

# Lyapunov 2.0

## Reference manual

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### Abstract

This Module obtains the sufficient conditions for the existence of a Lyapunov function and computes the associated Lyapunov function.

## 1 Introduction

We present the version 2.0 of the MAPLE package LYAPUNOV with a thorough revision of the implemented algorithms. The package determine Lyapunov functions for quasi-polynomial systems. We implemented a new and more efficient algorithm for the solution or simplification of such conditions, and a new implementation of its numerical solution for the case with no free parameters.

## 2 Package Commands

- **LyapunovFunction(eqs,vars,params,fixed-point)** - Determines the Lyapunov functions for a set of ODEs of the Quasi-Polynomial form and a given fixed point by solving, if possible, the admissibility conditions. In the arguments **eqs** stand for a list giving the flow of the set of first order ODEs, **vars** is the set of variables in the ODE set, **params** the set of free parameters in the equation and **fixed-point** is a list with the coordinates of the fixed point. If the equations depend on free parameters, returns a set with each element containing the Lyapunov function and the conditions on the variables **a** and the free parameters. If the algorithm is not capable of obtaining a solution it return the generic form of the Lyapunov function and the admissibility conditions. If no solution exists it returns *false*. If the equations do not depend on any free parameter the calling returns a numeric form for the Lyapunov function.

- **lyap\_func(eqs,vars,params,fixed-point)** - Returns the form of the Lyapunov function for a set of ODEs of the Quasi-Polynomial form and a given fixed point without solving the admissibility conditions. The arguments are the same as for **LyapunovFunction**.
- **AdmissibilityConditions(eqs,vars,params)** - Determines the admissibility conditions for a given set of ODEs. The arguments have the same meaning as in **LyapunovFunction**.
- **SolveAdmissCond(conds,params)** - Solves the admissibility conditions. The arguments are: **conds** the set of conditions as obtain from **AdmissibilityConditions** and **params** the set of free parameters in the set of ODEs.
- **SolveSingleIneq(ineq,as,params)** - Solves or simplifies a single inequation in **ineq** in the variables  $a_1, a_2, \dots$  in **as** and in the free parameters in **params**.
- **SolveManyIneqs(ineqs,as,params)** - Solves or simplifies a set of inequations in **ineqs** in the variables  $a_1, a_2, \dots$  in **as** and in the free parameters in **param**.

### 3 Examples

We present below examples of use of the package commands, for a few systems in a Maple worksheet.

#### Examples of use of the Lypunov 2.0 Package

```
> restart;
> with(lyapunov);
[AdmissibilityConditions, LyapunovFunction, SolveAdmissCond,
SolveManyIneqs, SolveSingleIneq, init, lyap_func] (1)
```

#### Two-Dimensional May-Leonard System

```
> # Defining the flow, the unknowns in the differential
    systems, and the free parameters, respectively.
> eqs:=[x1*(alpha1-beta1*x1-beta2*x2),x2*(alpha2-beta3*x1-
    beta4*x2)];
    variables:=[x1,x2];
    params:={alpha1,alpha2,beta1,beta2,beta3,beta4};
    eqs:= [x1 (α1 − β1 x1 − β2 x2), x2 (α2 − β3 x1 − β4 x2)]
    variables:= [x1, x2]
    params:= {α1, α2, β1, β2, β3, β4} (1.1)
```

```
> # fixed points:
> uu:= [solve ({op (eqs) }, {op (variables) }) ];
uu:= [ {x1 = 0, x2 = 0}, {x1 = 0, x2 = α2/β4}, {x1 = α1/β1, x2 = 0}, {x1
= α1 β4 − β2 α2 / (−β3 β2 + β4 β1), x2 = − (−α2 β1 + β3 α1) / (−β3 β2 + β4 β1)} ] (1.2)
```

```
> # Chosing a fixed point with non-vanishing components.
    # Later we will check if it is inside the positive quadrant.
> q:=uu[4];
    q2:=subs (q,variables);
    q:= {x1 = α1 β4 − β2 α2 / (−β3 β2 + β4 β1), x2 = − (−α2 β1 + β3 α1) / (−β3 β2 + β4 β1)}
    q2:= [ α1 β4 − β2 α2 / (−β3 β2 + β4 β1), − (−α2 β1 + β3 α1) / (−β3 β2 + β4 β1) ] (1.3)
```

```
> # Obtainin the admissibility conditions
> s0:=AdmissibilityConditions (eqs,variables,params);
s0:= { {a1^2 β3^2 + 2 a1 β3 α2 β2 + a2^2 β2^2 − 4 a1 β4 a2 β1 = 0, −β4 < 0}, {−β4 (1.4)
```

$$\{ -a_1 \beta_4 = 0, -a_2 \beta_1 = 0, -a_1 \beta_3 - a_2 \beta_2 = 0 \}, \{ -a_1 \beta_4 = 0, -a_1 \beta_3 - a_2 \beta_2 = 0, -\beta_1 < 0 \}$$

```
> #
# Determining the Lyapunov functions and the simplified
conditions on the parameters
#
t1:=time():
r1:=LyapunovFunction(eqs,variables,params,q2):
time()-t1;
```

$$0.085 \quad (1.5)$$

```
> nops(r1);
```

$$11 \quad (1.6)$$

```
> # As an example, let us consider the third case:
```

```
> r12:=r1[3];
```

$$r12 := \left[ \frac{1}{-\beta_3 \beta_2 + \beta_4 \beta_1} \left( -a_2 \alpha_1 \beta_4 - a_2 \alpha_1 \beta_4 \ln \left( \frac{x_1 (-\beta_3 \beta_2 + \beta_4 \beta_1)}{\alpha_1 \beta_4 - \beta_2 \alpha_2} \right) \right. \right. \quad (1.7)$$

$$+ a_1 \alpha_1 \beta_3 + a_1 \beta_3 \alpha_1 \ln \left( -\frac{x_2 (-\beta_3 \beta_2 + \beta_4 \beta_1)}{-\alpha_2 \beta_1 + \beta_3 \alpha_1} \right) + \beta_4 a_1 x_2 \beta_1$$

$$+ \beta_4 a_2 x_1 \beta_1 - a_2 x_1 \beta_3 \beta_2 - a_1 x_2 \beta_3 \beta_2$$

$$+ a_2 \beta_2 \alpha_2 \ln \left( \frac{x_1 (-\beta_3 \beta_2 + \beta_4 \beta_1)}{\alpha_1 \beta_4 - \beta_2 \alpha_2} \right) - a_1 \alpha_2 \beta_1 \ln \left( -\frac{x_2 (-\beta_3 \beta_2 + \beta_4 \beta_1)}{-\alpha_2 \beta_1 + \beta_3 \alpha_1} \right) - a_1 \alpha_2 \beta_1 + a_2 \alpha_2 \beta_2 \Bigg], \left\{ \right.$$

$$\left. -\frac{1}{4} \frac{a_1^2 \beta_3^2 + 2 a_1 \beta_3 a_2 \beta_2 + a_2^2 \beta_2^2 - 4 a_1 \beta_4 a_2 \beta_1}{a_1 \beta_4 a_2} = 0, -\beta_4 < 0 \right\} \Bigg]$$

```
> # These are the simplified conditions on the free parameters
and the unknowns a's:
```

```
qq1:=r12[2];
```

$$qq1 := \left\{ -\frac{1}{4} \frac{a_1^2 \beta_3^2 + 2 a_1 \beta_3 a_2 \beta_2 + a_2^2 \beta_2^2 - 4 a_1 \beta_4 a_2 \beta_1}{a_1 \beta_4 a_2} = 0, -\beta_4 < 0 \right\} \quad (1.8)$$

```
> # We then chose some values for the free variables:
```

```
vals:={a1=1,a2=2,beta2=5,beta3=4,beta4=3,alpha1=2,alpha2=1};
vals:={a1=1,a2=2,alpha1=2,alpha2=1,beta2=5,beta3=4,beta4=3}
```

$$(1.9)$$

```
> qq2:=subs(vals,qq1);
```

$$qq2 := \left\{ -\frac{49}{6} + \beta_1 = 0, -3 < 0 \right\} \quad (1.10)$$

```
> vals2:=vals union {beta1=49/6};
vals2:= {a1 = 1, a2 = 2, α1 = 2, α2 = 1, β1 =  $\frac{49}{6}$ , β2 = 5, β3 = 4, β4 = 3} (1.11)
```

```
> # Testing for the positivity of the fixed point components
> q3:=simplify(subs(vals2,q2));
q3:=  $\left[\frac{2}{9}, \frac{1}{27}\right]$  (1.12)
```

```
> # We then have the following Lyapunov function:
llf:=evalf(subs(vals2,r12[1]));
llf:= -0.4814814815 - 0.4444444444 ln(4.500000000 x1) (1.13)
- 0.03703703704 ln(27. x2) + x2 + 2. x1
```

```
> # And testing this Lyapunov function by performing a
numerical integration of the ODE system.
> diff_sys:=simplify(subs(vals2,convert(subs(x1=x1(t), x2=x2(t),
{Diff(x1,t)=eqs[1], Diff(x2,t)=eqs[2]}), diff)));
diff_sys:= {  $\frac{d}{dt} x1(t) = -\frac{1}{6} x1(t) (-12 + 49 x1(t) + 30 x2(t))$ ,  $\frac{d}{dt} x2(t) =$  (1.14)
 $-x2(t) (-1 + 4 x1(t) + 3 x2(t))$  }
```

```
> diff_sys2:=subs(vals2,diff_sys);
diff_sys2:= {  $\frac{d}{dt} x1(t) = -\frac{1}{6} x1(t) (-12 + 49 x1(t) + 30 x2(t))$ ,  $\frac{d}{dt} x2(t) =$  (1.15)
 $-x2(t) (-1 + 4 x1(t) + 3 x2(t))$  }
```

```
> # Here we chose x1(0)=1.0 and x2(0)=1.0 as initial condition.
> dsn := dsolve(diff_sys2 union {x1(0)=1.0,x2(0)=1.0}, numeric)
;
dsn:=proc(x_rkf45) ... end proc (1.16)
```

```
> Digits:=20:

t_ini:=0.0:
t_fin:=850.0:
nt:=100:

dt:=(t_fin-t_ini)/(nt-1):
gr:=1:
tt:=t_ini:
for i from 1 to nt do
vv:=subs(dsn(tt), [x1(t), x2(t)]):
tt:=tt+dt:
gr:=gr, [tt, evalf(log10(subs(x1=vv[1], x2=vv[2], llf)))]
od:

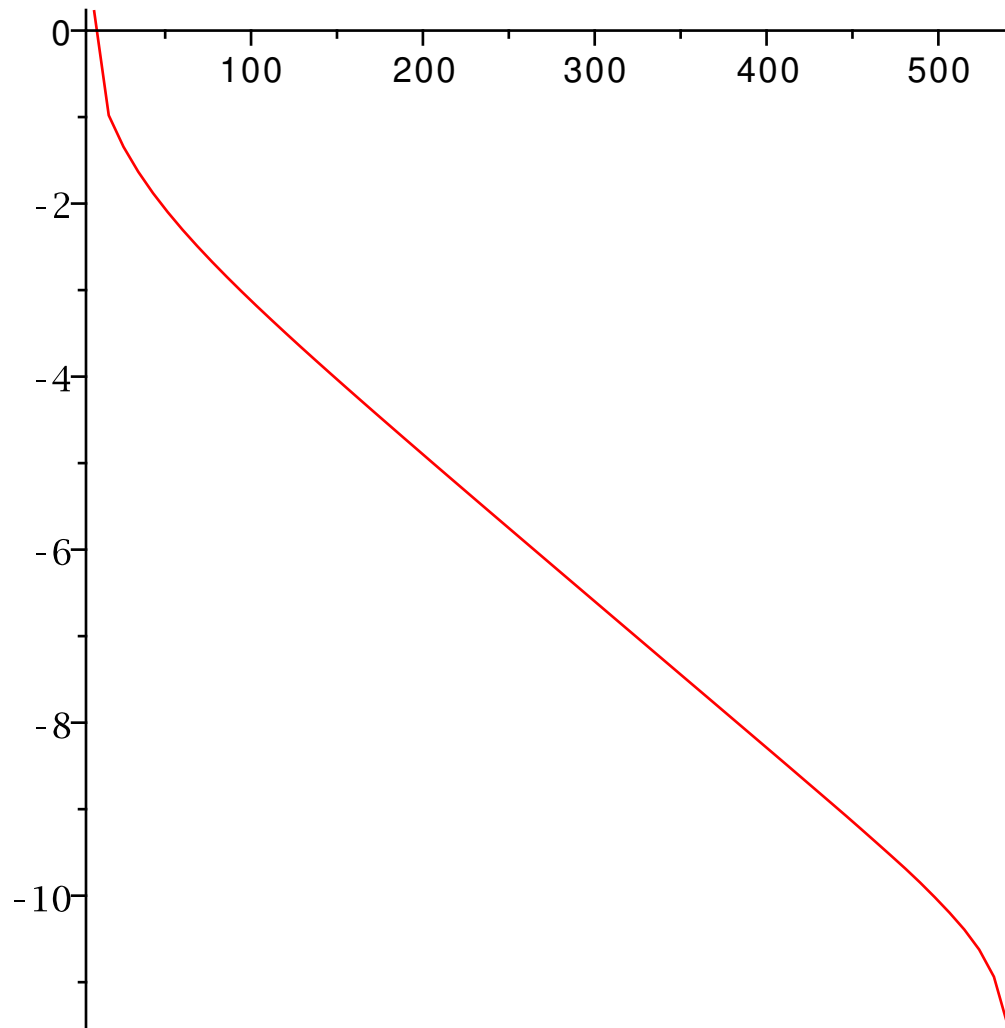
```

```
gr:=[gr]:
gr:=gr[2..nops(gr)]:
```

```
Digits:=10:
```

```
> # And finally plotting the value of the Lyapunov function as a  
function of time
```

```
> plot(gr);
```



## Three-Dimensional May-Leonard System

R.M. May, W.J. Leonard, Nonlinear aspects of competition between three species, SIAM J. Appl. Math. 29 (1975) 243-253

```
> # The system, its unknown variables and free parameters:
> eqs:=[l1*x1-x1*(x1+alpha*x2+beta*x3), l2*x2-x2*(beta*x1+x2+
alpha*x3), l3*x3-x3*(alpha*x1+beta*x2+x3)];
variables:=[x1,x2,x3];
params:={l1,l2,l3,alpha,beta};
```

$$\begin{aligned}
eqs &:= [l1 x1 - x1 (x1 + \alpha x2 + \beta x3), l2 x2 - x2 (\beta x1 + x2 + \alpha x3), l3 x3 \\
&\quad - x3 (\alpha x1 + \beta x2 + x3)] \\
variables &:= [x1, x2, x3] \\
params &:= \{\alpha, \beta, l1, l2, l3\}
\end{aligned} \tag{2.1}$$

$$\begin{aligned}
&> \# \\
&\# \text{ Determining the fixed points of the original system:} \\
&\# \\
&uu := [\text{solve}(\{\text{op}(eqs)\}, \{\text{op}(variables)\})]; \\
uu &:= \left[ \{x1 = 0, x2 = 0, x3 = 0\}, \{x1 = 0, x2 = 0, x3 = l3\}, \{x1 = 0, x2 = l2, x3 \right. \\
&\quad = 0\}, \left\{ x1 = 0, x2 = \frac{-l2 + \alpha l3}{\beta \alpha - 1}, x3 = \frac{-l3 + \beta l2}{\beta \alpha - 1} \right\}, \{x1 = l1, x2 = 0, x3 = 0\}, \\
&\quad \left\{ x1 = \frac{-l1 + \beta l3}{\beta \alpha - 1}, x2 = 0, x3 = \frac{-l3 + \alpha l1}{\beta \alpha - 1} \right\}, \left\{ x1 = \frac{-l1 + \alpha l2}{\beta \alpha - 1}, x2 \right. \\
&\quad = \frac{-l2 + \beta l1}{\beta \alpha - 1}, x3 = 0 \left. \right\}, \left\{ x1 = \frac{-l1 \beta \alpha + l1 + \alpha^2 l3 - \alpha l2 + \beta^2 l2 - \beta l3}{\alpha^3 - 3 \beta \alpha + \beta^3 + 1}, x2 \right. \\
&\quad = \frac{l3 \beta^2 - \alpha l3 + \alpha^2 l1 - l2 \beta \alpha + l2 - \beta l1}{\alpha^3 - 3 \beta \alpha + \beta^3 + 1}, x3 \\
&\quad = \left. \frac{\alpha^2 l2 - \beta \alpha l3 - \alpha l1 - \beta l2 + l1 \beta^2 + l3}{\alpha^3 - 3 \beta \alpha + \beta^3 + 1} \right]
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
&> \# \\
&\# \text{ Chosing a fixed point inside the positive orthant} \\
&\# \\
&q := uu[8]; \\
&q2 := \text{subs}(q, variables); \\
q &:= \left\{ x1 = \frac{-l1 \beta \alpha + l1 + \alpha^2 l3 - \alpha l2 + \beta^2 l2 - \beta l3}{\alpha^3 - 3 \beta \alpha + \beta^3 + 1}, x2 \right. \\
&\quad = \frac{l3 \beta^2 - \alpha l3 + \alpha^2 l1 - l2 \beta \alpha + l2 - \beta l1}{\alpha^3 - 3 \beta \alpha + \beta^3 + 1}, x3 \\
&\quad = \left. \frac{\alpha^2 l2 - \beta \alpha l3 - \alpha l1 - \beta l2 + l1 \beta^2 + l3}{\alpha^3 - 3 \beta \alpha + \beta^3 + 1} \right\} \\
q2 &:= \left[ \frac{-l1 \beta \alpha + l1 + \alpha^2 l3 - \alpha l2 + \beta^2 l2 - \beta l3}{\alpha^3 - 3 \beta \alpha + \beta^3 + 1}, \right. \\
&\quad \left. \frac{l3 \beta^2 - \alpha l3 + \alpha^2 l1 - l2 \beta \alpha + l2 - \beta l1}{\alpha^3 - 3 \beta \alpha + \beta^3 + 1}, \right]
\end{aligned} \tag{2.3}$$

$$\left[ \frac{\alpha^2 l2 - \beta \alpha l3 - \alpha l1 - \beta l2 + l1 \beta^2 + l3}{\alpha^3 - 3 \beta \alpha + \beta^3 + 1} \right]$$

> # The admissibility conditions.

> s0:=AdmissibilityConditions(eqs,variables,params);

$$\begin{aligned} s0 := \{ \{ 16 a1^2 a2^2 \alpha^2 + 16 a1^2 a3^2 \beta^2 + 16 a1^3 \alpha^2 a3 + 16 a1^3 a2 \beta^2 \\ - 64 a1^2 a2 a3 - 16 a2^2 \beta a3 \alpha^2 a1 - 16 a2 \beta^2 a3^2 \alpha a1 + 96 a1^2 a2 \alpha a3 \beta \\ + 16 a2^2 \beta^2 a1 a3 + 16 a1 a2 a3^2 \alpha^2 - 16 a1^2 \beta^2 a2^2 \alpha - 16 a1^2 \beta^3 a2 a3 \\ - 16 a1^2 \alpha^2 a3^2 \beta - 16 a1^2 \alpha^3 a3 a2 - 16 a1^3 \alpha^2 \beta a2 - 16 a1^3 \alpha \beta^2 a3 = 0, \\ a1^2 \alpha^2 + 2 a1 \alpha a2 \beta + a2^2 \beta^2 - 4 a1 a2 < 0 \}, \{ a1^2 \alpha^2 + 2 a1 \alpha a2 \beta + a2^2 \beta^2 \\ - 4 a1 a2 < 0, 16 a1 a2^2 \alpha^2 + 16 a1 a3^2 \beta^2 + 16 a1^2 \alpha^2 a3 + 16 a1^2 a2 \beta^2 \\ - 64 a1 a2 a3 - 16 a2^2 \beta a3 \alpha^2 - 16 a2 \beta^2 a3^2 \alpha + 96 a1 a2 \alpha a3 \beta \\ + 16 a2^2 \beta^2 a3 + 16 a2 a3^2 \alpha^2 - 16 a1 \beta^2 a2^2 \alpha - 16 a1 \beta^3 a2 a3 \\ - 16 a1 \alpha^2 a3^2 \beta - 16 a1 \alpha^3 a3 a2 - 16 a1^2 \alpha^2 \beta a2 - 16 a1^2 \alpha \beta^2 a3 < 0 \}, \\ \{ a1^2 \alpha^2 + 2 a1 \alpha a2 \beta + a2^2 \beta^2 - 4 a1 a2 = 0, a1^2 \beta^2 + 2 a1 \beta a3 \alpha + a3^2 \alpha^2 \\ - 4 a1 a3 = 0, 2 a1^2 \alpha \beta + 2 a1 \alpha^2 a3 + 2 a1 \beta^2 a2 + 2 a2 \alpha a3 \beta - 4 a1 \alpha a2 \\ - 4 a1 \beta a3 = 0 \}, \{ a1^2 \alpha^2 + 2 a1 \alpha a2 \beta + a2^2 \beta^2 - 4 a1 a2 = 0, 2 a1^2 \alpha \beta \\ + 2 a1 \alpha^2 a3 + 2 a1 \beta^2 a2 + 2 a2 \alpha a3 \beta - 4 a1 \alpha a2 - 4 a1 \beta a3 = 0, a1^2 \beta^2 \\ + 2 a1 \beta a3 \alpha + a3^2 \alpha^2 - 4 a1 a3 < 0 \} \} \end{aligned} \quad (2.4)$$

> # And solving (or simplifying) them:

> t1:=time();

ss:=SolveAdmissCond(s0,params);  
time()-t1;

14.753

(2.5)

> # We obtain 21 dferent solutions

> nops(ss);

21

(2.6)

> # Some are quite simple, other more complicated:

> ss[1];

$$\left\{ a1 - \frac{1}{2} 2^{1/3} a3 = 0, a2 - 2^{2/3} a3 = 0, \alpha - \frac{2}{3} 2^{2/3} = 0, \beta - \frac{2}{3} 2^{1/3} = 0 \right\} \quad (2.7)$$

> ss[2];

$$\left\{ a1 - \frac{1}{4} 2^{2/3} a3 = 0, a2 - 2 2^{1/3} a3 = 0, \alpha + \frac{4}{3} 2^{1/3} = 0, \beta - \frac{2}{3} 2^{2/3} = 0 \right\} \quad (2.8)$$

> ss[10];

$$\left\{ (-2 a1 \alpha^2 a3 - 2 a1 \beta^2 a2 + 2 a1 \alpha^2 \beta a2 + 2 a1 \alpha \beta^2 a3 - 6 a2 \alpha a3 \beta - a2^2 \alpha^2 \right. \quad (2.9)$$



$$\begin{aligned}
& + \beta^2 a^2 \alpha + \beta^3 a^2 a^3 + 4 a^2 a^3 - a^3 \beta^2 + \alpha^2 a^3 \beta + \alpha^3 a^3 a^2 \\
& + \left( (-2 \beta^2 a^2 \alpha + a^2 \beta^4 + a^2 \alpha^2 - 2 \alpha^3 a^3 a^2 - 2 \beta^2 \alpha^2 a^2 a^3 \right. \\
& + 10 a^2 \alpha a^3 \beta - 2 \beta^3 a^2 a^3 - 4 a^2 a^3 + \alpha^4 a^3 \beta^2 + a^3 \beta^2 - 2 \alpha^2 a^3 \beta) (a^3 \beta^2 \\
& - 4 a^2 a^3 + 2 a^2 \alpha a^3 \beta + a^2 \alpha^2) \left. \right)^{1/2} \left( -\alpha^2 a^3 - a^2 \beta^2 + \alpha^2 \beta a^2 + \alpha \beta^2 a^3 \right) \\
& < 0, \left( 2 a^1 \alpha^2 a^3 + 2 a^1 \beta^2 a^2 - 2 a^1 \alpha^2 \beta a^2 - 2 a^1 \alpha \beta^2 a^3 + 6 a^2 \alpha a^3 \beta \right. \\
& + a^2 \alpha^2 - \beta^2 a^2 \alpha - \beta^3 a^2 a^3 - 4 a^2 a^3 + a^3 \beta^2 - \alpha^2 a^3 \beta - \alpha^3 a^3 a^2 \\
& + \left( (-2 \beta^2 a^2 \alpha + a^2 \beta^4 + a^2 \alpha^2 - 2 \alpha^3 a^3 a^2 - 2 \beta^2 \alpha^2 a^2 a^3 \right. \\
& + 10 a^2 \alpha a^3 \beta - 2 \beta^3 a^2 a^3 - 4 a^2 a^3 + \alpha^4 a^3 \beta^2 + a^3 \beta^2 - 2 \alpha^2 a^3 \beta) (a^3 \beta^2 \\
& - 4 a^2 a^3 + 2 a^2 \alpha a^3 \beta + a^2 \alpha^2) \left. \right)^{1/2} \left( -\alpha^2 a^3 - a^2 \beta^2 + \alpha^2 \beta a^2 + \alpha \beta^2 a^3 \right) \\
& < 0, -a^1 \alpha - a^2 \beta - 2 \sqrt{a^1} \sqrt{a^2} < 0, -a^1 \alpha - a^2 \beta + 2 \sqrt{a^1} \sqrt{a^2} < 0 \}
\end{aligned}$$

```

> #
# Lyapunov function
#
t1:=time():
r1:=LyapunovFunction(eqs,variables,params,q2):
time()-t1;

```

5.615 (2.10)

```

> nops(r1);

```

21 (2.11)

```

> # Let us consider one of the cases:
> r2:=r1[5];

```

$$\begin{aligned}
r2:= & \left[ \frac{1}{(\alpha + 1 + \beta) (\alpha^2 - \alpha - \beta \alpha - \beta + 1 + \beta^2)} \left( a^1 x^3 + a^3 x^2 + a^2 x^1 - a^3 \alpha^2 l^1 \right. \right. \\
& - a^1 l^1 \beta^2 - a^1 \alpha^2 l^2 \\
& - a^2 \alpha^2 l^3 \ln \left( \frac{x^1 (\alpha + 1 + \beta) (\alpha^2 - \alpha - \beta \alpha - \beta + 1 + \beta^2)}{-l^1 \beta \alpha + l^1 + \alpha^2 l^3 - \alpha l^2 + \beta^2 l^2 - \beta l^3} \right) \\
& - a^2 \beta^2 l^2 \ln \left( \frac{x^1 (\alpha + 1 + \beta) (\alpha^2 - \alpha - \beta \alpha - \beta + 1 + \beta^2)}{-l^1 \beta \alpha + l^1 + \alpha^2 l^3 - \alpha l^2 + \beta^2 l^2 - \beta l^3} \right) \\
& \left. - a^1 l^1 \beta^2 \ln \left( \frac{x^3 (\alpha + 1 + \beta) (\alpha^2 - \alpha - \beta \alpha - \beta + 1 + \beta^2)}{\alpha^2 l^2 - \beta \alpha l^3 - \alpha l^1 - \beta l^2 + l^1 \beta^2 + l^3} \right) \right] \quad (2.12)
\end{aligned}$$

$$\begin{aligned}
& -a1 \alpha^2 l2 \ln \left( \frac{x3 (\alpha + 1 + \beta) (\alpha^2 - \alpha - \beta \alpha - \beta + 1 + \beta^2)}{\alpha^2 l2 - \beta \alpha l3 - \alpha l1 - \beta l2 + l1 \beta^2 + l3} \right) \\
& + a3 \alpha l3 \ln \left( \frac{x2 (\alpha + 1 + \beta) (\alpha^2 - \alpha - \beta \alpha - \beta + 1 + \beta^2)}{l3 \beta^2 - \alpha l3 + \alpha^2 l1 - l2 \beta \alpha + l2 - \beta l1} \right) \\
& + a2 \alpha l2 \ln \left( \frac{x1 (\alpha + 1 + \beta) (\alpha^2 - \alpha - \beta \alpha - \beta + 1 + \beta^2)}{-l1 \beta \alpha + l1 + \alpha^2 l3 - \alpha l2 + \beta^2 l2 - \beta l3} \right) \\
& + a2 \beta l3 \ln \left( \frac{x1 (\alpha + 1 + \beta) (\alpha^2 - \alpha - \beta \alpha - \beta + 1 + \beta^2)}{-l1 \beta \alpha + l1 + \alpha^2 l3 - \alpha l2 + \beta^2 l2 - \beta l3} \right) \\
& + a1 \alpha l1 \ln \left( \frac{x3 (\alpha + 1 + \beta) (\alpha^2 - \alpha - \beta \alpha - \beta + 1 + \beta^2)}{\alpha^2 l2 - \beta \alpha l3 - \alpha l1 - \beta l2 + l1 \beta^2 + l3} \right) \\
& + a1 \beta l2 \ln \left( \frac{x3 (\alpha + 1 + \beta) (\alpha^2 - \alpha - \beta \alpha - \beta + 1 + \beta^2)}{\alpha^2 l2 - \beta \alpha l3 - \alpha l1 - \beta l2 + l1 \beta^2 + l3} \right) \\
& - a3 \alpha^2 l1 \ln \left( \frac{x2 (\alpha + 1 + \beta) (\alpha^2 - \alpha - \beta \alpha - \beta + 1 + \beta^2)}{l3 \beta^2 - \alpha l3 + \alpha^2 l1 - l2 \beta \alpha + l2 - \beta l1} \right) \\
& - a3 l3 \beta^2 \ln \left( \frac{x2 (\alpha + 1 + \beta) (\alpha^2 - \alpha - \beta \alpha - \beta + 1 + \beta^2)}{l3 \beta^2 - \alpha l3 + \alpha^2 l1 - l2 \beta \alpha + l2 - \beta l1} \right) \\
& + a3 \beta l1 \ln \left( \frac{x2 (\alpha + 1 + \beta) (\alpha^2 - \alpha - \beta \alpha - \beta + 1 + \beta^2)}{l3 \beta^2 - \alpha l3 + \alpha^2 l1 - l2 \beta \alpha + l2 - \beta l1} \right) \\
& + a2 l1 \beta \alpha \ln \left( \frac{x1 (\alpha + 1 + \beta) (\alpha^2 - \alpha - \beta \alpha - \beta + 1 + \beta^2)}{-l1 \beta \alpha + l1 + \alpha^2 l3 - \alpha l2 + \beta^2 l2 - \beta l3} \right) \\
& + a1 \beta \alpha l3 \ln \left( \frac{x3 (\alpha + 1 + \beta) (\alpha^2 - \alpha - \beta \alpha - \beta + 1 + \beta^2)}{\alpha^2 l2 - \beta \alpha l3 - \alpha l1 - \beta l2 + l1 \beta^2 + l3} \right) \\
& + a3 l2 \beta \alpha \ln \left( \frac{x2 (\alpha + 1 + \beta) (\alpha^2 - \alpha - \beta \alpha - \beta + 1 + \beta^2)}{l3 \beta^2 - \alpha l3 + \alpha^2 l1 - l2 \beta \alpha + l2 - \beta l1} \right) + a2 \beta l3 \\
& - a2 \beta^2 l2 + a2 \alpha l2 - a2 \alpha^2 l3 + a3 \alpha l3 + a3 \beta l1 + a1 \alpha l1 + a1 \beta l2 \\
& + a1 \beta^3 x3 + a3 \beta^3 x2 + a2 x1 \beta^3 - a3 l3 \beta^2 - 3 \alpha a3 \beta x2 - 3 \alpha a2 x1 \beta \\
& - 3 \alpha a1 x3 \beta - a3 l2 - a2 l1 - a1 l3 + \alpha^3 a2 x1 + \alpha^3 a3 x2 + \alpha^3 a1 x3 \\
& - a3 l2 \ln \left( \frac{x2 (\alpha + 1 + \beta) (\alpha^2 - \alpha - \beta \alpha - \beta + 1 + \beta^2)}{l3 \beta^2 - \alpha l3 + \alpha^2 l1 - l2 \beta \alpha + l2 - \beta l1} \right)
\end{aligned}$$

$$\begin{aligned}
& -a_2 l_1 \ln \left( \frac{x_1 (\alpha + 1 + \beta) (\alpha^2 - \alpha - \beta \alpha - \beta + 1 + \beta^2)}{-l_1 \beta \alpha + l_1 + \alpha^2 l_3 - \alpha l_2 + \beta^2 l_2 - \beta l_3} \right) + a_3 l_2 \beta \alpha \\
& + a_2 l_1 \beta \alpha + a_1 \beta \alpha l_3 \\
& - a_1 l_3 \ln \left( \frac{x_3 (\alpha + 1 + \beta) (\alpha^2 - \alpha - \beta \alpha - \beta + 1 + \beta^2)}{\alpha^2 l_2 - \beta \alpha l_3 - \alpha l_1 - \beta l_2 + l_1 \beta^2 + l_3} \right), \{-a_3 + a_1 = 0, \\
& -a_3 + a_2 = 0, \alpha - 2 + \beta = 0\}
\end{aligned}$$

```

> # And chosing numerical values for the different free
parameters and variables
# and using the solutions on the conditions:
> vals:={l1=1,l2=1,l3=1,alpha=3,a3=1};
      vals:={a3=1, alpha=3, l1=1, l2=1, l3=1} (2.13)

```

```

> vals2:=vals union evalf(subs(vals,r2[2]));
vals2:={a3=1, alpha=3, l1=1, l2=1, l3=1, -1.+a1=0., -1.+a2=0., 1.+beta
=0.} (2.14)

```

```

> vals3:=solve(map(z->if type(z,`= `) then z fi,vals2));
vals3:={a1=1., a2=1., a3=1., alpha=3., beta=-1., l1=1., l2=1., l3=1.} (2.15)

```

```

> # The obtained Lyapunov function
> llf:=evalf(subs(vals3,r2[1]));
llf:=1.000000000 x1 + 1.000000000 x2 + 1.000000000 x3 - 1.000000000 (2.16)
      - 0.3333333334 ln(3.000000000 x1)
      - 0.3333333334 ln(3.000000000 x3)
      - 0.3333333334 ln(3.000000000 x2)

```

```

> # Now testing its validity by integrating the ODE system:
> diff_sys:=simplify(subs(vals3,convert(subs(x1=x1(t),x2=x2(t),
x3=x3(t),{Diff(x1,t)=eqs[1],Diff(x2,t)=eqs[2],Diff(x3,t)=eqs
[3]}),diff)));
diff_sys:= { d/dt x1(t) = x1(t) - 1. x1(t)^2 - 3. x1(t) x2(t) + x1(t) x3(t), d/dt x2(t) (2.17)
      = x2(t) + x1(t) x2(t) - 1. x2(t)^2 - 3. x2(t) x3(t), d/dt x3(t) = x3(t)
      - 3. x1(t) x3(t) + x2(t) x3(t) - 1. x3(t)^2 }

```

```

> dsn := dsolve(diff_sys union {x1(0)=5.0,x2(0)=5.0,x3(0)=5.0},
numeric);
      dsn:=proc(x_rkf45) ... end proc (2.18)

```

```

> Digits:=20:

```

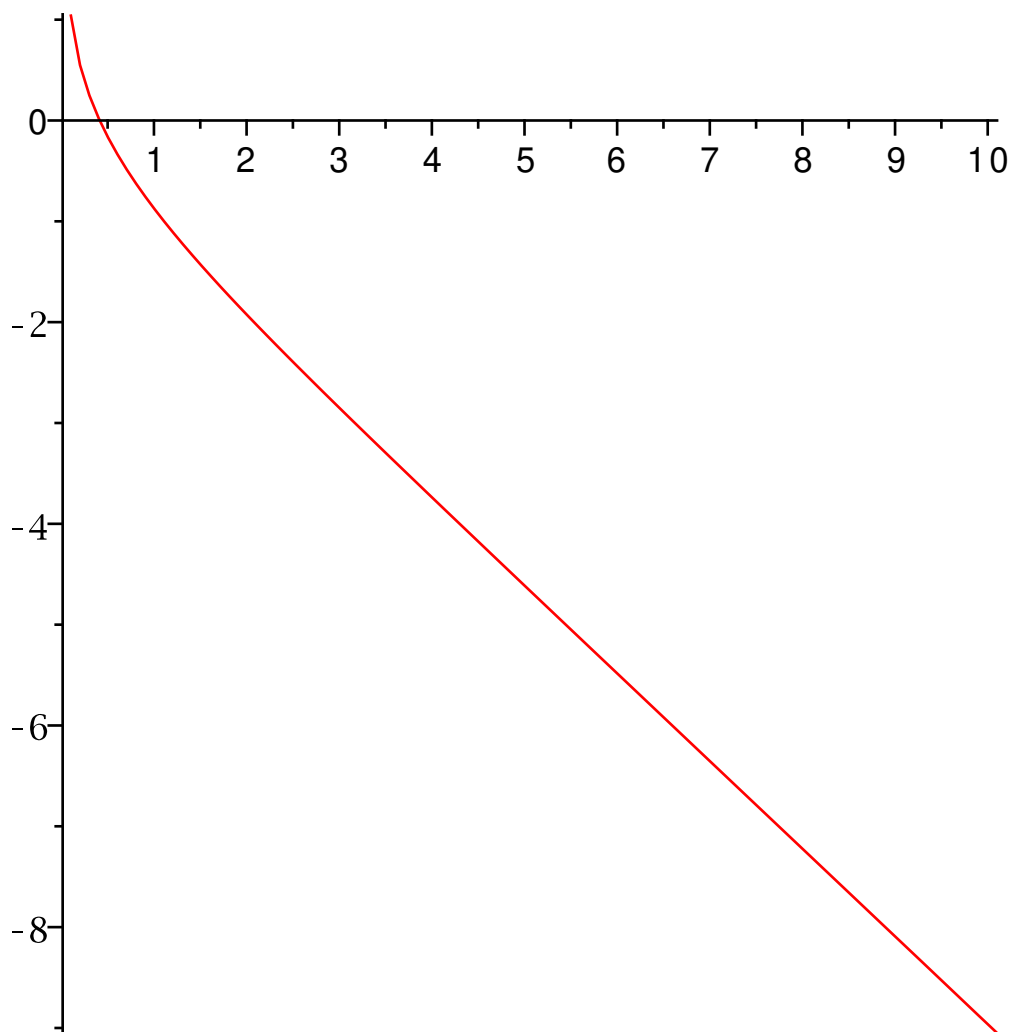
```

t_ini:=0.0:
t_fin:=10.0:
nt:=100:
dt:=(t_fin-t_ini)/(nt-1):

gr:=1:
tt:=t_ini:
for i from 1 to nt do
    vv:=subs(dsn(tt),[x1(t),x2(t),x3(t)]):
    tt:=tt+dt:
    gr:=gr,[tt,evalf(log10(subs(x1=vv[1],x2=vv[2],x3=vv[3],
11f))))]
od:
gr:=[gr]:
gr:=gr[2..nops(gr)]:

Digits:=10:
> plot(gr);

```



```

> # Using the values obtained above to illustrate the numerical
    determination of the Lyapunov function:
> eqs2:=subs(vals3,eqs);
eqs2:= [1.x1 - x1 (x1 + 3.x2 - 1.x3), 1.x2 - x2 (-1.x1 + x2 + 3.x3), 1.x3      (2.19)
        - x3 (3.x1 - 1.x2 + x3)]

```

```

> q3:=evalf(subs(vals3,q2));
        q3:= [0.3333333333, 0.3333333333, 0.3333333333]      (2.20)

```

```

> t1:=time();
    llf:=LyapunovFunction(eqs2,variables,params,q3);
    time()-t1;
llf:= 0.4999999998 x3 - 0.1666666666 ln(3.0000000000 x3) - 0.4999999998
      + 0.4999999998 x1 - 0.1666666666 ln(3.0000000000 x1)
      + 0.4999999998 x2 - 0.1666666666 ln(3.0000000000 x2)
                                0.009      (2.21)

```

```

> # Ans testing again with a numerical solution
> diff_sys2:=subs(vals,diff_sys);
diff_sys2:= { d/dt x1(t) = x1(t) - 1.x1(t)2 - 3.x1(t) x2(t) + x1(t) x3(t),      (2.22)
              d/dt x2(t) = x2(t) + x1(t) x2(t) - 1.x2(t)2 - 3.x2(t) x3(t), d/dt x3(t) = x3(t)
              - 3.x1(t) x3(t) + x2(t) x3(t) - 1.x3(t)2 }

```

```

> dsn := dsolve(diff_sys2 union {x1(0)=5.0,x2(0)=5.0,x3(0)=5.0}
, numeric);
        dsn:=proc(x_rkf45) ... end proc      (2.23)

```

```

> Digits:=20:

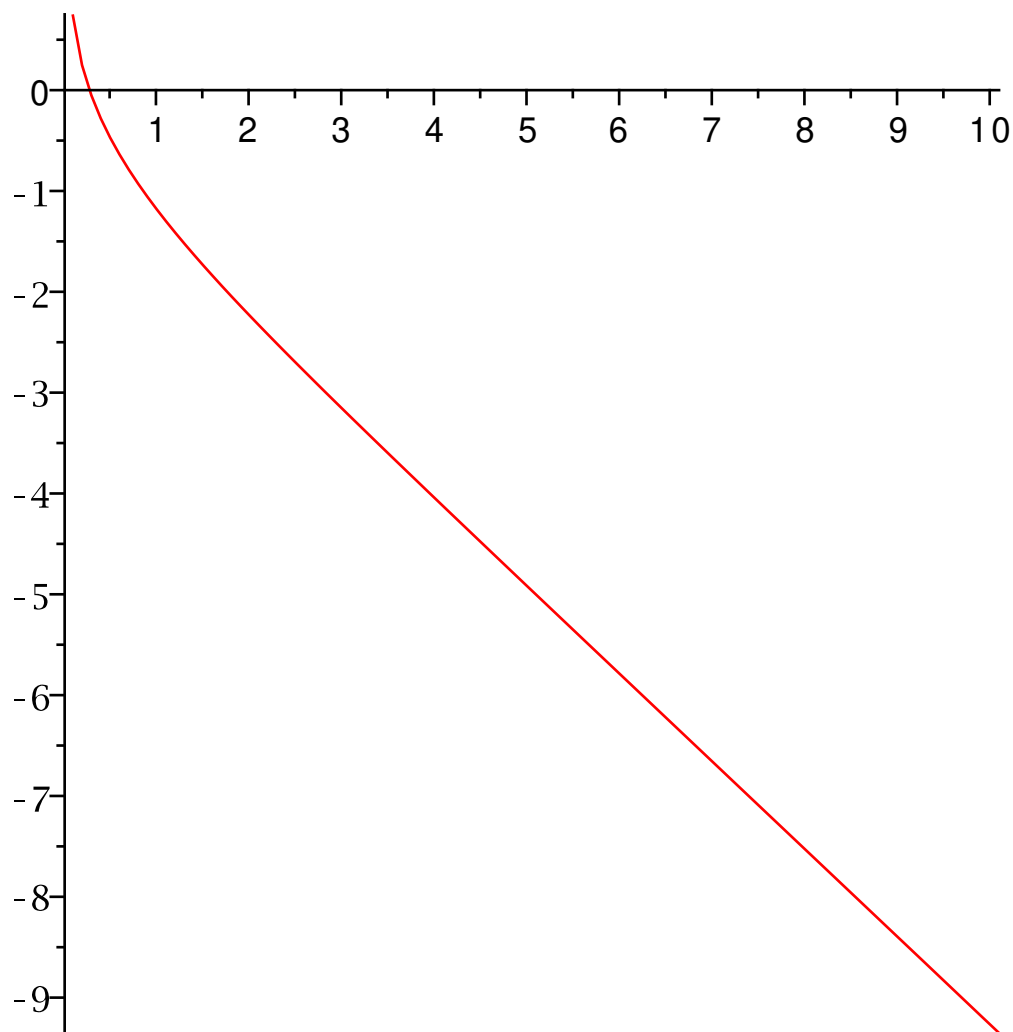
t_ini:=0.0:
t_fin:=10.0:
nt:=100:

dt:=(t_fin-t_ini)/(nt-1):
gr:=1:
tt:=t_ini:
for i from 1 to nt do
    vv:=subs(dsn(tt),[x1(t),x2(t),x3(t)]):
    tt:=tt+dt:
    gr:=gr,[tt,evalf(log10(subs(x1=vv[1],x2=vv[2],x3=vv[3],
llf)))])
od:
gr:=[gr]:
gr:=gr[2..nops(gr)]:

Digits:=10:

```

```
> # And here the the evolution of the Lyapunov function with
time
> plot(gr);
```



## Generalized mass action system

D.H. Irving, E.O. Voit, M.A. Savageau, Analysis of complex dynamic networks with ESSYNS, in: E.O. Voit (Ed.), Canonical Non-Linear Modelling S-systems Approach to Understanding Complexity, Van Nostrand Reinhold, 1991.

```
> # Defining the system
> eqs:=[l1*x1-alpha1*x1^b3*x2^b1,-l2*x2+alpha2*x1^b1*x3^b2,-l3*
x3+alpha3*x2^b1];
variables:=[x1,x2,x3];
params:={l1,l2,l3,b1,b2,b3,alpha1,alpha2,alpha3};
eqs:= [l1 x1 - α1 x1b3 x2b1, -l2 x2 + α2 x1b1 x3b2, -l3 x3 + α3 x2b1]
variables:= [x1, x2, x3]
```

$$params := \{\alpha 1, \alpha 2, \alpha 3, b 1, b 2, b 3, l 1, l 2, l 3\} \quad (3.1)$$

```
> # Determining the fixed points of the original system:
uu:=[solve({op(eqs)}, {op(variables)})];
```

$$uu := \left[ \begin{aligned} & \left\{ \begin{aligned} & x1 = e^{\frac{\ln\left(\frac{l1}{\alpha 1}\right) - b1 b2 \ln\left(\frac{l1}{\alpha 1}\right) + b1 b2 \ln\left(\frac{l3}{\alpha 3}\right) + b1 \ln\left(\frac{l2}{\alpha 2}\right)}{-1 + b1 b2 + b3 - b3 b1 b2 + b1^2}}, x2 \\ & - \frac{-b1 \ln\left(\frac{l1}{\alpha 1}\right) - b2 \ln\left(\frac{l3}{\alpha 3}\right) + b2 b3 \ln\left(\frac{l3}{\alpha 3}\right) - \ln\left(\frac{l2}{\alpha 2}\right) + \ln\left(\frac{l2}{\alpha 2}\right) b3}{-1 + b1 b2 + b3 - b3 b1 b2 + b1^2}, x3 \\ & - \frac{-b1^2 \ln\left(\frac{l1}{\alpha 1}\right) + b1^2 \ln\left(\frac{l3}{\alpha 3}\right) - \ln\left(\frac{l3}{\alpha 3}\right) - b1 \ln\left(\frac{l2}{\alpha 2}\right) + b3 \ln\left(\frac{l3}{\alpha 3}\right) + b3 b1 \ln\left(\frac{l2}{\alpha 2}\right)}{-1 + b1 b2 + b3 - b3 b1 b2 + b1^2} \end{aligned} \right\}, \{x1 \\ = 0, x2 = 0, x3 = 0\} \end{aligned} \right] \quad (3.2)$$

```
> # Chosing a fixed point inside the positive orthant
q:=uu[1];
```

$$q := \left\{ \begin{aligned} & x1 = e^{\frac{\ln\left(\frac{l1}{\alpha 1}\right) - b1 b2 \ln\left(\frac{l1}{\alpha 1}\right) + b1 b2 \ln\left(\frac{l3}{\alpha 3}\right) + b1 \ln\left(\frac{l2}{\alpha 2}\right)}{-1 + b1 b2 + b3 - b3 b1 b2 + b1^2}}, x2 \\ & - \frac{-b1 \ln\left(\frac{l1}{\alpha 1}\right) - b2 \ln\left(\frac{l3}{\alpha 3}\right) + b2 b3 \ln\left(\frac{l3}{\alpha 3}\right) - \ln\left(\frac{l2}{\alpha 2}\right) + \ln\left(\frac{l2}{\alpha 2}\right) b3}{-1 + b1 b2 + b3 - b3 b1 b2 + b1^2}, x3 \\ & - \frac{-b1^2 \ln\left(\frac{l1}{\alpha 1}\right) + b1^2 \ln\left(\frac{l3}{\alpha 3}\right) - \ln\left(\frac{l3}{\alpha 3}\right) - b1 \ln\left(\frac{l2}{\alpha 2}\right) + b3 \ln\left(\frac{l3}{\alpha 3}\right) + b3 b1 \ln\left(\frac{l2}{\alpha 2}\right)}{-1 + b1 b2 + b3 - b3 b1 b2 + b1^2} \end{aligned} \right\} \quad (3.3)$$

```
> q2:=subs(q,variables);
```

$$q2 := \left[ \begin{aligned} & e^{\frac{\ln\left(\frac{l1}{\alpha 1}\right) - b1 b2 \ln\left(\frac{l1}{\alpha 1}\right) + b1 b2 \ln\left(\frac{l3}{\alpha 3}\right) + b1 \ln\left(\frac{l2}{\alpha 2}\right)}{-1 + b1 b2 + b3 - b3 b1 b2 + b1^2}}, \\ & - \frac{-b1 \ln\left(\frac{l1}{\alpha 1}\right) - b2 \ln\left(\frac{l3}{\alpha 3}\right) + b2 b3 \ln\left(\frac{l3}{\alpha 3}\right) - \ln\left(\frac{l2}{\alpha 2}\right) + \ln\left(\frac{l2}{\alpha 2}\right) b3}{-1 + b1 b2 + b3 - b3 b1 b2 + b1^2}, \\ & - \frac{-b1^2 \ln\left(\frac{l1}{\alpha 1}\right) + b1^2 \ln\left(\frac{l3}{\alpha 3}\right) - \ln\left(\frac{l3}{\alpha 3}\right) - b1 \ln\left(\frac{l2}{\alpha 2}\right) + b3 \ln\left(\frac{l3}{\alpha 3}\right) + b3 b1 \ln\left(\frac{l2}{\alpha 2}\right)}{-1 + b1 b2 + b3 - b3 b1 b2 + b1^2} \end{aligned} \right] \quad (3.4)$$

```
> # Determining the possible Lyapunov functions
t1:=time();
rl:=LyapunovFunction(eqs,variables,params,q2):
time()-t1;
```

$$4.616 \quad (3.5)$$

```
> nops(rl);
```

$$(3.6)$$

15

(3.6)

```

> # Eliminating solutions with b1=0:
> rl2:=map(z->if not (has(z,b1=0)) then z fi,rl):
> nops(rl2);

```

10

(3.7)

```

> # Now considering the first case
> rl3:=rl2[1];

```

rl3:=

(3.8)

$$\left[ \left( a1 x1^{b1} x3^{b2} \left( \frac{l1}{\alpha1} \right)^{\frac{b1}{-1 + b1 b2 + b3 - b3 b1 b2 + b1^2}} \right. \right.$$

$$\left( \frac{l3}{\alpha3} \right)^{\frac{b2}{-1 + b1 b2 + b3 - b3 b1 b2 + b1^2}} \left( \frac{l2}{\alpha2} \right)^{\frac{1}{-1 + b1 b2 + b3 - b3 b1 b2 + b1^2}} x3$$

$$\left( \frac{l1}{\alpha1} \right)^{\frac{b1^2}{-1 + b1 b2 + b3 - b3 b1 b2 + b1^2}} \left( \frac{l3}{\alpha3} \right)^{\frac{1}{-1 + b1 b2 + b3 - b3 b1 b2 + b1^2}}$$

$$\left( \frac{l2}{\alpha2} \right)^{\frac{b1}{-1 + b1 b2 + b3 - b3 b1 b2 + b1^2}} x1 \left( \frac{l1}{\alpha1} \right)^{\frac{1}{-1 + b1 b2 + b3 - b3 b1 b2 + b1^2}}$$

$$\left( \frac{l3}{\alpha3} \right)^{\frac{b1 b2}{-1 + b1 b2 + b3 - b3 b1 b2 + b1^2}}$$



$$-a1\left(\frac{l3}{\alpha3}\right)^{\frac{b2b3}{-1+b1b2+b3-b3b1b2+bl^2}}\left(\frac{l2}{\alpha2}\right)^{\frac{b3}{-1+b1b2+b3-b3b1b2+bl^2}}$$

$$\left(\frac{1}{\left(\frac{ll}{\alpha l}\right)^{\frac{1}{-1+b1b2+b3-b3b1b2+bl^2}}}\right)$$

$$\left(\frac{l3}{\alpha3}\right)^{\frac{b1b2}{-1+b1b2+b3-b3b1b2+bl^2}}\left(\frac{l2}{\alpha2}\right)^{\frac{b1}{-1+b1b2+b3-b3b1b2+bl^2}}\right)$$

$$b1$$

$$\left(\left(\frac{ll}{\alpha l}\right)^{\frac{bl^2}{-1+b1b2+b3-b3b1b2+bl^2}}\left(\frac{l3}{\alpha3}\right)^{\frac{1}{-1+b1b2+b3-b3b1b2+bl^2}}\right)$$

$$\left(\frac{l2}{\alpha2}\right)^{\frac{b1}{-1+b1b2+b3-b3b1b2+bl^2}}\Bigg/$$

$$\begin{aligned}
& \left( \left( \frac{l3}{\alpha3} \right)^{\frac{b1^2}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}} \left( \frac{l3}{\alpha3} \right)^{\frac{b3}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}} \right. \\
& \left. \left( \frac{l2}{\alpha2} \right)^{\frac{b3\,b1}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}} \right) \\
& b2 \\
& \ln \left( \left( x1^{b1} x3^{b2} e^{-\frac{-b1\ln\left(\frac{l1}{\alpha1}\right) - b2\ln\left(\frac{l3}{\alpha3}\right) + b2\,b3\ln\left(\frac{l3}{\alpha3}\right) - \ln\left(\frac{l2}{\alpha2}\right) + \ln\left(\frac{l2}{\alpha2}\right)\,b3}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}} \right) \right) / \\
& \left( x2 \left( e^{\frac{\ln\left(\frac{l1}{\alpha1}\right) - b1\,b2\ln\left(\frac{l1}{\alpha1}\right) + b1\,b2\ln\left(\frac{l3}{\alpha3}\right) + b1\ln\left(\frac{l2}{\alpha2}\right)}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}} \right)^{b1} \right. \\
& \left. \left( e^{-\frac{-b1^2\ln\left(\frac{l1}{\alpha1}\right) + b1^2\ln\left(\frac{l3}{\alpha3}\right) - \ln\left(\frac{l3}{\alpha3}\right) - b1\ln\left(\frac{l2}{\alpha2}\right) + b3\ln\left(\frac{l3}{\alpha3}\right) + b3\,b1\ln\left(\frac{l2}{\alpha2}\right)}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}} \right)^{b2} \right) \\
& x2\,x3 \left( \frac{l1}{\alpha1} \right)^{\frac{b1^2}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}} \left( \frac{l3}{\alpha3} \right)^{\frac{1}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}}
\end{aligned}$$

$$\left(\frac{l2}{\alpha2}\right)^{\frac{b1}{-1+b1b2+b3-b3b1b2+b1^2}}x1\left(\frac{l1}{\alpha1}\right)^{\frac{1}{-1+b1b2+b3-b3b1b2+b1^2}}$$

$$\left(\frac{l3}{\alpha3}\right)^{\frac{b1b2}{-1+b1b2+b3-b3b1b2+b1^2}}$$

$$-a1\left(\frac{l3}{\alpha3}\right)^{\frac{b2b3}{-1+b1b2+b3-b3b1b2+b1^2}}\left(\frac{l2}{\alpha2}\right)^{\frac{b3}{-1+b1b2+b3-b3b1b2+b1^2}}$$

$$\left(\frac{l1}{\alpha1}\right)^{\frac{1}{-1+b1b2+b3-b3b1b2+b1^2}}\left(\frac{l1}{\alpha1}\right)^{\frac{1}{-1+b1b2+b3-b3b1b2+b1^2}}$$

$$\left(\frac{l3}{\alpha3}\right)^{\frac{b1b2}{-1+b1b2+b3-b3b1b2+b1^2}}\left(\frac{l2}{\alpha2}\right)^{\frac{b1}{-1+b1b2+b3-b3b1b2+b1^2}}\right)$$

$$b1$$

$$\left(\left(\left(\frac{l1}{\alpha1}\right)^{\frac{b1^2}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\left(\frac{l3}{\alpha3}\right)^{\frac{1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\right.\right.$$

$$\left.\left(\frac{l2}{\alpha2}\right)^{\frac{b1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\right)/$$

$$\left(\left(\frac{l3}{\alpha3}\right)^{\frac{b1^2}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\left(\frac{l3}{\alpha3}\right)^{\frac{b3}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\right.$$

$$\left.\left(\frac{l2}{\alpha2}\right)^{\frac{b3\,b1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\right)\right)$$

$$b2$$

$$x2\,x3\left(\frac{l1}{\alpha1}\right)^{\frac{b1^2}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\left(\frac{l3}{\alpha3}\right)^{\frac{1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}$$

$$\left(\frac{l2}{\alpha2}\right)^{\frac{b1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}x1\left(\frac{l1}{\alpha1}\right)^{\frac{1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}$$

$$\left(\frac{l3}{\alpha3}\right)^{\frac{b1b2}{-1+b1b2+b3-b3b1b2+b1^2}}$$

$$+a2x2^{b1}x2\left(\frac{l1}{\alpha1}\right)^{\frac{b1}{-1+b1b2+b3-b3b1b2+b1^2}}$$

$$\left(\frac{l3}{\alpha3}\right)^{\frac{b2}{-1+b1b2+b3-b3b1b2+b1^2}}\left(\frac{l2}{\alpha2}\right)^{\frac{1}{-1+b1b2+b3-b3b1b2+b1^2}}$$

$$\left(\frac{l1}{\alpha1}\right)^{\frac{b1^2}{-1+b1b2+b3-b3b1b2+b1^2}}\left(\frac{l3}{\alpha3}\right)^{\frac{1}{-1+b1b2+b3-b3b1b2+b1^2}}$$

$$\left(\frac{l2}{\alpha2}\right)^{\frac{b1}{-1+b1b2+b3-b3b1b2+b1^2}}x1\left(\frac{l1}{\alpha1}\right)^{\frac{1}{-1+b1b2+b3-b3b1b2+b1^2}}$$

$$\left(\frac{l3}{\alpha3}\right)^{\frac{b1b2}{-1+b1b2+b3-b3b1b2+b1^2}}$$

$$-a2\left(\frac{l3}{\alpha3}\right)^{\frac{b1^2}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\left(\frac{l3}{\alpha3}\right)^{\frac{b3}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}$$

$$\left(\frac{l2}{\alpha2}\right)^{\frac{b3\,b1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}$$

$$\left(\left(\frac{l1}{\alpha1}\right)^{\frac{b1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\left(\frac{l3}{\alpha3}\right)^{\frac{b2}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\right)$$

$$\left(\frac{l2}{\alpha2}\right)^{\frac{1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\Bigg/$$

$$\left(\left(\frac{l3}{\alpha3}\right)^{\frac{b2\,b3}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\left(\frac{l2}{\alpha2}\right)^{\frac{b3}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\right)$$

$$^{b1}\ln\bigg(\bigg(x2^{b1}$$

$$e^{-\frac{-b1^2\ln\left(\frac{l1}{\alpha1}\right)+b1^2\ln\left(\frac{l3}{\alpha3}\right)-\ln\left(\frac{l3}{\alpha3}\right)-b1\ln\left(\frac{l2}{\alpha2}\right)+b3\ln\left(\frac{l3}{\alpha3}\right)+b3b1\ln\left(\frac{l2}{\alpha2}\right)}{-1+b1b2+b3-b3b1b2+b1^2}}\Bigg)\Bigg/$$

$$\left(x3\left(e^{-\frac{-b1\ln\left(\frac{l1}{\alpha1}\right)-b2\ln\left(\frac{l3}{\alpha3}\right)+b2b3\ln\left(\frac{l3}{\alpha3}\right)-\ln\left(\frac{l2}{\alpha2}\right)+\ln\left(\frac{l2}{\alpha2}\right)b3}{-1+b1b2+b3-b3b1b2+b1^2}}\right)^{b1}\right)\right)$$

$$x2\left(\frac{l1}{\alpha1}\right)^{\frac{b1}{-1+b1b2+b3-b3b1b2+b1^2}}\left(\frac{l3}{\alpha3}\right)^{\frac{b2}{-1+b1b2+b3-b3b1b2+b1^2}}$$

$$\left(\frac{l2}{\alpha2}\right)^{\frac{1}{-1+b1b2+b3-b3b1b2+b1^2}}x3x1\left(\frac{l1}{\alpha1}\right)^{\frac{1}{-1+b1b2+b3-b3b1b2+b1^2}}$$

$$\left(\frac{l3}{\alpha3}\right)^{\frac{b1b2}{-1+b1b2+b3-b3b1b2+b1^2}}$$

$$-a2\left(\frac{l3}{\alpha3}\right)^{\frac{b1^2}{-1+b1b2+b3-b3b1b2+b1^2}}\left(\frac{l3}{\alpha3}\right)^{\frac{b3}{-1+b1b2+b3-b3b1b2+b1^2}}$$

$$\left(\frac{l2}{\alpha2}\right)^{\frac{b3b1}{-1+b1b2+b3-b3b1b2+b1^2}}$$

$$\left(\left(\frac{l1}{\alpha1}\right)^{\frac{b1}{-1+b1b2+b3-b3b1b2+b1^2}}\left(\frac{l3}{\alpha3}\right)^{\frac{b2}{-1+b1b2+b3-b3b1b2+b1^2}}\right)$$

$$\left(\frac{l2}{\alpha2}\right)^{\frac{1}{-1+b1b2+b3-b3b1b2+b1^2}}\Bigg/$$

$$\left(\left(\frac{l3}{\alpha3}\right)^{\frac{b2b3}{-1+b1b2+b3-b3b1b2+b1^2}}\left(\frac{l2}{\alpha2}\right)^{\frac{b3}{-1+b1b2+b3-b3b1b2+b1^2}}\right)$$

$$b1$$

$$\times2\left(\frac{l1}{\alpha1}\right)^{\frac{b1}{-1+b1b2+b3-b3b1b2+b1^2}}\left(\frac{l3}{\alpha3}\right)^{\frac{b2}{-1+b1b2+b3-b3b1b2+b1^2}}$$

$$\left(\frac{l2}{\alpha2}\right)^{\frac{1}{-1+b1b2+b3-b3b1b2+b1^2}}\times3\times1\left(\frac{l1}{\alpha1}\right)^{\frac{1}{-1+b1b2+b3-b3b1b2+b1^2}}$$



$$\left(\frac{l3}{\alpha3}\right)^{\frac{b1b2}{-1+b1b2+b3-b3b1b2+b1^2}}$$

$$+a3x1^{b3}x2^{b1}x2\left(\frac{l1}{\alpha1}\right)^{\frac{b1}{-1+b1b2+b3-b3b1b2+b1^2}}$$

$$\left(\frac{l3}{\alpha3}\right)^{\frac{b2}{-1+b1b2+b3-b3b1b2+b1^2}}\left(\frac{l2}{\alpha2}\right)^{\frac{1}{-1+b1b2+b3-b3b1b2+b1^2}}x3$$

$$\left(\frac{l1}{\alpha1}\right)^{\frac{b1^2}{-1+b1b2+b3-b3b1b2+b1^2}}\left(\frac{l3}{\alpha3}\right)^{\frac{1}{-1+b1b2+b3-b3b1b2+b1^2}}$$

$$\left(\frac{l2}{\alpha2}\right)^{\frac{b1}{-1+b1b2+b3-b3b1b2+b1^2}}\left(\frac{l1}{\alpha1}\right)^{\frac{1}{-1+b1b2+b3-b3b1b2+b1^2}}$$

$$\left(\frac{l3}{\alpha3}\right)^{\frac{b1b2}{-1+b1b2+b3-b3b1b2+b1^2}}$$

$$-a3\left(\frac{l1}{\alpha1}\right)^{\frac{b1b2}{-1+b1b2+b3-b3b1b2+b1^2}}$$

$$\left(\frac{1}{\left(\frac{l1}{\alpha1}\right)^{\frac{b1b2}{-1+b1b2+b3-b3b1b2+b1^2}}}\left(\frac{l1}{\alpha1}\right)^{\frac{1}{-1+b1b2+b3-b3b1b2+b1^2}}\right.\\ \left.\left(\frac{l3}{\alpha3}\right)^{\frac{b1b2}{-1+b1b2+b3-b3b1b2+b1^2}}\left(\frac{l2}{\alpha2}\right)^{\frac{b1}{-1+b1b2+b3-b3b1b2+b1^2}}\right)$$

$$b3$$

$$\left(\left(\frac{l1}{\alpha1}\right)^{\frac{b1}{-1+b1b2+b3-b3b1b2+b1^2}}\left(\frac{l3}{\alpha3}\right)^{\frac{b2}{-1+b1b2+b3-b3b1b2+b1^2}}\right)$$

$$\left(\frac{l2}{\alpha2}\right)^{\frac{1}{-1+b1b2+b3-b3b1b2+b1^2}}\Bigg/$$

$$\left(\left(\frac{l3}{\alpha3}\right)^{\frac{b2b3}{-1+b1b2+b3-b3b1b2+b1^2}}\left(\frac{l2}{\alpha2}\right)^{\frac{b3}{-1+b1b2+b3-b3b1b2+b1^2}}\right)$$

$$b1\ln\left(\left(x1^{b3}x2^{b1}e^{\frac{\ln\left(\frac{l1}{\alpha1}\right)-b1b2\ln\left(\frac{l1}{\alpha1}\right)+b1b2\ln\left(\frac{l3}{\alpha3}\right)+b1\ln\left(\frac{l2}{\alpha2}\right)}{-1+b1b2+b3-b3b1b2+b1^2}}\right)\right)/$$

$$\left(x1\left(e^{\frac{\ln\left(\frac{l1}{\alpha1}\right)-b1b2\ln\left(\frac{l1}{\alpha1}\right)+b1b2\ln\left(\frac{l3}{\alpha3}\right)+b1\ln\left(\frac{l2}{\alpha2}\right)}{-1+b1b2+b3-b3b1b2+b1^2}}\right)^{b3}\right)$$

$$\left(e^{-\frac{-b1\ln\left(\frac{l1}{\alpha1}\right)-b2\ln\left(\frac{l3}{\alpha3}\right)+b2b3\ln\left(\frac{l3}{\alpha3}\right)-\ln\left(\frac{l2}{\alpha2}\right)+\ln\left(\frac{l2}{\alpha2}\right)b3}{-1+b1b2+b3-b3b1b2+b1^2}}\right)^{b1}\right)$$

$$x2\left(\frac{l1}{\alpha1}\right)^{\frac{b1}{-1+b1b2+b3-b3b1b2+b1^2}}\left(\frac{l3}{\alpha3}\right)^{\frac{b2}{-1+b1b2+b3-b3b1b2+b1^2}}$$

$$\left(\frac{l2}{\alpha2}\right)^{\frac{1}{-1+b1b2+b3-b3b1b2+b1^2}}x3\left(\frac{l1}{\alpha1}\right)^{\frac{b1^2}{-1+b1b2+b3-b3b1b2+b1^2}}$$

$$\left(\frac{l3}{\alpha3}\right)^{\frac{1}{-1+b1b2+b3-b3b1b2+b1^2}}x1$$

$$-a3\left(\frac{l1}{\alpha1}\right)^{\frac{b1b2}{-1+b1b2+b3-b3b1b2+b1^2}}$$

$$\left(\frac{1}{\left(\frac{l1}{\alpha1}\right)^{\frac{b1b2}{-1+b1b2+b3-b3b1b2+b1^2}}}\left(\frac{l1}{\alpha1}\right)^{\frac{1}{-1+b1b2+b3-b3b1b2+b1^2}}\right)$$

$$\left(\frac{l3}{\alpha3}\right)^{\frac{b1b2}{-1+b1b2+b3-b3b1b2+b1^2}}\left(\frac{l2}{\alpha2}\right)^{\frac{b1}{-1+b1b2+b3-b3b1b2+b1^2}}\right)$$

$$b3$$

$$\left(\left(\frac{l1}{\alpha1}\right)^{\frac{b1}{-1+b1b2+b3-b3b1b2+b1^2}}\left(\frac{l3}{\alpha3}\right)^{\frac{b2}{-1+b1b2+b3-b3b1b2+b1^2}}\right)$$

$$\left(\frac{l2}{\alpha2}\right)^{\frac{1}{-1+b1b2+b3-b3b1b2+b1^2}}\Bigg/$$

$$\left(\left(\frac{l3}{\alpha3}\right)^{\frac{b2b3}{-1+b1b2+b3-b3b1b2+b1^2}}\left(\frac{l2}{\alpha2}\right)^{\frac{b3}{-1+b1b2+b3-b3b1b2+b1^2}}\right)$$

$b1$

$$\times2\left(\frac{l1}{\alpha1}\right)^{\frac{b1}{-1+b1b2+b3-b3b1b2+b1^2}}\left(\frac{l3}{\alpha3}\right)^{\frac{b2}{-1+b1b2+b3-b3b1b2+b1^2}}$$

$$\left(\frac{l2}{\alpha2}\right)^{\frac{1}{-1+b1b2+b3-b3b1b2+b1^2}}\times3\left(\frac{l1}{\alpha1}\right)^{\frac{b1^2}{-1+b1b2+b3-b3b1b2+b1^2}}$$

$$\left(\frac{l3}{\alpha3}\right)^{\frac{1}{-1+b1b2+b3-b3b1b2+b1^2}}\times1\Bigg/$$

$$\left(\times2\left(\frac{l1}{\alpha1}\right)^{\frac{b1}{-1+b1b2+b3-b3b1b2+b1^2}}\left(\frac{l3}{\alpha3}\right)^{\frac{b2}{-1+b1b2+b3-b3b1b2+b1^2}}\right)$$

$$\left(\frac{l2}{\alpha2}\right)^{\frac{1}{-1+b1b2+b3-b3b1b2+b1^2}}\times3\left(\frac{l1}{\alpha1}\right)^{\frac{b1^2}{-1+b1b2+b3-b3b1b2+b1^2}}$$

$$\left( \frac{l3}{\alpha3} \right)^{\frac{1}{-1 + b1b2 + b3 - b3b1b2 + b1^2}} \left( \frac{l2}{\alpha2} \right)^{\frac{b1}{-1 + b1b2 + b3 - b3b1b2 + b1^2}} x1$$

$$\left( \frac{l1}{\alpha1} \right)^{\frac{1}{-1 + b1b2 + b3 - b3b1b2 + b1^2}} \left( \frac{l3}{\alpha3} \right)^{\frac{b1b2}{-1 + b1b2 + b3 - b3b1b2 + b1^2}} \right),$$

$$\left\{ \frac{a2b1\alpha2 + a1b2\alpha3 + 2\sqrt{a1}\sqrt{a2}\sqrt{\alpha2\alpha3}}{a2\alpha2} = 0, -\frac{a1\alpha1 - \alpha2a3}{\alpha2} = 0, -1 \right.$$

$$\left. + b3 = 0, -\alpha2 < 0 \right\}$$

> # The corresponding solution for the admissibility conditions:

> r13[2];

$$\left\{ \frac{a2b1\alpha2 + a1b2\alpha3 + 2\sqrt{a1}\sqrt{a2}\sqrt{\alpha2\alpha3}}{a2\alpha2} = 0, -\frac{a1\alpha1 - \alpha2a3}{\alpha2} = 0, -1 \right. \quad (3.9)$$

$$\left. + b3 = 0, -\alpha2 < 0 \right\}$$

> # Some manipulations to obtaining compatible values for the free parameters and the unknowns a's

> r14:=map(z->if type(z,`= `) then z fi,subs(alpha2=2,r13[2]));

$$r14 := \left\{ \frac{1}{2} \frac{2a2b1 + a1b2\alpha3 + 2\sqrt{a1}\sqrt{a2}\sqrt{2}\sqrt{\alpha3}}{a2} = 0, -1 + b3 = 0, \right. \quad (3.10)$$

$$\left. -\frac{1}{2}a1\alpha1 + a3 = 0 \right\}$$

> r15:=map(z->if lhs(z)<>rhs(z) then z fi,solve(r14));

$$r15 := \left\{ a3 = \frac{1}{2}a1\alpha1, b1 = -\frac{1}{2} \frac{\sqrt{a1}\sqrt{\alpha3}(\sqrt{a1}\sqrt{\alpha3}b2 + 2\sqrt{a2}\sqrt{2})}{a2}, b3 \right. \quad (3.11)$$

$$\left. = 1 \right\}$$

> # Considering a possible set of values:

> vals:={a1=1,a2=1/2,l1=1,l2=1,l3=1,alpha1=3,alpha2=2,alpha3=4,b1=6};

$$vals := \left\{ a1 = 1, a2 = \frac{1}{2}, \alpha1 = 3, \alpha2 = 2, \alpha3 = 4, b1 = 6, l1 = 1, l2 = 1, l3 = 1 \right\} \quad (3.12)$$

> # And obtaining the remaining values;

> r16:=subs(vals,r15);

(3.13)

$$rl6 := \left\{ 6 = -\sqrt{4} (\sqrt{4} b2 + 2), a3 = \frac{3}{2}, b3 = 1 \right\} \quad (3.13)$$

> vals2:=solve(rl6) union vals;

$$vals2 := \left\{ a1 = 1, a2 = \frac{1}{2}, a3 = \frac{3}{2}, \alpha1 = 3, \alpha2 = 2, \alpha3 = 4, b1 = 6, b2 = -\frac{1}{2} \sqrt{4} - \frac{3}{2}, b3 = 1, l1 = 1, l2 = 1, l3 = 1 \right\} \quad (3.14)$$

> # Verifying that they indeed satisfy the obtained solution:

$$\text{evalf}(\text{subs}(vals2, rl3[2])); \quad \{-1.10^{-9} = 0., 0. = 0., -2. < 0.\} \quad (3.15)$$

> # And verifying for the Lyapunov function for given initial conditions

> eqsf:=evalf(subs(vals2, eqs));

$$eqsf := \left[ x1 - 3. x1 x2^6, -1. x2 + \frac{2. x1^6}{x3^{2.500000000}}, -1. x3 + 4. x2^6 \right] \quad (3.16)$$

> eqsf2:={Diff(x1,t)=eqsf[1], Diff(x2,t)=eqsf[2], Diff(x3,t)=eqsf[3]};

$$eqsf2 := \left\{ \frac{\partial}{\partial t} x1 = x1 - 3. x1 x2^6, \frac{\partial}{\partial t} x2 = -1. x2 + \frac{2. x1^6}{x3^{2.500000000}}, \frac{\partial}{\partial t} x3 = -1. x3 + 4. x2^6 \right\} \quad (3.17)$$

> diff\_sys:=convert(subs(x1=x1(t), x2=x2(t), x3=x3(t), eqsf2), diff);

$$diff\_sys := \left\{ \frac{d}{dt} x1(t) = x1(t) - 3. x1(t) x2(t)^6, \frac{d}{dt} x2(t) = -1. x2(t) + \frac{2. x1(t)^6}{x3(t)^{2.500000000}}, \frac{d}{dt} x3(t) = -1. x3(t) + 4. x2(t)^6 \right\} \quad (3.18)$$

> llf:=evalf(subs(vals2, rl3[1]));

$$llf := \frac{1}{x2 x3 x1} \left( 2.251563917 \left( \frac{0.4441357375 x1^7}{x3^{1.500000000}} - 0.2220678685 \ln \left( \frac{2.000000000 x1^6}{x2 x3^{2.500000000}} \right) x2 x3 x1 - 0.4996527043 x2 x3 x1 + 0.2220678687 x2^7 x1 - 0.05551696715 \ln \left( \frac{4.000000001 x2^6}{x3} \right) x2 x3 x1 + 0.6662036061 x1 x2^7 x3 - 0.2220678687 \ln(3.000000001 x2^6) x2 x3 x1 \right) \right) \quad (3.19)$$

```
> diff_sys2:=evalf(subs(vals,diff_sys));
```

$$\text{diff\_sys2} := \left\{ \frac{d}{dt} x1(t) = x1(t) - 3. x1(t) x2(t)^6, \frac{d}{dt} x2(t) = -1. x2(t) + \frac{2. x1(t)^6}{x3(t)^{2.5000000000}}, \frac{d}{dt} x3(t) = -1. x3(t) + 4. x2(t)^6 \right\} \quad (3.20)$$

```
> dsn := dsolve(diff_sys2 union {x1(0)=5.0,x2(0)=5.0,x3(0)=5.0}
, numeric);
dsn:=proc(x_rkf45) ... end proc (3.21)
```

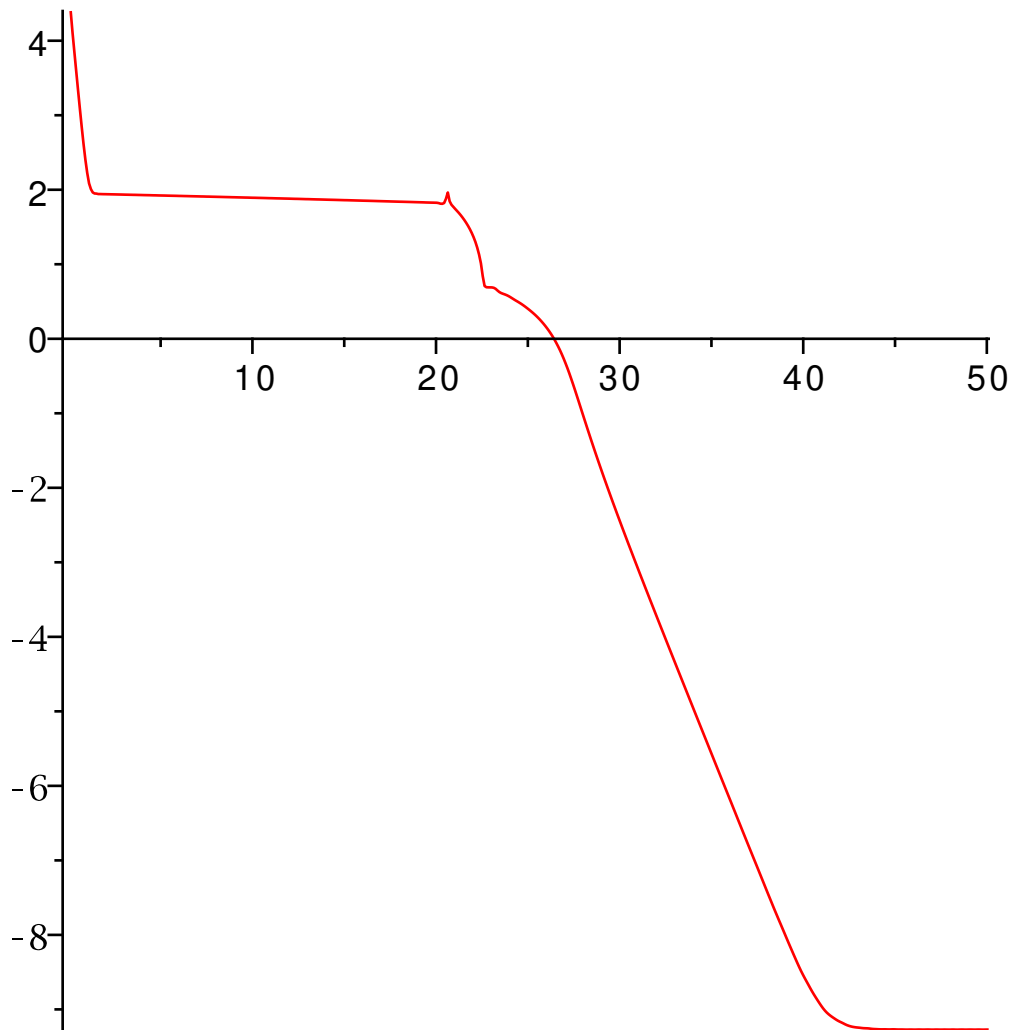
```
> Digits:=20:

t_ini:=0.0:
t_fin:=50.0:
nt:=500:

dt:=(t_fin-t_ini)/(nt-1):
gr:=1:
tt:=t_ini:
for i from 1 to nt do
    vv:=subs(dsn(tt),[x1(t),x2(t),x3(t)]):
    tt:=tt+dt:
    gr:=gr,[tt,evalf(log10(subs(x1=vv[1],x2=vv[2],x3=vv[3],
llf)))])
od:
gr:=[gr]:
gr:=gr[2..nops(gr)]:

Digits:=10:
> plot(gr);
```





## ▼ Three-Waves System

H. Hakken, Light, Vol. 2 Laser Light Dynamics, North-Holland (New York, 1985).  
 J. Weiland and H. Wilhelmsson, Coherent Non-Linear Interaction of Waves in Plasmas, Pergamon, Oxford (1977).

```
> # Writing down the system:
> eqs:=1:
  params:={g}:
  for i from 1 to 3 do
    pr:=cat(lambda,i)*cat(x,i):
    if i=1 then
      pr:=pr+g*x2*x3
    fi:
    pr:=pr+cat(x,i)*sum('cat'(N,i,jj)*'cat'(x,jj)^2,jj=1..3):
    eqs:=eqs,pr:
```

```

        params:=params union {seq('cat'(N,i,jj),jj=1..3)}:
    od:
    eqs:=[eqs]:
    eqs:=eqs[2..nops(eqs)]:
> eqs;
[λ1 x1 + g x2 x3 + x1 (N11 x12 + N12 x22 + N13 x32), λ2 x2 + x2 (N21 x12
+ N22 x22 + N23 x32), λ3 x3 + x3 (N31 x12 + N32 x22 + N33 x32) ] (4.1)
> variables:=[x1,x2,x3];
variables:= [x1, x2, x3] (4.2)
> params;
{N11, N12, N13, N21, N22, N23, N31, N32, N33, g} (4.3)

> # Giving some numerical values for the parameters
> vals:={N11=-1,N12=1,N13=7,N21=-10,N22=-10,N23=7,N31=-10,N32=
-1,N33=-4,g=2};
vals:= {N11 = -1, N12 = 1, N13 = 7, N21 = -10, N22 = -10, N23 = 7, N31 =
-10, N32 = -1, N33 = -4, g = 2} (4.4)

> eqs2:=subs(vals,eqs);
eqs2:= [λ1 x1 + 2 x2 x3 + x1 (-x12 + x22 + 7 x32), λ2 x2 + x2 (-10 x12
- 10 x22 + 7 x32), λ3 x3 + x3 (-10 x12 - x22 - 4 x32) ] (4.5)
> r1:=solve(subs(x1=1,x2=1,x3=1,eqs2),{lambda1,lambda2,lambda3}
);
r1:= {λ1 = -9, λ2 = 13, λ3 = 15} (4.6)
> eqs3:=subs(r1,eqs2);
eqs3:= [-9 x1 + 2 x2 x3 + x1 (-x12 + x22 + 7 x32), 13 x2 + x2 (-10 x12
- 10 x22 + 7 x32), 15 x3 + x3 (-10 x12 - x22 - 4 x32) ] (4.7)
> # Determining the Lyapunov function from the numerical method
> t1:=time():
llf:=LyapunovFunction(eqs3,variables,params,[1,1,1]);
time()-t1;
llf:=  $\frac{0.2349193456 x_2 x_3}{x_1} - 0.2349193456 \ln\left(\frac{x_2 x_3}{x_1}\right) - 2.152062513$ 
+ 0.9833984375 x12 - 0.9833984375 ln(x12) + 0.4201864672 x22
- 0.4201864672 ln(x22) + 0.5135582630 x32 - 0.5135582630 ln(x32)
1.327 (4.8)
> # And verifying it:
> diff_sys:=simplify(subs(vals,convert(subs(x1=x1(t),x2=x2(t),
x3=x3(t),{Diff(x1,t)=eqs3[1],Diff(x2,t)=eqs3[2],Diff(x3,t)=
eqs3[3]}),diff)));

```

$$\begin{aligned} \text{diff\_sys} := \left\{ \begin{aligned} \frac{d}{dt} x1(t) &= -9 x1(t) + 2 x2(t) x3(t) - x1(t)^3 + x1(t) x2(t)^2 \\ &+ 7 x1(t) x3(t)^2, \quad \frac{d}{dt} x2(t) = 13 x2(t) - 10 x2(t) x1(t)^2 - 10 x2(t)^3 \\ &+ 7 x2(t) x3(t)^2, \quad \frac{d}{dt} x3(t) = 15 x3(t) - 10 x3(t) x1(t)^2 - x3(t) x2(t)^2 \\ &- 4 x3(t)^3 \end{aligned} \right\} \end{aligned} \quad (4.9)$$

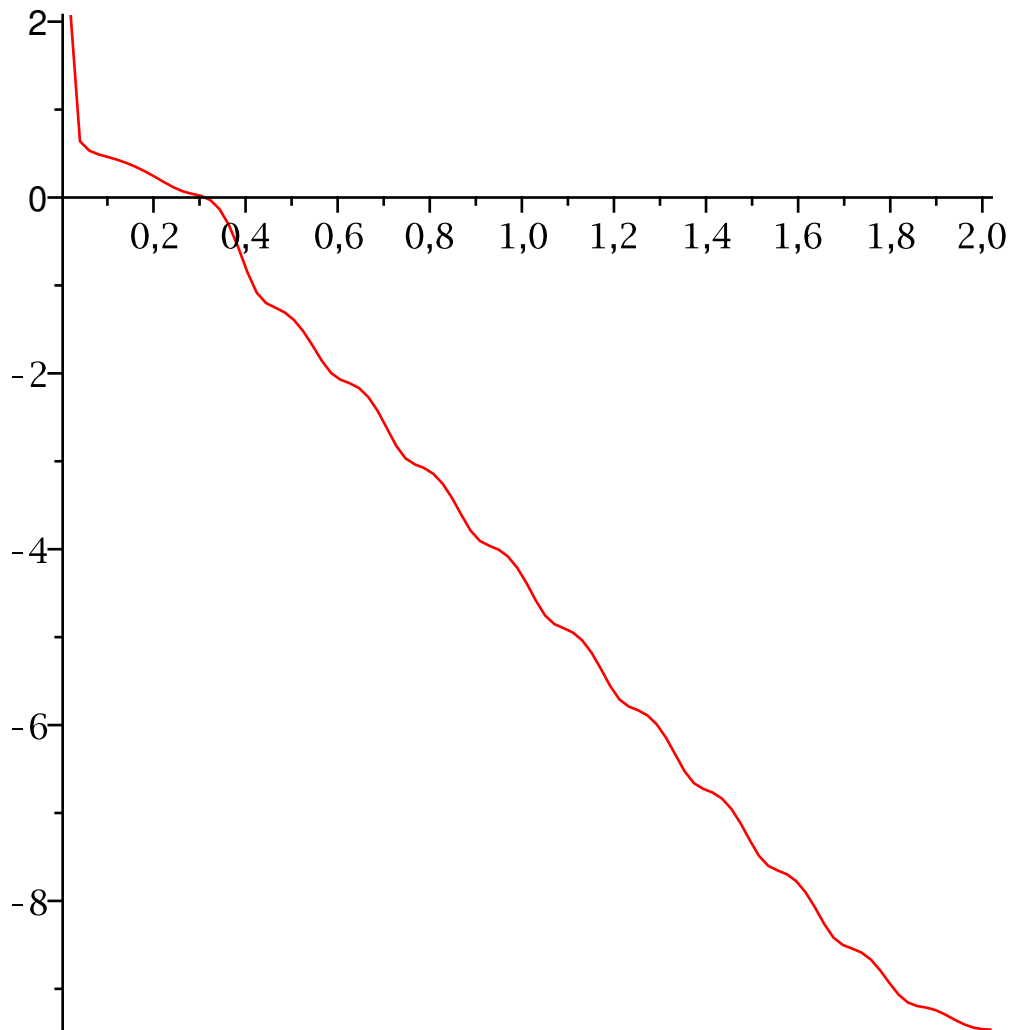
```
> dsn := dsolve(diff_sys union {x1(0)=1.0,x2(0)=15.0,x3(0)=5.0}
, numeric);
      dsn:=proc(x_rkf45) ... end proc
```

(4.10)

```
> Digits:=20:

t_ini:=0.0:
t_fin:=2.0:
nt:=100:

dt:=(t_fin-t_ini)/(nt-1):
gr:=1:
tt:=t_ini:
for i from 1 to nt do
    vv:=subs(dsn(tt),[x1(t),x2(t),x3(t)]):
    tt:=tt+dt:
    gr:=gr,[tt,evalf(log10(subs(x1=vv[1],x2=vv[2],x3=vv[3],
11f)))]:
od:
gr:=gr:
gr:=gr[2..nops(gr)]:
Digits:=10:
> plot(gr);
```



## Modified Verhulst-Solow model

Journal of Statistical Mechanics, "Modified Verhulst-Solow model for long-term population and economic growth"

<https://doi.org/10.1088/1742-5468/ad267a>

```
> eqs:=[x1*(1-alpha1+b*alpha2-(1-alpha1)*x1-alpha2*x2),
        x2*(-beta*x1+b*beta*alpha2+beta*alpha1*x1-(k+beta*
        alpha2)*x2+x3),
        x3*(-beta*alpha1+b*beta*alpha2+beta*alpha1*x1-beta*
        alpha2*x2)];
variables:=[x1,x2,x3];
params:={b,k,beta,alpha1,alpha2};
eqs:=[x1*(1-alpha1+b*alpha2-(1-alpha1)*x1-alpha2*x2), x2*(-beta*x1+b*beta*alpha2+beta*alpha1*x1-
        (k+beta*alpha2)*x2+x3), x3*(-beta*alpha1+b*beta*alpha2+beta*alpha1*x1-beta*alpha2*x2)]
variables:=[x1,x2,x3]
```

(5.1)

$$params := \{\alpha 1, \alpha 2, b, \beta, k\} \quad (5.1)$$

> **q:=solve(eqs,variables);**

$$q := \left[ [x1 = 0, x2 = 0, x3 = 0], \left[ x1 = 0, x2 = -\frac{\alpha 1 - b \alpha 2}{\alpha 2}, x3 = -\frac{\alpha 1 k + \alpha 1 \beta \alpha 2 - b \alpha 2 k}{\alpha 2} \right], \left[ x1 = 0, x2 = \frac{b \beta \alpha 2}{k + \beta \alpha 2}, x3 = 0 \right], [x1 = 1, x2 = b, x3 = k b + \beta - \beta \alpha 1], \left[ x1 = \frac{-k + \alpha 1 k - b \alpha 2 k - \beta \alpha 2 + \alpha 1 \beta \alpha 2}{k(-1 + \alpha 1)}, x2 = \frac{\beta(-1 + \alpha 1)}{k}, x3 = 0 \right], \left[ x1 = \frac{-1 + \alpha 1 - b \alpha 2}{-1 + \alpha 1}, x2 = 0, x3 = 0 \right] \right] \quad (5.2)$$

> **q2:=q[4];**

$$q2 := [x1 = 1, x2 = b, x3 = k b + \beta - \beta \alpha 1] \quad (5.3)$$

> **q3:=subs(q2,variables);**

$$q3 := [1, b, k b + \beta - \beta \alpha 1] \quad (5.4)$$

> **t1:=time();**

**rl:=LyapunovFunction(eqs,variables,params,q3);**

**time()-t1;**

$$\begin{aligned} rl := & \left[ \left[ -a1 k b \ln\left(\frac{x3}{k b + \beta}\right) - a1 k b - a3 b \ln\left(\frac{x2}{b}\right) - a3 b - a1 \beta + a2 x1 \right. \right. \\ & + a3 x2 - a2 \ln(x1) - a1 \beta \ln\left(\frac{x3}{k b + \beta}\right) - a2 + a1 x3, \left. \left\{ \alpha 1 = 0, \right. \right. \\ & - \frac{1}{4} \frac{a1^2 \beta^4 \alpha 2 - 4 k a1 a2 \beta + a2^2 \alpha 2 - 2 a2 a1 \beta^2 \alpha 2}{a1 \beta a2} = 0, -a1 \beta \alpha 2 + a3 \\ & = 0 \left. \right\} \right], \left[ -a1 k b \ln\left(\frac{x3}{k b + \beta}\right) - a1 k b - a3 b \ln\left(\frac{x2}{b}\right) - a3 b - a1 \beta + a2 x1 \right. \\ & + a3 x2 - a2 \ln(x1) - a1 \beta \ln\left(\frac{x3}{k b + \beta}\right) - a2 + a1 x3, \left. \left\{ \alpha 1 = 0, -a1 \beta \alpha 2 \right. \right. \\ & + a3 = 0, \alpha 2 < 0, \beta < 0, \alpha 2 (a1^2 \beta^4 \alpha 2 - 4 k a1 a2 \beta + a2^2 \alpha 2 \\ & - 2 a2 a1 \beta^2 \alpha 2) < 0 \left. \right\} \right], \left[ -a1 k b \ln\left(\frac{x3}{k b + \beta}\right) - a1 k b - a3 b \ln\left(\frac{x2}{b}\right) \right. \\ & - a3 b - a1 \beta + a2 x1 + a3 x2 - a2 \ln(x1) - a1 \beta \ln\left(\frac{x3}{k b + \beta}\right) - a2 \\ & + a1 x3, \left. \left\{ \alpha 1 = 0, -a1 \beta \alpha 2 + a3 = 0, \alpha 2 < 0, \alpha 2 (a1^2 \beta^4 \alpha 2 - 4 k a1 a2 \beta \right. \right. \end{aligned}$$

$$\begin{aligned}
& + a^2 \alpha_2 - 2 a_2 a_1 \beta^2 \alpha_2) < 0, -\beta < 0\} \Big], \Big[ -a_1 k b \ln\left(\frac{x_3}{k b + \beta}\right) - a_1 k b \\
& - a_3 b \ln\left(\frac{x_2}{b}\right) - a_3 b - a_1 \beta + a_2 x_1 + a_3 x_2 - a_2 \ln(x_1) \\
& - a_1 \beta \ln\left(\frac{x_3}{k b + \beta}\right) - a_2 + a_1 x_3, \{\alpha_1 = 0, -a_1 \beta \alpha_2 + a_3 = 0, \beta < 0, \\
& \alpha_2 (a_1^2 \beta^4 \alpha_2 - 4 k a_1 a_2 \beta + a_2^2 \alpha_2 - 2 a_2 a_1 \beta^2 \alpha_2) < 0, -\alpha_2 < 0\} \Big], \Big[ \\
& -a_1 k b \ln\left(\frac{x_3}{k b + \beta}\right) - a_1 k b - a_3 b \ln\left(\frac{x_2}{b}\right) - a_3 b - a_1 \beta + a_2 x_1 + a_3 x_2 \\
& - a_2 \ln(x_1) - a_1 \beta \ln\left(\frac{x_3}{k b + \beta}\right) - a_2 + a_1 x_3, \{\alpha_1 = 0, -a_1 \beta \alpha_2 + a_3 = 0, \\
& \alpha_2 (a_1^2 \beta^4 \alpha_2 - 4 k a_1 a_2 \beta + a_2^2 \alpha_2 - 2 a_2 a_1 \beta^2 \alpha_2) < 0, -\alpha_2 < 0, -\beta < 0\} \\
& \Big] \Big]
\end{aligned}$$

$$0.125 \quad (5.5)$$

**> nops(r1);**

$$5 \quad (5.6)$$

**> r12:=r1[1];**

$$\begin{aligned}
r12 := & \Big[ -a_1 k b \ln\left(\frac{x_3}{k b + \beta}\right) - a_1 k b - a_3 b \ln\left(\frac{x_2}{b}\right) - a_3 b - a_1 \beta + a_2 x_1 \\
& + a_3 x_2 - a_2 \ln(x_1) - a_1 \beta \ln\left(\frac{x_3}{k b + \beta}\right) - a_2 + a_1 x_3, \left\{ \alpha_1 = 0, \right. \\
& \left. -\frac{1}{4} \frac{a_1^2 \beta^4 \alpha_2 - 4 k a_1 a_2 \beta + a_2^2 \alpha_2 - 2 a_2 a_1 \beta^2 \alpha_2}{a_1 \beta a_2} = 0, -a_1 \beta \alpha_2 + a_3 \right. \\
& \left. = 0 \right\} \Big]
\end{aligned} \quad (5.7)$$

**> r13:=map(z->if lhs(z)<>rhs(z) then z fi,solve(r12[2],{a1,a2,a3,alpha1,alpha2}));**

$$r13 := \left\{ a_3 = \frac{4 a_1^2 \beta^2 k a_2}{(a_1 \beta^2 - a_2)^2}, \alpha_1 = 0, \alpha_2 = \frac{4 k a_1 a_2 \beta}{(a_1 \beta^2 - a_2)^2} \right\} \quad (5.8)$$

**> vals:={a1=1/2,a2=1/3,beta=3,k=2,b=7};**

$$vals := \left\{ a_1 = \frac{1}{2}, a_2 = \frac{1}{3}, b = 7, \beta = 3, k = 2 \right\} \quad (5.9)$$

```
> vals2:=vals union subs(vals,r13);
```

$$vals2 := \left\{ a1 = \frac{1}{2}, a2 = \frac{1}{3}, a3 = \frac{216}{625}, \alpha1 = 0, \alpha2 = \frac{144}{625}, b = 7, \beta = 3, k = 2 \right\} \quad (5.10)$$

```
> diff_sys:=simplify(subs(vals2,convert(subs(x1=x1(t),x2=x2(t),
x3=x3(t),{Diff(x1,t)=eqs[1],Diff(x2,t)=eqs[2],Diff(x3,t)=eqs
[3]}),diff)));
```

$$diff\_sys := \left\{ \frac{d}{dt} x1(t) = -\frac{1}{625} x1(t) (-1633 + 625 x1(t) + 144 x2(t)), \frac{d}{dt} x2(t) \right. \quad (5.11)$$

$$= -\frac{1}{625} x2(t) (1875 x1(t) - 3024 + 1682 x2(t) - 625 x3(t)), \frac{d}{dt} x3(t) =$$

$$\left. -\frac{432}{625} x3(t) (-7 + x2(t)) \right\}$$

```
> llf:=evalf(subs(vals2,subs(vals,r1[1][1])));
```

$$llf := -8.500000000 \ln(0.05882352941 x3) - 11.25253333 \quad (5.12)$$

$$- 2.419200000 \ln(0.1428571429 x2) + 0.3333333333 x1$$

$$+ 0.3456000000 x2 - 0.3333333333 \ln(x1) + 0.5000000000 x3$$

```
> dsn := dsolve(diff_sys union {x1(0)=1.0,x2(0)=1.0,x3(0)=2.0},
numeric);
```

$$dsn := \text{proc}(x\_rkf45) \dots \text{end proc} \quad (5.13)$$

```
> Digits:=20;
```

$$Digits := 20 \quad (5.14)$$

```
> t_ini:=0.0:
t_fin:=7.0:
nt:=100:
dt:=(t_fin-t_ini)/(nt-1):
gr:=1:
tt:=t_ini:
for i from 1 to nt do
    vv:=subs(dsn(tt),[x1(t),x2(t),x3(t)]):
    tt:=tt+dt:
    gr:=gr,[tt,evalf(log10(subs(x1=vv[1],x2=vv[2],x3=vv[3],
llf)))]
od:
gr:=[gr]:
gr:=gr[2..nops(gr)]:
Digits:=10:
> plot(gr);
```

