

THE ASYMPTOTIC BEHAVIOR OF THE TOTAL NUMBER OF PARTICLES IN A CRITICAL BRANCHING PROCESS WITH IMMIGRATION

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ABSTRACT. A sequence of branching processes with immigration is considered in the case where the mean number of descendants of a particle tends to unity. The rate of growth and asymptotic behavior of the total number of particles in the population are found.

1. INTRODUCTION

Assume that $\{\xi_{k,j}^{(n)}, k, j \in \mathbb{N}\}$ and $\{\varepsilon_k^{(n)}, k \in \mathbb{N}\}$ are two independent families of independent nonnegative integer valued and identically distributed random variables for all $n \in \mathbb{N}$. Let $\{X_k^{(n)}, k = 0, 1, \dots\}$, $n \in \mathbb{N}$, be a sequence of branching processes with immigration defined by the following recurrence relations:

$$(1) \quad X_0^{(n)} = 0, \quad X_k^{(n)} = \sum_{j=1}^{X_{k-1}^{(n)}} \xi_{k,j}^{(n)} + \varepsilon_k^{(n)}, \quad k, n \in \mathbb{N}.$$

If the random variables $\xi_{k,j}^{(n)}$ and $\varepsilon_k^{(n)}$, $n \geq 1$, are treated as the number of descendants of a j th particle in a $(k-1)$ th generation of a certain population of particles and the number of particles immigrating into the k th generation, respectively, then $X_k^{(n)}$ is the total number of particles in the k th generation of the population.

A sequence of branching processes with immigration (1) is called almost critical if $E\xi_{1,1}^{(n)} \rightarrow 1$ as $n \rightarrow \infty$. Consider random stepwise processes Z_n , $n \in \mathbb{N}$, defined by

$$Z_n(t) = \sum_{k=1}^{[nt]} X_k^{(n)}, \quad t \geq 0, \quad n \in \mathbb{N}.$$

It is clear that the trajectories of the processes Z_n , $n \in \mathbb{N}$, belong to the Skorokhod space $D[0, \infty)$. The variable $Z_n(t)$ is the total number of particles (counted up to the moment $[nt]$) in the branching process with immigration $X_k^{(n)}$, $k \geq 0$.

Pakes [1] studies the rate of growth and asymptotic behavior (as $n \rightarrow \infty$) of the fluctuation of the total number of descendants of a particle in a branching Galton–Watson process up to the moment n under the assumption that the process does not vanish. Karpenko and Nagaev [2] investigate the limit behavior of the conditional distribution of the total number of descendants of a particle in the Galton–Watson process under the condition that the process vanishes at the moment n and in the case where the

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expectation of the number of particles generated by a single particle tends to unity as $n \rightarrow \infty$. A number of papers, say [3, 4, 8, 9], are devoted to functional limit theorems for a sequence of branching processes with immigration. On the other hand, the asymptotic behavior of the total number of particles Z_n has been studied quite a bit.

The aim of this paper is to study the asymptotic behavior of the process Z_n as well as that of its deviation from the mean value, $Z_n - \mathbf{E}Z_n$, for the case where the mean value of the number of descendants of a single particle tends to unity from the left with a rate being slower than n^{-1} as $n \rightarrow \infty$.

2. MAINSTREAM

We assume that

$$m_n = \mathbf{E}\xi_{1,1}^{(n)}, \quad \sigma_n^2 = \text{Var} \xi_{1,1}^{(n)}, \quad \lambda_n = \mathbf{E}\varepsilon_1^{(n)}, \quad b_n^2 = \text{Var} \varepsilon_1^{(n)}$$

are finite for all $n \in \mathbb{N}$. Throughout below d_n , $n \in \mathbb{N}$, denotes a certain sequence of positive numbers such that $d_n \rightarrow \infty$ and $n^{-1}d_n \rightarrow 0$ as $n \rightarrow \infty$; $W(t)$, $t \geq 0$, is a standard Wiener process in the space $D[0, \infty)$; $I(A)$ is the indicator of an event A ; and the symbol \xrightarrow{P} denotes the convergence in probability of random variables.

Theorem 1 provides some information on the rate of growth of the process Z_n .

Theorem 1. *Assume that*

- (1) $m_n = 1 + \alpha d_n^{-1} + o(d_n^{-1})$ as $n \rightarrow \infty$ for some fixed $\alpha < 0$;
- (2) the limit $\lim_{n \rightarrow \infty} \sigma_n^2 = \sigma^2 \geq 0$ exists and is finite;
- (3) the limits $\lim_{n \rightarrow \infty} \lambda_n = \lambda \geq 0$ and $\lim_{n \rightarrow \infty} b_n^2 = b^2 \geq 0$ exist and are finite.

Then

$$\frac{Z_n}{nd_n} \rightarrow Z \quad \text{as } n \rightarrow \infty$$

in the space $D[0, \infty)$ equipped with the Skorokhod J -topology, where the limit process Z is defined by

$$Z(t) = |\alpha|^{-1} \lambda t, \quad t \geq 0.$$

Theorem 2 describes the asymptotic behavior of the deviation of the process Z_n from its mean value.

Theorem 2. *Assume that*

- (1) $m_n = 1 + \alpha d_n^{-1} + o(d_n^{-1})$ as $n \rightarrow \infty$ for some fixed $\alpha < 0$;
- (2) the limit $\lim_{n \rightarrow \infty} d_n \sigma_n^2 = \sigma^2 \geq 0$ exists and is finite;
- (3) the limits $\lim_{n \rightarrow \infty} \lambda_n = \lambda \geq 0$ and $\lim_{n \rightarrow \infty} b_n^2 = b^2 \geq 0$ exist and are finite;
- (4) for all $\varepsilon > 0$,

$$d_n \mathbf{E}(\xi_{1,1}^{(n)} - m_n)^2 I(|\xi_{1,1}^{(n)} - m_n| > \varepsilon \sqrt{n}) \rightarrow 0, \quad n \rightarrow \infty;$$

- (5) for all $\varepsilon > 0$,

$$\mathbf{E}(\varepsilon_1^{(n)} - \lambda_n)^2 I(|\varepsilon_1^{(n)} - \lambda_n| > \varepsilon \sqrt{n}) \rightarrow 0, \quad n \rightarrow \infty.$$

Then

$$(d_n \sqrt{n})^{-1} (Z_n - \mathbf{E}Z_n) \rightarrow Y \quad \text{as } n \rightarrow \infty$$

in the space $D[0, \infty)$ equipped with the Skorokhod J -topology, where the limit process Y is defined by

$$Y(t) = |\alpha|^{-1} (|\alpha|^{-1} \lambda \sigma^2 + b^2)^{1/2} W(t), \quad t \geq 0.$$

Remark 1. The asymptotic behavior of the process Z_n and that of its deviation are easy to extract from the results of papers [3, 4, 8] and Theorem 5.1 of [5] in the case where $m_n = 1 + \alpha n^{-1} + o(n^{-1})$.

Remark 2. Theorems 1 and 2 show that the rate of convergence of m_n to unity influences essentially the rate of growth and asymptotic behavior of the deviation of the process Z_n .

Remark 3. In general, the conditions $b_n^2 \rightarrow b^2 > 0$ and $d_n \sigma^2 \rightarrow \sigma^2 > 0$ do not imply assumptions (4) and (5) of Theorem 2, respectively. For example, let $\varepsilon_k^{(n)}$ assume values 0, 1, and n with probabilities n^{-2} , $1 - 2n^{-2}$, and n^{-2} , respectively. Then $\lambda_n = 1 + n^{-1} + o(n^{-1})$ and $b_n^2 = 1 - 2n^{-1} + o(n^{-1})$ as $n \rightarrow \infty$. For every $\varepsilon > 0$,

$$\mathbb{E} \left(\varepsilon_1^{(n)} - \lambda_n \right)^2 I \left(\left| \varepsilon_1^{(n)} - \lambda_n \right| > \varepsilon \sqrt{n} \right) \approx \frac{(n-1)^2}{n^2} \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

Thus $b_n^2 \rightarrow 1$ as $n \rightarrow \infty$ in this case, but assumption 5 does not hold.

3. PROOFS

Proof of Theorem 1. It is easy to see that

$$(2) \quad \mathbb{E} X_k^{(n)} = \frac{1 - m_n^k}{1 - m_n} \lambda_n, \quad k = 0, 1, 2, \dots$$

Put $G_n(t) = (nd_n)^{-1} Z_n(t)$. The assumptions of Theorem 1 imply that

$$(3) \quad \mathbb{E} G_n(t) \rightarrow |\alpha|^{-1} \lambda t \quad \text{as } n \rightarrow \infty.$$

We are going to estimate $\text{Var } G_n(t)$. Equality (2.13) of the paper [4] yields

$$\text{Var } G_n(t) = (nd_n)^{-2} (U_n(t) b_n^2 + V_n(t) \lambda_n \sigma_n^2),$$

where

$$U_n(t) = \sum_{k=1}^{[nt]+1} \frac{1 - m_n^{2(k-1)}}{1 - m_n^2} \left(2 \frac{1 - m_n^{[nt]-k+2}}{1 - m_n} - 1 \right),$$

$$V_n(t) = \sum_{k=1}^{[nt]+1} \frac{(1 - m_n^{k-1})(1 - m_n^{k-2})}{(1 - m_n)(1 - m_n^2)} \left(2 \frac{1 - m_n^{[nt]-k+2}}{1 - m_n} - 1 \right).$$

It is easy to see that

$$U_n(t) \leq \frac{2(1 + nt)}{(1 - m_n)^2}, \quad V_n(t) \leq \frac{2(1 + nt)}{(1 - m_n)^3}.$$

Then

$$(4) \quad \text{Var } G_n(t) \leq 2\alpha^{-2} \left(\frac{b_n^2}{n} + \frac{\lambda_n \sigma_n^2 d_n}{\alpha n} \right) t \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

for all $t \geq 0$. Applying the Chebyshev inequality we conclude from here that $G_n(t) \xrightarrow{\mathbb{P}} Z(t)$ as $n \rightarrow \infty$ for all $t \geq 0$ in view of relation (3). Since the limit process Z is continuous and nonrandom, it remains to show that the sequence of processes $\{G_n(t), t \geq 0\}$, $n \in \mathbb{N}$, is dense (see Theorem 15.1 of [5]). Indeed, relations (3) and (4) imply that

$$\begin{aligned} \mathbb{E} (G_n(t) - G_n(s))^2 &\leq 3 \left(\text{Var } G_n(s) + \text{Var } G_n(t) + (\mathbb{E} G_n(t) - \mathbb{E} G_n(s))^2 \right) \\ &\leq 4\alpha^{-2} \lambda^2 (t - s)^2 \end{aligned}$$

for all $t, s \geq 0$ and all sufficiently large n . Hence the sequence of processes $\{G_n(t), t \geq 0\}$, $n \in \mathbb{N}$, is dense by the density criteria 15.5 and 12.3 of [5]. \square

Now we are going to prove the following three auxiliary results and then use them to derive the statement of Theorem 2. Put

$$M_k^{(n)} = \sum_{j=1}^{X_{k-1}^{(n)}} \left(\xi_{k,j}^{(n)} - m_n \right) + \varepsilon_k^{(n)} - \lambda_k, \quad k = 1, 2, \dots$$

Denote by $F_k^{(n)}$ the σ -algebra generated by random variables $\{X_0^{(n)}, X_1^{(n)}, \dots, X_k^{(n)}\}$. It is clear that $\{M_k^{(n)}, k \geq 0\}$ is a martingale-difference with respect to the flow of σ -algebras $F_k^{(n)}, k \geq 0$.

Lemma 1. *The representation*

$$W_n(t) = [d_n(1 - m_n)]^{-1} \left(\widetilde{M}_n^{(1)}(t) - m_n \widetilde{M}_n^{(2)}(t) \right)$$

holds, where

$$W_n(t) = (d_n \sqrt{n})^{-1} (Z_n(t) - \mathbb{E} Z_n(t)),$$

$$\widetilde{M}_n^{(1)}(t) = n^{-1/2} \sum_{j=1}^{[nt]} M_j^{(n)}, \quad \widetilde{M}_n^{(2)}(t) = n^{-1/2} \sum_{j=1}^{[nt]} m_n^{[nt]-j} M_j^{(n)}.$$

Proof of Lemma 1. The variable $X_k^{(n)}$ is represented as

$$X_k^{(n)} = m_n X_{k-1}^{(n)} + \lambda_n + M_k^{(n)}, \quad k = 1, 2, \dots,$$

in view of equality (1), whence $\mathbb{E} X_k^{(n)} = m_n \mathbb{E} X_{k-1}^{(n)} + \lambda_n, k = 1, 2, \dots$. Therefore the random variables $X_k^{(n)} - \mathbb{E} X_k^{(n)}, k \geq 0$, satisfy the recurrence equation

$$X_k^{(n)} - \mathbb{E} X_k^{(n)} = m_n \left(X_{k-1}^{(n)} - \mathbb{E} X_{k-1}^{(n)} \right) + M_k^{(n)}, \quad k = 1, 2, \dots$$

The solution of the latter recurrence equation is given by

$$X_k^{(n)} - \mathbb{E} X_k^{(n)} = \sum_{j=1}^k m_n^{k-j} M_j^{(n)}, \quad k = 1, 2, \dots$$

Summing up the latter equalities with respect to k from 1 to $[nt]$ and normalizing the result appropriately, we complete the proof of the lemma. \square

Lemma 2. *If assumptions (1)–(3) of Theorem 2 hold, then*

$$\widetilde{M}_n^{(2)}(t) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

in the Skorokhod space $D[0, \infty)$.

Proof of Lemma 2. It is easy to see that

$$(5) \quad \begin{aligned} & n^{-1} \sum_{j=1}^{[nt]} m_n^{2([nt]-1)} \mathbb{E} \left(\left(M_k^{(n)} \right)^2 / F_{j-1}^{(n)} \right) \\ &= \frac{\sigma_n^2}{n} \sum_{j=1}^{[nt]} m_n^{2([nt]-1)} X_{j-1}^{(n)} + \frac{b_n^2}{n} \cdot \frac{1 - m_n^{2[nt]}}{1 - m_n^2}. \end{aligned}$$

By assumptions (1)–(3) of Theorem 2 and by using relation (2) we get

$$\frac{b_n^2}{n} \cdot \frac{1 - m_n^{2[nt]}}{1 - m_n^2} \sim \frac{b^2}{2|\alpha|} \cdot \frac{d_n}{n} \rightarrow 0, \quad \frac{\sigma_n^2}{n} \sum_{j=1}^{[nt]} m_n^{2([nt]-j)} \mathbb{E} X_{j-1}^{(n)} \sim \frac{\lambda \sigma^2}{2\alpha^2} \cdot \frac{d_n}{n} \rightarrow 0$$

as $n \rightarrow \infty$. This together with (5) implies that

$$n^{-1} \sum_{j=1}^{[nt]} m_n^{2([nt]-j)} \mathbb{E} \left(\left(M_j^{(n)} \right)^2 / F_{j-1}^{(n)} \right) \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \rightarrow \infty.$$

Then

$$n^{-1} \sum_{j=1}^{[nt]} m_n^{2([nt]-j)} \mathbb{E} \left(\left(M_j^{(n)} \right)^2 I \left(m_n^{[nt]-j} \left| M_j^{(n)} \right| > \varepsilon \sqrt{n} \right) / F_{j-1}^{(n)} \right) \xrightarrow{\mathbb{P}} 0$$

as $n \rightarrow \infty$. Therefore all the assumptions of Theorem 7.1.11 of [6] hold, and we complete the proof of Lemma 2. \square

Lemma 3. *Assume that all the assumptions of Theorem 2 hold. Then the weak convergence*

$$(6) \quad \widetilde{M}_n^{(1)} \rightarrow \left(|\alpha|^{-1} \lambda \sigma^2 + b^2 \right)^{1/2} W \quad \text{as } n \rightarrow \infty$$

holds in the Skorokhod space $D[0, \infty)$.

Proof of Lemma 3. Since the sequence $(M_k^{(n)}, F_k^{(n)})$, $k \geq 1$, is a martingale-difference, Theorem 7.1.11 of [6] implies that we only need to show that

$$(7) \quad n^{-1} \sum_{j=1}^{[nt]} \mathbb{E} \left(\left(M_j^{(n)} \right)^2 / F_{j-1}^{(n)} \right) \xrightarrow{\mathbb{P}} \left(|\alpha|^{-1} \lambda \sigma^2 + b^2 \right) t$$

and that, for all $\varepsilon > 0$,

$$(8) \quad R_n(\varepsilon, t) = n^{-1} \sum_{j=1}^{[nt]} \mathbb{E} \left(\left(M_j^{(n)} \right)^2 I \left(\left| M_j^{(n)} \right| > \varepsilon \sqrt{n} \right) / F_{j-1}^{(n)} \right) \xrightarrow{\mathbb{P}} 0$$

as $n \rightarrow \infty$. Relation (7) follows in view of

$$\mathbb{E} \left(\left(M_j^{(n)} \right)^2 / F_k^{(n)} \right) = \sigma_n^2 X_{k-1}^{(n)} + b_n^2,$$

Theorem 1, and assumptions (1)–(3) of Theorem 2.

Now we pass to the proof of relation (8). Put

$$N_{n,k}^{(1)} = \sum_{j=1}^{X_{k-1}^{(n)}} (\xi_{k,j} - m_n), \quad N_{n,k}^{(2)} = \varepsilon_k^{(n)} - \lambda_n.$$

Note that

$$(9) \quad I(|X + Y| > 2\varepsilon) \leq I(|X| > \varepsilon) + I(|Y| > \varepsilon)$$

for all random variables X and Y and for every $\varepsilon > 0$. This together with the elementary inequality $(a + b)^2 \leq 2(a^2 + b^2)$ implies that

$$R_n(2\varepsilon, t) \leq 2 \sum_{i,j=1}^2 R_{i,j}^{(n)}(\varepsilon, t)$$

with probability one, since $M_k^{(n)} = N_{n,k}^{(1)} + N_{n,k}^{(2)}$, where

$$R_{i,j}^{(n)}(\varepsilon, t) = n^{-1} \sum_{k=1}^{[nt]} \mathbb{E} \left(\left(N_{n,k}^{(i)} \right)^2 I \left(\left| N_{n,k}^{(j)} \right| > \varepsilon \sqrt{n} \right) / F_{k-1}^{(n)} \right), \quad i, j = 1, 2.$$

Therefore relation (8) follows if

$$(10) \quad R_{i,j}^{(n)}(\varepsilon, t) \xrightarrow{P} 0 \quad \text{as } n \rightarrow \infty$$

for $i, j = 1, 2$ and for all $t > 0, \varepsilon > 0$.

First we treat the case of $i = j = 1$ in (10). We have $(N_{n,k}^{(1)})^2 = J_k^{(n)} + L_k^{(n)}$, where

$$J_k^{(n)} = \sum_{j=1}^{X_{k-1}^{(n)}} \left(\xi_{k,j}^{(n)} - m_n \right)^2, \quad L_k^{(n)} = 2 \sum_{i=1}^{X_{k-1}^{(n)}} \sum_{j=i+1}^{X_{k-1}^{(n)}} \left(\xi_{k,i}^{(n)} - m_n \right) \left(\xi_{k,j}^{(n)} - m_n \right).$$

Now we introduce the random variables

$$S_{k,j}^{(n)} = N_{n,k}^{(1)} - \left(\xi_{k,j}^{(n)} - m_n \right), \quad j = 1, 2, \dots, X_{k-1}^{(n)}.$$

Using inequality (9) we get

$$(11) \quad \begin{aligned} n^{-1} \sum_{k=1}^{[nt]} \mathbb{E} \left(J_k^{(n)} I \left(\left| N_{n,k}^{(1)} \right| > 2\varepsilon\sqrt{n} \right) / F_{k-1}^{(n)} \right) \\ \leq n^{-1} \sum_{k=1}^{[nt]} \mathbb{E} \left(\sum_{j=1}^{X_{k-1}^{(n)}} \left(\xi_{k,j}^{(n)} - m_n \right)^2 I \left(\left| \xi_{k,j}^{(n)} - m_n \right| > \varepsilon\sqrt{n} \right) / F_{k-1}^{(n)} \right) \\ + n^{-1} \sum_{k=1}^{[nt]} \mathbb{E} \left(\sum_{j=1}^{X_{k-1}^{(n)}} \left(\xi_{k,j}^{(n)} - m_n \right)^2 I \left(\left| S_{k,j}^{(n)} \right| > \varepsilon\sqrt{n} \right) / F_{k-1}^{(n)} \right) \\ = A_n + B_n. \end{aligned}$$

Since the random variables $\xi_{k,j}^{(n)}$, $k, j \in \mathbb{N}$, are independent and identically distributed, assumptions (3) in Theorem 2 and in Theorem 1 yield,

$$(12) \quad A_n \stackrel{P}{\sim} |\alpha|^{-1} \lambda t d_n \mathbb{E} \left(\left(\xi_{1,1}^{(n)} - m_n \right)^2 I \left(\left| \xi_{1,1}^{(n)} - m_n \right| > \varepsilon\sqrt{n} \right) \right) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

for all $t > 0$, where $\varphi \stackrel{P}{\sim} \psi$ means that $\varphi\psi^{-1} \xrightarrow{P} 1$ as $n \rightarrow \infty$.

Next we consider B_n . Since the random variables $\xi_{k,j}^{(n)} - m_n$ and $S_{k,j}^{(n)}$ are independent, we apply the Chebyshev inequality for conditional probabilities to make sure that

$$(13) \quad B_n \leq \frac{4}{\varepsilon^2 n^2} \sigma_n^4 \sum_{k=1}^{[nt]} \left(X_k^{(n)} \right)^2$$

with probability one. Applying Lemma 2.1 of [2] we obtain

$$(14) \quad \sigma_n^4 n^{-2} \sum_{k=1}^{[nt]} \mathbb{E} \left(X_k^{(n)} \right)^2 \leq 2\alpha^{-2} (d_n \sigma_n^2)^2 (|\alpha| b_n^2 + \lambda_n^2) [nt] n^{-2} \rightarrow 0$$

as $n \rightarrow \infty$. This together with (13) and Markov's inequality implies that

$$B_n \xrightarrow{P} 0 \quad \text{as } n \rightarrow \infty.$$

The latter relation together with (12) leads to

$$(15) \quad n^{-1} \sum_{k=1}^{[nt]} \mathbb{E} \left(J_k^{(n)} I \left(\left| N_{n,k}^{(1)} \right| > \varepsilon\sqrt{n} \right) / F_{k-1}^{(n)} \right) \xrightarrow{P} 0 \quad \text{as } n \rightarrow \infty$$

in view of (11). Next, we apply the Cauchy–Bunyakovskiĭ and Chebyshev inequalities for conditional probabilities and conclude that

$$(16) \quad n^{-1} \sum_{k=1}^{[nt]} \mathbb{E} \left(\left| L_k^{(n)} \right| I \left(\left| N_{n,k}^{(1)} \right| > \varepsilon \sqrt{n} \right) / F_{k-1}^{(n)} \right) \leq 2^{1/2} \varepsilon^{-1} n^{-3/2} \sigma_n^3 \sum_{k=1}^{[nt]} \left(X_{k-1}^{(n)} \right)^2$$

with probability one. Similarly to inequality (14),

$$(nd_n)^{-3/2} \sum_{k=1}^{[nt]} \mathbb{E} \left(X_{k-1}^{(n)} \right)^2 \leq 2 (n^{-1} d_n)^{1/2} \alpha^{-2} (|\alpha| b_n^2 + \lambda_n^2) t \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This together with (16) and (15) implies (10) for the case of $i = j = 1$.

Next we consider the case of $i = 1, j = 2$. Since $N_{n,k}^{(1)}$ and $N_{n,k}^{(2)}$ are independent, the Chebyshev inequality for conditional probabilities and Theorem 1 imply that

$$R_{1,2}^{(n)}(\varepsilon, t) \leq \frac{b_n^2 \sigma_n^2}{\varepsilon^2 n^2} \sum_{k=1}^{[nt]} X_{k-1}^{(n)} \stackrel{P}{\sim} \frac{\lambda \sigma^2 b^2}{\varepsilon^2 |\alpha|} t \cdot \frac{1}{n} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Similarly we have

$$R_{2,1}^{(n)}(\varepsilon, t) \xrightarrow{P} 0 \quad \text{as } n \rightarrow \infty.$$

Relation (10) follows directly from assumption (4) in the case of $i = j = 2$, since the random variables $\varepsilon_k^{(n)}$, $k \in \mathbb{N}$, are independent and identically distributed. The proof of Lemma 3 is completed. \square

The proof of Theorem 2 follows directly from Lemmas 1–3 and Theorem 4.1 of [5].

4. EXAMPLES

Below are two examples of sequences of branching processes with immigration for which assumptions of Theorems 1 and 2 hold.

Example 1. Let $\xi_{1,1}^{(n)}$ have the Bernoulli distribution with success probability p_n such that $p_n = 1 + \alpha d_n^{-1} + o(d_n^{-1})$ as $n \rightarrow \infty$, where $\alpha < 0$ is a fixed number, the immigration process is governed by a Poisson law with parameter $\lambda_n \geq 0$, and there exists a finite nonnegative number λ such that $\lambda_n \rightarrow \lambda$ as $n \rightarrow \infty$. It is easy to check that all the assumptions of Theorems 1 and 2 hold. In this case,

$$Z(t) = |\alpha|^{-1} \lambda t, \quad Y(t) = |\alpha|^{-1} (2\lambda)^{1/2} W(t).$$

If both $\xi_{1,1}^{(n)}$ and $\varepsilon_1^{(n)}$ have the Bernoulli distribution with the same success probability $p_n = 1 + \alpha d_n^{-1}$, $\alpha < 0$, then

$$Z(t) = |\alpha|^{-1} t, \quad Y(t) = |\alpha|^{-1} W(t).$$

Example 2. Let $\xi_{1,1}^{(n)}$ assume three values 0, 1, and 2 with probabilities $2d_n^{-1}$, $1 - 3d_n^{-1}$, and d_n^{-1} , respectively, and let the random variable $\varepsilon_1^{(n)}$ have the geometric distribution with success probability p_n , that is,

$$\mathbb{P} \left(\varepsilon_1^{(n)} = k \right) = p_n (1 - p_n)^{k-1}, \quad k = 1, 2, \dots$$

We have

$$m_n = 1 - d_n^{-1}, \quad \sigma_n^2 = d_n^{-1} (3 - d_n^{-1}), \quad \lambda_n = p_n^{-1}, \quad b_n^2 = (1 - p_n) p_n^{-2}.$$

Let $p_n \rightarrow p > 0$. It is clear that assumptions (1), (2), and (3) of Theorems 1 and 2 hold and moreover that

$$\alpha = -1, \quad \sigma^2 = 3, \quad \lambda = p^{-1}, \quad b^2 = (1 - p) p^{-2}.$$

Assumptions (4) and (5) of Theorem 2 can also be easily checked. In the case under consideration,

$$Z(t) = p^{-1}t, \quad Y(t) = (p^{-1}(2 + p^{-1}))^{1/2}W(t).$$

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