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The Uniform Convergence of Matrix Powers

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1. Introduction

In [OLDENBURGER (1940)] it was shown that for a given matrix A the sequence A^n converges to A^∞, say, as $n \to \infty$ if and only if the eigenvalues of A are less than 1 in magnitude except possibly for some eigenvalues equal to 1 each of which corresponds to a linear elementary divisor.

Here we consider a family F of matrices A and obtain sufficient conditions for the uniform convergence in F of A^n as $n \to \infty$.

We do not specify any particular norm since we consider only matrices of finite order. We also note when the limits A^∞ are uniformly bounded.

2. Preliminaries

By SCHUR's theorem any matrix is unitarily similar to an upper triangular matrix. Moreover the eigenvalues may be placed in any desired order along the diagonal (though this is difficult to achieve in practice). When convenient we shall group the unit eigenvalues together above the others and so take our matrices in the partitioned form

$$A = \left(\begin{array}{c:c} E & C \\ \hdashline 0 & T \end{array}\right) \tag{1}$$

where E and T are square and upper triangular with the diagonal elements of E being 1, and those of T being less than 1 in magnitude. Note that the orders of E and T need not be fixed for all A in F.

Definition. If A^n converges then the *convergence factor* x_A of A is given by

$$x_A = \max_{|\lambda_i|<1} |\lambda_i|$$

where $\lambda_1, \ldots, \lambda_K$ are the eigenvalues of the K^{th} order matrix A.

For matrices of form (1) x_A is the spectral radius R_T of T.

Lemma. *If A is of form (1) then A^n converges if and only if $E = I$ the identity matrix.*

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Proof. If A^n converges then E^n converges and, by OLDENBURGER'S result there exists S such that $E=SIS^{-1}=I$. Conversely if $E=I$ then, as $n\to\infty$,

$$A^n=\left(\begin{array}{c:c} I & C\sum_{i=0}^{n-1}T^i \\ \hdashline 0 & T^n \end{array}\right)\to\left(\begin{array}{c:c} I & C(I-T)^{-1} \\ \hdashline 0 & 0 \end{array}\right). \tag{2}$$

3. Uniform Convergence

We consider an infinite family F of $K\times K$ matrices A. A necessary condition for the uniform convergence of A^n is the uniform convergence of the diagonal elements λ_i^n, i.e. the existence of x, depending on F, such that $x_A\leqq x<1$. We might call x the *convergence factor* of F.

This condition is not sufficient as consideration of the following family (with $x=\frac{1}{2}$) shows:

$$A(k)=\begin{pmatrix} k^{-1} & e^k \\ 0 & k^{-1} \end{pmatrix}, \qquad k=2,3,\ldots.$$

Thus some extra restriction must be imposed to ensure uniform convergence.

From now on we shall write bounded to mean bounded uniformly in F.

It is too severe to require boundedness of the A since this would exclude families of unbounded nilpotent matrices.

The first of the authors obtained in [1] the following condition for matrices given in the form (1). If $A=(a_{ij})$ let $|A|=(|a_{ij}|)$. To have uniform convergence of the A^n to bounded limits it suffices that $|A|^N$ be bounded for some N depending on F.

We remark that $|A|^N$ cannot be replaced by A^N as is shown by the family $A(k)=\begin{pmatrix} \frac{1}{2} & k \\ 0 & -\frac{1}{2} \end{pmatrix}$, $k=1,2,\ldots$ for which $A^{2n}=2^{-2n}I$ and yet A^n does not converge uniformly.

The conditions given below do not require A to be in a special form and are quite easy to verify in practice.

Theorem. *Let F be an infinite family of $K\times K$ matrices. The sequences $\{A^n\}$ converge uniformly if* OLDENBURGER'S *condition holds for each A and there exist x, N depending only on F such that* (i) $x_A\leqq x<1$ *and* (ii) $(x_A)^N A$ *are uniformly bounded.*

Proof. The hypotheses are invariant under unitary similarity transformations and so we may consider A to be in form (1). From the expression (2) for A^n we see that uniform convergence of T^n and $C\sum_{i=n}^{\infty}T^i$ as $n\to\infty$ imply the same for A^n. Let us write $T=D+U$ where D is $p\times p$ and diagonal and $U^p=0$. Note that p may depend on A but $p\leqq K$. We shall work with any minimal norm $\|\ \|$, i.e. one in which $\|D\|=\max_i|d_i|=x_A$.

Since the product $(D_1U)(D_2U)\ldots(D_pU)=0$ for any diagonal matrices D_i it follows by straight multiplication that

$$\|(D+U)^n\|\leqq\sum_{j=0}^{p-1}\binom{n}{j}\|D\|^{n-j}\|U\|^j$$

since there are $\binom{n}{j}$ terms involving j Us, all with the same bound. But (ii) implies that $x_A^N\|U\| \leqq u$, depending on F, and so

$$\begin{aligned}
\|(D+U)^n\| &\leqq \sum_0^{K-1} \binom{n}{j} x_A^{n-j-jN} u^j \\
&\leqq \mu K \binom{n}{K-1} x_A^{n-(K-1)(N+1)}, \quad [\mu = \max(1, u^{K-1})] \\
&\qquad\qquad \text{for} \quad n > (K-1)(N+1) \geqq 2(K-1) \\
&\leqq \gamma n^{K-1} x^n, \; \gamma^{-1} = (K-1)!\, x^{(K-1)(N+1)}/\mu K, \\
&\to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

In particular $\|T^n\| \leqq \delta n^{K-1} x_A^{n-(K-1)(N+1)}$; $\delta = \mu K$. Thus

$$\begin{aligned}
\left\| C \sum_{i=n}^{\infty} T^i \right\| &\leqq \|C\| \delta \sum_{i=n}^{\infty} i^{K-1} x_A^{i-(K-1)(N+1)} \\
&\leqq c\delta \sum_{i=n}^{\infty} i^{K-1} x_A^{i-(K-1)(N+1)-N} \quad \text{since} \quad x_A^N\|C\| \leqq c \quad \text{by (ii).}
\end{aligned}$$

Let $\alpha = NK+K-1$. When $n-\alpha > \left[\left(\frac{1+x}{2x}\right)^{1/(K-1)} - 1\right]^{-1}$ the series above is majorized by a geometric series with ratio $\frac{1}{2}(1+x)$ and

$$\left\| C \sum_{i=n}^{\infty} T^i \right\| \leqq 2c\delta(1-x)^{-1} n^{K-1} x^{n-\alpha} \to 0, \quad \text{as} \quad n \to \infty.$$

This completes the proof.

The conditions of the theorem are not necessary as is shown by the family:

$$A(k) = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ & k^{-1} & k \\ & & 0 \end{pmatrix}, \qquad k = 2, 3, \ldots, \quad \text{since } (\tfrac{1}{2})^n k \text{ is not convergent.}$$

Even when A is in form (1) we cannot replace condition (ii) by the requirements that $D^N U$ and CD^N be bounded. See, for example,

$$A(k) = \begin{pmatrix} \frac{1}{2} & 1 & 1 \\ 0 & 0 & k \\ 0 & 0 & 0 \end{pmatrix}, \qquad k = 1, 2, \ldots,$$

for which convergence of A^n is not uniform since the (1, 3) element is $(1+2k)/2^{n-3}$ for $n \geqq 2$.

4. Bounded Limits

When will the limit matrices A^∞ be bounded in F? The example $A(k) = \begin{pmatrix} 1 & k \\ 0 & k^{-1} \end{pmatrix}$, $k = 2, 3, \ldots$ shows that convergence may be uniform and yet the limits unbounded.

By considering the solutions of triangular systems of equations we see that, with the necessary condition $x_A \leqq x \leqq 1$, T is bounded if and only if $(I-T)^{-1}$ is bounded. Now the (1, 2) block of A^∞ is $C(I-T)^{-1}$ and is bounded if C, T (and therefore A) are bounded. However when $C = 0$ then A^∞ is bounded independently of T.

Conversely if $C(I-T)^{-1}=B$, bounded in F, then

$$Q^*AQ=\begin{pmatrix} I & C \\ 0 & T \end{pmatrix}=\begin{pmatrix} I & -B \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & T \end{pmatrix}\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}$$

for some unitary matrix Q. Thus we have

Theorem. *A^∞ is bounded in F if and only if A is boundedly similar to a direct sum of an identity matrix and a convergent matrix.*

References

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