Lyapunov 2.0
Reference manual

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Abstract

This Module obtains the sufficient conditions for the existence of a Lyapunov function and computes the associated Lyapunov function.

1 Introduction

We present the version 2.0 of the MAPLE package LYAPUNOV with a thorough revision of the implemented algorithms. The package determine Lyapunov functions for quasi-polynomial systems. We implemented a new and more efficient algorithm for the solution or simplification of such conditions, and a new implementation of its numerical solution for the case with no free parameters.

2 Package Commands

- **LyapunovFunction(eqs,vars,params,fixed-point)** - Determines the Lyapunov functions for a set of ODEs of the Quasi-Polynomial form and a given fixed point by solving, if possible, the admissibility conditions. In the arguments **eqs** stand for a list giving the flow of the set of first order ODEs, **vars** is the set of variables in the ODE set, **params** the set of free parameters in the equation and **fixed-point** is a list with the coordinates of the fixed point. If the equations depend on free parameters, returns a set with each element containing the Lyapunov function and the conditions on the variables **a** and the free parameters. If the algorithm is not capable of obtaining a solution it return the generic form of the Lyapunov function and the admissibility conditions. If no solution exists it returns *false*. If the equations do not depend on any free parameter the calling returns a numeric form for the Lyapunov function.

- **lyap_func(eqs,vars,params,fixed-point)** - Returns the form of the Lyapunov function for a set of ODEs of the Quasi-Polynomial form and a given fixed point without solving the admissibility conditions. The arguments are the same as for **LyapunovFunction**.

- **AdmissibilityConditions(eqs,vars,params)** - Determines the admissibility conditions for a given set of ODEs. The arguments have the same meaning as in **LyapunovFunction**.

- **SolveAdmissCond(conds,params)** - Solves the admissibility conditions. The arguments are: **conds** the set of conditions as obtain from **AdmissibilityConditions** and **params** the set of free parameters in the set of ODEs.

- **SolveSingleIneq(ineq,as,params)** - Solves or simplifies a single inequation in **ineq** in the variables a_1, a_2, ... in **as** and in the free parameters in **params**.

- **SolveManyIneqs(ineqs,as,params)** - Solves or simplifies a set of inequations in **ineqs** in the variables a_1, a_2, ... in **as** and in the free parameters in **param**.

3 Global control variables

A few global variables are used to control the behaviour of the implemented algorithms. Theysvalues can be accessed and changed using the Maple command `LYAPUNOV[Name] := Nalue:`. Their meaning and default values are given in Table 1 for each `Name`.

Name	**Description**	**Default value**
_eps	Value for the parameter ϵ in the numeric algorithm	10^{-12}
_maxiter	Maximal number of iterations in the numeric algorithm	1000
_incogname	variable name for the a_i variables in the admissibility condition	a
_maxiter	variable name for the w_i variables in the admissibility condition	w
_timelim	time limit for some computation, e.g. Gröbner basis	20.0
_nummaxQP	limits the number of subsequent decompositions when determining the admissibility conditions	6
_groebner	if a Gröbner basis is computed before solving equalities	true
_maxsize	maximal number of terms) for factorization operations	500

Table 1: Global control variables

3 Examples

We present below examples of use of the package commands, for a few systems in a Maple worksheet.

Examples of use of the Lypunov 2.0 Package

```
> restart;
> with(lyapunov);
```

$$[AdmissibilityConditions, LyapunovFunction, SolveAdmissCond, SolveManyIneqs, SolveSingleIneq, init, lyap\_func] \quad \textbf{(1)}$$

Two-Dimensional May-Leonard System

```
> # Defining the flow, the unknowns in the differential systems, and the free parameters, respectively.
> eqs:=[x1*(alpha1-beta1*x1-beta2*x2),x2*(alpha2-beta3*x1-beta4*x2)];
  variables:=[x1,x2];
  params:={alpha1,alpha2,beta1,beta2,beta3,beta4};
```

$$eqs := [x1\,(\alpha 1 - \beta 1\,x1 - \beta 2\,x2),\ x2\,(\alpha 2 - \beta 3\,x1 - \beta 4\,x2)]$$

$$variables := [x1, x2]$$

$$params := \{\alpha 1, \alpha 2, \beta 1, \beta 2, \beta 3, \beta 4\} \quad \textbf{(1.1)}$$

```
> # fixed points:
> uu:=[solve({op(eqs)},{op(variables)})];
```

$$uu := \left[\{x1 = 0, x2 = 0\}, \left\{x1 = 0, x2 = \frac{\alpha 2}{\beta 4}\right\}, \left\{x1 = \frac{\alpha 1}{\beta 1}, x2 = 0\right\}, \left\{x1 = \frac{\alpha 1\,\beta 4 - \beta 2\,\alpha 2}{-\beta 3\,\beta 2 + \beta 4\,\beta 1}, x2 = -\frac{-\alpha 2\,\beta 1 + \beta 3\,\alpha 1}{-\beta 3\,\beta 2 + \beta 4\,\beta 1}\right\}\right] \quad \textbf{(1.2)}$$

```
> # Chosing a fixed point with non-vanising components.
  # Later we will check if it is inside the positive quadrant.
> q:=uu[4];
  q2:=subs(q,variables);
```

$$q := \left\{x1 = \frac{\alpha 1\,\beta 4 - \beta 2\,\alpha 2}{-\beta 3\,\beta 2 + \beta 4\,\beta 1}, x2 = -\frac{-\alpha 2\,\beta 1 + \beta 3\,\alpha 1}{-\beta 3\,\beta 2 + \beta 4\,\beta 1}\right\}$$

$$q2 := \left[\frac{\alpha 1\,\beta 4 - \beta 2\,\alpha 2}{-\beta 3\,\beta 2 + \beta 4\,\beta 1}, -\frac{-\alpha 2\,\beta 1 + \beta 3\,\alpha 1}{-\beta 3\,\beta 2 + \beta 4\,\beta 1}\right] \quad \textbf{(1.3)}$$

```
> # Obtainin the admissibility conditions
> s0:=AdmissibilityConditions(eqs,variables,params);
```

$$s0 := \{\{a1^2\,\beta 3^2 + 2\,a1\,\beta 3\,a2\,\beta 2 + a2^2\,\beta 2^2 - 4\,a1\,\beta 4\,a2\,\beta 1 = 0,\ -\beta 4 < 0\},\ \{-\beta 4 \quad \textbf{(1.4)}$$

$< 0,\ a1^2\,\beta3^2 + 2\,a1\,\beta3\,a2\,\beta2 + a2^2\,\beta2^2 - 4\,a1\,\beta4\,a2\,\beta1 < 0\},\ \{-a1\,\beta4 = 0,$
$-a2\,\beta1 = 0,\ -a1\,\beta3 - a2\,\beta2 = 0\},\ \{-a1\,\beta4 = 0,\ -a1\,\beta3 - a2\,\beta2 = 0,\ -\beta1$
$< 0\}\}$

```
> #
  # Determining the Lyapunov functions and the simplified
  conditions on the parameters
  #
  t1:=time():
  rl:=LyapunovFunction(eqs,variables,params,q2):
  time()-t1;
```

$$0.085 \qquad (1.5)$$

```
> nops(rl);
```

$$11 \qquad (1.6)$$

```
> # As an example, let us consider the third case:
> rl2:=rl[3];
```

$$rl2 := \Bigg[\frac{1}{-\beta3\,\beta2 + \beta4\,\beta1}\Bigg(-a2\,\alpha1\,\beta4 - a2\,\alpha1\,\beta4\ln\Big(\frac{x1\,(-\beta3\,\beta2 + \beta4\,\beta1)}{\alpha1\,\beta4 - \beta2\,\alpha2}\Big) + a1\,\alpha1\,\beta3 + a1\,\beta3\,\alpha1\ln\Big(-\frac{x2\,(-\beta3\,\beta2 + \beta4\,\beta1)}{-\alpha2\,\beta1 + \beta3\,\alpha1}\Big) + \beta4\,a1\,x2\,\beta1 + \beta4\,a2\,x1\,\beta1 - a2\,x1\,\beta3\,\beta2 - a1\,x2\,\beta3\,\beta2 + a2\,\beta2\,\alpha2\ln\Big(\frac{x1\,(-\beta3\,\beta2 + \beta4\,\beta1)}{\alpha1\,\beta4 - \beta2\,\alpha2}\Big) - a1\,\alpha2\,\beta1\ln\Big(-\frac{x2\,(-\beta3\,\beta2 + \beta4\,\beta1)}{-\alpha2\,\beta1 + \beta3\,\alpha1}\Big) - a1\,\alpha2\,\beta1 + a2\,\alpha2\,\beta2\Bigg),\ \Big\{-\frac{1}{4}\,\frac{a1^2\,\beta3^2 + 2\,a1\,\beta3\,a2\,\beta2 + a2^2\,\beta2^2 - 4\,a1\,\beta4\,a2\,\beta1}{a1\,\beta4\,a2} = 0,\ -\beta4 < 0\Big\}\Bigg] \qquad (1.7)$$

```
> # These are the simplified conditions on the free parameters
  and the unknowns a's:
  qq1:=rl2[2];
```

$$qq1 := \Big\{-\frac{1}{4}\,\frac{a1^2\,\beta3^2 + 2\,a1\,\beta3\,a2\,\beta2 + a2^2\,\beta2^2 - 4\,a1\,\beta4\,a2\,\beta1}{a1\,\beta4\,a2} = 0,\ -\beta4 < 0\Big\} \qquad (1.8)$$

```
> # We then chose some values for the free variables:
  vals:={a1=1,a2=2,beta2=5,beta3=4,beta4=3,alpha1=2,alpha2=1};
```

$$vals := \{a1 = 1,\ a2 = 2,\ \alpha1 = 2,\ \alpha2 = 1,\ \beta2 = 5,\ \beta3 = 4,\ \beta4 = 3\} \qquad (1.9)$$

```
> qq2:=subs(vals,qq1);
```

$$qq2 := \Big\{-\frac{49}{6} + \beta1 = 0,\ -3 < 0\Big\} \qquad (1.10)$$

```
> vals2:=vals union {beta1=49/6};
```

$$vals2 := \left\{a1 = 1,\ a2 = 2,\ \alpha 1 = 2,\ \alpha 2 = 1,\ \beta 1 = \frac{49}{6},\ \beta 2 = 5,\ \beta 3 = 4,\ \beta 4 = 3\right\} \quad \textbf{(1.11)}$$

```
> # Testing for the positivity of the fixed point components
> q3:=simplify(subs(vals2,q2));
```

$$q3 := \left[\frac{2}{9}, \frac{1}{27}\right] \quad \textbf{(1.12)}$$

```
> # We then have the following Lyapunov function:
  llf:=evalf(subs(vals2,rl2[1]));
```

$$llf := -0.4814814815 - 0.4444444444 \ln(4.500000000\, x1) - 0.03703703704 \ln(27.\, x2) + x2 + 2.\, x1 \quad \textbf{(1.13)}$$

```
> # And testing this Lyapunov function by performing a
  numerical integration of the ODE system.
> diff_sys:=simplify(subs(vals2,convert(subs(x1=x1(t),x2=x2(t),
  {Diff(x1,t)=eqs[1],Diff(x2,t)=eqs[2]}),diff)));
```

$$diff\_sys := \left\{\frac{d}{dt}\, x1(t) = -\frac{1}{6}\, x1(t)\, (-12 + 49\, x1(t) + 30\, x2(t)),\ \frac{d}{dt}\, x2(t) = -x2(t)\, (-1 + 4\, x1(t) + 3\, x2(t))\right\} \quad \textbf{(1.14)}$$

```
> diff_sys2:=subs(vals2,diff_sys);
```

$$diff\_sys2 := \left\{\frac{d}{dt}\, x1(t) = -\frac{1}{6}\, x1(t)\, (-12 + 49\, x1(t) + 30\, x2(t)),\ \frac{d}{dt}\, x2(t) = -x2(t)\, (-1 + 4\, x1(t) + 3\, x2(t))\right\} \quad \textbf{(1.15)}$$

```
> # Here we chose x1(0)=1.0 and x2(0)=1.0 as initial condition.
> dsn := dsolve(diff_sys2 union {x1(0)=1.0,x2(0)=1.0}, numeric)
  ;
```

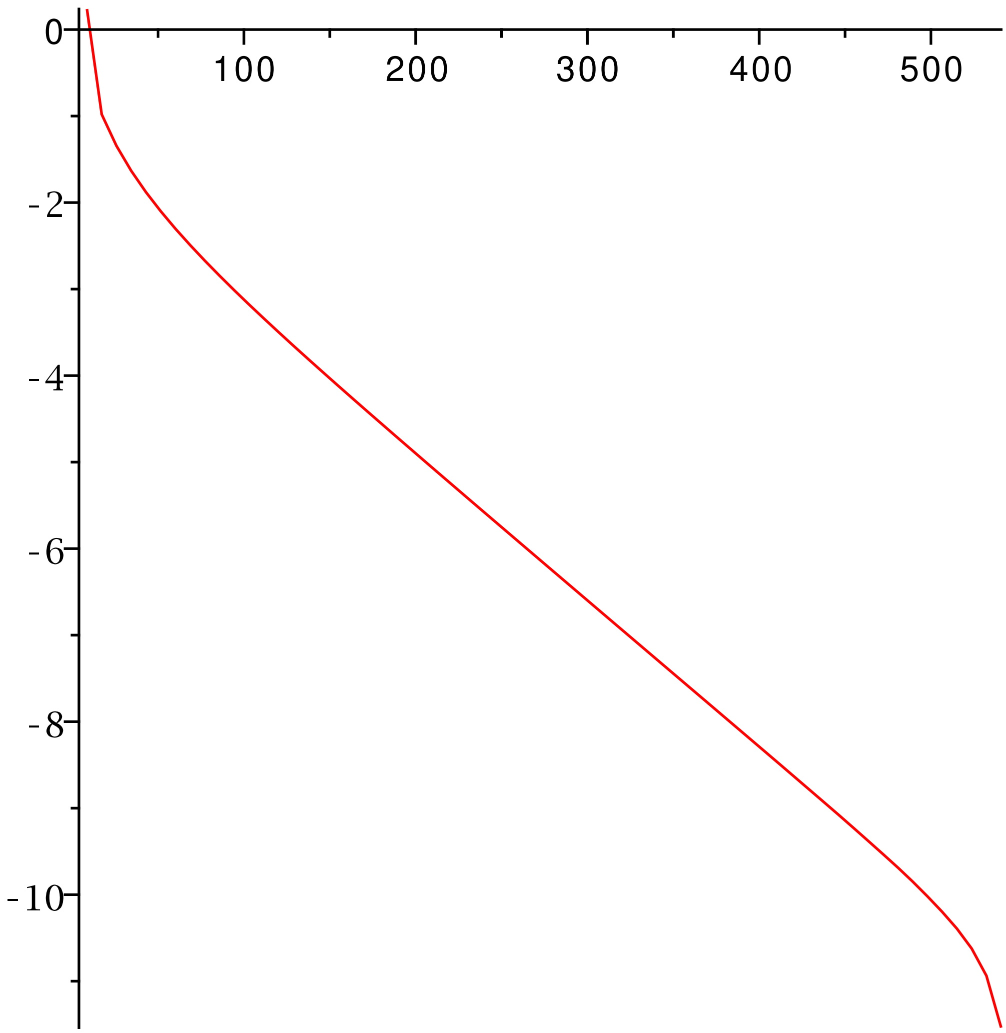
$$dsn := \textbf{proc}(x\_rkf45)\ \ldots\ \textbf{end proc} \quad \textbf{(1.16)}$$

```
> Digits:=20:

  t_ini:=0.0:
  t_fin:=850.0:
  nt:=100:

  dt:=(t_fin-t_ini)/(nt-1):
  gr:=1:
  tt:=t_ini:
  for i from 1 to nt do
      vv:=subs(dsn(tt),[x1(t),x2(t)]):
      tt:=tt+dt:
      gr:=gr,[tt,evalf(log10(subs(x1=vv[1],x2=vv[2],llf)))]
  od:
```

```
gr:=[gr]:
gr:=gr[2..nops(gr)]:

Digits:=10:
```

```
> # And finally ploting the value of the Lyapunov function as a
function of time
```

```
> plot(gr);
```



Three-Dimensional May-Leonard System

R.M. May, W.J. Leonard, Nonlinear aspects of competition between three species, SIAM J. Appl. Math. 29 (1975) 243-253

```
> # The system, its unknown variables and free parameters:
```

```
> eqs:=[l1*x1-x1*(x1+alpha*x2+beta*x3),l2*x2-x2*(beta*x1+x2+
alpha*x3),l3*x3-x3*(alpha*x1+beta*x2+x3)];
variables:=[x1,x2,x3];
params:={l1,l2,l3,alpha,beta};
```

$$eqs := [l1\,x1 - x1\,(x1 + \alpha\,x2 + \beta\,x3),\ l2\,x2 - x2\,(\beta\,x1 + x2 + \alpha\,x3),\ l3\,x3 - x3\,(\alpha\,x1 + \beta\,x2 + x3)]$$

$$variables := [x1, x2, x3]$$

$$params := \{\alpha, \beta, l1, l2, l3\} \tag{2.1}$$

```
> #
  # Determining the fixed points of the original system:
  #
  uu:=[solve({op(eqs)},{op(variables)})];
```

$$uu := \left[\{x1 = 0, x2 = 0, x3 = 0\}, \{x1 = 0, x2 = 0, x3 = l3\}, \{x1 = 0, x2 = l2, x3 = 0\}, \left\{x1 = 0, x2 = \frac{-l2 + \alpha\,l3}{\beta\,\alpha - 1}, x3 = \frac{-l3 + \beta\,l2}{\beta\,\alpha - 1}\right\}, \{x1 = l1, x2 = 0, x3 = 0\}, \left\{x1 = \frac{-l1 + \beta\,l3}{\beta\,\alpha - 1}, x2 = 0, x3 = \frac{-l3 + \alpha\,l1}{\beta\,\alpha - 1}\right\}, \left\{x1 = \frac{-l1 + \alpha\,l2}{\beta\,\alpha - 1}, x2 = \frac{-l2 + \beta\,l1}{\beta\,\alpha - 1}, x3 = 0\right\}, \left\{x1 = \frac{-l1\,\beta\,\alpha + l1 + \alpha^2\,l3 - \alpha\,l2 + \beta^2\,l2 - \beta\,l3}{\alpha^3 - 3\,\beta\,\alpha + \beta^3 + 1}, x2 = \frac{l3\,\beta^2 - \alpha\,l3 + \alpha^2\,l1 - l2\,\beta\,\alpha + l2 - \beta\,l1}{\alpha^3 - 3\,\beta\,\alpha + \beta^3 + 1}, x3 = \frac{\alpha^2\,l2 - \beta\,\alpha\,l3 - \alpha\,l1 - \beta\,l2 + l1\,\beta^2 + l3}{\alpha^3 - 3\,\beta\,\alpha + \beta^3 + 1}\right\}\right] \tag{2.2}$$

```
> #
  # Chosing a fixed point inside the positive orthant
  #
  q:=uu[8];
  q2:=subs(q,variables);
```

$$q := \left\{x1 = \frac{-l1\,\beta\,\alpha + l1 + \alpha^2\,l3 - \alpha\,l2 + \beta^2\,l2 - \beta\,l3}{\alpha^3 - 3\,\beta\,\alpha + \beta^3 + 1}, x2 = \frac{l3\,\beta^2 - \alpha\,l3 + \alpha^2\,l1 - l2\,\beta\,\alpha + l2 - \beta\,l1}{\alpha^3 - 3\,\beta\,\alpha + \beta^3 + 1}, x3 = \frac{\alpha^2\,l2 - \beta\,\alpha\,l3 - \alpha\,l1 - \beta\,l2 + l1\,\beta^2 + l3}{\alpha^3 - 3\,\beta\,\alpha + \beta^3 + 1}\right\}$$

$$q2 := \left[\frac{-l1\,\beta\,\alpha + l1 + \alpha^2\,l3 - \alpha\,l2 + \beta^2\,l2 - \beta\,l3}{\alpha^3 - 3\,\beta\,\alpha + \beta^3 + 1}, \frac{l3\,\beta^2 - \alpha\,l3 + \alpha^2\,l1 - l2\,\beta\,\alpha + l2 - \beta\,l1}{\alpha^3 - 3\,\beta\,\alpha + \beta^3 + 1}, \right. \tag{2.3}$$

$$\left.\frac{\alpha^2 l2-\beta\alpha l3-\alpha l1-\beta l2+l1\beta^2+l3}{\alpha^3-3\beta\alpha+\beta^3+1}\right]$$

```
> # The admissibility conditions.
> s0:=AdmissibilityConditions(eqs,variables,params);
```

$$s0 := \{\{16\,a1^2\,a2^2\,\alpha^2+16\,a1^2\,a3^2\,\beta^2+16\,a1^3\,\alpha^2\,a3+16\,a1^3\,a2\,\beta^2 -64\,a1^2\,a2\,a3-16\,a2^2\,\beta\,a3\,\alpha^2\,a1-16\,a2\,\beta^2\,a3^2\,\alpha\,a1+96\,a1^2\,a2\,\alpha\,a3\,\beta +16\,a2^2\,\beta^2\,a1\,a3+16\,a1\,a2\,a3^2\,\alpha^2-16\,a1^2\,\beta^2\,a2^2\,\alpha-16\,a1^2\,\beta^3\,a2\,a3 -16\,a1^2\,\alpha^2\,a3^2\,\beta-16\,a1^2\,\alpha^3\,a3\,a2-16\,a1^3\,\alpha^2\,\beta\,a2-16\,a1^3\,\alpha\,\beta^2\,a3=0, a1^2\,\alpha^2+2\,a1\,\alpha\,a2\,\beta+a2^2\,\beta^2-4\,a1\,a2<0\}, \{a1^2\,\alpha^2+2\,a1\,\alpha\,a2\,\beta+a2^2\,\beta^2 -4\,a1\,a2<0, 16\,a1\,a2^2\,\alpha^2+16\,a1\,a3^2\,\beta^2+16\,a1^2\,\alpha^2\,a3+16\,a1^2\,a2\,\beta^2 -64\,a1\,a2\,a3-16\,a2^2\,\beta\,a3\,\alpha^2-16\,a2\,\beta^2\,a3^2\,\alpha+96\,a1\,a2\,\alpha\,a3\,\beta +16\,a2^2\,\beta^2\,a3+16\,a2\,a3^2\,\alpha^2-16\,a1\,\beta^2\,a2^2\,\alpha-16\,a1\,\beta^3\,a2\,a3 -16\,a1\,\alpha^2\,a3^2\,\beta-16\,a1\,\alpha^3\,a3\,a2-16\,a1^2\,\alpha^2\,\beta\,a2-16\,a1^2\,\alpha\,\beta^2\,a3<0\}, \{a1^2\,\alpha^2+2\,a1\,\alpha\,a2\,\beta+a2^2\,\beta^2-4\,a1\,a2=0, a1^2\,\beta^2+2\,a1\,\beta\,a3\,\alpha+a3^2\,\alpha^2 -4\,a1\,a3=0, 2\,a1^2\,\alpha\,\beta+2\,a1\,\alpha^2\,a3+2\,a1\,\beta^2\,a2+2\,a2\,\alpha\,a3\,\beta-4\,a1\,\alpha\,a2 -4\,a1\,\beta\,a3=0\}, \{a1^2\,\alpha^2+2\,a1\,\alpha\,a2\,\beta+a2^2\,\beta^2-4\,a1\,a2=0, 2\,a1^2\,\alpha\,\beta +2\,a1\,\alpha^2\,a3+2\,a1\,\beta^2\,a2+2\,a2\,\alpha\,a3\,\beta-4\,a1\,\alpha\,a2-4\,a1\,\beta\,a3=0, a1^2\,\beta^2 +2\,a1\,\beta\,a3\,\alpha+a3^2\,\alpha^2-4\,a1\,a3<0\}\} \quad \textbf{(2.4)}$$

```
> # And solving (or simplifying) them:
> t1:=time():
  ss:=SolveAdmissCond(s0,params):
  time()-t1;
```

$$14.753 \quad \textbf{(2.5)}$$

```
> # We obtain 21 dferent solutions
> nops(ss);
```

$$21 \quad \textbf{(2.6)}$$

```
> # Some are quite simple, other more complicated:
> ss[1];
```

$$\left\{a1-\frac{1}{2}\,2^{1/3}\,a3=0,\ a2-2^{2/3}\,a3=0,\ \alpha-\frac{2}{3}\,2^{2/3}=0,\ \beta-\frac{2}{3}\,2^{1/3}=0\right\} \quad \textbf{(2.7)}$$

```
> ss[2];
```

$$\left\{a1-\frac{1}{4}\,2^{2/3}\,a3=0,\ a2-2\,2^{1/3}\,a3=0,\ \alpha+\frac{4}{3}\,2^{1/3}=0,\ \beta-\frac{2}{3}\,2^{2/3}=0\right\} \quad \textbf{(2.8)}$$

```
> ss[10];
```

$$\{(-2\,a1\,\alpha^2\,a3-2\,a1\,\beta^2\,a2+2\,a1\,\alpha^2\,\beta\,a2+2\,a1\,\alpha\,\beta^2\,a3-6\,a2\,\alpha\,a3\,\beta-a2^2\,\alpha^2 \quad \textbf{(2.9)}$$

$$+\beta^2 a2^2 \alpha + \beta^3 a2\, a3 + 4\, a2\, a3 - a3^2 \beta^2 + \alpha^2 a3^2 \beta + \alpha^3 a3\, a2$$
$$+ \left(\left(-2\, \beta^2 a2^2 \alpha + a2^2 \beta^4 + a2^2 \alpha^2 - 2\, \alpha^3 a3\, a2 - 2\, \beta^2 \alpha^2 a2\, a3 \right.\right.$$
$$+ 10\, a2\, \alpha\, a3\, \beta - 2\, \beta^3 a2\, a3 - 4\, a2\, a3 + \alpha^4 a3^2 + a3^2 \beta^2 - 2\, \alpha^2 a3^2 \beta \big) \big(a3^2 \beta^2$$
$$\left.\left. - 4\, a2\, a3 + 2\, a2\, \alpha\, a3\, \beta + a2^2 \alpha^2 \big) \right)^{1/2} \right) \left(-\alpha^2 a3 - a2\, \beta^2 + \alpha^2 \beta\, a2 + \alpha\, \beta^2 a3 \right)$$
$$< 0,\ \big(2\, a1\, \alpha^2 a3 + 2\, a1\, \beta^2 a2 - 2\, a1\, \alpha^2 \beta\, a2 - 2\, a1\, \alpha\, \beta^2 a3 + 6\, a2\, \alpha\, a3\, \beta$$
$$+ a2^2 \alpha^2 - \beta^2 a2^2 \alpha - \beta^3 a2\, a3 - 4\, a2\, a3 + a3^2 \beta^2 - \alpha^2 a3^2 \beta - \alpha^3 a3\, a2$$
$$+ \left(\left(-2\, \beta^2 a2^2 \alpha + a2^2 \beta^4 + a2^2 \alpha^2 - 2\, \alpha^3 a3\, a2 - 2\, \beta^2 \alpha^2 a2\, a3 \right.\right.$$
$$+ 10\, a2\, \alpha\, a3\, \beta - 2\, \beta^3 a2\, a3 - 4\, a2\, a3 + \alpha^4 a3^2 + a3^2 \beta^2 - 2\, \alpha^2 a3^2 \beta \big) \big(a3^2 \beta^2$$
$$\left.\left. - 4\, a2\, a3 + 2\, a2\, \alpha\, a3\, \beta + a2^2 \alpha^2 \big) \right)^{1/2} \right) \left(-\alpha^2 a3 - a2\, \beta^2 + \alpha^2 \beta\, a2 + \alpha\, \beta^2 a3 \right)$$
$$< 0,\ -a1\, \alpha - a2\, \beta - 2\, \sqrt{a1}\, \sqrt{a2} < 0,\ -a1\, \alpha - a2\, \beta + 2\, \sqrt{a1}\, \sqrt{a2} < 0 \}$$

```
> #
  # Lyapunov function
  #
  t1:=time():
  rl:=LyapunovFunction(eqs,variables,params,q2):
  time()-t1;
```

$$5.615 \qquad \textbf{(2.10)}$$

```
> nops(rl);
```

$$21 \qquad \textbf{(2.11)}$$

```
> # Let us consider one of the cases:
> r2:=rl[5];
```

$$r2 := \left[\frac{1}{(\alpha + 1 + \beta)\, (\alpha^2 - \alpha - \beta\, \alpha - \beta + 1 + \beta^2)} \Big(a1\, x3 + a3\, x2 + a2\, x1 - a3\, \alpha^2\, l1 \right. \qquad \textbf{(2.12)}$$
$$- a1\, l1\, \beta^2 - a1\, \alpha^2\, l2$$
$$- a2\, \alpha^2\, l3 \ln\left(\frac{x1\, (\alpha + 1 + \beta)\, (\alpha^2 - \alpha - \beta\, \alpha - \beta + 1 + \beta^2)}{-l1\, \beta\, \alpha + l1 + \alpha^2\, l3 - \alpha\, l2 + \beta^2\, l2 - \beta\, l3} \right)$$
$$- a2\, \beta^2\, l2 \ln\left(\frac{x1\, (\alpha + 1 + \beta)\, (\alpha^2 - \alpha - \beta\, \alpha - \beta + 1 + \beta^2)}{-l1\, \beta\, \alpha + l1 + \alpha^2\, l3 - \alpha\, l2 + \beta^2\, l2 - \beta\, l3} \right)$$
$$- a1\, l1\, \beta^2 \ln\left(\frac{x3\, (\alpha + 1 + \beta)\, (\alpha^2 - \alpha - \beta\, \alpha - \beta + 1 + \beta^2)}{\alpha^2\, l2 - \beta\, \alpha\, l3 - \alpha\, l1 - \beta\, l2 + l1\, \beta^2 + l3} \right)$$

$$- a1\,\alpha^2\, l2 \ln\left(\frac{x3\,(\alpha+1+\beta)\,(\alpha^2-\alpha-\beta\alpha-\beta+1+\beta^2)}{\alpha^2\, l2-\beta\alpha\, l3-\alpha\, l1-\beta\, l2+l1\,\beta^2+l3}\right)$$

$$+ a3\,\alpha\, l3 \ln\left(\frac{x2\,(\alpha+1+\beta)\,(\alpha^2-\alpha-\beta\alpha-\beta+1+\beta^2)}{l3\,\beta^2-\alpha\, l3+\alpha^2\, l1-l2\,\beta\alpha+l2-\beta\, l1}\right)$$

$$+ a2\,\alpha\, l2 \ln\left(\frac{x1\,(\alpha+1+\beta)\,(\alpha^2-\alpha-\beta\alpha-\beta+1+\beta^2)}{-l1\,\beta\alpha+l1+\alpha^2\, l3-\alpha\, l2+\beta^2\, l2-\beta\, l3}\right)$$

$$+ a2\,\beta\, l3 \ln\left(\frac{x1\,(\alpha+1+\beta)\,(\alpha^2-\alpha-\beta\alpha-\beta+1+\beta^2)}{-l1\,\beta\alpha+l1+\alpha^2\, l3-\alpha\, l2+\beta^2\, l2-\beta\, l3}\right)$$

$$+ a1\,\alpha\, l1 \ln\left(\frac{x3\,(\alpha+1+\beta)\,(\alpha^2-\alpha-\beta\alpha-\beta+1+\beta^2)}{\alpha^2\, l2-\beta\alpha\, l3-\alpha\, l1-\beta\, l2+l1\,\beta^2+l3}\right)$$

$$+ a1\,\beta\, l2 \ln\left(\frac{x3\,(\alpha+1+\beta)\,(\alpha^2-\alpha-\beta\alpha-\beta+1+\beta^2)}{\alpha^2\, l2-\beta\alpha\, l3-\alpha\, l1-\beta\, l2+l1\,\beta^2+l3}\right)$$

$$- a3\,\alpha^2\, l1 \ln\left(\frac{x2\,(\alpha+1+\beta)\,(\alpha^2-\alpha-\beta\alpha-\beta+1+\beta^2)}{l3\,\beta^2-\alpha\, l3+\alpha^2\, l1-l2\,\beta\alpha+l2-\beta\, l1}\right)$$

$$- a3\, l3\,\beta^2 \ln\left(\frac{x2\,(\alpha+1+\beta)\,(\alpha^2-\alpha-\beta\alpha-\beta+1+\beta^2)}{l3\,\beta^2-\alpha\, l3+\alpha^2\, l1-l2\,\beta\alpha+l2-\beta\, l1}\right)$$

$$+ a3\,\beta\, l1 \ln\left(\frac{x2\,(\alpha+1+\beta)\,(\alpha^2-\alpha-\beta\alpha-\beta+1+\beta^2)}{l3\,\beta^2-\alpha\, l3+\alpha^2\, l1-l2\,\beta\alpha+l2-\beta\, l1}\right)$$

$$+ a2\, l1\,\beta\alpha \ln\left(\frac{x1\,(\alpha+1+\beta)\,(\alpha^2-\alpha-\beta\alpha-\beta+1+\beta^2)}{-l1\,\beta\alpha+l1+\alpha^2\, l3-\alpha\, l2+\beta^2\, l2-\beta\, l3}\right)$$

$$+ a1\,\beta\alpha\, l3 \ln\left(\frac{x3\,(\alpha+1+\beta)\,(\alpha^2-\alpha-\beta\alpha-\beta+1+\beta^2)}{\alpha^2\, l2-\beta\alpha\, l3-\alpha\, l1-\beta\, l2+l1\,\beta^2+l3}\right)$$

$$+ a3\, l2\,\beta\alpha \ln\left(\frac{x2\,(\alpha+1+\beta)\,(\alpha^2-\alpha-\beta\alpha-\beta+1+\beta^2)}{l3\,\beta^2-\alpha\, l3+\alpha^2\, l1-l2\,\beta\alpha+l2-\beta\, l1}\right) + a2\,\beta\, l3$$

$$- a2\,\beta^2\, l2 + a2\,\alpha\, l2 - a2\,\alpha^2\, l3 + a3\,\alpha\, l3 + a3\,\beta\, l1 + a1\,\alpha\, l1 + a1\,\beta\, l2$$

$$+ a1\,\beta^3\, x3 + a3\,\beta^3\, x2 + a2\, x1\,\beta^3 - a3\, l3\,\beta^2 - 3\,\alpha\, a3\,\beta\, x2 - 3\,\alpha\, a2\, x1\,\beta$$

$$- 3\,\alpha\, a1\, x3\,\beta - a3\, l2 - a2\, l1 - a1\, l3 + \alpha^3\, a2\, x1 + \alpha^3\, a3\, x2 + \alpha^3\, a1\, x3$$

$$- a3\, l2 \ln\left(\frac{x2\,(\alpha+1+\beta)\,(\alpha^2-\alpha-\beta\alpha-\beta+1+\beta^2)}{l3\,\beta^2-\alpha\, l3+\alpha^2\, l1-l2\,\beta\alpha+l2-\beta\, l1}\right)$$

$$-a2\,l1 \ln\left(\frac{x1\,(\alpha+1+\beta)\,(\alpha^2-\alpha-\beta\,\alpha-\beta+1+\beta^2)}{-l1\,\beta\,\alpha+l1+\alpha^2\,l3-\alpha\,l2+\beta^2\,l2-\beta\,l3}\right)+a3\,l2\,\beta\,\alpha$$

$$+a2\,l1\,\beta\,\alpha+a1\,\beta\,\alpha\,l3$$

$$-a1\,l3 \ln\left(\frac{x3\,(\alpha+1+\beta)\,(\alpha^2-\alpha-\beta\,\alpha-\beta+1+\beta^2)}{\alpha^2\,l2-\beta\,\alpha\,l3-\alpha\,l1-\beta\,l2+l1\,\beta^2+l3}\right)\Bigg),\ \{-a3+a1=0,$$

$$-a3+a2=0,\ \alpha-2+\beta=0\}\Bigg]$$

```
> # And chosing numerical values for the different free
  parameters and variables
  # and using the solutions on the conditions:
> vals:={l1=1,l2=1,l3=1,alpha=3,a3=1};
```

$$vals := \{a3=1,\ \alpha=3,\ l1=1,\ l2=1,\ l3=1\} \qquad \textbf{(2.13)}$$

```
> vals2:=vals union evalf(subs(vals,r2[2]));
```

$$vals2 := \{a3=1,\ \alpha=3,\ l1=1,\ l2=1,\ l3=1,\ -1.+a1=0.,\ -1.+a2=0.,\ 1.+\beta=0.\} \qquad \textbf{(2.14)}$$

```
> vals3:=solve(map(z->if type(z,`=`) then z fi,vals2));
```

$$vals3 := \{a1=1.,\ a2=1.,\ a3=1.,\ \alpha=3.,\ \beta=-1.,\ l1=1.,\ l2=1.,\ l3=1.\} \qquad \textbf{(2.15)}$$

```
> # The obtained Lyapunov function
> llf:=evalf(subs(vals3,r2[1]));
```

$$llf := 1.000000000\,x1+1.000000000\,x2+1.000000000\,x3-1.000000000 \qquad \textbf{(2.16)}$$
$$-0.3333333334\ln(3.000000000\,x1)$$
$$-0.3333333334\ln(3.000000000\,x3)$$
$$-0.3333333334\ln(3.000000000\,x2)$$

```
> # Now testing its validity by integrating the ODE system:
> diff_sys:=simplify(subs(vals3,convert(subs(x1=x1(t),x2=x2(t),
  x3=x3(t),{Diff(x1,t)=eqs[1],Diff(x2,t)=eqs[2],Diff(x3,t)=eqs
  [3]}),diff)));
```

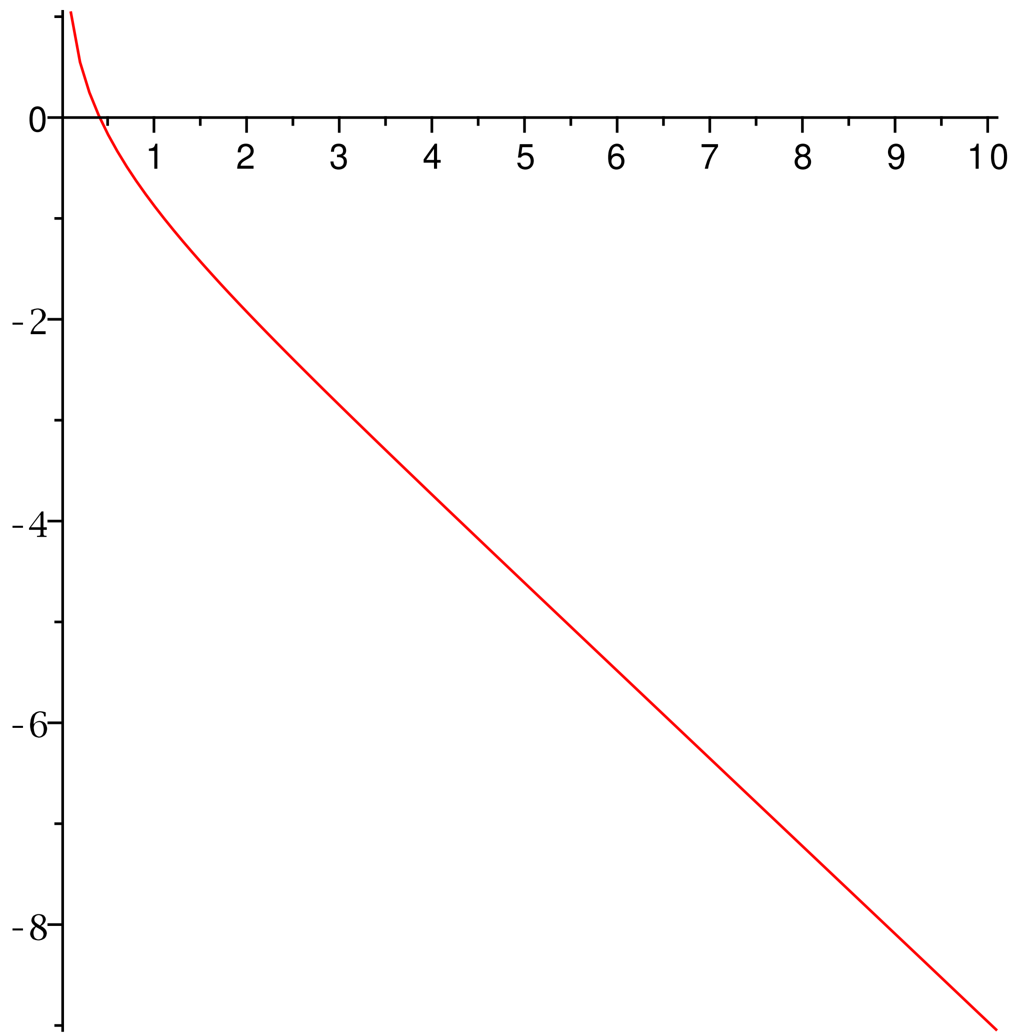
$$diff\_sys := \left\{\frac{d}{dt}\,x1(t)=x1(t)-1.\,x1(t)^2-3.\,x1(t)\,x2(t)+x1(t)\,x3(t),\ \frac{d}{dt}\,x2(t) \right. \qquad \textbf{(2.17)}$$
$$=x2(t)+x1(t)\,x2(t)-1.\,x2(t)^2-3.\,x2(t)\,x3(t),\ \frac{d}{dt}\,x3(t)=x3(t)$$
$$\left.-3.\,x1(t)\,x3(t)+x2(t)\,x3(t)-1.\,x3(t)^2\right\}$$

```
> dsn := dsolve(diff_sys union {x1(0)=5.0,x2(0)=5.0,x3(0)=5.0},
  numeric);
```

$$dsn := \mathbf{proc}(x\_rkf45)\ \ldots\ \mathbf{end\ proc} \qquad \textbf{(2.18)}$$

```
> Digits:=20:
```

```
t_ini:=0.0:
t_fin:=10.0:
nt:=100:
dt:=(t_fin-t_ini)/(nt-1):

gr:=1:
tt:=t_ini:
for i from 1 to nt do
    vv:=subs(dsn(tt),[x1(t),x2(t),x3(t)]):
    tt:=tt+dt:
    gr:=gr,[tt,evalf(log10(subs(x1=vv[1],x2=vv[2],x3=vv[3],
llf)))]
od:
gr:=[gr]:
gr:=gr[2..nops(gr)]:

Digits:=10:
> plot(gr);
```



```
> # Using the values obtained above to ilustrate the numerical
  determination of the Lyapunov function:
> eqs2:=subs(vals3,eqs);
```

$$eqs2 := [1.\, x1 - x1\, (x1 + 3.\, x2 - 1.\, x3),\ 1.\, x2 - x2\, (-1.\, x1 + x2 + 3.\, x3),\ 1.\, x3 - x3\, (3.\, x1 - 1.\, x2 + x3)] \tag{2.19}$$

```
> q3:=evalf(subs(vals3,q2));
```

$$q3 := [0.3333333333,\ 0.3333333333,\ 0.3333333333] \tag{2.20}$$

```
> t1:=time():
  llf:=LyapunovFunction(eqs2,variables,params,q3);
  time()-t1;
```

$$llf := 0.4999999998\, x3 - 0.1666666666 \ln(3.000000000\, x3) - 0.4999999998 + 0.4999999998\, x1 - 0.1666666666 \ln(3.000000000\, x1) + 0.4999999998\, x2 - 0.1666666666 \ln(3.000000000\, x2)$$

$$0.009 \tag{2.21}$$

```
> # Ans testing again with a numerical solution
> diff_sys2:=subs(vals,diff_sys);
```

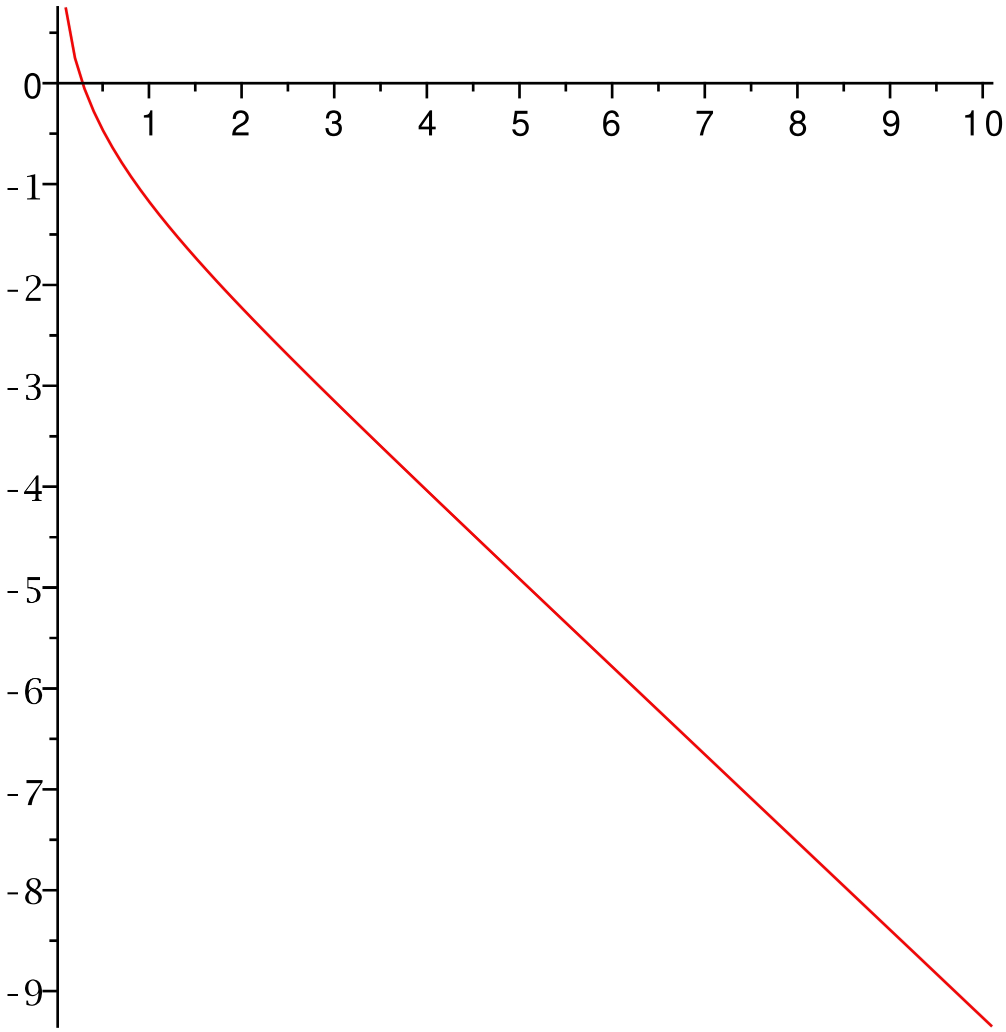
$$diff\_sys2 := \left\{ \frac{d}{dt}\, x1(t) = x1(t) - 1.\, x1(t)^2 - 3.\, x1(t)\, x2(t) + x1(t)\, x3(t),\ \frac{d}{dt}\, x2(t) = x2(t) + x1(t)\, x2(t) - 1.\, x2(t)^2 - 3.\, x2(t)\, x3(t),\ \frac{d}{dt}\, x3(t) = x3(t) - 3.\, x1(t)\, x3(t) + x2(t)\, x3(t) - 1.\, x3(t)^2 \right\} \tag{2.22}$$

```
> dsn := dsolve(diff_sys2 union {x1(0)=5.0,x2(0)=5.0,x3(0)=5.0}
  , numeric);
```

$$dsn := \mathbf{proc}(x\_rkf45)\ \ldots\ \mathbf{end\ proc} \tag{2.23}$$

```
> Digits:=20:

  t_ini:=0.0:
  t_fin:=10.0:
  nt:=100:

  dt:=(t_fin-t_ini)/(nt-1):
  gr:=1:
  tt:=t_ini:
  for i from 1 to nt do
      vv:=subs(dsn(tt),[x1(t),x2(t),x3(t)]):
      tt:=tt+dt:
      gr:=gr,[tt,evalf(log10(subs(x1=vv[1],x2=vv[2],x3=vv[3],
  llf)))]
  od:
  gr:=[gr]:
  gr:=gr[2..nops(gr)]:

  Digits:=10:
```

```
> # And here the the evolution of the Lyapunov function with
  time
> plot(gr);
```



Generalized mass action system

D.H. Irving, E.O. Voit, M.A. Savageau, Analysis of complex dynamic networks with ESSYNS, in: E.O. Voit (Ed.), Canonical Non-Linear Modelling S-systems Approach to Understanding Complexity, Van Nostrand Reinhold, 1991.

```
> # Defining the system
> eqs:=[l1*x1-alpha1*x1^b3*x2^b1,-l2*x2+alpha2*x1^b1*x3^b2,-l3*
  x3+alpha3*x2^b1];
  variables:=[x1,x2,x3];
  params:={l1,l2,l3,b1,b2,b3,alpha1,alpha2,alpha3};
```

$$eqs := \left[l1\, x1 - \alpha 1\, x1^{b3}\, x2^{b1},\ -l2\, x2 + \alpha 2\, x1^{b1}\, x3^{b2},\ -l3\, x3 + \alpha 3\, x2^{b1}\right]$$

$$variables := [x1, x2, x3]$$

$$params := \{\alpha 1, \alpha 2, \alpha 3, b1, b2, b3, l1, l2, l3\} \tag{3.1}$$

```
> # Determining the fixed points of the original system:
uu:=[solve({op(eqs)},{op(variables)})];
```

$$uu := \left[\left\{ x1 = e^{\frac{\ln\left(\frac{l1}{\alpha 1}\right) - b1\,b2\ln\left(\frac{l1}{\alpha 1}\right) + b1\,b2\ln\left(\frac{l3}{\alpha 3}\right) + b1\ln\left(\frac{l2}{\alpha 2}\right)}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}}, x2 = e^{-\frac{-b1\ln\left(\frac{l1}{\alpha 1}\right) - b2\ln\left(\frac{l3}{\alpha 3}\right) + b2\,b3\ln\left(\frac{l3}{\alpha 3}\right) - \ln\left(\frac{l2}{\alpha 2}\right) + \ln\left(\frac{l2}{\alpha 2}\right)b3}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}}, x3 = e^{-\frac{-b1^2\ln\left(\frac{l1}{\alpha 1}\right) + b1^2\ln\left(\frac{l3}{\alpha 3}\right) - \ln\left(\frac{l3}{\alpha 3}\right) - b1\ln\left(\frac{l2}{\alpha 2}\right) + b3\ln\left(\frac{l3}{\alpha 3}\right) + b3\,b1\ln\left(\frac{l2}{\alpha 2}\right)}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}} \right\}, \{x1 = 0, x2 = 0, x3 = 0\} \right] \tag{3.2}$$

```
> # Chosing a fixed point inside the positive orthant
q:=uu[1];
```

$$q := \left\{ x1 = e^{\frac{\ln\left(\frac{l1}{\alpha 1}\right) - b1\,b2\ln\left(\frac{l1}{\alpha 1}\right) + b1\,b2\ln\left(\frac{l3}{\alpha 3}\right) + b1\ln\left(\frac{l2}{\alpha 2}\right)}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}}, x2 = e^{-\frac{-b1\ln\left(\frac{l1}{\alpha 1}\right) - b2\ln\left(\frac{l3}{\alpha 3}\right) + b2\,b3\ln\left(\frac{l3}{\alpha 3}\right) - \ln\left(\frac{l2}{\alpha 2}\right) + \ln\left(\frac{l2}{\alpha 2}\right)b3}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}}, x3 = e^{-\frac{-b1^2\ln\left(\frac{l1}{\alpha 1}\right) + b1^2\ln\left(\frac{l3}{\alpha 3}\right) - \ln\left(\frac{l3}{\alpha 3}\right) - b1\ln\left(\frac{l2}{\alpha 2}\right) + b3\ln\left(\frac{l3}{\alpha 3}\right) + b3\,b1\ln\left(\frac{l2}{\alpha 2}\right)}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}} \right\} \tag{3.3}$$

```
> q2:=subs(q,variables);
```

$$q2 := \left[e^{\frac{\ln\left(\frac{l1}{\alpha 1}\right) - b1\,b2\ln\left(\frac{l1}{\alpha 1}\right) + b1\,b2\ln\left(\frac{l3}{\alpha 3}\right) + b1\ln\left(\frac{l2}{\alpha 2}\right)}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}}, e^{-\frac{-b1\ln\left(\frac{l1}{\alpha 1}\right) - b2\ln\left(\frac{l3}{\alpha 3}\right) + b2\,b3\ln\left(\frac{l3}{\alpha 3}\right) - \ln\left(\frac{l2}{\alpha 2}\right) + \ln\left(\frac{l2}{\alpha 2}\right)b3}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}}, e^{-\frac{-b1^2\ln\left(\frac{l1}{\alpha 1}\right) + b1^2\ln\left(\frac{l3}{\alpha 3}\right) - \ln\left(\frac{l3}{\alpha 3}\right) - b1\ln\left(\frac{l2}{\alpha 2}\right) + b3\ln\left(\frac{l3}{\alpha 3}\right) + b3\,b1\ln\left(\frac{l2}{\alpha 2}\right)}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}} \right] \tag{3.4}$$

```
> # Determing the possible Lyapunov functions
t1:=time():
rl:=LyapunovFunction(eqs,variables,params,q2):
time()-t1;
```

$$4.616 \tag{3.5}$$

```
> nops(rl);
```

(3.6)

$$15 \tag{3.6}$$

```
> # Eliminating solutions with b1=0:
> rl2:=map(z->if not(has(z,b1=0)) then z fi,rl):
> nops(rl2);
```

$$10 \tag{3.7}$$

```
> # Now considering the first case
> rl3:=rl2[1];
```

$$rl3 := \tag{3.8}$$

$$\Bigg[\Bigg(a1\, x1^{b1}\, x3^{b2} \left(\frac{l1}{\alpha 1}\right)^{\frac{b1}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}}$$

$$\left(\frac{l3}{\alpha 3}\right)^{\frac{b2}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}} \left(\frac{l2}{\alpha 2}\right)^{\frac{1}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}} x3$$

$$\left(\frac{l1}{\alpha 1}\right)^{\frac{b1^2}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}} \left(\frac{l3}{\alpha 3}\right)^{\frac{1}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}}$$

$$\left(\frac{l2}{\alpha 2}\right)^{\frac{b1}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}} x1 \left(\frac{l1}{\alpha 1}\right)^{\frac{1}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}}$$

$$\left(\frac{l3}{\alpha 3}\right)^{\frac{b1\, b2}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}}$$

$$- a1 \left(\frac{l3}{\alpha 3}\right)^{\frac{b2\,b3}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}} \left(\frac{l2}{\alpha 2}\right)^{\frac{b3}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}}$$

$$\left(\frac{1}{\left(\frac{l1}{\alpha 1}\right)^{\frac{b1\,b2}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}}} \left(\left(\frac{l1}{\alpha 1}\right)^{\frac{1}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}}\right.\right.$$

$$\left.\left.\left(\frac{l3}{\alpha 3}\right)^{\frac{b1\,b2}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}} \left(\frac{l2}{\alpha 2}\right)^{\frac{b1}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}}\right)\right)$$

$$b1$$

$$\left(\left(\left(\frac{l1}{\alpha 1}\right)^{\frac{b1^2}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}} \left(\frac{l3}{\alpha 3}\right)^{\frac{1}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}}\right.\right.$$

$$\left.\left(\frac{l2}{\alpha 2}\right)^{\frac{b1}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}}\right) \Big/$$

$$\left(\left(\frac{l3}{\alpha 3}\right)^{\frac{b1^2}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\left(\frac{l3}{\alpha 3}\right)^{\frac{b3}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\right.$$

$$\left.\left(\frac{l2}{\alpha 2}\right)^{\frac{b3\,b1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\right)\Bigg)$$

$$b2$$

$$\ln\left(\left(x1^{b1}\,x3^{b2}\,\mathrm{e}^{-\frac{-b1\ln\left(\frac{l1}{\alpha 1}\right)-b2\ln\left(\frac{l3}{\alpha 3}\right)+b2\,b3\ln\left(\frac{l3}{\alpha 3}\right)-\ln\left(\frac{l2}{\alpha 2}\right)+\ln\left(\frac{l2}{\alpha 2}\right)b3}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\right)\Bigg/\right.$$

$$\left(x2\left(\mathrm{e}^{\frac{\ln\left(\frac{l1}{\alpha 1}\right)-b1\,b2\ln\left(\frac{l1}{\alpha 1}\right)+b1\,b2\ln\left(\frac{l3}{\alpha 3}\right)+b1\ln\left(\frac{l2}{\alpha 2}\right)}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\right)^{b1}\right.$$

$$\left.\left.\left(\mathrm{e}^{-\frac{-b1^2\ln\left(\frac{l1}{\alpha 1}\right)+b1^2\ln\left(\frac{l3}{\alpha 3}\right)-\ln\left(\frac{l3}{\alpha 3}\right)-b1\ln\left(\frac{l2}{\alpha 2}\right)+b3\ln\left(\frac{l3}{\alpha 3}\right)+b3\,b1\ln\left(\frac{l2}{\alpha 2}\right)}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\right)^{b2}\right)\right)$$

$$x2\,x3\left(\frac{l1}{\alpha 1}\right)^{\frac{b1^2}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\left(\frac{l3}{\alpha 3}\right)^{\frac{1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}$$

$$\left(\frac{l2}{\alpha 2}\right)^{\frac{b1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}} x1 \left(\frac{l1}{\alpha 1}\right)^{\frac{1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}$$

$$\left(\frac{l3}{\alpha 3}\right)^{\frac{b1\,b2}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}$$

$$-a1 \left(\frac{l3}{\alpha 3}\right)^{\frac{b2\,b3}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}} \left(\frac{l2}{\alpha 2}\right)^{\frac{b3}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}$$

$$\left(\frac{1}{\left(\frac{l1}{\alpha 1}\right)^{\frac{b1\,b2}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}} \left(\left(\frac{l1}{\alpha 1}\right)^{\frac{1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\right.\right.$$

$$\left.\left.\left(\frac{l3}{\alpha 3}\right)^{\frac{b1\,b2}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}} \left(\frac{l2}{\alpha 2}\right)^{\frac{b1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\right)\right)$$

$b1$

$$\Bigg(\Bigg(\bigg(\frac{l1}{\alpha 1}\bigg)^{\frac{b1^2}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}} \bigg(\frac{l3}{\alpha 3}\bigg)^{\frac{1}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}}$$

$$\bigg(\frac{l2}{\alpha 2}\bigg)^{\frac{b1}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}}\Bigg) \Bigg/$$

$$\Bigg(\bigg(\frac{l3}{\alpha 3}\bigg)^{\frac{b1^2}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}} \bigg(\frac{l3}{\alpha 3}\bigg)^{\frac{b3}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}}$$

$$\bigg(\frac{l2}{\alpha 2}\bigg)^{\frac{b3\, b1}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}}\Bigg)\Bigg)$$

$$b2$$

$$x2\, x3 \bigg(\frac{l1}{\alpha 1}\bigg)^{\frac{b1^2}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}} \bigg(\frac{l3}{\alpha 3}\bigg)^{\frac{1}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}}$$

$$\bigg(\frac{l2}{\alpha 2}\bigg)^{\frac{b1}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}} x1 \bigg(\frac{l1}{\alpha 1}\bigg)^{\frac{1}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}}$$

$$\left(\frac{l3}{\alpha 3}\right)^{\frac{b1\,b2}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}$$

$$+\,a2\,x2^{b1}\,x2\left(\frac{l1}{\alpha 1}\right)^{\frac{b1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}$$

$$\left(\frac{l3}{\alpha 3}\right)^{\frac{b2}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\left(\frac{l2}{\alpha 2}\right)^{\frac{1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}$$

$$\left(\frac{l1}{\alpha 1}\right)^{\frac{b1^2}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\left(\frac{l3}{\alpha 3}\right)^{\frac{1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}$$

$$\left(\frac{l2}{\alpha 2}\right)^{\frac{b1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}x1\left(\frac{l1}{\alpha 1}\right)^{\frac{1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}$$

$$\left(\frac{l3}{\alpha 3}\right)^{\frac{b1\,b2}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}$$

$$- a2 \left(\frac{l3}{\alpha 3}\right)^{\frac{b1^2}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}} \left(\frac{l3}{\alpha 3}\right)^{\frac{b3}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}}$$

$$\left(\frac{l2}{\alpha 2}\right)^{\frac{b3\,b1}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}}$$

$$\Bigg(\Bigg(\left(\frac{l1}{\alpha 1}\right)^{\frac{b1}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}} \left(\frac{l3}{\alpha 3}\right)^{\frac{b2}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}}$$

$$\left(\frac{l2}{\alpha 2}\right)^{\frac{1}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}}\Bigg) \Bigg/$$

$$\Bigg(\left(\frac{l3}{\alpha 3}\right)^{\frac{b2\,b3}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}} \left(\frac{l2}{\alpha 2}\right)^{\frac{b3}{-1 + b1\,b2 + b3 - b3\,b1\,b2 + b1^2}}\Bigg)\Bigg)$$

$$b1$$

$$\ln\Bigg(\Bigg(x2^{b1}$$

$$e^{-\frac{-b1^2 \ln\left(\frac{l1}{\alpha 1}\right) + b1^2 \ln\left(\frac{l3}{\alpha 3}\right) - \ln\left(\frac{l3}{\alpha 3}\right) - b1 \ln\left(\frac{l2}{\alpha 2}\right) + b3 \ln\left(\frac{l3}{\alpha 3}\right) + b3\, b1 \ln\left(\frac{l2}{\alpha 2}\right)}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}}\Bigg) \Bigg/$$

$$\left(x3 \left(e^{-\frac{-b1 \ln\left(\frac{l1}{\alpha 1}\right) - b2 \ln\left(\frac{l3}{\alpha 3}\right) + b2\, b3 \ln\left(\frac{l3}{\alpha 3}\right) - \ln\left(\frac{l2}{\alpha 2}\right) + \ln\left(\frac{l2}{\alpha 2}\right) b3}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}}\right)^{b1}\right)\Bigg)$$

$$x2 \left(\frac{l1}{\alpha 1}\right)^{\frac{b1}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}} \left(\frac{l3}{\alpha 3}\right)^{\frac{b2}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}}$$

$$\left(\frac{l2}{\alpha 2}\right)^{\frac{1}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}} x3\, x1 \left(\frac{l1}{\alpha 1}\right)^{\frac{1}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}}$$

$$\left(\frac{l3}{\alpha 3}\right)^{\frac{b1\, b2}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}}$$

$$- a2 \left(\frac{l3}{\alpha 3}\right)^{\frac{b1^2}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}} \left(\frac{l3}{\alpha 3}\right)^{\frac{b3}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}}$$

$$\left(\frac{l2}{\alpha 2}\right)^{\frac{b3\,b1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}$$

$$\left(\left(\left(\frac{l1}{\alpha 1}\right)^{\frac{b1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\left(\frac{l3}{\alpha 3}\right)^{\frac{b2}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\right.\right.$$

$$\left.\left(\frac{l2}{\alpha 2}\right)^{\frac{1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\right)\Bigg/$$

$$\left.\left(\left(\frac{l3}{\alpha 3}\right)^{\frac{b2\,b3}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\left(\frac{l2}{\alpha 2}\right)^{\frac{b3}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\right)\right)$$

$$b1$$

$$x2\left(\frac{l1}{\alpha 1}\right)^{\frac{b1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\left(\frac{l3}{\alpha 3}\right)^{\frac{b2}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}$$

$$\left(\frac{l2}{\alpha 2}\right)^{\frac{1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}x3\,x1\left(\frac{l1}{\alpha 1}\right)^{\frac{1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}$$

$$\left(\frac{l3}{\alpha 3}\right)^{\frac{b1\,b2}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}$$

$$+a3\,x1^{b3}\,x2^{b1}\,x2\left(\frac{l1}{\alpha 1}\right)^{\frac{b1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}$$

$$\left(\frac{l3}{\alpha 3}\right)^{\frac{b2}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\left(\frac{l2}{\alpha 2}\right)^{\frac{1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}x3$$

$$\left(\frac{l1}{\alpha 1}\right)^{\frac{b1^2}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\left(\frac{l3}{\alpha 3}\right)^{\frac{1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}$$

$$\left(\frac{l2}{\alpha 2}\right)^{\frac{b1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\left(\frac{l1}{\alpha 1}\right)^{\frac{1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}$$

$$\left(\frac{l3}{\alpha 3}\right)^{\frac{b1\,b2}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}$$

$$- a3 \left(\frac{l1}{\alpha 1} \right)^{\frac{b1\, b2}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}}$$

$$\left(\frac{1}{\left(\frac{l1}{\alpha 1} \right)^{\frac{b1\, b2}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}}} \left(\left(\frac{l1}{\alpha 1} \right)^{\frac{1}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}} \right.\right.$$

$$\left.\left. \left(\frac{l3}{\alpha 3} \right)^{\frac{b1\, b2}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}} \left(\frac{l2}{\alpha 2} \right)^{\frac{b1}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}} \right) \right)$$

$$b3$$

$$\left(\left(\left(\frac{l1}{\alpha 1} \right)^{\frac{b1}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}} \left(\frac{l3}{\alpha 3} \right)^{\frac{b2}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}} \right.\right.$$

$$\left. \left(\frac{l2}{\alpha 2} \right)^{\frac{1}{-1 + b1\, b2 + b3 - b3\, b1\, b2 + b1^2}} \right) \Big/$$

$$\left(\left(\frac{l3}{\alpha 3}\right)^{\frac{b2\,b3}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\left(\frac{l2}{\alpha 2}\right)^{\frac{b3}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\right)\right)$$

$$b1\,\ln\left(\left(x1^{b3}\,x2^{b1}\,\mathrm{e}^{\frac{\ln\left(\frac{l1}{\alpha 1}\right)-b1\,b2\ln\left(\frac{l1}{\alpha 1}\right)+b1\,b2\ln\left(\frac{l3}{\alpha 3}\right)+b1\ln\left(\frac{l2}{\alpha 2}\right)}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\right)\right/$$

$$\left(x1\left(\mathrm{e}^{\frac{\ln\left(\frac{l1}{\alpha 1}\right)-b1\,b2\ln\left(\frac{l1}{\alpha 1}\right)+b1\,b2\ln\left(\frac{l3}{\alpha 3}\right)+b1\ln\left(\frac{l2}{\alpha 2}\right)}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\right)^{b3}\right.$$

$$\left.\left.\left(\mathrm{e}^{-\frac{-b1\ln\left(\frac{l1}{\alpha 1}\right)-b2\ln\left(\frac{l3}{\alpha 3}\right)+b2\,b3\ln\left(\frac{l3}{\alpha 3}\right)-\ln\left(\frac{l2}{\alpha 2}\right)+\ln\left(\frac{l2}{\alpha 2}\right)b3}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\right)^{b1}\right)\right)$$

$$x2\left(\frac{l1}{\alpha 1}\right)^{\frac{b1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\left(\frac{l3}{\alpha 3}\right)^{\frac{b2}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}$$

$$\left(\frac{l2}{\alpha 2}\right)^{\frac{1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}x3\left(\frac{l1}{\alpha 1}\right)^{\frac{b1^2}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}$$

$$\left(\frac{l3}{\alpha 3}\right)^{\frac{1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}} x1$$

$$-a3\left(\frac{l1}{\alpha 1}\right)^{\frac{b1\,b2}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}$$

$$\left(\frac{1}{\left(\frac{l1}{\alpha 1}\right)^{\frac{b1\,b2}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}}\left(\left(\frac{l1}{\alpha 1}\right)^{\frac{1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\right.\right.$$

$$\left.\left.\left(\frac{l3}{\alpha 3}\right)^{\frac{b1\,b2}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\left(\frac{l2}{\alpha 2}\right)^{\frac{b1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\right)\right)$$

$$b3$$

$$\left(\left(\left(\frac{l1}{\alpha 1}\right)^{\frac{b1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\left(\frac{l3}{\alpha 3}\right)^{\frac{b2}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\right.\right.$$

$$\left(\frac{l2}{\alpha 2}\right)^{\frac{1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^{2}}}\Bigg) \Bigg/$$

$$\left(\left(\frac{l3}{\alpha 3}\right)^{\frac{b2\,b3}{-1+b1\,b2+b3-b3\,b1\,b2+b1^{2}}}\left(\frac{l2}{\alpha 2}\right)^{\frac{b3}{-1+b1\,b2+b3-b3\,b1\,b2+b1^{2}}}\right)\Bigg)$$

$b1$

$$x2\left(\frac{l1}{\alpha 1}\right)^{\frac{b1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^{2}}}\left(\frac{l3}{\alpha 3}\right)^{\frac{b2}{-1+b1\,b2+b3-b3\,b1\,b2+b1^{2}}}$$

$$\left(\frac{l2}{\alpha 2}\right)^{\frac{1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^{2}}} x3\left(\frac{l1}{\alpha 1}\right)^{\frac{b1^{2}}{-1+b1\,b2+b3-b3\,b1\,b2+b1^{2}}}$$

$$\left(\frac{l3}{\alpha 3}\right)^{\frac{1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^{2}}} x1\Bigg) \Bigg/$$

$$\Bigg(x2\left(\frac{l1}{\alpha 1}\right)^{\frac{b1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^{2}}}\left(\frac{l3}{\alpha 3}\right)^{\frac{b2}{-1+b1\,b2+b3-b3\,b1\,b2+b1^{2}}}$$

$$\left(\frac{l2}{\alpha 2}\right)^{\frac{1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^{2}}} x3\left(\frac{l1}{\alpha 1}\right)^{\frac{b1^{2}}{-1+b1\,b2+b3-b3\,b1\,b2+b1^{2}}}$$

$$\left(\frac{l3}{\alpha 3}\right)^{\frac{1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\left(\frac{l2}{\alpha 2}\right)^{\frac{b1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}x1$$

$$\left(\frac{l1}{\alpha 1}\right)^{\frac{1}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\left(\frac{l3}{\alpha 3}\right)^{\frac{b1\,b2}{-1+b1\,b2+b3-b3\,b1\,b2+b1^2}}\Bigg),$$

$$\left\{\frac{a2\,b1\,\alpha 2+a1\,b2\,\alpha 3+2\sqrt{a1}\sqrt{a2}\sqrt{\alpha 2\,\alpha 3}}{a2\,\alpha 2}=0,\ -\frac{a1\,\alpha 1-\alpha 2\,a3}{\alpha 2}=0,\ -1+b3=0,\ -\alpha 2<0\right\}\Bigg]$$

```
> # The corresponding solution for the admissibility
  conditions:
> rl3[2];
```

$$\left\{\frac{a2\,b1\,\alpha 2+a1\,b2\,\alpha 3+2\sqrt{a1}\sqrt{a2}\sqrt{\alpha 2\,\alpha 3}}{a2\,\alpha 2}=0,\ -\frac{a1\,\alpha 1-\alpha 2\,a3}{\alpha 2}=0,\ -1+b3=0,\ -\alpha 2<0\right\} \quad \textbf{(3.9)}$$

```
> # Some manipulations to obtaing compatible values for the
  free parameters and the unknowns a's
> rl4:=map(z->if type(z,`=`) then z fi,subs(alpha2=2,rl3[2]));
```

$$rl4:=\left\{\frac{1}{2}\,\frac{2\,a2\,b1+a1\,b2\,\alpha 3+2\sqrt{a1}\sqrt{a2}\sqrt{2}\sqrt{\alpha 3}}{a2}=0,\ -1+b3=0,\ -\frac{1}{2}\,a1\,\alpha 1+a3=0\right\} \quad \textbf{(3.10)}$$

```
> rl5:=map(z->if lhs(z)<>rhs(z) then z fi,solve(rl4));
```

$$rl5:=\left\{a3=\frac{1}{2}\,a1\,\alpha 1,\ b1=-\frac{1}{2}\,\frac{\sqrt{a1}\sqrt{\alpha 3}\left(\sqrt{a1}\sqrt{\alpha 3}\,b2+2\sqrt{a2}\sqrt{2}\right)}{a2},\ b3=1\right\} \quad \textbf{(3.11)}$$

```
> # Considering a possible set of values:
> vals:={a1=1,a2=1/2,l1=1,l2=1,l3=1,alpha1=3,alpha2=2,alpha3=4,
  b1=6};
```

$$vals:=\left\{a1=1,\ a2=\frac{1}{2},\ \alpha 1=3,\ \alpha 2=2,\ \alpha 3=4,\ b1=6,\ l1=1,\ l2=1,\ l3=1\right\} \quad \textbf{(3.12)}$$

```
> # And obtainig the remaining values;
> rl6:=subs(vals,rl5);
```

(3.13)

$$rl6 := \left\{6 = -\sqrt{4}\,\left(\sqrt{4}\;b2+2\right),\; a3 = \frac{3}{2},\; b3 = 1\right\} \tag{3.13}$$

```
> vals2:=solve(rl6) union vals;
```

$$vals2 := \left\{a1 = 1,\; a2 = \frac{1}{2},\; a3 = \frac{3}{2},\; \alpha 1 = 3,\; \alpha 2 = 2,\; \alpha 3 = 4,\; b1 = 6,\; b2 = -\frac{1}{2}\sqrt{4} - \frac{3}{2},\; b3 = 1,\; l1 = 1,\; l2 = 1,\; l3 = 1\right\} \tag{3.14}$$

```
> # Verifying that they indeed satisfy the obtained solution:
> evalf(subs(vals2,rl3[2]));
```

$$\{-1.\,10^{-9} = 0.,\; 0. = 0.,\; -2. < 0.\} \tag{3.15}$$

```
> # And verifying for the Lyapunov function for given initial
  conditions
> eqsf:=evalf(subs(vals2,eqs));
```

$$eqsf := \left[x1 - 3.\,x1\,x2^6,\; -1.\,x2 + \frac{2.\,x1^6}{x3^{2.500000000}},\; -1.\,x3 + 4.\,x2^6\right] \tag{3.16}$$

```
> eqsf2:={Diff(x1,t)=eqsf[1],Diff(x2,t)=eqsf[2],Diff(x3,t)=eqsf
  [3]};
```

$$eqsf2 := \left\{\frac{\partial}{\partial t}\,x1 = x1 - 3.\,x1\,x2^6,\; \frac{\partial}{\partial t}\,x2 = -1.\,x2 + \frac{2.\,x1^6}{x3^{2.500000000}},\; \frac{\partial}{\partial t}\,x3 = -1.\,x3 + 4.\,x2^6\right\} \tag{3.17}$$

```
> diff_sys:=convert(subs(x1=x1(t),x2=x2(t),x3=x3(t),eqsf2),
  diff);
```

$$diff\_sys := \left\{\frac{d}{dt}\,x1(t) = x1(t) - 3.\,x1(t)\,x2(t)^6,\; \frac{d}{dt}\,x2(t) = -1.\,x2(t) + \frac{2.\,x1(t)^6}{x3(t)^{2.500000000}},\; \frac{d}{dt}\,x3(t) = -1.\,x3(t) + 4.\,x2(t)^6\right\} \tag{3.18}$$

```
> llf:=evalf(subs(vals2,rl3[1]));
```

$$llf := \frac{1}{x2\,x3\,x1}\left(2.251563917\left(\frac{0.4441357375\,x1^7}{x3^{1.500000000}} - 0.2220678685\ln\left(\frac{2.000000000\,x1^6}{x2\,x3^{2.500000000}}\right)x2\,x3\,x1 - 0.4996527043\,x2\,x3\,x1 + 0.2220678687\,x2^7\,x1 - 0.05551696715\ln\left(\frac{4.000000001\,x2^6}{x3}\right)x2\,x3\,x1 + 0.6662036061\,x1\,x2^7\,x3 - 0.2220678687\ln\left(3.000000001\,x2^6\right)x2\,x3\,x1\right)\right) \tag{3.19}$$

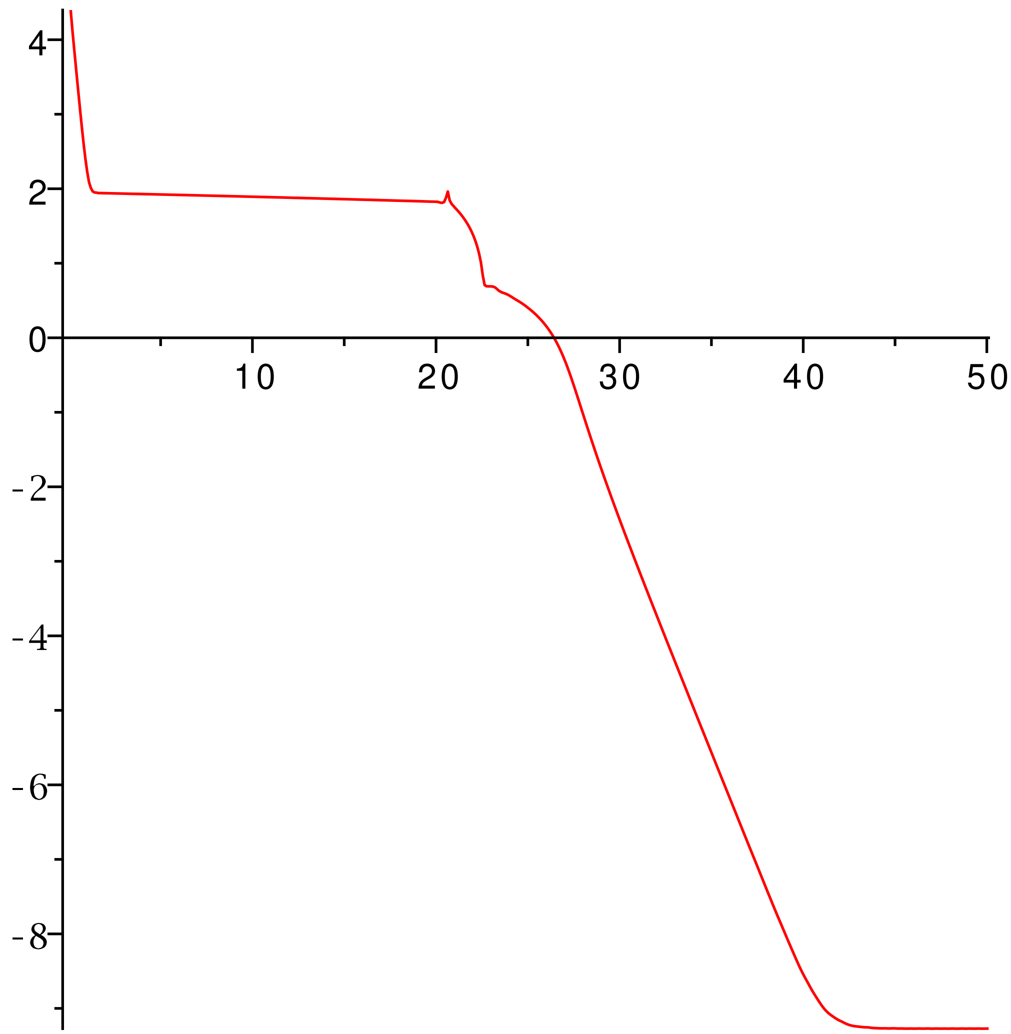
```
> diff_sys2:=evalf(subs(vals,diff_sys));
```

$$diff\_sys2 := \left\{ \frac{d}{dt}\, x1(t) = x1(t) - 3.\, x1(t)\, x2(t)^6,\ \frac{d}{dt}\, x2(t) = -1.\, x2(t) + \frac{2.\, x1(t)^6}{x3(t)^{2.500000000}},\ \frac{d}{dt}\, x3(t) = -1.\, x3(t) + 4.\, x2(t)^6 \right\} \qquad (3.20)$$

```
> dsn := dsolve(diff_sys2 union {x1(0)=5.0,x2(0)=5.0,x3(0)=5.0}
, numeric);
```

$$dsn := \textbf{proc}(x\_rkf45)\ \ldots\ \textbf{end proc} \qquad (3.21)$$

```
> Digits:=20:

  t_ini:=0.0:
  t_fin:=50.0:
  nt:=500:

  dt:=(t_fin-t_ini)/(nt-1):
  gr:=1:
  tt:=t_ini:
  for i from 1 to nt do
      vv:=subs(dsn(tt),[x1(t),x2(t),x3(t)]):
      tt:=tt+dt:
      gr:=gr,[tt,evalf(log10(subs(x1=vv[1],x2=vv[2],x3=vv[3],
  llf)))]
  od:
  gr:=[gr]:
  gr:=gr[2..nops(gr)]:

  Digits:=10:
> plot(gr);
```



Three-Waves System

H. Hakken, Light, Vol. 2 Laser Light Dynamics, North-Holland (New York,1985).
J. Weiland and H. Wilhelmsson, Coherent Non-Linear Interaction of Waves in Plasmas, Pergamon, Oxford (1977).

```
> # Writing down the system:
> eqs:=1:
  params:={g}:
  for i from 1 to 3 do
      pr:=cat(lambda,i)*cat(x,i):
      if i=1 then
         pr:=pr+g*x2*x3
      fi:
      pr:=pr+cat(x,i)*sum('cat'(N,i,jj)*'cat'(x,jj)^2,jj=1..3):
      eqs:=eqs,pr:
```

```
    params:=params union {seq('cat'(N,i,jj),jj=1..3)}:
  od:
  eqs:=[eqs]:
  eqs:=eqs[2..nops(eqs)]:
> eqs;
```

$$[\lambda 1\, x1 + g\, x2\, x3 + x1\,(N11\, x1^2 + N12\, x2^2 + N13\, x3^2),\ \lambda 2\, x2 + x2\,(N21\, x1^2 + N22\, x2^2 + N23\, x3^2),\ \lambda 3\, x3 + x3\,(N31\, x1^2 + N32\, x2^2 + N33\, x3^2)] \tag{4.1}$$

```
> variables:=[x1,x2,x3];
```

$$variables := [x1, x2, x3] \tag{4.2}$$

```
> params;
```

$$\{N11, N12, N13, N21, N22, N23, N31, N32, N33, g\} \tag{4.3}$$

```
> # Giving some numerical values for the parameters
> vals:={N11=-1,N12=1,N13=7,N21=-10,N22=-10,N23=7,N31=-10,N32=
  -1,N33=-4,g=2};
```

$$vals := \{N11 = -1, N12 = 1, N13 = 7, N21 = -10, N22 = -10, N23 = 7, N31 = -10, N32 = -1, N33 = -4, g = 2\} \tag{4.4}$$

```
> eqs2:=subs(vals,eqs);
```

$$eqs2 := [\lambda 1\, x1 + 2\, x2\, x3 + x1\,(-x1^2 + x2^2 + 7\, x3^2),\ \lambda 2\, x2 + x2\,(-10\, x1^2 - 10\, x2^2 + 7\, x3^2),\ \lambda 3\, x3 + x3\,(-10\, x1^2 - x2^2 - 4\, x3^2)] \tag{4.5}$$

```
> r1:=solve(subs(x1=1,x2=1,x3=1,eqs2),{lambda1,lambda2,lambda3}
  );
```

$$r1 := \{\lambda 1 = -9, \lambda 2 = 13, \lambda 3 = 15\} \tag{4.6}$$

```
> eqs3:=subs(r1,eqs2);
```

$$eqs3 := [-9\, x1 + 2\, x2\, x3 + x1\,(-x1^2 + x2^2 + 7\, x3^2),\ 13\, x2 + x2\,(-10\, x1^2 - 10\, x2^2 + 7\, x3^2),\ 15\, x3 + x3\,(-10\, x1^2 - x2^2 - 4\, x3^2)] \tag{4.7}$$

```
> # Determining the Lyapunov function from the numerical method
> t1:=time():
  llf:=LyapunovFunction(eqs3,variables,params,[1,1,1]);
  time()-t1;
```

$$llf := \frac{0.2349193456\, x2\, x3}{x1} - 0.2349193456 \ln\left(\frac{x2\, x3}{x1}\right) - 2.152062513 + 0.9833984375\, x1^2 - 0.9833984375 \ln(x1^2) + 0.4201864672\, x2^2 - 0.4201864672 \ln(x2^2) + 0.5135582630\, x3^2 - 0.5135582630 \ln(x3^2)$$

$$1.327 \tag{4.8}$$

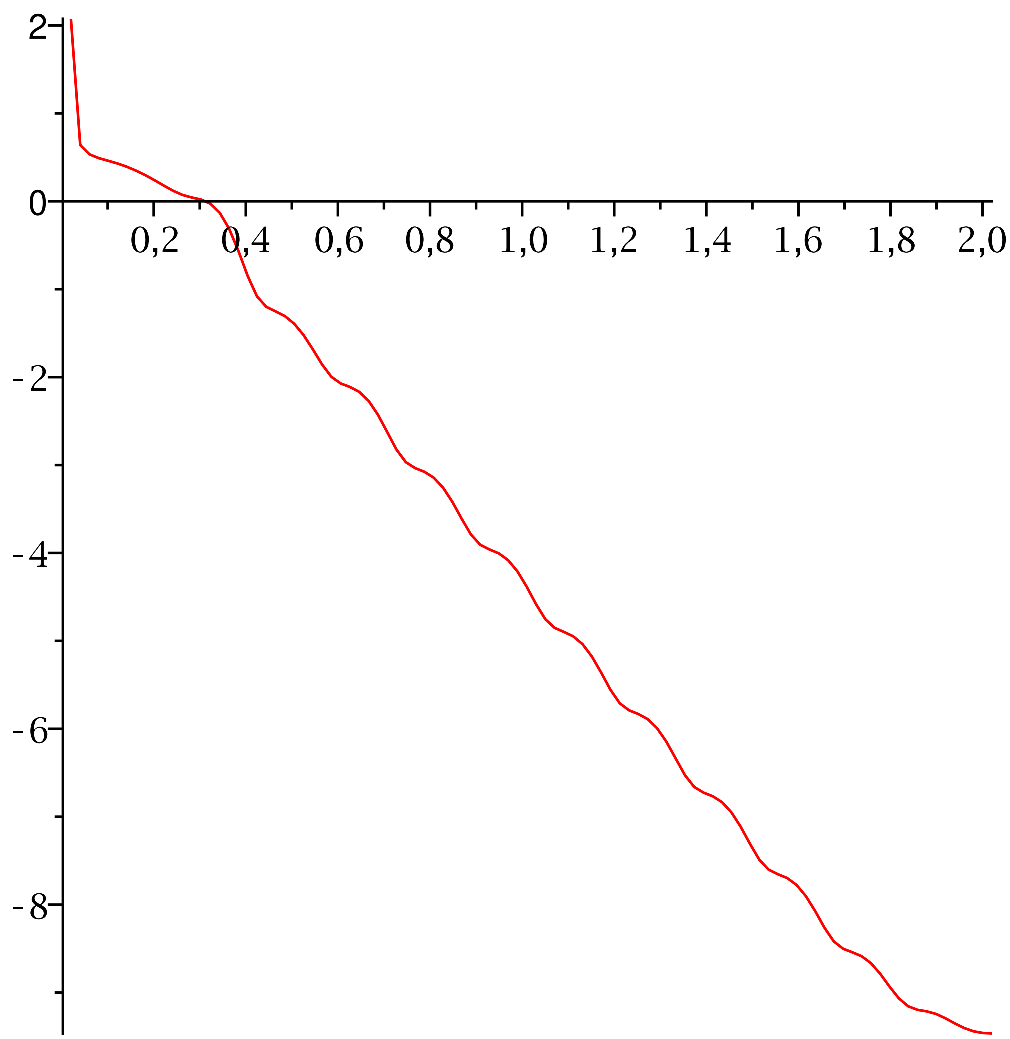
```
> # And verifying it:
> diff_sys:=simplify(subs(vals,convert(subs(x1=x1(t),x2=x2(t),
  x3=x3(t),{Diff(x1,t)=eqs3[1],Diff(x2,t)=eqs3[2],Diff(x3,t)=
  eqs3[3]}),diff)));
```

$$diff\_sys := \left\{ \frac{d}{dt}\, x1(t) = -9\, x1(t) + 2\, x2(t)\, x3(t) - x1(t)^3 + x1(t)\, x2(t)^2 + 7\, x1(t)\, x3(t)^2,\ \frac{d}{dt}\, x2(t) = 13\, x2(t) - 10\, x2(t)\, x1(t)^2 - 10\, x2(t)^3 + 7\, x2(t)\, x3(t)^2,\ \frac{d}{dt}\, x3(t) = 15\, x3(t) - 10\, x3(t)\, x1(t)^2 - x3(t)\, x2(t)^2 - 4\, x3(t)^3 \right\} \tag{4.9}$$

```
> dsn := dsolve(diff_sys union {x1(0)=1.0,x2(0)=15.0,x3(0)=5.0}
, numeric);
```

$$dsn := \mathbf{proc}(x\_rkf45)\ \ldots\ \mathbf{end\ proc} \tag{4.10}$$

```
> Digits:=20:

t_ini:=0.0:
t_fin:=2.0:
nt:=100:

dt:=(t_fin-t_ini)/(nt-1):
gr:=1:
tt:=t_ini:
for i from 1 to nt do
    vv:=subs(dsn(tt),[x1(t),x2(t),x3(t)]):
    tt:=tt+dt:
    gr:=gr,[tt,evalf(log10(subs(x1=vv[1],x2=vv[2],x3=vv[3],
llf)))]
od:
gr:=[gr]:
gr:=gr[2..nops(gr)]:
Digits:=10:
> plot(gr);
```



Modified Verhulst-Solow model

Journal of Statistical Mechanics, "Modified Verhulst-Solow model for long-term population and economic growth"
https://doi.org/10.1088/1742-5468/ad267a

```
> eqs:=[x1*(1-alpha1+b*alpha2-(1-alpha1)*x1-alpha2*x2),
        x2*(-beta*x1+b*beta*alpha2+beta*alpha1*x1-(k+beta*
  alpha2)*x2+x3),
        x3*(-beta*alpha1+b*beta*alpha2+beta*alpha1*x1-beta*
  alpha2*x2)];
  variables:=[x1,x2,x3];
  params:={b,k,beta,alpha1,alpha2};
```

$$eqs := [x1\,(1-\alpha 1+b\,\alpha 2-(1-\alpha 1)\,x1-\alpha 2\,x2),\ x2\,(-\beta\,x1+b\,\beta\,\alpha 2+\beta\,\alpha 1\,x1-(k+\beta\,\alpha 2)\,x2+x3),\ x3\,(-\beta\,\alpha 1+b\,\beta\,\alpha 2+\beta\,\alpha 1\,x1-\beta\,\alpha 2\,x2)]$$

$$variables := [x1,\ x2,\ x3] \tag{5.1}$$

$$params := \{\alpha 1, \alpha 2, b, \beta, k\} \quad \textbf{(5.1)}$$

```
> q:=solve(eqs,variables);
```

$$q := \left[[x1 = 0, x2 = 0, x3 = 0], \left[x1 = 0, x2 = -\frac{\alpha 1 - b\,\alpha 2}{\alpha 2}, x3 = -\frac{\alpha 1\,k + \alpha 1\,\beta\,\alpha 2 - b\,\alpha 2\,k}{\alpha 2}\right], \left[x1 = 0, x2 = \frac{b\,\beta\,\alpha 2}{k + \beta\,\alpha 2}, x3 = 0\right], [x1 = 1, x2 = b, x3 = k\,b + \beta - \beta\,\alpha 1], \left[x1 = \frac{-k + \alpha 1\,k - b\,\alpha 2\,k - \beta\,\alpha 2 + \alpha 1\,\beta\,\alpha 2}{k\,(-1 + \alpha 1)}, x2 = \frac{\beta\,(-1 + \alpha 1)}{k}, x3 = 0\right], \left[x1 = \frac{-1 + \alpha 1 - b\,\alpha 2}{-1 + \alpha 1}, x2 = 0, x3 = 0\right] \right] \quad \textbf{(5.2)}$$

```
> q2:=q[4];
```

$$q2 := [x1 = 1, x2 = b, x3 = k\,b + \beta - \beta\,\alpha 1] \quad \textbf{(5.3)}$$

```
> q3:=subs(q2,variables);
```

$$q3 := [1, b, k\,b + \beta - \beta\,\alpha 1] \quad \textbf{(5.4)}$$

```
> t1:=time():
  rl:=LyapunovFunction(eqs,variables,params,q3);
  time()-t1;
```

$$rl := \Bigg[\Bigg[\Bigg[-a1\,k\,b \ln\left(\frac{x3}{k\,b + \beta}\right) - a1\,k\,b - a3\,b \ln\left(\frac{x2}{b}\right) - a3\,b - a1\,\beta + a2\,x1 + a3\,x2 - a2 \ln(x1) - a1\,\beta \ln\left(\frac{x3}{k\,b + \beta}\right) - a2 + a1\,x3, \Big\{ \alpha 1 = 0, -\frac{1}{4}\,\frac{a1^2\,\beta^4\,\alpha 2 - 4\,k\,a1\,a2\,\beta + a2^2\,\alpha 2 - 2\,a2\,a1\,\beta^2\,\alpha 2}{a1\,\beta\,a2} = 0, -a1\,\beta\,\alpha 2 + a3 = 0 \Big\} \Bigg] \Bigg], \Bigg[-a1\,k\,b \ln\left(\frac{x3}{k\,b + \beta}\right) - a1\,k\,b - a3\,b \ln\left(\frac{x2}{b}\right) - a3\,b - a1\,\beta + a2\,x1 + a3\,x2 - a2 \ln(x1) - a1\,\beta \ln\left(\frac{x3}{k\,b + \beta}\right) - a2 + a1\,x3, \{\alpha 1 = 0, -a1\,\beta\,\alpha 2 + a3 = 0, \alpha 2 < 0, \beta < 0, \alpha 2\,(a1^2\,\beta^4\,\alpha 2 - 4\,k\,a1\,a2\,\beta + a2^2\,\alpha 2 - 2\,a2\,a1\,\beta^2\,\alpha 2) < 0\} \Bigg], \Bigg[-a1\,k\,b \ln\left(\frac{x3}{k\,b + \beta}\right) - a1\,k\,b - a3\,b \ln\left(\frac{x2}{b}\right) - a3\,b - a1\,\beta + a2\,x1 + a3\,x2 - a2 \ln(x1) - a1\,\beta \ln\left(\frac{x3}{k\,b + \beta}\right) - a2 + a1\,x3, \{\alpha 1 = 0, -a1\,\beta\,\alpha 2 + a3 = 0, \alpha 2 < 0, \alpha 2\,(a1^2\,\beta^4\,\alpha 2 - 4\,k\,a1\,a2\,\beta$$

$$+ a2^2\,\alpha 2 - 2\,a2\,a1\,\beta^2\,\alpha 2\big) < 0,\ -\beta < 0\}\Big],\ \Big[-a1\,k\,b\ln\left(\frac{x3}{k\,b+\beta}\right) - a1\,k\,b$$

$$- a3\,b\ln\left(\frac{x2}{b}\right) - a3\,b - a1\,\beta + a2\,x1 + a3\,x2 - a2\ln(x1)$$

$$- a1\,\beta\ln\left(\frac{x3}{k\,b+\beta}\right) - a2 + a1\,x3,\ \{\alpha 1 = 0,\ -a1\,\beta\,\alpha 2 + a3 = 0,\ \beta < 0,$$

$$\alpha 2\left(a1^2\,\beta^4\,\alpha 2 - 4\,k\,a1\,a2\,\beta + a2^2\,\alpha 2 - 2\,a2\,a1\,\beta^2\,\alpha 2\right) < 0,\ -\alpha 2 < 0\}\Big],\ \Big[$$

$$-a1\,k\,b\ln\left(\frac{x3}{k\,b+\beta}\right) - a1\,k\,b - a3\,b\ln\left(\frac{x2}{b}\right) - a3\,b - a1\,\beta + a2\,x1 + a3\,x2$$

$$- a2\ln(x1) - a1\,\beta\ln\left(\frac{x3}{k\,b+\beta}\right) - a2 + a1\,x3,\ \{\alpha 1 = 0,\ -a1\,\beta\,\alpha 2 + a3 = 0,$$

$$\alpha 2\left(a1^2\,\beta^4\,\alpha 2 - 4\,k\,a1\,a2\,\beta + a2^2\,\alpha 2 - 2\,a2\,a1\,\beta^2\,\alpha 2\right) < 0,\ -\alpha 2 < 0,\ -\beta < 0\}$$

$$\Big]\Big]$$

$$0.125 \tag{5.5}$$

```
> nops(rl);
```

$$5 \tag{5.6}$$

```
> rl2:=rl[1];
```

$$rl2 := \Big[-a1\,k\,b\ln\left(\frac{x3}{k\,b+\beta}\right) - a1\,k\,b - a3\,b\ln\left(\frac{x2}{b}\right) - a3\,b - a1\,\beta + a2\,x1 \tag{5.7}$$

$$+ a3\,x2 - a2\ln(x1) - a1\,\beta\ln\left(\frac{x3}{k\,b+\beta}\right) - a2 + a1\,x3,\ \Big\{\alpha 1 = 0,$$

$$-\frac{1}{4}\,\frac{a1^2\,\beta^4\,\alpha 2 - 4\,k\,a1\,a2\,\beta + a2^2\,\alpha 2 - 2\,a2\,a1\,\beta^2\,\alpha 2}{a1\,\beta\,a2} = 0,\ -a1\,\beta\,\alpha 2 + a3$$

$$= 0\Big\}\Big]$$

```
> rl3:=map(z->if lhs(z)<>rhs(z) then z fi,solve(rl2[2],{a1,a2,
  a3,alpha1,alpha2}));
```

$$rl3 := \left\{a3 = \frac{4\,a1^2\,\beta^2\,k\,a2}{\left(a1\,\beta^2 - a2\right)^2},\ \alpha 1 = 0,\ \alpha 2 = \frac{4\,k\,a1\,a2\,\beta}{\left(a1\,\beta^2 - a2\right)^2}\right\} \tag{5.8}$$

```
> vals:={a1=1/2,a2=1/3,beta=3,k=2,b=7};
```

$$vals := \left\{a1 = \frac{1}{2},\ a2 = \frac{1}{3},\ b = 7,\ \beta = 3,\ k = 2\right\} \tag{5.9}$$

```
> vals2:=vals union subs(vals,rl3);
```

$$vals2 := \left\{a1 = \frac{1}{2},\ a2 = \frac{1}{3},\ a3 = \frac{216}{625},\ \alpha 1 = 0,\ \alpha 2 = \frac{144}{625},\ b = 7,\ \beta = 3,\ k = 2\right\} \quad \textbf{(5.10)}$$

```
> diff_sys:=simplify(subs(vals2,convert(subs(x1=x1(t),x2=x2(t),
  x3=x3(t),{Diff(x1,t)=eqs[1],Diff(x2,t)=eqs[2],Diff(x3,t)=eqs
  [3]}),diff)));
```

$$diff\_sys := \left\{\frac{d}{dt}\,x1(t) = -\frac{1}{625}\,x1(t)\,(-1633 + 625\,x1(t) + 144\,x2(t)),\ \frac{d}{dt}\,x2(t) = -\frac{1}{625}\,x2(t)\,(1875\,x1(t) - 3024 + 1682\,x2(t) - 625\,x3(t)),\ \frac{d}{dt}\,x3(t) = -\frac{432}{625}\,x3(t)\,(-7 + x2(t))\right\} \quad \textbf{(5.11)}$$

```
> llf:=evalf(subs(vals2,subs(vals,rl[1][1])));
```

$$llf := -8.500000000\ln(0.05882352941\,x3) - 11.25253333 - 2.419200000\ln(0.1428571429\,x2) + 0.3333333333\,x1 + 0.3456000000\,x2 - 0.3333333333\ln(x1) + 0.5000000000\,x3 \quad \textbf{(5.12)}$$

```
> dsn := dsolve(diff_sys union {x1(0)=1.0,x2(0)=1.0,x3(0)=2.0},
  numeric);
```

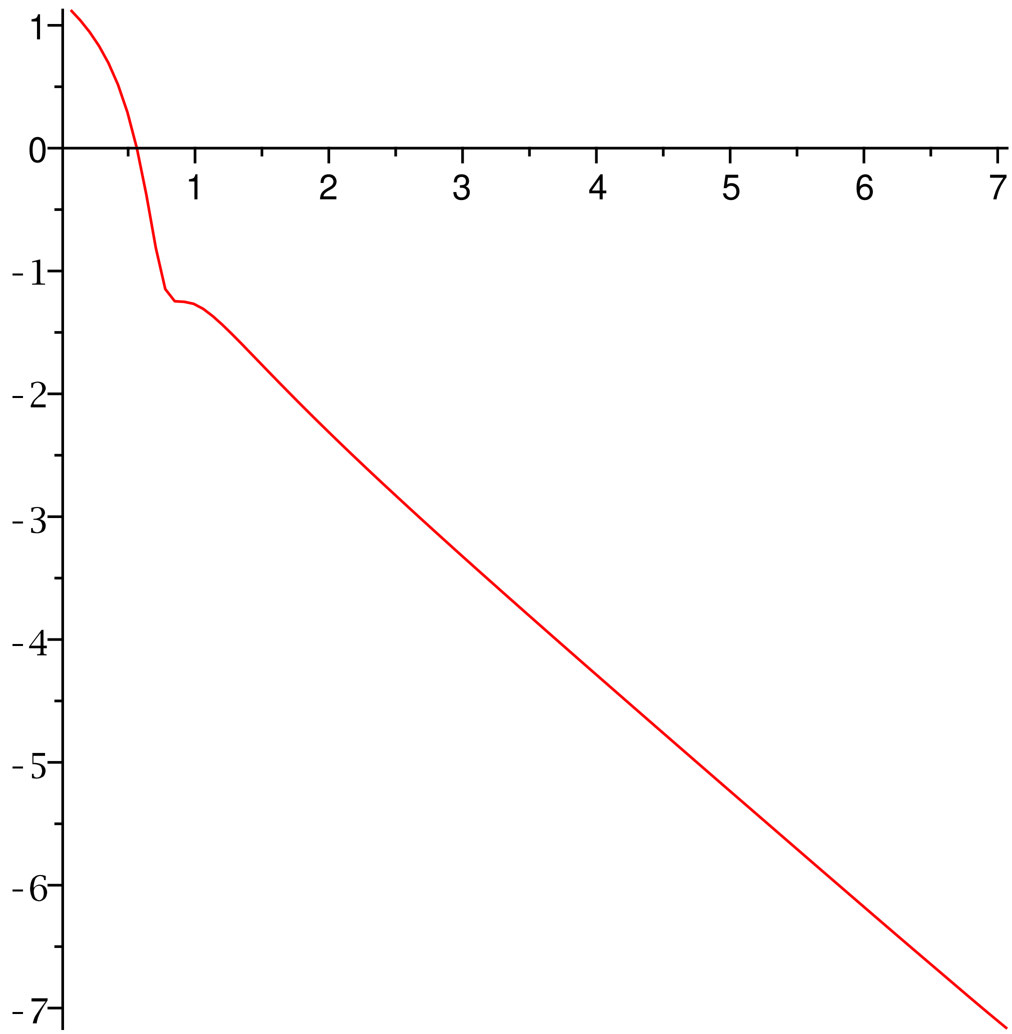
$$dsn := \textbf{proc}(x\_rkf45)\ \ldots\ \textbf{end proc} \quad \textbf{(5.13)}$$

```
> Digits:=20;
```

$$Digits := 20 \quad \textbf{(5.14)}$$

```
> t_ini:=0.0:
  t_fin:=7.0:
  nt:=100:
  dt:=(t_fin-t_ini)/(nt-1):
  gr:=1:
  tt:=t_ini:
  for i from 1 to nt do
      vv:=subs(dsn(tt),[x1(t),x2(t),x3(t)]):
      tt:=tt+dt:
      gr:=gr,[tt,evalf(log10(subs(x1=vv[1],x2=vv[2],x3=vv[3],
  llf)))]
  od:
  gr:=[gr]:
  gr:=gr[2..nops(gr)]:
  Digits:=10:
```

```
> plot(gr);
```


1
0
-1
-2
-3
-4
-5
-6
-7
1
2
3
4
5
6
7