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Non-linear integral equation approach for the boundary reconstruction in double-connected planar domains

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Contents

Introduction

The mathematical modeling of electrostatic or thermal imaging methods in nondestructive testing and evaluation leads to inverse boundary value problems for the Laplace equation. In principle, in these applications an unknown inclusion within a conducting host medium with constant conductivity is resolved from overdetermined Cauchy data on the accessible part of the boundary of the medium.

The idea to reduce the problem of boundary reconstruction to the system of non-linear equations and to employ a regularized iterative procedure was firstly suggested in [13]. This approach was successfully extended in [2, 5, 8, 14, 13] for the case of the Laplace equation and in [6, 7, 9, 10, 11, 15, 16, 17] for the Helmholtz equation.

As an alternative to the reciprocity gap approach based on Green's integral theorem we propose iterative solutions methods based on the Green's function. Although the proposed methods are restricted to the class of domains for which the Green's function can be easily found the methods have several advantages over the reciprocity gap approach. In particular, the corresponding single layer potential is bounded at infinity and hence its modification is not needed. Moreover for the complicated boundary conditions such as generalized impedance the proposed methods will be easier to adopt.

In order to get Cauchy data and test our boundary reconstruction methods we need firstly solve direct problem and find normal derivative of solution on exterior boundary. It can be solved by mesh method, finite element method or the boundary integral equations method using single and double layer potentials. We use the latter because the reformulation of the problem in the form of integral equations reduces its dimension.

Chapter 1

Direct problem

1.1. Set up of the problem

Two problems are called inverse together if setting of one of them includes solution of another, and vice versa. A simple example of inverse problems include: addition and subtraction, differentiation and integration. The problem that is more simple, more studied, has better properties for numerical implementation, called the direct problem, and another - inverse. Generally, the study of inverse problems requires advanced knowledge of relevant direct problem. So to start we formulate the direct problem.

Definition 1.1 *The function is called regular in some region Z, if it is analytical in this area and unambiguous.*

Definition 1.2 *Unambiguous function $f : \mathbb{Z} \to \mathbb{Z}$ is called analytic at the point z_0, if it is expanded in a Taylor series in a neighborhood centered at this point, and the series converge to the function f in this neighborhood. The function analytic at each point of Z, is called analytic function in this area.*

We assume that D is a doubly connected bounded domain in $\mathbb{R}^2$ with the boundary ∂D consisting of two disjoint closed C^2 curves Γ and Λ such that Γ is contained in the interior of Λ.

The corresponding direct problem is: Given a function f on Γ and consider the Dirichlet problem for regular $u \in C^2(D) \cap C(\bar{D})$ satisfying the Laplace equation

$$\Delta u = 0 \quad \text{in } D \tag{1.1.1}$$

and the boundary conditions

$$u = f_0 \quad \text{on } \Gamma \tag{1.1.2}$$

and

$$u = f \quad \text{on } \Lambda, \tag{1.1.3}$$

where f_0 and f are specified functions on Γ and Λ respectively, that have enough smoothness to ensure the existence of the normal derivative on Λ.

1.2. Green's function

To this end, we denote the interior of Λ by B. Then, by G we denote the Green's function for B, that is, G is defined for all $x \neq y$ in $\overline{B}$ and of the form

$$G(x,y) = \frac{1}{2\pi} \ln \frac{1}{|x-y|} + \tilde{G}(x,y), \tag{1.2.4}$$

where, for fixed $y \in B$, the function $\tilde{G}$ is harmonic in B with respect to x such that $G(\cdot\,,y) = 0$ on Λ. We note that for Λ a circle of radius R centered at the origin $\tilde{G}$ is explicitly given by

$$\tilde{G}(x,y) = \frac{1}{4\pi} \ln \frac{R^4 + |x|^2|y|^2 - 2R^2\, x \cdot y}{R^2}\,.$$

An important feature of the Green's functions is that they dependent on the type of region D and the type of boundary conditions, and does not depend on value of conditions.

1.3. Single-layer and double-layer potentials

For function $\varphi \in C(\Gamma)$, $\quad \Gamma \in C^p \quad (p \geq 2)$ let introduse sigle-layer potential

$$u(x) = \int\limits_{\Gamma} \varphi(y) G(x,y) ds(y), \quad x \in B \setminus \Gamma \tag{1.3.5}$$

and double-layer potential

$$\upsilon(x) = \int\limits_{\Gamma} \varphi(y) \frac{G(x,y)}{\nu(y)} ds(y), \quad x \in B \setminus \Gamma \tag{1.3.6}$$

with density φ.

Here G is Green's function for Dirichlet problem of Laplace equation (1.2.4) in limited region B, ν - outward unit normal to Γ.

Based on the classical results [12] and taking into account the properties of the Green's function for the potentials (1.3.5) and (1.3.6) are true next statements.

Theorem 1.3 *Let $\Gamma \in C^p \quad (p \geq 2)$ and $\varphi \in C(\Gamma)$. Than single-layer potential (1.3.5) with density φ is regular function in B and satisfies*

$$\|u\|_{\infty,D} \leq C\|\varphi\|_{\infty,\Gamma},$$

where $\|\cdot\|_\infty$ denotes norm in the space of continuous functions , $C > 0$ - some constant. At the boundary Γ_0 we have the following representation

$$u(x) = \int\limits_{\Gamma} \varphi(y) G(x,y) ds(y), \quad x \in \Gamma.$$

Theorem 1.4 *Let* $D_0 = B\backslash D$. *Then double-layer potential(1.3.6) with density* $\varphi \in C(\Gamma)$ *can be extended continuously from the area* D_0 *to* $\overline{D_0}$ *and from* $B\backslash\overline{D_0}$ *to* $B\backslash D_0$ *with next value of derivative at the boundary* Γ

$$\frac{\partial u}{\partial \nu}(x) = \int\limits_{\Gamma} \varphi(y) \frac{\partial^2 G(x,y)}{\partial \nu(x) \partial \nu(y)} ds(y), \quad x \in \Gamma$$

and satisfies

$$\|v\|_{\infty,\overline{D_0}} \leq C\|\varphi\|_{\infty,\Gamma}, \quad \|v\|_{\infty,B\backslash D_0} \leq C\|\varphi\|_{\infty,\Gamma}.$$

From the properties of the Green function we obtain that the potential (1.3.5) and (1.3.6) are continuous on the boundary of B and a single-layer potential (1.3.5) satisfies homogeneous boundary conditions

$$u(x) = \int\limits_{\Gamma} \varphi(y) G(x,y) ds(y) = 0, \quad x \in \Lambda.$$

1.4. Reduction to boundary integral equation

Let ω - regular harmonic function in D, which on Λ satisfies (1.1.3)

$$\omega = f \quad on \quad \Lambda;$$

Using results above, we present the solution of the problem (1.1.1) - (1.1.3) as a combination of single-layer potential (1.3.5) and partial solution ω

$$u(x) = \int\limits_{\Gamma} \varphi(y) G(x,y) ds(y) + \omega, \quad x \in D \qquad (1.4.7)$$

where unknown is a density $\varphi \in C(\Gamma)$ and by G we denote Green's function for Dirichlet problem of Laplace equation (1.2.4)

It is easy to see, that function (1.4.7) is regular and harmonic in region

D, and also satisfies condition (1.1.3) on Λ.

In general, the function ω in D can be represented as a double-layer potential

$$\omega(x) = -\int\limits_{\Lambda} f(y)\frac{\partial G(x,y)}{\partial \nu(y)} ds(y), \quad x \in D. \tag{1.4.8}$$

To satisfy the boundary condition (1.1.2) according to the theorem (1.3) about continuity of the single-layer potential, seeking the unique solution of (1.1.1) - (1.1.3) leads to integral equations of the first kind for unknown density φ:

$$\int\limits_{\Gamma} \varphi(y) G(x,y) ds(y) = f_0(x) - \omega(x), \quad x \in \Gamma. \tag{1.4.9}$$

We name the integral equation (1.4.9) as a field equation.

To describe the algorithms conveniently a parametrization of boundary curves is required. Let

$$\lambda(s) = \{(x_1(s), x_2(s)) : s \in [0, 2\pi]\}$$

is the parametrization for the exterior curve Λ. For simplicity we consider only starlike interior curves, i.e., we choose a parametrization in polar coordinates of the form

$$\gamma_r(s) = \{r(s)c(s) : s \in [0, 2\pi]\}, \tag{1.4.10}$$

where $c(s) = (\cos s, \sin s)$ and $r : \mathbb{R} \to (0, \infty)$ is a 2π periodic function representing the radial distance from the origin. However, we wish to emphasize that the concepts described below, in principle, are not confined to starlike boundaries only. We indicate the dependence on r by denoting the curve with parametrization (1.4.10) by Γ_r. We introduce the parametrized density as $\varphi(t) := \varphi(\gamma_r(t))$ or $\phi(t) := \varphi(\gamma_r(t))|\gamma_r'(t)|$.

Integral equation of the first kind (1.4.9) is well-posed with a logarithmic singularity for an approximation of r. Since all functions in this equation are 2π periodical we implement the trigonometric quadrature method. To do this we rewrite the equation (2.2.6) in the following equivalent form

$$\frac{1}{2\pi}\int_0^{2\pi}\phi(\tau)\left[-\frac{1}{2}\ln\frac{4}{e}\sin^2\frac{t-\tau}{2}+K_r(t,\tau)\right]d\tau = f_0(t)-w_r(t), \quad t\in[0,2\pi], \tag{1.4.11}$$

where

$$K_r(t,\tau) := \frac{1}{2}\ln\frac{\frac{4}{e}\sin^2\frac{t-\tau}{2}}{|\gamma_r(t)-\gamma_r(\tau)|^2}+\tilde{G}(\gamma_r(t),\gamma_r(\tau)), \quad t\neq\tau$$

with the diagonal term

$$K_r(t,t) = \frac{1}{2}\ln\frac{1}{e|\gamma_r'(t)|^2}+\tilde{G}(\gamma_r(t),\gamma_r(t)).$$

For $0<\alpha<1$ by $C^{0,\alpha}$ and $C^{1,\alpha}$ we denote Holder space of continuous and continuously differentiable functions with relevant Holder norms ([12]). Let $C^{0,\alpha}_{2\pi}(\mathbb{R})$ and $C^{1,\alpha}_{2\pi}(\mathbb{R})$ - subspace of 2π-periodic functions in $C^{0,\alpha}(\mathbb{R})$ and $C^{1,\alpha}(\mathbb{R})$. Then based on [12] we next result about well-posedness.

Theorem 1.5 *For all right parts $f_0 - w_r \equiv F \in C^{1,\alpha}_{2\pi}(\mathbb{R})$ equation (1.4.11) has an unique solution $\phi\in C^{0,\alpha}_{2\pi}(\mathbb{R})$.*

Theorem 1.6 *For all piecewise continuous regular functions f_0 and f, that are defined on Γ and Λ respectively, exists an unique solution of Dirichlet problem* (1.1.1)-(1.1.3)*, which continuously depends on the input.*

1.5. Trigonometric quadrature method

The following trigonometric quadratures with equidistant points $t_j = \frac{j\pi}{n}$, $j=0,\ldots,2n-1$ are used

$$\frac{1}{2\pi}\int_0^{2\pi} f(\tau)\ln\left(\frac{4}{e}\sin^2\frac{t-\tau}{2}\right)d\tau \approx \sum_{j=0}^{2n-1} f(t_j)R_j^n(t) \tag{1.5.12}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{f(\tau)}{2\sin^2\frac{t-\tau}{2}}d\tau \approx \sum_{j=0}^{2n-1} f(t_j)T_j^n(t) \tag{1.5.13}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} f(\tau)\,d\tau \approx \frac{1}{2n}\sum_{k=0}^{2n-1} f(t_k) \tag{1.5.14}$$

with the weight functions

$$R_j^n(t) := -\frac{1}{2n}\Big\{1+2\sum_{m=1}^{n-1}\frac{1}{m}\cos m(t-t_j)+\frac{\cos n(t-t_j)}{n}\Big\},$$

$$T_j^n(t) := -\frac{1}{n}\sum_{m=1}^{n-1} m\cos m(t-t_j) - \frac{\cos n(t-t_j)}{2}.$$

It leads to the following system of linear equations with respect to $\phi_{ni} \approx \phi(t_i)$

$$\sum_{i=0}^{2n-1}\phi_{ni}[-\frac{1}{2}R_i^n(t_k)+\frac{1}{2n}K(t_k,t_i)] = F(t_k), \quad k=0,\dots,2n-1 \tag{1.5.15}$$

where

$$F(t) = f_0(t) + \frac{1}{2n}\sum_{i=0}^{2n-1} f(t_i)H(t,t_i),$$

$$H(t,\tau) := \frac{\partial G(\gamma_r(t),\lambda(\tau))}{\partial\nu(\lambda(\tau))}|\lambda'(t)|.$$

In [4] based on the theory of collectively compact operators was studied the convergence and the error of proposed method. In particular, if $\Lambda \in C^\infty$ and $F \in C^{p+1}$, then $\|\tilde{\phi}_n - \phi\|_{0,\alpha} \le Cn^{-p}$, where $\tilde{\phi}_n$ - trigonometric polynomial built on points ϕ_{ni}, which were found from system (1.5.15). Note that when Λ and F are analytical we obtain exponential convergence.

Chapter 2

Boundary reconstruction

We assume that D is a doubly connected bounded domain in $\mathbb{R}^2$ with the boundary ∂D consisting of two disjoint closed C^2 curves Γ and Λ such that Γ is contained in the interior of Λ.

The inverse problem we are concerned with is: Given the Dirichlet data f on Λ with $f \neq 0$ and the Neumann data

$$g := \frac{\partial u}{\partial \nu} \quad \text{on } \Lambda \tag{2.0.1}$$

determine the shape of the interior boundary Γ,where u is a solution of direct problem (1.1.1)-(1.1.3) where $f_0 \equiv 0$ in (1.1.2). Here, and in the sequel, by ν we denote the outward unit normal to Γ or to Λ.

We tacitly assume that f has enough smoothness, for example $f \in C^{1,\alpha}(\Lambda)$ for classical solutions or $f \in H^{1/2}(\Lambda)$ for weak solutions, to ensure the existence of the normal derivative on Λ. As opposed to the forward boundary value problem, the inverse problem is nonlinear and ill-posed.

The issue of uniqueness, i.e., identifiability of the unknown curve from the Cauchy data on , is settled by the following theorem.

Theorem 2.1 *Let Γ and $\widetilde{\Gamma}$ be two closed curves contained in the interior of Λ and denote by u and $\widetilde{u}$ the solutions to the Dirichlet problem* (1.1.1)–

(1.1.3) *for the interior boundaries* Γ *and* $\widetilde{\Gamma}$, *respectively. Assume that* $f \neq 0$ *and*

$$\frac{\partial u}{\partial \nu} = \frac{\partial \widetilde{u}}{\partial \nu}$$

on an open subset of Λ. *Then* $\Gamma = \widetilde{\Gamma}$.

2.1. Reduction to boundary integral equations

According chapter 1 seeking the unique solution of (1.1.1)–(1.1.3), where $f_0 \equiv 0$, leads to the integral equation of the first kind

$$\int_\Gamma G(x,y)\varphi(y)\,ds(y) = -w(x), \quad x \in \Gamma, \tag{2.1.1}$$

for the unknown density φ. The given flux g on Λ leads to the integral equation

$$\int_\Gamma \varphi(y)\frac{\partial G(x,y)}{\partial \nu(x)} ds(y) = g(x) - \frac{\partial w}{\partial \nu}(x), \quad x \in \Lambda, \tag{2.1.2}$$

which is named a data equation.

Let introduce the single-layer potential

$$(S\varphi)(x) := \int_\Gamma G(x,y)\varphi(y)\,ds(y), \quad x \in \Gamma, \tag{2.1.3}$$

and the operator

$$(A\varphi)(x) := \int_\Gamma \frac{\partial G(x,y)}{\partial \nu(x)}\,\varphi(y)\,ds(y), \quad x \in \Lambda, \tag{2.1.4}$$

for the normal derivative of the single-layer potential on Λ.

Theorem 2.2 *The inverse boundary value problem* (1.1.1)-(1.1.3),(2.0.1) *is equivalent to the system of integral equations*

$$S\varphi = -w \quad on \quad \Gamma, \tag{2.1.5}$$

$$A\varphi = g - \frac{\partial w}{\partial \nu} \quad on \quad \Lambda. \tag{2.1.6}$$

Proof. Analogously to [13]. $\square$

Theorem 2.3 *The operator $S : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is bijective and has bounded inverse. The operator $A : L^2(\Gamma) \to L^2(\Lambda)$ is injective and has dense range.*

Proof. The bijectivity of S is the classical result and can be found in [12]. The injectivity of A is proved in [3]. $\square$

After a parametrization of curves in the form (1.4.10) we denote the operators defined through (2.1.3) and (2.1.4) for $\Gamma = \Gamma_r$ are given by

$$(S_r\phi)(t) = \frac{1}{2\pi} \int_0^{2\pi} \phi(\tau) G(\gamma_r(t), \gamma_r(\tau)) d\tau,$$

$$(\tilde{S}_r\varphi)(t) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(\tau) G(\gamma_r(t), \gamma_r(\tau)) |\gamma_r'(\tau)| d\tau,$$

$$(A_r\phi)(t) = \frac{1}{2\pi} \int_0^{2\pi} \phi(\tau) \frac{\partial G}{\partial \nu(\lambda(t))}(\lambda(t), \gamma_r(\tau)) d\tau$$

and

$$(\tilde{A}_r\varphi)(t) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(\tau) \frac{\partial G}{\partial \nu(\lambda(t))}(\lambda(t), \gamma_r(\tau)) |\gamma_r'(\tau)| d\tau.$$

2.2. Iterative schemes.

Operators S_r, A_r and $\tilde{A}_r$ have the following Frechét derivatives with respect to the radial function r

$$(S'[r,\phi]q)(t) = \frac{1}{2\pi} \int_0^{2\pi} \phi(\tau) [q(\tau) L_r^{(1)}(t,\tau) + q(t) L_r^{(2)}(t,\tau)] d\tau,$$

$$(A'[r,\phi]q)(t) = \frac{1}{2\pi} \int_0^{2\pi} \phi(\tau) q(\tau) H_r^{(1)}(t,\tau) d\tau.$$

and

$$(\tilde{A}'[r,\varphi]q)(t) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(\tau)\left[q(\tau)H_r^{(1)}(t,\tau)|\gamma_r'(\tau)| + \frac{r(\tau)q(\tau)+r'(\tau)q'(\tau)}{|\gamma_r'(t)|}H_r^{(2)}(t,\tau)\right]d\tau. \tag{2.2.1}$$

Here we introduced the kernels

$$L_r^{(1)}(t,\tau) := -\frac{r(\tau)-r(t)\cos(t-\tau)}{|\gamma_r(t)-\gamma_r(\tau)|^2} + \operatorname{grad}_{\gamma_r(\tau)}\tilde{G}(\gamma_r(t),\gamma_r(\tau))\cdot c(\tau),$$

$$L_r^{(2)}(t,\tau) := -\frac{r(t)-r(\tau)\cos(t-\tau)}{|\gamma_r(t)-\gamma_r(\tau)|^2} + \operatorname{grad}_{\gamma_r(t)}\tilde{G}(\gamma_r(t),\gamma_r(\tau))\cdot c(t),$$

$$H_r^{(1)}(t,\tau) := \operatorname{grad}_{\gamma_r(\tau)}\frac{\partial G(\lambda(t),\gamma_r(\tau))}{\partial\nu(\lambda(t))}\cdot c(\tau)$$

and

$$H_r^{(2)}(t,\tau) := \frac{\partial G(\lambda(t),\gamma_r(\tau))}{\partial\nu(\lambda(t))}.$$

Note that

$$\lim_{\tau\to t}(q(\tau)L_r^{(1)}(t,\tau)+q(t)L_r^{(2)}(t,\tau)) = \frac{r(t)q(t)+r'(t)q'(t)}{|\gamma_r'(t)|^2}+2q(t)\operatorname{grad}_{\gamma_r(t)}\tilde{G}(\gamma_r(t),\gamma_r(t))\cdot c($$

These representation were obtained by standard differentiation procedure in (2.1.3) and (2.1.4). Also we will need the Frechét derivative for the function w

$$(w'[r]q)(t) = -\frac{1}{2\pi}\int_0^{2\pi} f(\tau)q(\tau)W_r(t,\tau)d\tau$$

with

$$W_r(t,\tau) := |\lambda'(\tau)|\operatorname{grad}_{\gamma_r(t)}\frac{\partial G(\gamma_r(t),\lambda(\tau))}{\partial\nu(\lambda(\tau))}\cdot c(t).$$

The linear operators $S'[r,\varphi]$ and $A'[r,\varphi]$ have the following properties.

Theorem 2.4 *Let r be the radial function of the interior boundary Γ_r and let ϕ be a solution to the integral equation* (2.1.5), *i.e.* $S_r\phi = -w$ *on* Γ_r. *Assume that $q\in C^2[0,2\pi]$ and $\psi\in L^2[0,2\pi]$ solve the homogeneous system*

$$S_r\psi + S'[r,\phi]q + w'[r]q = 0, \tag{2.2.2}$$

$$A_r\psi + A'[r,\phi]q = 0. \tag{2.2.3}$$

Then $q = 0$ and $\psi = 0$.

Proof. As it is shown in [8], for sufficiently small q, the perturbed interior curve as given in polar coordinates by

$$\Gamma_{r+q} = \{(r(t)+q(t))c(t) : t \in [0,2\pi]\}$$

can be represented in the form

$$\Gamma_{r+q} = \{r(t)c(t) + \widetilde{q}(t)\nu(t) : t \in [0,2\pi]\}$$

in terms of the unit normal vector

$$\nu(t) = r'(t)(-\sin t, \cos t) - r(t)(\cos t, \sin t)$$

to the unperturbed curve Γ_r and a function $\widetilde{q}$. Now in the Fréchet derivatives S', A' and w' we may replace the perturbation vector $\zeta(t) = q(t)c(t)$ by $\widetilde{\zeta} = \widetilde{q}\,\nu$. We introduce the function

$$V(x) := \int_0^{2\pi} \psi(\tau)G(x,\gamma_r(\tau))d\tau - \int_0^{2\pi} \operatorname{grad}_x G(x,\gamma_r(\tau)) \cdot \widetilde{\zeta}(\tau)\phi(\tau)\, d\tau, \quad x \in \mathbb{R}^2 \setminus \Gamma_r.$$

Then (2.2.3) implies that $\frac{\partial V}{\partial \nu} = 0$ on Λ. The function V satisfies the Laplace equation in the exterior of Λ, it decays at infinity, therefore by the uniqueness for the exterior Neumann problem we conclude that $V \equiv 0$ in the exterior of Λ. By analyticity we obtain $V \equiv 0$ in the exterior of Γ_r. Approaching Γ_r from the exterior by the jump relations we obtain

$$\begin{aligned} 0 = &\int_0^{2\pi} \psi(\tau)G(\gamma_r(t),\gamma_r(\tau)d\tau \\ &- \int_0^{2\pi} \operatorname{grad}_{\gamma_r(t)} G(\gamma_r(t),\gamma_r(\tau)) \cdot \widetilde{\zeta}(\tau)\phi(\tau)\, d\tau + \frac{1}{2}\widetilde{q}(t)\phi(t), \quad t \in [0,2\pi]. \end{aligned}$$

Employing the above equality and recalling the definition (**??**) of u we can rewrite (2.2.2) as follows

$$\widetilde{\zeta} \cdot \operatorname{grad} u \circ \gamma_r = 0.$$

Due to the definition of u and the condition on φ we have $u = 0$ on Γ_r, which is equivalent to

$$\widetilde{\zeta} \cdot \nu \circ \gamma_r \left(\frac{\partial u}{\partial \nu} \right) \circ \gamma_r = 0.$$

Since by Holmgren's theorem $\frac{\partial u}{\partial \nu}$ cannot vanish on open subsets of Γ_r we obtain $\widetilde{\zeta} \cdot \nu \circ \gamma_r = \widetilde{q} = 0$ and hence $q = 0$. Analogously to [13] by continuity of a single-layer potential and the uniqueness of the interior Dirichlet problem we obtain $V = 0$ in $\mathbb{R}^2$ and therefore the density $\psi = 0$. $\square$

Theorem 2.5 *Let r be the radial function of the interior boundary Γ_r and let ϕ be a solution to the integral equation* (2.1.6)*, i.e. $A_r\phi = g - \frac{\partial w}{\partial \nu}$ on Λ. Assume that $q \in C^2[0, 2\pi]$ solves the homogeneous equation*

$$S'[r, \phi]q + w'[r]q = 0. \tag{2.2.4}$$

Then $q = 0$.

Proof. Since ϕ is a solution to $A_r\phi = g - \frac{\partial w}{\partial \nu}$ on Λ it also satisfies $S_r\phi = -w$ on Γ_r. We represent the perturbed interior curve again as

$$\Gamma_{r+q} = \{r(t)c(t) + \widetilde{q}(t)\nu(t) : t \in [0, 2\pi]\}$$

and introduce the function

$$V(x) := \int_0^{2\pi} \phi(\tau)G(x, \gamma_r(\tau))d\tau - \int_\Lambda \frac{\partial G(x, y)}{\partial \nu(y)} f(y)\, ds(y), \quad x \in \mathbb{R}^2 \setminus \Gamma_r.$$

The function V is a solution to the interior Dirichlet boundary value problem with the homogeneous condition. In view of the unique solution we obtain $V \equiv 0$ in the interior of Γ_r and therefore $\left.\frac{\partial V}{\partial \nu}\right|_{\Gamma_r} = 0$, i.e.

$$\begin{aligned} 0 = &\widetilde{q}(t)\nu(t) \cdot \mathrm{grad}_{\gamma_r(t)} \int_0^{2\pi} G(\gamma_r(t), \gamma_r(\tau))\phi(\tau)\, d\tau + \frac{1}{2}\widetilde{q}(t)\phi(t) \\ &- \widetilde{q}(t)\nu(t) \cdot \mathrm{grad}_{\gamma_r(t)} \int_\Lambda \frac{\partial G(\gamma_r(t), y)}{\partial \nu(y)} f(y)\, ds(y), \quad t \in [0, 2\pi]. \end{aligned}$$

From (2.2.4) we find

$$0{=}-\frac{1}{2}\widetilde{q}(t)\phi(t)-\int_0^{2\pi}\operatorname{grad}_{\gamma_r(t)}G(\gamma_r(t),\gamma_r(\tau))\cdot\widetilde{\zeta}(\tau)\,\phi(\tau)\,d\tau,\quad t\in[0,2\pi]. \tag{2.2.5}$$

We define a double layer potential

$$W(x){:=}-\int_0^{2\pi}\operatorname{grad}_x G(x,\gamma_r(\tau))\cdot\nu(\tau)\widetilde{q}(\tau)\phi(\tau)\,d\tau,\quad x\in\mathbb{R}^2\setminus\Gamma_r.$$

Since the function W is harmonic in the interior of Γ_r and satisfies the homogeneous Dirichlet boundary condition, (2.2.5), it implies $W\equiv 0$ in the interior of Γ_r. One can show,similarly to [12, Theorem 6.21], that the operator $-I+K$ is injective, where

$$(K\psi)(t)=\int_0^{2\pi}\operatorname{grad}_{\gamma_r(t)}G(\gamma_r(t),\gamma_r(\tau))\cdot\nu(\tau)\psi(\tau)\,d\tau,\quad t\in[0,2\pi]$$

Hence from (2.2.5) we obtain

$$\widetilde{q}(t)\phi(t)=0,\quad t\in[0,2\pi]$$

Bu the jump relations for the function V we have

$$\frac{1}{|r'|}\phi=\left.\frac{\partial V^-}{\partial\nu}\right|_{\Gamma_r}-\left.\frac{\partial V^+}{\partial\nu}\right|_{\Gamma_r}=-\left.\frac{\partial V^+}{\partial\nu}\right|_{\Gamma_r}.$$

Since by Holmgren's theorem $\frac{\partial V^+}{\partial\nu}$ cannot vanish on open subsets of Γ_r and $|r'|\neq 0$ we obtain $\widetilde{q}=0$ and hence $q=0$. □

Remark (about the Algorithm 2).

If the interior boundary is a circle, then exists a nontrivial solution $q=$ *const* to the homogeneous equation $A'[r,\varphi]q=0$. Indeed, introducing the function

$$V(x)=-q\operatorname{grad}_x\int_0^{2\pi}G(x,\gamma_r(\tau))\cdot\nu(\tau)\varphi(\tau)d\tau,\quad x\in\mathbb{R}^2\setminus\Gamma_r$$

we obtain that V is a unique solution to the exterior Neumann boundary value problem with the homogeneous condition in the exterior of Λ, and hence $V^+|_{\Gamma_r} = 0$. Since the null-space of the operator of integral equation

$$\frac{1}{2}\varphi(t) - \mathrm{grad}_x \int_0^{2\pi} G(t, \gamma_r(\tau)) \cdot \nu(\tau)\varphi(\tau)d\tau = 0, \quad t \in [0, 2\pi]$$

is not empty, i.e. one can find $q \neq 0$ which solves $A'[r, \varphi]q = 0$.

In view of this remark we introduced the modified version $\widetilde{A}'[r, \varphi]$, (2.2.1), instead of the operator $A'[r, \varphi]$.

Now we describe three iterative algorithms for the numerical solution of (2.1.5)-(2.1.6).

Algorithm 1.

1. Choose some starting value r. Solve the well-posed integral equation

$$S_r\phi = -w_r. \tag{2.2.6}$$

2. For the given r and φ solve the system of linearized ill-posed integral equations

$$S_r\psi + S'[r, \phi]q + w'[r]q = -S_r\phi - w_r, \tag{2.2.7}$$

$$A_r\psi + A'[r, \phi]q = g - \frac{\partial w}{\partial \nu} - A_r\phi \tag{2.2.8}$$

with respect to functions ψ and q.

3. Calculate new approximations for the radial function $r = r + q$ and for the density $\phi = \phi + \psi$.

4. Repeat steps 2-3 until some stopping criterion is satisfied.

Algorithm 2.

1. Choose some starting value r.

2. Solve the well-posed integral equation

$$\tilde{S}_r\varphi = -w_r. \tag{2.2.9}$$

3. For the given r and φ solve the linearized ill-posed integral equation

$$\tilde{A}'[r,\varphi]q = g - \frac{\partial w}{\partial \nu} - \tilde{A}_r\varphi \tag{2.2.10}$$

with respect to function q.

4. Calculate new approximations for the radial function $r = r + q$.
5. Repeat steps 2-4 until some stopping criterion is satisfied.

Algorithm 3.

1. Choose some starting value r.
2. Solve the ill-posed integral equation

$$A_r\phi = g - \frac{\partial w}{\partial \nu}. \tag{2.2.11}$$

3. For given r and φ solve the linearized ill-posed integral equation

$$S'[r,\phi]q + \omega'[r]q = -S_r\phi - w_r \tag{2.2.12}$$

with respect to function q.

4. Calculate new approximations for the radial function $r = r + q$.
5. Repeat steps 2-4 until some stopping criteria is satisfied.

Note here that we need to use some regularization methods in the case of ill-posed integral equations. According to properties of the corresponding integral operators an application of the Tikhonov regularization is justified for the algorithms 1,3.

2.3. Implementation.

Algorithm 1.

Step1. On the first step of this algorithm we need to solve the well posed integral equation of the first kind (2.2.6) with a logarithmic singularity for a current approximation of r. Analogously to the approach in previous chapter (2.2.6) leads to the following system of linear equations with respect to $\phi_{ni} \approx \phi(t_i)$

$$\sum_{i=0}^{2n-1} \phi_{ni}[-\frac{1}{2}R_i(t_k) + \frac{1}{2n}K(t_k, t_i)] = -\tilde{w}_r(t_k), \quad k = 0, \dots, 2n-1$$

with

$$\tilde{w}_r(t) = -\frac{1}{2n}\sum_{i=0}^{2n-1} f(t_i)H(t, t_i),$$

where

$$H(t, \tau) := \frac{\partial G(\gamma_r(t), \lambda(\tau))}{\partial \nu(\lambda(\tau))}|\lambda'(t)|.$$

The convergence and error analysis for this method can be found in [12].

Step2. We search the unknown corrections in the system (2.2.7)-(2.2.8) as

$$\psi_n = \sum_{i=0}^{2n-1} \psi_{ni} l_i^1, \quad q_m = \sum_{i=0}^{2m} q_{mi} l_i^2,$$

where l_i^1, $i = 0, \dots, 2n-1$ are basic Lagrangian trigonometric polynomials and l_i^2, $i = 0, \dots, 2m$ are known basic functions. The quadrature method applied to (2.2.7)-(2.2.8) give us the linear system

$$\sum_{i=0}^{2n-1} \psi_{ni}\mathcal{A}_{ki}^{(11)} + \sum_{i=0}^{2m} q_{mi}\mathcal{A}_{ki}^{(12)} = b_k^{(1)}, \quad k = 0, \dots, 2n-1,$$

$$\sum_{i=0}^{2n-1} \psi_{ni}\mathcal{A}_{ki}^{(21)} + \sum_{i=0}^{2m} q_{mi}\mathcal{A}_{ki}^{(22)} = b_k^{(2)}, \quad k = 0, \dots, 2n-1$$

with matrix coefficients

$$\mathcal{A}_{ki}^{(11)} = -\frac{1}{2}R_i(t_k) + \frac{1}{2n}K_r(t_k, t_i), \quad \mathcal{A}_{ki}^{(21)} = \frac{1}{2n}H_r^{(2)}(t_k, t_i),$$

$$\mathcal{A}_{ki}^{(12)} = \frac{1}{2n}\sum_{j=0}^{2n-1}\{\phi_{nj}[l_i^2(t_j)L_r^{(1)}(t_k, t_j) + l_i^2(t_k)L_r^{(2)}(t_k, t_j)] + l_i^2(t_j)f(t_i)W_r(t_k, t_j)\},$$

$$\mathcal{A}_{ki}^{(22)} = \frac{1}{2n}\sum_{j=0}^{2n-1} \phi_{nj} l_i^2(t_j) H_r^{(1)}(t_k, t_j)$$

and right hand side

$$b_k^{(1)} = \sum_{i=0}^{2n-1} \phi_{ni}[-\frac{1}{2}R_i(t_k) - \frac{1}{2n}K_r(t_k, t_i)] - \tilde{w}_r(t_k),$$

$$b_k^{(2)} = g(t_k) - \frac{\partial \tilde{w}_r}{\partial \nu}(t_k) - \frac{1}{2n}\sum_{i=0}^{2n-1} \phi_{ni} H_r^{(2)}(t_k, t_i).$$

Here $2n \geq 2m+1$.

Thus the received ill-posed linear system is overdetermined and therefore we reduce it to the least-squares problem. The following cost functional needs to be minimized

$$F(\psi_{n0}, \ldots, \psi_{n,2n-1}, q_{m0}, \ldots, q_{m,2m}) = \sum_{i=0}^{2n-1} \left| \sum_{j=0}^{2n-1} \psi_{nj}\mathcal{A}_{ij}^{(11)} + \sum_{j=0}^{2m} q_{mj}\mathcal{A}_{ij}^{(12)} - b_i^{(1)} \right|^2 +$$

$$\sum_{i=0}^{2n-1} \left| \sum_{j=0}^{2n-1} \psi_{nj}\mathcal{A}_{ij}^{(21)} + \sum_{j=0}^{2m} q_{mj}\mathcal{A}_{ij}^{(22)} - b_i^{(2)} \right|^2 +$$

$$\alpha \sum_{j=0}^{2n-1} \omega_{1j}\psi_{nj}^2 + \beta \sum_{j=0}^{2m} \omega_{2j} q_{mj}^2$$

with the regularization parameters $\alpha > 0$ and $\beta > 0$ and weight coefficients ω_{1j} and ω_{2j}. Clearly, the final linear system has the form

$$\alpha\omega_{1i}\psi_{ni} + \sum_{j=0}^{2n-1} \psi_{nj}\mathbf{a}_{ij}^{(11)} + \sum_{j=0}^{2m} q_{mj}\mathbf{a}_{ij}^{(12)} = \mathbf{b}_i^{(1)}, \quad i = 0, \ldots, 2n-1,$$

$$\beta\omega_{2i}q_{mi} + \sum_{j=0}^{2n-1} \psi_{nj}\mathbf{a}_{ij}^{(21)} + \sum_{j=0}^{2m} q_{mj}\mathbf{a}_{ij}^{(22)} = \mathbf{b}_i^{(2)}, \quad i = 0, \ldots, 2m,$$

where

$$\mathbf{a}_{ij}^{(\ell p)} = \sum_{k=0}^{2n-1} \mathcal{A}_{ki}^{(\ell 1)}\mathcal{A}_{kj}^{(p1)} + \sum_{k=0}^{2m} \mathcal{A}_{ki}^{(\ell 2)}\mathcal{A}_{kj}^{(p2)}$$

and
$$\mathbf{b}_i^{(\ell)} = \sum_{k=0}^{2n-1} \mathcal{A}_{ki}^{(\ell 1)} b_k^{(1)} + \sum_{k=0}^{2m} \mathcal{A}_{ki}^{(\ell 2)} b_k^{(2)}.$$

Step 3. Now we can evaluate the new values for the radial function $r_m = r_m + q_m$ and for the density $\phi_n = \phi_n + \psi_n$.

The following stopping criterion can be used

$$\frac{\|q_m\|_{L^2[0,2\pi]\|}}{\|r_m\|_{L^2[0,2\pi]}} < \epsilon$$

with sufficiently small $\epsilon > 0$, or a discrepancy principle, as well.

Algorithm 2.

Step2. It is analogous to the *Step 1* from the Algorithm 1.

Step3. To find the correction q from (2.2.10) we make the discretization by the quadrature method and due to its ill-posednes we minimize the following Tikhonov functional

$$F(q_{m0}, \dots, q_{m,2m}) = \sum_{i=0}^{2n-1} \left| \sum_{j=0}^{2m} q_{mj} \mathcal{A}_{ij}^{(22)} - b_i^{(2)} \right|^2 + \beta \sum_{j=0}^{2m} \omega_{2j} q_{mj}^2, \ 2n \geq 2m+1.$$

The corresponding linear system has the form

$$\beta \omega_{2i} q_{mi} + \sum_{j=0}^{2m} q_{mj} \mathbf{a}_{ij} = \mathbf{b}_i, \quad i = 0, \dots, 2m$$

with
$$\mathbf{a}_{ij} = \sum_{k=0}^{2n-1} \mathcal{A}_{ki}^{(22)} \mathcal{A}_{kj}^{(22)}, \quad \mathbf{b}_i = \sum_{k=0}^{2n-1} \mathcal{A}_{ki}^{(22)} b_k^{(2)}.$$

Algorithm 3.

Step2. The discretization in (2.2.11) and ill-posednes of the received linear system lead to the minimization of the following Tikhonov functional

$$F(\psi_{n0}, \dots, \psi_{n,2n-1}) = \sum_{i=0}^{2n-1} \left| \sum_{j=0}^{2n-1} \psi_{nj} \mathcal{A}_{ij}^{(21)} - \tilde{b}_i^{(2)} \right|^2 + \alpha \sum_{j=0}^{2n-1} \omega_{1j} \psi_{nj}^2$$

with

$$\tilde{b}_i^{(2)} = g(t_k) - \frac{\partial \tilde{w}_r}{\partial \nu}(t_k).$$

which is equivalent to solving the linear system

$$\alpha\omega_{1i}\psi_{ni} + \sum_{j=0}^{2n-1} \psi_{nj}\mathbf{a}_{ij}^{(1)} = \mathbf{b}_i^{(1)}, \quad i = 0, \ldots, 2n-1,$$

where

$$\mathbf{a}_{ij}^{(1)} = \sum_{k=0}^{2n-1} \mathcal{A}_{ki}^{(21)}\mathcal{A}_{kj}^{(21)}, \quad \mathbf{b}_i^{(1)} = \sum_{k=0}^{2n-1} \mathcal{A}_{ki}^{(21)}\tilde{b}_k^{(2)}.$$

Step3. To find the correction q from (2.2.12) we make the discretization by quadrature method and due to its ill-posednes we minimize the following Tikhonov functional

$$F(q_{m0}, \ldots, q_{m,2m}) = \sum_{i=0}^{2n-1} \left| \sum_{j=0}^{2m} q_{mj}\mathcal{A}_{ij}^{(12)} - b_i^{(1)} \right|^2 + \beta \sum_{j=0}^{2m} \omega_{2j} q_{mj}^2, \; 2n \geq 2m+1.$$

Thus the corresponding linear system has the form

$$\beta\omega_{2i}q_{mi} + \sum_{j=0}^{2m} q_{mj}\mathbf{a}_{ij}^{(2)} = \mathbf{b}_i^{(2)}, \quad i = 0, \ldots, 2m,$$

where

$$\mathbf{a}_{ij}^{(2)} = \sum_{k=0}^{2n-1} \mathcal{A}_{ki}^{(12)}\mathcal{A}_{kj}^{(12)}, \quad \mathbf{b}_i^{(2)} = \sum_{k=0}^{2n-1} \mathcal{A}_{ki}^{(12)} b_k^{(1)}.$$

Chapter 3

Numerical examples

3.1. Direct problem

For testing of this method of solving the direct problem we use the following test examples.

3.1.1. Example 1

Let for (1.1.1)-(1.1.3)

$$f_0 = 1,$$

$$f = 1.$$

Also we note that Λ is a circle of radius $R = 1$ centered at the origin

$$\Lambda = \{(x, y)|x^2 + y^2 = 1\},$$

and

$$\Gamma = \{(x, y)|x^2 + y^2 = 0.3^2\}.$$

Then domain $D = \{(x, y)|0.3^2 \leq x^2 + y^2 \leq 1\}$. Obviously, the exact solution to this problem is function $u \equiv 1$ and its normal derivative on the boundary is equal zero. Then we find a solution u_n at the point $(0.5, 0)$

and compare with the exact solution at the same point . Also we calculate the normal derivative of the solution on the boundary Λ and its error. The results of calculations with different number of points n are given in the table (3.1) .

Table 3.1: Result for example 1

n	$\|\frac{\partial u}{\partial \nu} - \frac{\partial u_n}{\partial \nu}\|_{\infty,\Lambda}$	$\lvert u(0.5,0) - u_n(0.5,0)\rvert$
8	$1,86 \times 10^{-4}$	$7,71 \times 10^{-3}$
16	$1,22 \times 10^{-8}$	$3,05 \times 10^{-5}$
32	$1,25 \times 10^{-13}$	$4,66 \times 10^{-10}$
64	$6,19 \times 10^{-13}$	$6,66 \times 10^{-16}$

3.1.2. Example 2

As an exact solution of (1.1.1)-(1.1.3) in D we choose the restriction of the fundamental solution

$$u_{ex}(x) = \ln |\, x - y^* \,|, \quad x \in D, y = (10,0) \in \mathbb{R}^2 \backslash D$$

Then boundary conditions we choose like

$$f_0 = u_{ex}|_{\Gamma_0},$$

$$f = u_{ex}|_{\Lambda}.$$

After solving this problem for different number of points n we obtain the next results in table (3.2) .

Table 3.2: Result for example 2

n	$\|\frac{\partial u_{ex}}{\partial \nu} - \frac{\partial u_n}{\partial \nu}\|_{\infty,\Lambda}$	$\|u_{ex}(0.5,0) - u_n(0.5,0)\|$
8	4.43×10^{-4}	$1,67 \times 10^{-2}$
16	2.92×10^{-8}	6.61×10^{-5}
32	2.74×10^{-13}	1.01×10^{-9}
64	1.38×10^{-13}	1.33×10^{-15}

3.2. Boundary reconstruction

We demonstrate the feasibility of the proposed methods for the inverse problem (1.1.1)-(1.1.3),(2.0.1) with following boundaries $\lambda(t) = \{Rc(t), t \in [0, 2\pi]\}$ with $R = 2$, and

$$\gamma_r(t) = \left\{ \sqrt{\cos^2 t + 0.25 \sin^2 t}\, c(t), t \in [0, 2\pi] \right\}.$$

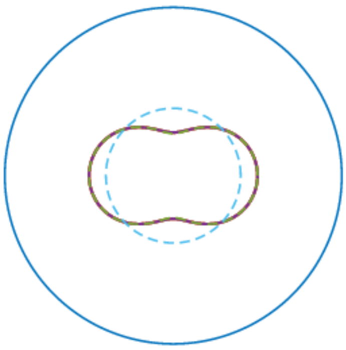
The Cauchy data on Λ were generated by solving the direct problem (1.1.1)-(1.1.3) for $f = 1$ on Λ and calculating g as the normal derivative on Λ. The noisy data were formed as

$$g^\delta = g + \delta(2\eta - 1)\|g\|_{L_2(\Lambda)}$$

with the noise level δ and the random value $\eta \in (0, 1)$. The results of the numerical experiments for exact and noisy data with $\delta = 5\%$ are reflected on Fig. 3.1. Here we used the following discretization parameters $n = 16$, $m = 4$ and $\epsilon = 0.0001$. The values of regularization parameters, numbers of iterations and errors are given in Tabl. 3.2..

	δ	It.	E	α	β
Algorithm 1	0%	7	0.00561	10^{-13}	10^{-5}
	5%	8	0.07367	10^{-10}	10^{-3}
Algorithm 2	0%	21	0.00614		10^{-2}
	5%	17	0.03843		10^{-1}
Algorithm 3	0%	21	0.00322	10^{-14}	10^{-7}
	5%	15	0.04714	10^{-5}	10^{-1}

Table 3.3: Errors and regularization parameters



a). Reconstruction for exact data

b). Reconstruction for 5% of noise in the data

Graphic 3.1: Reconstruction of the boundary Γ

Conclusions

In this course paper we consider the reconstruction of an interior curve from the given Cauchy data of a harmonic function on the exterior boundary of the given planar domain. With the help of Green's function and potential theory the non-linear boundary problem is reduced to the system of non-linear boundary integral equations. The three iterative algorithms are developed for its numerical solution. We found the Frechét derivatives for the corresponding operators and showed unique solvability of the linearized systems. Full discretization of the systems is realized by a trigonometric quadrature method. Due to the inherited ill-posedness in the obtained system of linear equations we apply the Tikhonov regularization.

Also to obtain normal derivative of the solution of the direct problem on the exterior boundary for the inverse problem we considered the boundary integral equations method using single-layer and double-layer potentials.

The numerical results show that the proposed methods give good accuracy of reconstructions with an economical computational cost.

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