Intersection Sequences and the Number of Intersections of Two Polygons

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Abstract

We discuss the number of proper intersections that is possible between two polygons. We prove a new bound for two odd-sided polygons, and show that the bound is tight when one of the polygons has no more than 7 sides.

1 Introduction

We discuss the maximum number of intersections between two polygons with a given number of sides. In the case when m and n are both odd ($n \leq m$), we prove an improved upper bound of $mn - m - \frac{n}{3}$ (Theorem 1).

The study of the complexity of various geometric objects has a long history in computational geometry. The study of arrangements of hyperplanes, for example, has been widely studied, and the number of geometric components involved in the intersection of hyperplanes is well known ***REF***

2 Background

Following [Kar+03] we use the notation $f(m, n)$ to denote the maximum number of intersections possible between an m-gon and an n-gon arranged in the plane ($m, n \geq 3$). For example, the bound $f(3, 3) = 6$ is realized by arranging two triangles to make a six-pointed star. As noted in [DMS93] and [Kar+03], it is easy to see that when m and n are both even then $f(m, n) = mn$ (see Figure 1) and when m is odd but n is even then $f(m, n) = n(m - 1)$ (see Figure 2). The latter bound is a result of the fact that if m is odd then a single line segment can intersect at most $m - 1$ sides of an m-gon. The more difficult case is when m and n are both odd integers. This last case is still unresolved. In [DMS93] there is the conjecture:

Conjecture 1 *If m and n are both odd then $f(m,n) = mn - m - n + 3$.*

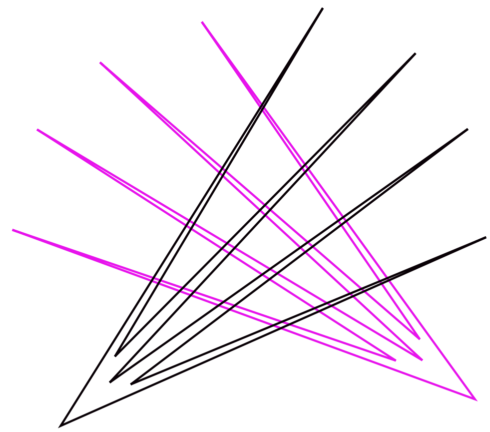

Figure 1: An optimal arrangement of two even-sided polygons, with mn intersections.

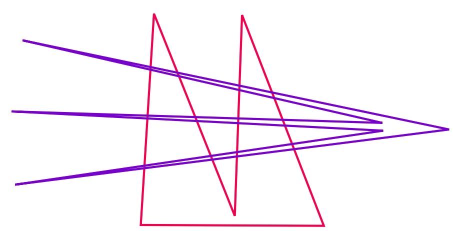

Figure 2: An optimal arrangement of an even-sided and an odd-sided polygon, with $(n-1)m$ intersections.

Given odd integers n and m it is straightforward to construct an arrangement of an n-gon and m-gon with exactly $mn - m - n + 3$ intersections. See Figure 3. Hence it remains to prove that Conjecture 1 gives the correct tight upper bound. The proof given in [DMS93] has a fatal error.

In [Kar+03] progress is made toward proving the conjecture: it is shown that for $n \leq m$, we have $f(m,n) \leq mn - n - \left\lceil \frac{n}{6} \right\rceil$. This weaker bound suffices to prove Conjecture 1 when the smaller integer n is at most 5.

In the present article we use a different strategy – *intersection sequences* – to improve the bound to $f(m,n) \leq mn - m - \frac{n}{3}$. This allows us to prove Conjecture 1 when $n \leq 7$.

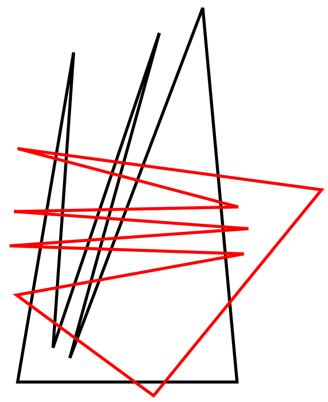

Figure 3: An optimal arrangement of two odd-sided polygons, with $mn - m - n + 3$ intersections.

3 Tools

Suppose that we have an arrangement of the polygons P and Q. Each edge q of Q intersects a set E of edges of P, and there is a natural ordering on E induced by q: if we place an orientation on Q then we order the edges of E based on the order in which they are struck by q.

For example, in Figure 4, let Q be the dashed polygon, let P be the solid polygon, and let the edges of P be labeled by the integers 1 to 7, consecutively. Beginning at the vertex of Q marked as "x" and proceeding counterclockwise around Q, the induced intersection sequences are:
754321,
123457,
754321,
123456,
6543,
21,
123467.

We will simplify this notation in two ways. First, we note that the ordering determined by an edge q is unique up to reversal of the sequence, since it depends only on the choice of one of two possible orientations for q. Hence we shall use the notation $[p_1 \dots p_n]$ to represent the two orderings $p_1 \dots p_n$ and $p_n \dots p_1$. Second, since several consecutive edges of Q may have the same intersection sequence (up to orientation), we choose to list a sequence only once if it has length $n - 1$ and appears several times in a row in a list of the sequences to represent the class of a chain of consecutive edges of q with the same sequence.

Hence, a simplified representation of the intersection sequences in Figure 4 is: [754321]
[654321]
[6543]

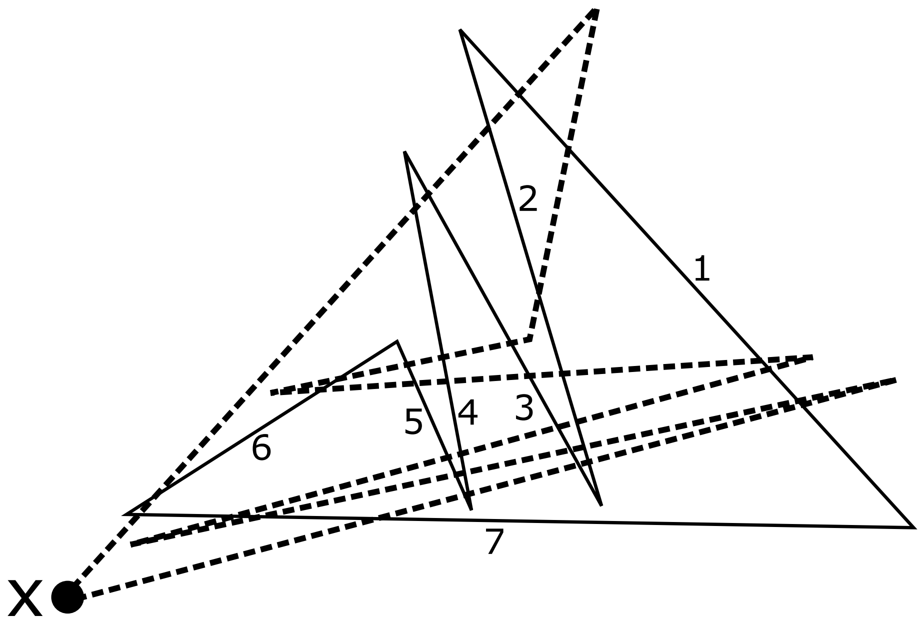

Figure 4: An arrangement of polygons.

[21]
[764321].

We shall study the properties of the list of intersection sequences of an arrangement. To this end, it is useful to compare the different ways in which a single set E of edges of a polygon P can be intersected by distinct edges q_1 and q_2 of a second polygon. It turns out that there are two cases, depending on whether or not q_1 or q_2 "points at" the other. Consider Figure 5, in which q_1 and q_2 are drawn along with their line extensions. Assuming that these extensions are not parallel, they partition the plane into four regions (labeled I, II, III, and IV in Figure 5). We say that q_1 "points at" q_2 if the line extension of q_1 intersects the interior of q_2.

Lemma 1 *Let q_i and q_j be directed line segments which do not intersect except possibly at their endpoints. Let E be any set of line segments which do not intersect each other except possibly at their endpoints. Then*

1. *If neither q_i nor q_j points at the other, then q_i and q_j induce the same or opposite intersection orderings on E.*

2. *If q_j points at q_i then there exists a partition of E into subsets E_1 and E_2 such that all of E_1 is before all of E_2 on q_i and such that E_1 and E_2 each have either identical or opposite orderings on q_i and q_j.*

Proof. This is apparent from Figure 5. ■

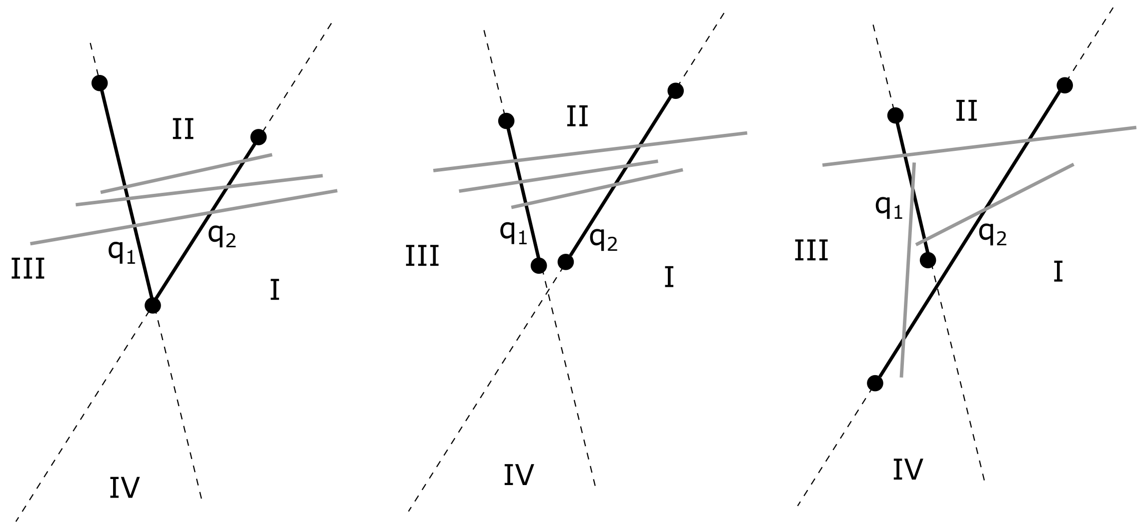

Figure 5: Here, segments q_1 and q_2 are edges of a single polygon Q and the grey segments form a non-intersecting set of edges.

When a directed line segment q intersects an edge p of a polygon P, it either moves from the outside of P to the inside of P, or vice versa. That is, q either strikes p from the outside or from the inside of P. We introduce some notation to reflect these two possibilities: $\overrightarrow{p}$ and $\overleftarrow{p}$. If p is struck on one side by S_1 and the other side by S_2, then we write $\overrightarrow{p}$ in one of the sequences and $\overleftarrow{p}$ in the other. It is not always convenient to establish which of $\overrightarrow{p}$ and $\overleftarrow{p}$ is which; it is sufficient to know that they represent p being struck from opposite sides. Similarly, if each edge in a set of edges β is struck oppositely by S_1 and S_2 then we will write $\overleftarrow{\beta}$ and $\overrightarrow{\beta}$ for these two possibilities. For example, if $S_1 = [\overrightarrow{a}\,\overrightarrow{b_1}\,\overrightarrow{b_2}\,\overrightarrow{b_3}\dots\overrightarrow{b_n}]$ and $S_2 = [\overrightarrow{a}\,\overleftarrow{b_n}\dots\overleftarrow{b_3}\,\overleftarrow{b_2}\,\overleftarrow{b_1}]$ then we could write $\beta = b_1b_2b_3\dots b_n$ and then $S_1 = [\overrightarrow{a}\,\overrightarrow{\beta}]$ and $S_2 = [\overrightarrow{a}\,\overleftarrow{\beta}]$.

Observe that if p_1 and p_2 are consecutive edges of P, then any edge that strikes them both must strike them from opposite directions. Also observe that if p_i and p_j are consecutive in the intersection sequence for q, then q must strike them from opposite directions.

Corollary 1 *If a and b have the same relative directions on q_1 and q_2 and c is between a and b on both q_1 and q_2 then c has the same relative direction with respect to a and b on q_1 and q_2.*

Proof. This follows from Lemma 1. ■

Lemma 2 *If S_1 and S_2 are consecutive intersection sequences for Q such that $|S_1| = |S_2| = n-1$, then S_1 and S_2 miss consecutive edges of P: $S_1 = \widehat{a}$ and $S_2 = \widehat{a+1}$. Furthermore S_2 differs from S_1 only by replacing $a+1$ with a in* *the same position. That is, there exist (possibly empty) sequences of edges β_1 and β_2 such that $S_1 = [\beta_1 a \beta_2], S_2 = [\beta_1 (a+1) \beta_2]$.*

Proof. Since P has n edges, $|S_1 \cap S_2| = n - 2$. Let a and b be the missing edges: $S_1 = \widehat{b}, S_2 = \widehat{a}$. Since S_1 and S_2 are consecutive sequences, there exist consecutive edges q_1 and q_2 of Q with intersection sequences S_1 and S_2, respectively. See the first diagram in Figure 5, where the line extensions of q_1 and q_2 partition the plane into four regions. Each edge of $S_1 \cap S_2$ has one endpoint in region I and one endpoint in region III. Consider the multigraph G which has I, II, III, and IV as its vertex set and the edges $\{p_i\}$ of P as its edges. Since P is a closed polygon, the p_i form a closed circuit in G; hence each vertex has even degree and G is connected. If a and b were omitted from G, then I and III would each have odd degree $n-2$ in G. For G to be connected and I and III to each have even degree, one of a or b has endpoint I and the other has endpoint III. But then the other endpoint of each edge is still unaccounted for – the two edges must share an endpoint in region II or IV. (In fact an edge with endpoints III and IV does not intersect q_1 or q_2, so the shared endpoint must be II.) Hence a and b are consecutive edges of P.

If a enters region II by intersecting q_1 between p_i and p_j, then b must enter region II by intersecting q_2 between p_i and p_j, since neither a nor b can intersect other edges of P. See Figure 6. Therefore, $S_1 = [\beta_1 a \beta_2]$ and $S_2 = [\beta_1 b \beta_2]$. ■

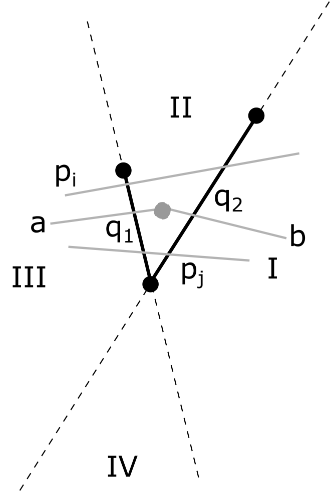

Figure 6: If a is between p_i and p_j on q_1, then b is between p_i and p_j on q_2.

Lemma 3 *$S_1 = [\beta_1 \overrightarrow{a}\,\overrightarrow{\alpha}\, b \overrightarrow{\beta_2}]$ and $S_2 = [\beta_3 \overrightarrow{a}\,\overleftarrow{\alpha}\, c \overrightarrow{\beta_4}]$ intersect if $|S_1| \geq n-2$ and $|S_2| \geq n-2$.*

Proof. Let q_1 and q_2 have intersection sequences S_1 and S_2 respectively. Since α changes orientations relative to a between S_1 and S_2, q_1 points at q_2 or vice versa. Without loss of generality, suppose that q_2 points at q_1. See Figure 7. Since $S_1 = [\beta_1 \overrightarrow{a}\,\overrightarrow{\alpha}\, b\overrightarrow{\beta_2}]$, there are no edges of P intersecting q_1 between a and α. It follows that c does not intersect q_1. Let z be the endpoint of c that is in the region bounded by some edge of α, q_1, and the line extension of q_2. The other edge of P with endpoint z cannot intersect q_1 either. Thus $|S_1| \le n-2$. However, the line extension of q_1 must intersect an even number of edges of P, so since n is odd we see that in fact $|S_1| \le n-3$, which is a contradiction. ■

Lemma 4 *If S_1, S_2, and S_3 are distinct consecutive intersection sequences of length $n-1$ for Q then for some chain of edges abc, the sequences are either $S_1 = [bc\beta]$, $S_2 = [ac\beta]$, and $S_3 = [ab\beta]$ or else $S_1 = [ab\beta]$, $S_2 = [ac\beta]$, and $S_3 = [bc\beta]$.*

Proof. Suppose that $S_1 = \widehat{a}$, $S_2 = \widehat{b}$, and $S_3 = \widehat{c}$. By Lemma 2, a, b, and c form a chain of edges of P, with b being the middle edge. We have $S_1 = [\overrightarrow{\beta_1}\,\overrightarrow{b}\,\overrightarrow{\beta_2}\,\overrightarrow{c}\,\overrightarrow{\beta_3}]$, $S_2 = [\overrightarrow{\beta_1}\,\overrightarrow{a}\,\overrightarrow{\beta_2}\,\overrightarrow{c}\,\overrightarrow{\beta_3}]$, and $S_3 = [\overrightarrow{\beta_1}\,\overrightarrow{a}\,\overrightarrow{\beta_2}\,\overleftarrow{b}\,\overrightarrow{\beta_3}]$. Since b and c are consecutive on P, they have opposite directions on S_1. Hence, the direction of b on S_3 is the opposite of its direction on S_1. But if β_1 and β_3 are both nonempty, this would contradict Corollary 1; thus β_1 or β_3 is empty, and we see that b is on an end of either S_1 or S_3. Without loss of generality, say that β_1 is empty. Then we have $S_1 = [\overrightarrow{b}\,\overrightarrow{\beta_2}\,\overrightarrow{c}\,\overrightarrow{\beta_3}]$, $S_2 = [\overrightarrow{a}\,\overrightarrow{\beta_2}\,\overrightarrow{c}\,\overrightarrow{\beta_3}]$, and $S_3 = [\overrightarrow{a}\,\overrightarrow{\beta_2}\,\overleftarrow{b}\,\overrightarrow{\beta_3}]$. Now S_3 can be written as $S_3 = [\overleftarrow{\beta_3}\,\overrightarrow{b}\,\overleftarrow{\beta_2}\,\overleftarrow{a}]$, which is incompatible with S_1 by Lemma 3 unless β_2 is empty, since the direction of β_2 is reversed relative to b between S_1 and S_3. So β_2 is empty as well, and we have $S_1 = [\overrightarrow{b}\,\overrightarrow{c}\,\overrightarrow{\beta_3}]$, $S_2 = [\overrightarrow{a}\,\overrightarrow{c}\,\overrightarrow{\beta_3}]$, and $S_3 = [\overrightarrow{a}\,\overleftarrow{b}\,\overrightarrow{\beta_3}]$. ■

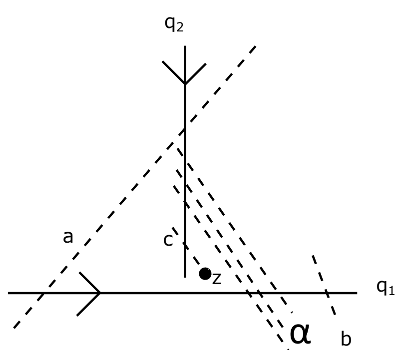

Figure 7:

Figure 4 gives an example of Lemma 4, with $a = 7$, $b = 6$, $c = 5$, and $\beta = [4321]$. In fact this example is maximal in the sense that it is impossible to construct an arrangement with four consecutive distinct sequences of length $n-1$:

Lemma 5 *Four consecutive distinct sequences of length $(n-1)$ is impossible.*

Proof. Suppose that Q has four consecutive sequences of length $n-1$: $S_1 = \widehat{a}$, $S_2 = \widehat{b}$, $S_3 = \widehat{c}$, and $S_4 = \widehat{d}$. By Lemma 2, $abcd$ is a chain of edges of P. By Lemma 4, WLOG $S_1 = [bc\beta]$, $S_2 = [ac\beta]$, and $S_3 = [ab\beta]$. But also by Lemma 4 then, if $S_2 = [ac\beta]$ then since c is the middle edge in the chain bcd we must have $S_4 = [cb\beta]$. On the other hand, writing $\beta = \beta_1 d \beta_2$, Lemma 2 implies that $S_4 = [ab\beta_1 c\beta_2]$. This is a contradiction. ■

Lemma 6 *Let P be a polygon. Every possible intersection sequence of edges of P of odd length k is a truncation of a possible intersection sequence of length $k+1$.*

Proof. Let q be a line segment which intersects k edges of P, where k is odd. Since P is a simple closed curve, it follows from the Jordan curve theorem that one endpoint of q lies inside P. We can extend q in the direction of its inner endpoint until the resulting extended segment exits P, creating one additional intersection that will be recorded on an end of the resulting intersection sequence of length $k+1$. ■

Corollary 2 *If S_1 and S_2 are consecutive intersection sequences, where $|S_1| = n-2$ and $|S_2| = n-1$, then either S_1 is a truncation of S_2 or else S_1 is a truncation of $\widehat{a}$ and $S_2 = \widehat{b}$, where a and b are adjacent.*

Proof.

■

Lemma 7 *If an edge q intersects an odd number of edges of P then one of the two edges of Q adjacent to q intersects less than $n-1$ edges of P.*

4 Results

We begin by proving that Conjecture 1 is true for $n \leq 7$. Let Q have m edges and P have n edges (n,m both odd). First suppose that $n = 3$. Since each edge of Q can intersect at most 2 sides of P, there are at most $2m$ intersections, which matches the conjecture.

Next suppose that $n = 5$. The conjecture is that $f(m,5) = 4m - 2$. Hence a counterexample to Conjecture 1 would require that each edge of Q intersects exactly 4 edges of P. But then since each edge of P must be missed by a different intersection sequence, this leads to five consecutive intersection sequences, each of length 4. This is a contradiction, by Lemma 5.

Finally, suppose that $n = 7$. The conjecture is that $f(m,7) = 6m - 4$. Hence a counterexample entails at most two extra missed edges.

Theorem 1 *When m and n are odd, with $m \geq n \geq 5$ we have that the number of intersections of an m-gon and an n-gon ($n \leq m$) cannot exceed $mn - m - \dfrac{n}{3}$.*

Proof. Consider the reduced list $S_1 \ldots S_t$ of intersection sequences of Q, listed in the order they appear around Q. Let $X = x_1, \ldots, x_t$ be the corresponding list of lengths of those sequences. Observe that $x_i \leq n-1$ for each i, so $f \leq m(n-1)$. We count the number of "extra" missed edges. Suppose we write $x_i = n-1-d_i$, where d_i measures the number of "extra" edges missed by each edge of Q with sequence S_i. Write $D = \sum d_i$. Then $f(m,n) \leq m(n-1) - D$.

By Lemma 5, no more than three consecutive terms of X can be $n-1$. Since each edge of P must be missed by some S_i, then for $n > 3$ some term of X is less than $n-1$.

Without loss of generality we can choose our starting edge of Q so that $x_1 < n-1$. Then we partition X into subsequences $\sigma_1, \tau_1, \sigma_2, \tau_2, \ldots \sigma_s, \tau_s$ such that each τ_i has length at most 3 and consists only of $n-1$ terms. We allow for the possibility that some τ_i could be empty. Each σ_i has one of the following forms: either it consists of a single even term, or else it has length at least two and its first and last terms are odd numbers corresponding to edges of Q that are on opposite ends of a chain of edges that lies entirely inside P other than its two end edges.

First suppose that some σ_i consists of a single even $x_j = n-1-2k$ for some positive integer k. Then the edges of Q corresponding to σ_i and τ_i miss $2k$ "extra" edges and at most $2k+1+3$ distinct edges of P. Hence the ratio of extra edges to distinct edges is bounded below by $\dfrac{2k}{2k+4} \geq \dfrac{1}{3}$.

Next suppose that σ_i has the second form described above. Suppose that its first term is $n-1-(2k_1+1)$ and its last term is $n-1-(2k_2+1)$ for non-negative integers k_1 and k_2. The total number of extra missed edges by σ_i and τ_i is $2k_1+1+2k_2+1+d$, where d is the number of extra edges missed by the interior edges of σ_i. The total number of distinct edges of P missed by σ_i and τ_i cannot exceed $2k_1+2+2k_2+2+d+t+3$, where t is the number of interior edges of σ_i. Note that $t < d$. Therefore the ratio of extra edges to distinct edges is bounded below by

$$\frac{2k_1+2k_2+2+d}{2k_1+2k_2+7+2d} \tag{1}$$

.

We see that the bound (1) is at least $\dfrac{1}{3}$ unless $k_1 = k_2 = d = 0$. But Lemma 7 tells us that if $k_2 = 0$ then the last edge of σ_i and the first edge of τ_i miss at most two distinct edges of P (instead of 3). This reduces the denominator of 1 by 1, and again we get that the ratio of extra edges to distinct edges is bounded below by $\dfrac{1}{3}$.

Therefore the total number of extra missed edges for the polygonal arrangement is at least $\dfrac{n}{3}$, which means that $f(m,n) \leq m(n-1) - \dfrac{n}{3}$.
■

Corollary 3 *Conjecture 1 is true for $n \leq 7$.*

Proof. By Theorem 1, $f(m,5) \leq 4m - \dfrac{5}{3}$ and $f(m,7) \leq 6m - \dfrac{7}{3}$. Since $f(m,n)$ must be an even integer, this gives $f(m,5) \leq 4m-2$ and $f(m,7) \leq 6m-4$. ■

5 Attacking the General Case

Here we suggest a strategy for proving Conjecture 1 in its entirety. Suppose that we normalize

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