.
Here the $\mathcal{O}(\alpha)$ correction again involves the Sudakov double logarithm.

12 Renormalization

The procedure we follow in yielding a "renormalized" perturbation theory formulated in terms of physically measurable parameters are:

1. Absorb the field-strength renormalizations into the Lagrangian by rescaling the fields.

2. Split each term of the Lagrangian into two pieces, absorbing the infinite and unobservable shifts into counterterms.
3. Specify the renormalization conditíons, which define the physical masses and coupling constants and keep the field-strength renormalizations equal to 1.
4. Compute amplitudes with the new Feynman rules, adjusting the counterterms as necessary to maintain the renormalization conditions.

The QED Lagrangian is

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + \overline{\psi}(i\partial - m_0)\psi - e_0\bar{\psi}\gamma^\mu\psi A_\mu.$$

The four conditions we will use are

$$\Sigma(\not{p} = m) = 0;$$

$$\left.\frac{d}{dp}\Sigma(p)\right|_{p=m} = 0;$$

$$\Pi(q^2 = 0) = 0;$$

$$-ie\Gamma^\mu(p' - p = 0) = -ie\gamma^\mu.$$

The first condition fixes the electron mass at m, while the next two fix the residues of the electron and photon propagators at 1. Given these conditions, the final condition fixes the electron charge to be e.

The four conditions allow us to determine the four counterterms in all of the diagrams required to carry out this determination to one-loop order. We will now collect these results and find explicit expressions for the renormalization constants of QED to order α. For overall consistency, we will use dimensional regularization to control ultraviolet divergences, and a photon mass μ to control infrared divergences. We computed the vertex and self-energy diagrams using the Pauli-Villars regularization scheme.

The first two conditions involve the electron self-energy. We evaluated the one-loop diagram contributing to $\Sigma(p)$, using a Pauli-Villars regulator. If we re-evaluate the diagram in dimensional regularization, we find some additional terms in the Dirac algebra from the modified contraction identities. Taking these terms into account, we find for this diagram ($\epsilon = 4 - d$)

$$-i\Sigma_2(p) = -i\frac{e^2}{(4\pi)^{d/2}}\int\limits_0^1 dx\,\frac{\Gamma(2-\frac{d}{2})}{((1-x)m^2 + x\mu^2 - x(1-x)p^2)^{2-d/2}}\times((4-\epsilon)m-(2-\epsilon)x\ \not{p}).$$

Therefore, according to the first of conditions,

$$m\delta_2 - \delta_m = \Sigma_2(m) = \frac{e^2 m}{(4\pi)^{d/2}}\int\limits_0^1 dx\ \frac{\Gamma(2-\frac{d}{2})\cdot(4-2x-\epsilon(1-x))}{((1-x)^2m^2 + x\mu^2)^{2-d/2}}.$$

Similarly, the second of conditions determines δ_2:

$$\delta_2 = \frac{d}{dp}\Sigma_2(m)$$
$$= -\frac{e^2}{(4\pi)^{d/2}} \int_0^1 dx \, \frac{\Gamma(2-\frac{d}{2})}{((1-x)^2m^2 + x\mu^2)^2 - d/2}$$
$$\times \Big[(2-\epsilon)x - \frac{\epsilon}{2}\frac{2x(1-x)m^2}{(1-x)^2m^2 + x\mu^2}(4 - 2x - \epsilon(1-x))\Big].$$

Notice that the second term in the brackets gives a finite result as $\epsilon \to 0$, because it multiplies the divergent gamma function.

The third condition of requires the value of the photon self-energy diagram: : $\frac{\Gamma(2-\frac{d}{2})}{(m^2 - x(1-x)q^2)^{2-d/2}}\,(8x(1-x))$. Then

$$\delta_3 = \Pi_2(0) = -\frac{e^2}{(4\pi)^{d/2}} \int_0^1 dx \, \frac{\Gamma(2-\frac{d}{2})}{(m^2)^{2-d/2}}\big(8x(1-x)\big).$$

The last condition requires the value of the electron vertex function. Again, we will rework the diagram in dimensional regularization. Then the shift in the form factor $F_1(q^2)$becomes

$$\delta F_1(q^2) = \frac{e^2}{(4\pi)^{d/2}} \int dx\, dy\, dz\, \delta(x+y+z-1)\left[\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}\frac{(2-\epsilon)^2}{2}\right]$$
$$+ \frac{\Gamma(3-\frac{d}{2})}{\Delta^{3-d/2}}\big(q^2[2(1-x)(1-y) - \epsilon xy] + m^2[2(1-4z+z^2) - \epsilon(1-z)^2]\big)\Big],$$

where $\Delta = (1-z)^2m^2 + z\mu^2 - xyq^2$ as before. The fourth renormalization condition then determines

$$\delta_1 = -\delta F_1(0) = -\frac{e^2}{(4\pi)^{d/2}} \int dz\,(1-z)\Big[\frac{\Gamma(2-\frac{d}{2})}{((1-z)^2m^2 + z\mu^2)^{2-d/2}}\frac{(2-\epsilon)^2}{2}$$
$$+ \frac{\Gamma(3-\frac{d}{2})}{((1-z)^2m^2 + z\mu^2)^{3-d/2}}[2(1-4z+z^2) - \epsilon(1-z)^2]m^2\Big].$$

13 Infrared Finiteness

We studied two types of vertex correction:
1. **UV Divergence:** Which we resolve by renormalisation.
2. **Infrared Divergence:** Which we get in bot real and virtual correction. We have two types of IR divergence, i.e. soft and collinear divergence. For massive case there will be no collinear divergence.

13.1 KLN Theorem

The Kinoshita-Lee-Nauenberg theorem, also known as the KLN theorem, asserts that perturbatively, the whole of the standard model remains infrared (IR) finite.
Infrared divergences seem only to cancel at the cross-section level for sufficiently inclusive quantities.
KLN theorem provides two revelations:
1) The processes that contribute to assure the cancellation include exactly forward scattering and
2) infrared divergences cancel when summing over final states alone for fixed initial state or summing over initial states for a fixed final state.
Infrared divergences arise from photons with "soft" momenta: real photons with energy less than some cutoff E_ℓ, and virtual photons with (after Wick rotation) $k^2 < E_\rho^2$.
If the soft photon is emitted last the denominator of the propagator is $(p'+k)^2 - m^2 = 2p' \cdot k$, which vanishes as $k \to 0$.
For each virtual photon we obtain the expression

$$\frac{e^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{-i}{k^2+i\epsilon}\Big(\frac{p'}{p'\cdot k}-\frac{p}{p\cdot k}\Big)\cdot\Big(\frac{p'}{-p'\cdot k}-\frac{p}{-p\cdot k}\Big) \equiv \mathbf{X}.$$

The factor of 1/2 is required because our procedure has counted each Feynman diagram twice: interchanging k_i and k_j gives back the same diagram.
We notice that this approximation scheme assigns to the diagram with one loop and no external photons the value

$$\bar{u}(p')\big(i\mathcal{M}_{\text{hard}}\big)u(p)\cdot \mathbf{X}.$$

Thus, X must be precisely the infrared limit of the one-loop correction to the form factor:

$$\mathbf{X} = -\frac{\alpha}{2\pi} f_{\text{IR}}(q^2)\log\Big(\frac{-q^2}{\mu^2}\Big).$$

For emission of real photon wee get additional factor

$$\int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2k}\,e^2(-g_{\mu\nu})\Big(\frac{p'^{\mu}}{p'\cdot k}-\frac{p^{\mu}}{p\cdot k}\Big)\Big(\frac{p'^{\nu}}{p'\cdot k}-\frac{p^{\nu}}{p\cdot k}\Big) \equiv \mathbf{Y}$$

in the cross section. Assuming that the energy of the photon is greater than μ and less than E_ℓ (the detector threshold), this expression is simply

$$\mathbf{Y} = \frac{\alpha}{\pi}\mathcal{I}(\mathbf{v},\mathbf{v}')\log\Big(\frac{E_\ell}{\mu}\Big) = \frac{\alpha}{\pi} f_{\text{IR}}(q^2)\log\Big(\frac{E_\ell^2}{\mu^2}\Big).$$

Combining the results for virtual and real photons gives our final result for the measured cross section, to all orders in α,for the process$\mathbf{p} \to \mathbf{p}' +$ (any number of photons with $k < E_\ell$):

$$\begin{aligned}
\left(\frac{d\sigma}{d\Omega}\right)_{\text{meas.}} &= \left(\frac{d\sigma}{d\Omega}\right)_0 \times \exp(2\mathbf{X}) \times \exp(\mathbf{Y}) \\
&= \left(\frac{d\sigma}{d\Omega}\right)_0 \times \exp\left[-\frac{\alpha}{\pi} f_{\text{IR}}(q^2)\log\left(\frac{-q^2}{\mu^2}\right)\right] \times \exp\left[\frac{\alpha}{\pi} f_{\text{IR}}(q^2)\log\left(\frac{E_\ell^2}{\mu^2}\right)\right] \\
&= \left(\frac{d\sigma}{d\Omega}\right)_0 \times \exp\left[-\frac{\alpha}{\pi} f_{\text{IR}}(q^2)\log\left(\frac{-q^2}{E_\ell^2}\right)\right].
\end{aligned}$$

The correction factor depends on the detector sensitivity E_ℓ, but is independent of the infrared cutoff μ.
Now, however, the correction factor is controlled in magnitude–always between 0 and 1.

In the limit $-q^2 \gg m^2$, our result becomes

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{meas.}} = \left(\frac{d\sigma}{d\Omega}\right)_0 \times \left|\exp\left[-\frac{\alpha}{2\pi}\log\left(\frac{-q^2}{m^2}\right)\log\left(\frac{-q^2}{E_\ell^2}\right)\right]\right|^2\Bigg|$$

In this limit, the probability of scattering without emitting a hard photon decreases faster than any power of q^2.

14 Conclusion

In this project, we have undertaken a comprehensive study of next-to-leading-order (NLO) quantum electrodynamics (QED) calculations for pair production processes at lepton colliders. Through theoretical developments, computational implementations, and numerical analyses, we have aimed to improve the precision of theoretical predictions for these fundamental processes and their relevance to experimental measurements.

We have developed a rigorous theoretical framework for NLO QED calculations, encompassing the formalism for treating virtual and real photon emissions, interference effects, and the renormalization of divergent contributions. By systematically including higher-order corrections, we have achieved a more accurate description of pair production processes at lepton colliders.

Our study underscores the importance of NLO QED calculations in precision physics at lepton colliders. By accounting for radiative corrections at higher orders, we have obtained a deeper understanding of the underlying dynamics of pair production processes and their implications for fundamental physics.

Future prospects includes numerical calculation of finite cross-section and analyse the results numerically.

15 Appendix

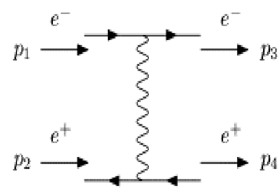
15.1 Feynman rules for QED

initial-state particle:	$u(p)$	
final-state particle:	$\overline{u}(p)$	
initial-state antiparticle:	$\overline{v}(p)$	
final-state antiparticle:	$v(p)$	
initial-state photon:	$\varepsilon_\mu(p)$	
final-state photon:	$\varepsilon^*_\mu(p)$	
photon propagator:	$-\frac{ig_{\mu\nu}}{q^2}$	
fermion propagator:	$-\frac{i(\gamma^\mu q_\mu + m)}{q^2 - m^2}$	
QED vertex:	$-iQe\gamma^\mu$	

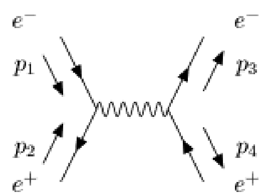
15.2 Calculation of Matrix Elements

15.2.1 $e^+ + e^- \longrightarrow e^+ + e^-$: Bhabha Scattering

The Feynman Amplitude has two terms:



$$
\begin{aligned}
M_1 &= -\bar{v}(p_2)(ie\gamma^\mu)v(p_4)\frac{-ig^{\mu\nu}}{(p_2-p_4)^2}\bar{u}(p_3)(-ie\gamma^\nu)u(p_1) \\
&= \frac{ie^2}{(p_2-p_4)^2}(\bar{v}(p_2)\gamma^\mu v(p_4))(\bar{u}(p_3)\gamma_\mu u(p_1))
\end{aligned} \tag{1}
$$



$$
\begin{aligned}
M_2 &= \bar{v}(p_2)(ie\gamma^\nu)u(p_1)\frac{-ig^{\mu\nu}}{(p_1+p_2)^2}\bar{u}(p_3)(-ie\gamma^\mu)v(p_4) \\
&= \frac{-ie^2}{(p_1+p_2)^2}(\bar{v}(p_2)\gamma^\nu u(p_1))(\bar{u}(p_3)\gamma_\nu v(p_4))
\end{aligned} \tag{2}
$$

Therefore, **the Feynman amplitude** is $M_1 + M_2$. Feynman ampltude square is:

$$
\begin{aligned}
|M|^2 &= (M_1+M_2)^*(M_1+M_2) \\
&= M_1^*M_1 + M_1^*M_2 + M_1M_2^* + M_2^*M_2 \\
&= |M_1|^2 + M_1^*M_2 + M_1M_2^* + |M_2|^2
\end{aligned} \tag{3}
$$

To calculate the cross section, we will average over the spins of the incoming particles (se- and se+ possible values) and sum over the spins of the outgoing particles. That is,

$$
\begin{aligned}
|\bar{M}|^2 &= \frac{1}{(2s_{e-}+1)(2s_{e+}+1)}\sum_{spins}|M|^2 \\
&= \frac{1}{4}\sum_{s_1=1}^{2}\sum_{s_2=1}^{2}\sum_{s_3=1}^{2}\sum_{s_4=1}^{2}|M|^2
\end{aligned} \tag{4}
$$

Now calculating each term separately:
$|M_1|^2$:

$$
\begin{aligned}
|M_1|^2 &= \sum_{s_i}[\frac{ie^2}{(p_2-p_4)^2}(\bar{v}(p_2)\gamma^\mu v(p_4))(\bar{u}(p_3)\gamma_\mu u(p_1))]^*[\frac{ie^2}{(p_2-p_4)^2}(\bar{v}(p_2)\gamma^\mu v(p_4))(\bar{u}(p_3)\gamma_\mu u(p_1))] \\
&= \sum_{s_i}\frac{e^4}{(p_2-p_4)^4}(\bar{v}(p_2)\gamma^\mu v(p_4))^*(\bar{u}(p_3)\gamma_\mu u(p_1))^*(\bar{v}(p_2)\gamma^\mu v(p_4))(\bar{u}(p_3)\gamma_\mu u(p_1)) \\
&= \sum_{s_i}\frac{e^4}{(p_2-p_4)^4}(\bar{v}(p_4)\gamma^{\mu'} v(p_2))(\bar{u}(p_1)\gamma_{\mu'} u(p_3))(\bar{v}(p_2)\gamma^\mu v(p_4))(\bar{u}(p_3)\gamma_\mu u(p_1)) \\
&= \sum_{s_i}\frac{e^4}{(p_2-p_4)^4}(\bar{v}(p_4)\gamma^{\mu'} v(p_2))(\bar{v}(p_2)\gamma^\mu v(p_4))(\bar{u}(p_1)\gamma_{\mu'} u(p_3))(\bar{u}(p_3)\gamma_\mu u(p_1)) \\
&= \frac{e^4}{(p_2-p_4)^4}\sum_{s_i}Tr[((v(p_4)\bar{v}(p_4))\gamma^{\mu'}(v(p_2)\bar{v}(p_2))\gamma^\mu]Tr[(u(p_1)\bar{u}(p_1))\gamma_{\mu'}(u(p_3)\bar{u}(p_3))\gamma_\mu] \\
&= \frac{e^4}{(p_2-p_4)^4}Tr[(\not{p}_4-m)\gamma^{\mu'}(\not{p}_2-m)\gamma^\mu]Tr[(\not{p}_1+m)\gamma_{\mu'}(\not{p}_3+m)\gamma_\mu] \\
&= \frac{32e^4}{(p_2-p_4)^4}((p_4.p_3)(p_2.p_1)+(p_4.p_1)(p_3.p_2))
\end{aligned} \tag{5}
$$

substituting respective Mandelstam terms :

$$
\begin{aligned}
\frac{1}{4}|M_1|^2 &= \frac{1}{4}\frac{32e^4}{(p_2-p_4)^4}((p_4.p_3)(p_2.p_1)+(p_4.p_1)(p_3.p_2)) \\
&= \frac{8e^4}{t^2}(\frac{s}{2}\frac{s}{2}+\frac{-u}{2}\frac{-u}{2}) \\
&= 2e^4\frac{s^2+u^2}{t^2}
\end{aligned} \tag{6}
$$

$M_1^*M_2$:

$$\begin{aligned}
M_1^*M_2 &= \sum_{s_i}[\frac{ie^2}{(p_2-p_4)^2}(\bar{v}(p_2)\gamma^\mu v(p_4))(\bar{u}(p_3)\gamma_\mu u(p_1))]^*[\frac{-ie^2}{(p_1+p_2)^2}(\bar{v}(p_2)\gamma^\nu u(p_1))(\bar{u}(p_3)\gamma_\nu v(p_4))] \\
&= \sum_{s_i}\frac{-e^4}{(p_2-p_4)^2(p_1+p_2)^2}(\bar{v}(p_2)\gamma^\mu v(p_4))^*(\bar{u}(p_3)\gamma_\mu u(p_1))^*(\bar{v}(p_2)\gamma^\nu u(p_1))(\bar{u}(p_3)\gamma_\nu v(p_4)) \\
&= \sum_{s_i}\frac{-e^4}{(p_2-p_4)^2(p_1+p_2)^2}(\bar{v}(p_4)\gamma^{\mu'} v(p_2))(\bar{u}(p_1)\gamma_{\mu'} u(p_3))(\bar{v}(p_2)\gamma^\nu u(p_1))(\bar{u}(p_3)\gamma_\nu v(p_4)) \\
&= \frac{-e^4}{(p_2-p_4)^2(p_1+p_2)^2}\sum_{s_i}Tr[\bar{v}(p_4)\gamma^{\mu'} v(p_2)\bar{u}(p_1)\gamma_{\mu'} u(p_3)\bar{v}(p_2)\gamma^\nu u(p_1)\bar{u}(p_3)\gamma_\nu v(p_4)] \\
&= \frac{-e^4}{(p_2-p_4)^2(p_1+p_2)^2}\sum_{s_i}Tr[\gamma^{\mu'} v(p_2)\bar{v}(p_2)\gamma^\nu u(p_1)\bar{u}(p_1)\gamma_{\mu'} u(p_3)\bar{u}(p_3)\gamma_\nu v(p_4)\bar{v}(p_4)] \\
&= \frac{-e^4}{(p_2-p_4)^2(p_1+p_2)^2}Tr[\gamma^{\mu'}(\not{p}_2-m)\gamma^\nu(\not{p}_1+m)\gamma_{\mu'}(\not{p}_3+m)\gamma_\nu(\not{p}_4-m)] \\
&= \frac{-e^4}{(p_2-p_4)^2(p_1+p_2)^2}Tr[\gamma^{\mu'}\not{p}_2\gamma^\nu\not{p}_1\gamma_{\mu'}\not{p}_3\gamma_\nu\not{p}_4] \\
&= \frac{-e^4}{(p_2-p_4)^2(p_1+p_2)^2}(-2)Tr[\not{p}_1\gamma^\nu\not{p}_2\not{p}_3\gamma_\nu\not{p}_4] \\
&= \frac{-e^4}{(p_2-p_4)^2(p_1+p_2)^2}(-8.p_2.p_3)Tr[\not{p}_1\not{p}_4] \\
&= \frac{-e^4}{(p_2-p_4)^2(p_1+p_2)^2}(-8p_2.p_3)(4p_1.p_4)
\end{aligned} \tag{7}$$

substituting respective Mandelstam terms :

$$\begin{aligned}
\frac{1}{4}M_1^*M_2 &= \frac{1}{4}\frac{-32e^4}{(p_2-p_4)^2(p_1+p_2)^2}(p_2.p_3)(p_1.p_4) \\
&= (-2e^4)\frac{u^2}{st}
\end{aligned} \tag{8}$$

$M_1M_2^*$:

$$
\begin{aligned}
M_1M_2^* &= \sum_{s_i}[\frac{ie^2}{(p_2-p_4)^2}(\bar{v}(p_2)\gamma^\mu v(p_4))(\bar{u}(p_3)\gamma_\mu u(p_1))][\frac{-ie^2}{(p_1+p_2)^2}(\bar{v}(p_2)\gamma^\nu u(p_1))(\bar{u}(p_3)\gamma_\nu v(p_4))]^* \\
&= \sum_{s_i}\frac{-e^4}{(p_1+p_2)^2(p_2-p_4)^2}(\bar{v}(p_2)\gamma^\mu v(p_4))(\bar{u}(p_3)\gamma_\mu u(p_1))(\bar{v}(p_2)\gamma^\nu u(p_1))^*(\bar{u}(p_3)\gamma_\nu v(p_4))^* \\
&= \sum_{s_i}\frac{-e^4}{(p_1+p_2)^2(p_2-p_4)^2}(\bar{v}(p_2)\gamma^\mu v(p_4))(\bar{u}(p_3)\gamma_\mu u(p_1))(\bar{u}(p_1)\gamma^{\nu'} v(p_2))(\bar{v}(p_4)\gamma_{\nu'} u(p_3)) \\
&= \frac{-e^4}{(p_1+p_2)^2(p_2-p_4)^2}\sum_{s_i}Tr[(\bar{v}(p_2)\gamma^\mu v(p_4))(\bar{u}(p_3)\gamma_\mu u(p_1))(\bar{u}(p_1)\gamma^{\nu'} v(p_2))(\bar{v}(p_4)\gamma_{\nu'} u(p_3))] \\
&= \frac{-e^4}{(p_1+p_2)^2(p_2-p_4)^2}\sum_{s_i}Tr[\gamma^\mu v(p_4)\bar{v}(p_4)\gamma_{\nu'} u(p_3)\bar{u}(p_3)\gamma_\mu u(p_1)\bar{u}(p_1)\gamma^{\nu'} v(p_2)\bar{v}(p_2)] \\
&= \frac{-e^4}{(p_1+p_2)^2(p_2-p_4)^2}Tr[\gamma^\mu(\not{p}_4-m)\gamma_{\nu'}(\not{p}_3+m)\gamma_\mu(\not{p}_1+m)\gamma^{\nu'}(\not{p}_2-m)] \\
&= \frac{-e^4}{(p_1+p_2)^2(p_2-p_4)^2}Tr[\gamma^\mu\not{p}_4\gamma_{\nu'}\not{p}_3\gamma_\mu\not{p}_1\gamma^{\nu'}\not{p}_2)] \\
&= \frac{-e^4}{(p_1+p_2)^2(p_2-p_4)^2}(32(p_1.p_4)(p_2.p_3))
\end{aligned}
\tag{9}
$$

substituting respective Mandelstam terms :

$$
\begin{aligned}
\frac{1}{4}M_1M_2^* &= \frac{1}{4}\frac{-32e^4}{(p_1+p_2)^2(p_2-p_4)^2}(p_1.p_4)(p_2.p_3) \\
&= (-2e^4)\frac{u^2}{st}
\end{aligned}
\tag{10}
$$

$|M_2|^2$:

$$
\begin{aligned}
|M_2|^2 &= \sum_{s_i}[\frac{-ie^2}{(p_1+p_2)^2}(\bar{v}(p_2)\gamma^\nu u(p_1))(\bar{u}(p_3)\gamma_\nu v(p_4))]^*[\frac{-ie^2}{(p_1+p_2)^2}(\bar{v}(p_2)\gamma^\nu u(p_1))(\bar{u}(p_3)\gamma_\nu v(p_4))] \\
&= \sum_{s_i}\frac{e^4}{(p_1+p_2)^4}(\bar{v}(p_2)\gamma^\nu u(p_1)) * (\bar{u}(p_3)\gamma_\nu v(p_4)) * (\bar{v}(p_2)\gamma^\nu u(p_1))(\bar{u}(p_3)\gamma_\nu v(p_4)) \\
&= \sum_{s_i}\frac{e^4}{(p_1+p_2)^4}(\bar{u}(p_1)\gamma^{\nu'} v(p_2))(\bar{v}(p_4)\gamma_{\nu'} u(p_3))(\bar{v}(p_2)\gamma^\nu u(p_1))(\bar{u}(p_3)\gamma_\nu v(p_4)) \\
&= \sum_{s_i}\frac{e^4}{(p_1+p_2)^4}(\bar{u}(p_1)\gamma^{\nu'} v(p_2))(\bar{v}(p_2)\gamma^\nu u(p_1))(\bar{v}(p_4)\gamma_{\nu'} u(p_3))(\bar{u}(p_3)\gamma_\nu v(p_4)) \\
&= \frac{e^4}{(p_1+p_2)^4}\sum_{s_i} Tr[u(p_1)\bar{u}(p_1)\gamma^{\nu'} v(p_2)\bar{v}(p_2)\gamma^\nu]Tr[v(p_4)\bar{v}(p_4)\gamma_{\nu'} u(p_3)\bar{u}(p_3)\gamma_\nu] \\
&= \frac{e^4}{(p_1+p_2)^4} Tr[(\not{p}_1+m)\gamma^{\nu'}(\not{p}_2-m)\gamma^\nu(\not{p}_4-m)\gamma_{\nu'}(\not{p}_3+m)\gamma_\nu] \\
&= \frac{e^4}{(p_1+p_2)^4}((p_1.p_4)(p_2.p_3)+(p_1.p_3)(p_2.p_4)
\end{aligned}
\tag{11}
$$

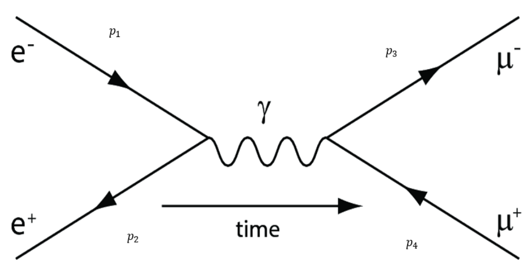
substituting respective Mandelstam terms :

$$
\begin{aligned}
\frac{1}{4}|M_2|^2 &= \frac{e^4}{(p_1+p_2)^4}((p_1.p_4)(p_2.p_3)+(p_1.p_3)(p_2.p_4) \\
&= \frac{8e^4}{s^2}(\frac{-u}{2}\frac{-u}{2}+\frac{-t}{2}\frac{-t}{2}) \\
&= 2e^4\frac{u^2+t^2}{s^2}
\end{aligned}
\tag{12}
$$

Therefore the $|\bar{M}|^2$

$$
\begin{aligned}
|\bar{M}|^2 &= 2e^4\frac{s^2+u^2}{t^2} + (-2e^4)\frac{u^2}{st} + (-2e^4)\frac{u^2}{st} + 2e^4\frac{u^2+t^2}{s^2} \\
&= 2e^4(\frac{s^2+u^2}{t^2} - \frac{2u^2}{st} + \frac{u^2+t^2}{s^2})
\end{aligned}
\tag{13}
$$

15.2.2 Process : $e^- + e^+ \longrightarrow \mu^- + \mu^+$



The Feynman Amplitude here:

$$\begin{aligned} M &= \bar{v}(p_2)(ie\gamma^{\nu})u(p_1)\frac{-ig^{\mu\nu}}{(p_1+p_2)^2}\bar{u}(p_3)(-ie\gamma^{\mu})v(p_4) \\ &= \frac{-ie^2}{(p_1+p_2)^2}(\bar{v}(p_2)\gamma^{\nu}u(p_1))(\bar{u}(p_3)\gamma_{\nu}v(p_4)) \end{aligned} \tag{14}$$

The average Feynman Amplitude square is then:

$$\begin{aligned} |\bar{M}|^2 &= (\frac{1}{2})^2\sum_{s_i}|M|^2 \\ &= (\frac{1}{2})^2\sum_{s_i}M^*.M \\ &= \sum_{s_i}[\frac{-ie^2}{4(p_1+p_2)^2}(\bar{v}(p_2)\gamma^{\nu}u(p_1))(\bar{u}(p_3)\gamma_{\nu}v(p_4))]^*[\frac{-ie^2}{(p_1+p_2)^2}(\bar{v}(p_2)\gamma^{\nu}u(p_1))(\bar{u}(p_3)\gamma_{\nu}v(p_4))] \\ &= \sum_{s_i}\frac{e^4}{4(p_1+p_2)^4}(\bar{v}(p_2)\gamma^{\nu}u(p_1))*(\bar{u}(p_3)\gamma_{\nu}v(p_4))*(\bar{v}(p_2)\gamma^{\nu}u(p_1))(\bar{u}(p_3)\gamma_{\nu}v(p_4)) \\ &= \sum_{s_i}\frac{e^4}{4(p_1+p_2)^4}(\bar{u}(p_1)\gamma^{\nu'}v(p_2))(\bar{v}(p_4)\gamma_{\nu'}u(p_3))(\bar{v}(p_2)\gamma^{\nu}u(p_1))(\bar{u}(p_3)\gamma_{\nu}v(p_4)) \\ &= \sum_{s_i}\frac{e^4}{4(p_1+p_2)^4}(\bar{u}(p_1)\gamma^{\nu'}v(p_2))(\bar{v}(p_2)\gamma^{\nu}u(p_1))(\bar{v}(p_4)\gamma_{\nu'}u(p_3))(\bar{u}(p_3)\gamma_{\nu}v(p_4)) \\ &= \frac{e^4}{4(p_1+p_2)^4}\sum_{s_i}Tr[u(p_1)\bar{u}(p_1)\gamma^{\nu'}v(p_2)\bar{v}(p_2)\gamma^{\nu}]Tr[v(p_4)\bar{v}(p_4)\gamma_{\nu'}u(p_3)\bar{u}(p_3)\gamma_{\nu}] \\ &= \frac{e^4}{4(p_1+p_2)^4}Tr[(\not{p}_1+m)\gamma^{\nu'}(\not{p}_2-m)\gamma^{\nu}(\not{p}_4-m)\gamma_{\nu'}(\not{p}_3+m)\gamma_{\nu}] \\ &= \frac{e^4}{4(p_1+p_2)^4}((p_1.p_4)(p_2.p_3)+(p_1.p_3)(p_2.p_4)) \end{aligned} \tag{15}$$

substituting respective Mandelstam terms :

$$\begin{aligned} |\bar{M}|^2 &= \frac{e^4}{4(p_1+p_2)^4}((p_1.p_4)(p_2.p_3)+(p_1.p_3)(p_2.p_4)) \\ &= \frac{8e^4}{s^2}(\frac{-u}{2}\frac{-u}{2}+\frac{-t}{2}\frac{-t}{2}) \\ &= 2e^4\frac{u^2+t^2}{s^2} \end{aligned} \tag{16}$$

References

[1] An Introduction To Quantum Field Theory *Michael E. Peskin and Daniel V. Schroeder*

[2] Modern Particle Physics *Mark Thomson*

[3] Complete QED NLO contributions and their implementation in the event generator PHOKHARA *F. Campanario et al.*

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