Lauro Cabral

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CECS 229

HW1

Section 4.1

5) Show that if *a* | *b* and *b* | *a*, where *a* and *b* are integers, then *a* = *b* or *a* = -*b*.

$a|b \Leftrightarrow b = ak_0 \qquad b|a \Leftrightarrow a = bk_1$

we can combine these two equations and we get:

$$b = (bk_0)k_2$$

$$b = bk,\ k = 1 \text{ because } b = b(1)$$

since $k = 1$, we get:

$$b = a(1) \qquad a=b(1)$$

$$b = a \qquad a=b$$

8) Prove or disprove that if *a* | *bc*, where a, b, and c are positive integers and a ≠ 0 , then *a* | *b* or *a* | *c*.

$$a\,|bc \Leftrightarrow a\,|\,b\,,\ a\,|c$$

$$b = ak_0 \qquad c = ak_1$$

we can combine these two equations and we get:

$$c = \left(\frac{b}{k0}\right)k_1 \qquad \text{or} \qquad b = \left(\frac{c}{k1}\right)k_0$$

Since *a* is not part of the equation it proves that *a* CANNOT divide into *b* or *c*

13) Suppose that *a* and *b* are integers, $a \equiv 4(\text{mod } 13)$, and $b \equiv 9(\text{mod } 13)$. Find the integer *c* with 0≤c≤12 such that

a) $c \equiv 9a(\text{mod}13)$

$$9a = 9(4\text{mod}13) = 36 \text{ mod } 13 = 10$$
$$9a(\text{mod}13) = 10 \text{ mod } 13 = 10$$
$$c = 10$$

Prove:

1) $10 \equiv 36 \text{ mod } 13 \Leftrightarrow 10 \text{ mod } 13 = 36 \text{ mod } 13$ (equivalence)

2) $10 - 36 = -26$
$-26/13 = -2$

b) $c \equiv 11b(\text{mod}13)$

$$11b = 11(9\text{mod}13) = 99 \text{ mod } 13 = 8$$
$$11b(\text{mod}13) = 8 \text{ mod } 13 = 8$$
$$c = 8$$

Prove:

1) $8 \equiv 99 \text{ mod } 13 \Leftrightarrow 8 \text{ mod } 13 = 99 \text{ mod } 13$(equivalence)

2) $8 - 99 = -91$
$-91/13 = -7$

c) $c \equiv a+b(\text{mod}13)$

$$a + b = (4\text{mod}13) + (9\text{mod}13) = 13 \text{ mod } 13 = 0$$
$$a+b(\text{mod}13) = 0 \text{ mod } 13 = 0$$
$$c = 0$$

Prove:

1) $0 \equiv 13 \text{ mod } 13 \Leftrightarrow 0 \text{ mod } 13 = 13 \text{ mod } 13$ (equivalence)

2) $13 – 0 = 13 / 13 = 1$

d) $c \equiv 2a+3b(\text{mod}13)$

$$2a + 3b = 2(4\text{mod}13)+3(9\text{mod}13) = 8\text{mod}13 + 27\text{mod}13 = 35\text{mod}13 = 9$$
$$2a + 3b(\text{mod}13) = 9 \text{ mod } 13 = 9$$
$$c = 9$$

Prove:

1) $9 \equiv 35 \text{ mod } 13 \Leftrightarrow 9 \text{ mod } 13 = 35 \text{ mod } 13$(equivalence)

2) $35 – 9 = 26 / 13 = 2$

e) $c \equiv a^2+b^2(\text{mod}13)$

$$a^2+b^2 = (4\text{mod}13)^2+(9\text{mod}13)^2 = 16 + 81 = 97$$
$$a^2+b^2(\text{mod } 13) = 97 \text{ mod } 13 = 6$$
$$c = 6$$

Prove:

1) $97 \equiv 6 \text{ mod } 13 \Leftrightarrow 97 \text{ mod } 13 = 6 \text{ mod } 13$(equivalence)

2) $97 – 6 = 91 / 13 = 7$

f) $c \equiv a^3-b^3(\text{mod}13)$

$$a^3-b^3 = (4\text{mod}13)^3-(9\text{mod}13)^3 = 64 - 729 = -665$$
$$a^3-b^3 \text{ (mod } 13) = -665 \text{ mod } 13 = 11$$
$$c = 11$$

Prove:

1) -665≡11 mod 13 ⇔ -665 mod 13 = 11 mod 13(equivalence)

2) 11 – (-665) = 676 / 13 = 52

17) Show that if *n* and *k* are positive integers , then $\lceil \frac{n}{k} \rceil = \lfloor \frac{n-1}{k} \rfloor + 1$

Since n=kq+r , where 0 ≤ r < k

Case 1: r ≠ 0 (remainder is not 0)

$$n = kq + r \Leftrightarrow \frac{n}{k} = q + \frac{r}{k}$$

$\frac{r}{k}$ is an improper fraction (not 0 because we stated not zero) in this case because,

$$0 \le r < k \Leftrightarrow 0 \le \frac{r}{k} < 1$$

Therefore,

$\frac{n}{k}$ = q +(some number less then 1)⇔ $\lceil \frac{n}{k} \rceil = q+1$, '+1' because we round up

Then,

$$n-1 = kq + r - 1 \Leftrightarrow \frac{n-1}{k} = q + \frac{r-1}{k}$$

$\frac{r-1}{k}$ is an improper fraction (not 0 because we stated not zero) in this case because,

$$0 \le r-1 < k \Leftrightarrow 0 \le \frac{r-1}{k} < 1$$

Therefore,

$\lfloor \frac{n-1}{k} \rfloor = q$, no r because we stated r < 1

If we add $\lfloor \frac{n-1}{k} \rfloor$ plus 1 to both sides we get

$$\lfloor \frac{n-1}{k} \rfloor + 1 = q + 1 \Leftrightarrow \lfloor \frac{n-1}{k} \rfloor + 1 = \lceil \frac{n}{k} \rceil$$

Case 2: r = 0 (remainder is 0)

n =kq + r , where r = 0 -> n = kq

Therefore ,

$$n = kq \Leftrightarrow \frac{n}{k} = q \Leftrightarrow \lceil \frac{n}{k} \rceil = q$$

Then,

$$n\text{-}1 = kq\text{-}1 \Leftrightarrow \frac{n-1}{k} = q - \frac{1}{k}$$

Since , $-\frac{1}{k}$ adds with q(q is an integer) make q less then original value by fractions not whole numbers. Ie. If q = 60 , then it would be 59 ≤ q < 60.

So,

$$\lfloor\frac{n-1}{k}\rfloor = q - 1$$

If we add $\lfloor\frac{n-1}{k}\rfloor$ plus 1 to both sides we get

$$\lfloor\frac{n-1}{k}\rfloor + 1 = q - 1 + 1$$

$$\Rightarrow \lfloor\frac{n-1}{k}\rfloor + 1 = q$$

$$\Rightarrow \lfloor\frac{n-1}{k}\rfloor + 1 = \lceil\frac{n}{k}\rceil$$

18) Show that if *a* is an integer and *d* is an integer greater then 1, then the quotient and remainder obtained when *a* is divided by *d* are $\lfloor\frac{a}{d}\rfloor$ and $a\text{-}d\lfloor\frac{a}{d}\rfloor$, respectively.

* Note: 0≤ r < d , such that a = dq+r *

$$a = dq + r$$

$$\Rightarrow \frac{a}{d} = q + \frac{r}{d}, \text{ since } 0 \le \frac{r}{d} < 1$$

$$\Rightarrow \lfloor\frac{a}{d}\rfloor = q + (\text{some decimal}) \Leftrightarrow \lfloor\frac{a}{d}\rfloor = q \checkmark$$

$$\Rightarrow a = d(\lfloor\frac{a}{d}\rfloor) + r$$

$$\Rightarrow r = a - d(\lfloor\frac{a}{d}\rfloor)\checkmark$$

29) Decide whether each of these integers is congruent to 5 modulo 17.

a) 80

80-5 = 75 / 17 ≈ 4.411, no

b) 103

$$103 - 5 = 98/17 \approx 5.764, \text{ no}$$

c) -29

$$-29 - 5 = -34 / 17 = -2, \text{ yes}$$

d) -122

$$-122 - 5 = -127 / 17 \approx -7.470 , \text{ no}$$

34) Show that if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, where a , b , c , d , and m are integers with $m \geq 2$, then $a - c \equiv b - d \pmod{m}$.

$$a \equiv b \pmod{m} \iff a = b + m , b = a + m$$

$$c \equiv d \pmod{m} \iff c = d + m , d = c + m$$

$$a - c = b - d ✓$$

$$b - d = a - c ✓$$

40) Prove that if n is an odd positive integer, then $n^2 \equiv 1 \pmod{8}$.

$$n = 2k + 1, \text{ where } k > 0$$

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 4k(k+1) + 1$$

if $k = 0$,

$$n^2 = 1$$

$$1 \equiv 1 \pmod{8}$$

If $k > 0$,

$$n^2 = 4k(k+1)+1$$

Sinc, $k(k+1)$ is always even we can say that is $k(k+1) = 2k$

$$n^2 = 4(2k) +1$$

$$n^2 = 8k + 1$$

Hence, $n^2 \equiv 1 \pmod{8}$

Section 4.2

5. Convert the octal expansion of each of these integers to a binary expansion.

a) $(572)_8$

101111010

b) $(1604)_8$

1110000100

c) $(423)_8$

100010011

d) $(2417)_8$

10100001111

7. Convert the hex expansion of each of these integers to a binary expansion.

a) $(80E)_{16}$

100000001110

b) $(135AB)_{16}$

10011010110101011

c) $(ABBA)_{16}$

1010101110111010

d) $(DEFACED)_{16}$

1101111011111010110011101101

14. Show that the binary expansions of a positive integer can be obtained from its hexadecimal expansion by translating each hexadecimal digit into a block of four binary digits.

If h_i is a hexadecimal digit then we know that

$$h_i = 2^0 b_{i0} + 2^1 b_{i1} + 2^2 b_{i2} + 2^3 b_{i3}$$

Therefore,

$$(h_3h_2h_1h_0) = (2^0 b_{00} + 2^1 b_{01} + 2^2 b_{02} + 2^3 b_{03}) + (2^0 b_{10} + 2^1 b_{11} + 2^2 b_{12} + 2^3 b_{13}) + (2^0 b_{20} + 2^1 b_{21} + 2^2 b_{22} + 2^3 b_{23}) + (2^0 b_{30} + 2^1 b_{31} + 2^2 b_{32} + 2^3 b_{33})$$

19. Give a procedure for converting from the octal expansion of an integer to its hexadecimal expansion using binary notation as an intermediate step.

First, we convert octal expansion to binary which can be done in 9 steps:

Step 1: Let the given number have n number of digits
Step 2: Multiply each digit of the number with 8^{n-1}, when the digit is in the nth position from the right end of the number. If the number has decimal part the multiply each digit in the decimal part by $\frac{1}{8^m}$ when the digit is in the m^{th} position from the decimal point.
Step 3: Add all terms after multiplication
Step 4: The obtained value is the equivalent decimal number
Step 5: Consider the decimal number, divide it by 2
Step 6: Note the remainder
Step 7: Continue the above two steps for the quotient till the quotient is zero
Step 8: Write the remainders in the reverse order
Step 9: The obtained number is the equivalent binary number for the given octal number.

ie.

Convert 41_8 to a binary number.

$$41_8 = (4 * 8^1) + (1 * 8^0)$$

$$= 4 * 8 + 1 * 1$$

$$= 32+1$$

$$= 33 \text{(Decimal number)}$$

Now convert this decimal number to a binary number.

2 | 33
2 | 16 -- 1
2 | 8 -- 0
2 | 4 --0
2 | 2 -- 0
1 -- 0

The binary number is 100001_2

$41_8 = 100001_2$

Then we convert binary to hexadecimal which can be done in 5 steps:

1) Working from right to left, split the binary number into groups of 4 digits. If the left-most grouping has less than 4 digits, make up the difference with zeros.

```
1.   0010 | 0001 // Binary of number 33
```

2) Populate the next row with "8 4 2 1" under each grouping. The 8-4-2-1 represents the binary place values for each of the four positions, the combination of which total up to 15 (the highest digit that can be used in a base 16 number)

```
1.  0010 | 0001 // Binary of number 33
2.  8421 | 8421
```

3) Multiply each digit in Row 1 by the corresponding place value in Row 2 and place the result in Row 3.

```
1.  0010 | 0001 // Binary of number 33
2.  8421 | 8421
3.  0020 | 0001
```

4) Add the products in Row 3 for each group of 4 and place the total in Row 4.

```
1.  0010 | 0001 // Binary of number 33
2.  8421 | 8421
3.  0020 | 0001
4.   2   |   1
```

5) Change any values in Row D that are greater than 9 into the hexadecimal letter they are represented by.

```
1.  0010 | 0001 // Binary of number 33
2.  8421 | 8421
3.  0020 | 0001
4.   2   |   1
5.      21h
```

22. Find the sum and the product of each of these pairs of numbers. Express your answers as a base 3 expansion.

a) $(112)_3$, $(210)_3$

```
  112
+ 210
------
 1022
```

```
   112
x  210
-------
   000
  112
1001
-------
1001220
```

b) $(2112)_3$, $(12021)_3$

```
  1 11
   2112
+ 12021
-------
  21210
```

```
      2112
 x   12021
----------
      2112
    12001
   00000
  12001
 2112
-------
111020122
```

c) $(20001)_3$, $(1111)_3$

```

  20001
+  1111
-------
  21112
```

```
   20001
 x  1111
-----------
   20001
  20001
 20001
20001
-------
22221111
```

d) $(120021)_3$, $(2002)_3$

```
     11
  120021
+   2002
--------
  122100
```

```
  120021
 x  1111
-----------
   1010112
  0000000
 0000000
1010112
-----------
1011122112
```