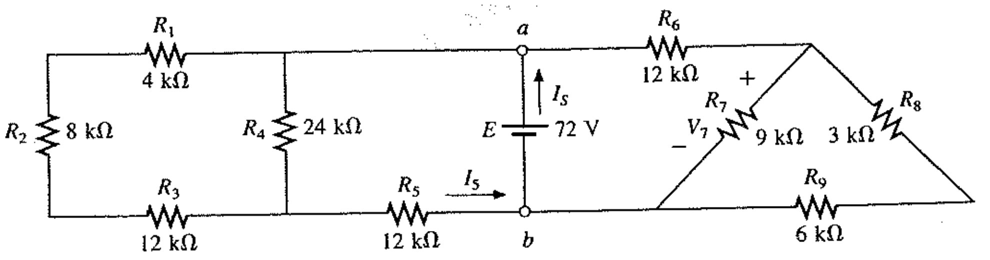
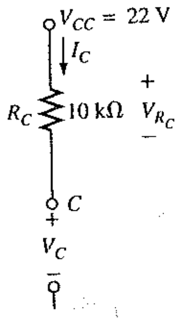
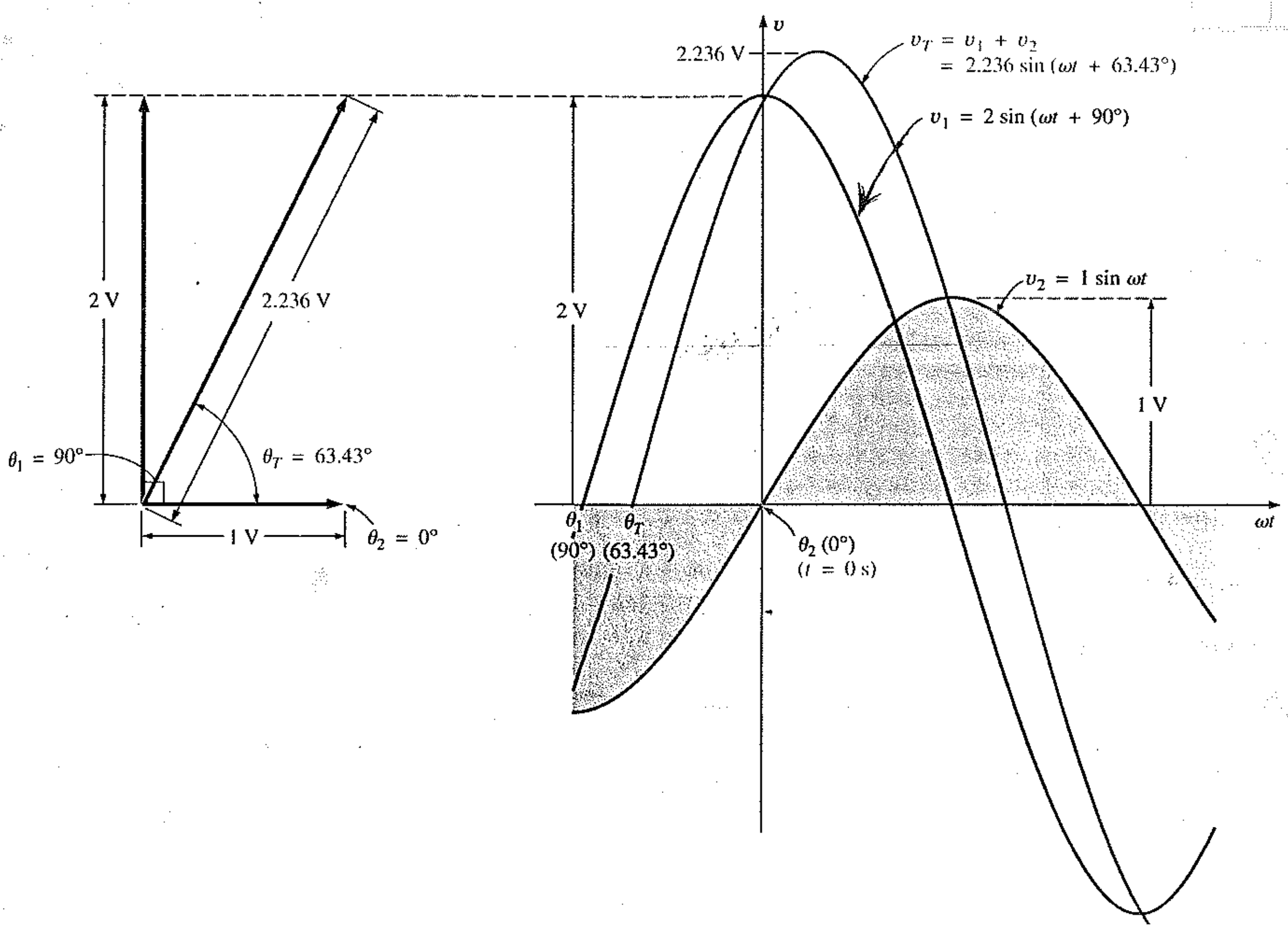
ELECTRIC CIRCUITS







BY

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بسم الله الرحمن الرحيم

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PREFACE

This is a new revised edition. The first edition appeared more than ten years ago, while the second one appeared more than four years ago. Both editions are revised to help in updating the first one. A number of sections were reduced in length, while others were expanded to reflect recent trends and to respond to comments received from current users. As always, the classroom experience continues to provide insight into a number of areas that needed to be reexamined.

Electric circuits is one of the most important subjects for engineers and all those who want to study electrical engineering. Electricity is now influencing each aspect in our life. Can you imagine life without electricity ??. The answer is very easy, without electricity, life would be very difficult and even impossible.

The foundation of modern day society is particularly sensitive to a few areas of development, research and interest. In recent years, it has become obvious that the electrical and electronics industry is one area that will have a broad impact on future development in a host of activities that affect our life style, general health and capabilities. Electric circuits is very important to understand and develop all the above aspects.

This book represents an introduction to circuit analysis. It gives a simple explanation to basics of electric circuits. It helps the readers who have not read any books about this subject. This book consists of 10 chapters and a list of references.

Chapter (1), gives the units and dimensions used throughout the book.

Chapter (2), explains the meaning of electric charges, currents, voltage and power. Also the definition of electric circuit is given.

Chapter (3), Ohm's law, Kirchoff's current law (KCL) and Kirchoff's voltage law (KVL) are given. The color codes of resistors are explained. The definition of voltage and current sources is given.

Chapter (4), shows some methods for solution of DC electric circuits. Combination of elements (series or parallel) is explained. This chapter also explains star-delta and delta-star conversions. Loop current and Node voltage methods are also given in this chapter.

Various network theorems are given in chapter (5). Network theorems are in aid, and sometimes an essential one, in the analysis of networks. There are many network theorems-too many for us to consider. This chapter has all the important practical ones.

Up to this point of our study, the voltage and current sources have all been DC. Most of sources in practice are AC. Chapter (6) explains in detail sinusoidal alternating current and voltage. Since, cosine waves, maximum, instantaneous, average and effective values current and voltage are given in this chapter. Also the sinusoidal currents and voltages in ideal elements of electric circuits are given.

Complex algebra and phasors are useful tools needed in AC problems. Chapter (7) gives the main laws of these topics applied for transforming sinusoids into complex numbers called phasors.

Chapter (8) gives simple AC circuits (series and parallel). Impedance as well as admittance are explained. Power absorbed, complex power and apparent power are defined in this chapter.

In an AC circuit the energy stored in inductors and capacitors can interact at certain frequencies to produce resonance. In chapter (9), series and parallel resonance are explained. Quality factor is defined for various circuits.

Chapter (10), describes magnetic circuits. These circuits are important to the readers interested in electrical machines, transformer design, and similar topic like television, computers, tape recorders and telephones. Ohm's law for magnetic circuits, magnetizing force, hysteresis, series magnetic circuits and series-parallel magnetic circuits are given in this chapter.

Finally, this book is written such that the readers can real and understand the material efficiently-in a short time as possible. The English used is easy to understand: common words, short words, short sentences and short paragraphs. Also, examples follow the introduction of all major topics and most minus ones. It is hoped that these will give a better understanding and illustration. References for more knowledge are given in the end of this book.

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CONTENTS

Chapter One

A BRIEF HISTORY

The phenomenon of static electricity has been toyed with since antiquity. The Greeks called the fossil resin substance so often used to demonstrate the effects of static electricity elektron, but no extensive study was made of the subject until William Gilbert researched the event in 1600. In the years to follow, there was a continuing investigation of electrostatic charge by a number of individuals such as Otto von Guericke, who developed the first machine to generate large amounts of charge, and Stephen Gray, who was able to transmit electrical charge over long distances on silk threads. Charles DuFay demonstrated that charges either attract or repel each other, leading him to believe that there were two types of charge-a theory we subscribe to today with our defined positive and negative charges.

There are many who believe the true beginnings of the electrical era lie with the effects of Pieter van Musschenbroek and Benjamin Franklin. In 1745, van Musschenbroek introduced the Leyden jar for the storage of electrical charge (the first capacitor) and demonstrated electrical shock (and therefore the power of this new form of energy). Franklin used the leyden jar some seven years later to establish that lightning is simply an electrical discharge, and expanded on a number of other important theories including the definition of the two types of charge as positive and negative. From this point on, new discoveries and theories seemed to occur at an increasing rate as the number of individuals performing research in the area grew .

In 1784, Charles Coulomb demonstrated in Paris that the force between charges is inversely related to the square of the distance between the charges. In 1791. Luigi Galvani, Professor of Anatomy at the University of Bologna, in Itly, performed experiments revealing that electricity is present in every animal. The first voltaic cell with its ability to produce electricity through the chemical action of a metal dissolving in acid was developed by another Italian. Alessandro Volta, in 1799.

The fever pitch continued into the early 1800, with Christian Oersted, a Swedish professor of physics, announcing in 1820 a relation-ship between magnetism and electricity that serves as the foundation for the theory of electromagnetism as we know it today. In the same year, a French physicist, Andre Ampere, demonstrated that there are magnetic effects around every current-carrying conductor and that current-carrying conductor can attract and repel each other just like magnates. In the period 1826 to 1827, a German physicist, Georg Ohm, introduced an important relationship between potential, current, and resistance which we now refer to as Ohm's law. In 1831, an English physicist, Michael Faraday, demonstrated his theory of electromagnetic induction, whereby a changing current in one coil can induce a changing current in another coil even though the two coils are not directly connected. Professor Faraday also did extensive work on a storage device he called the condenser, which we refer to as a capacitor today. He introduced the idea of adding a dielectric between the plates of a capacitor to increase the storage capacity James Clerk Maxwell, a Scottish professor of natural philosophy, performed extensive mathematical analyses to develop what are currently called Maxwell's equations which support the efforts of Faraday linking electric and magnetic effects. Maxwell also developed the electromagnetic theory of light in 1862, which among other things revealed that electromagnetic waves travel through air at the velocity of light (186,000 miles per second or $3X10^{8}$ meters per second). In 1888, a German physicist, Heinrich Rudolph Hertz, through experimentation with lower-frequency electromagnetic waves (microwaves), substantiated Maxwell's predictions and equations. In the mid 1800s, Professor Gustav Robert Kirchhoff introduced a series of laws of voltages and currents that find application at every level and area of this field. In 1895, another German physicist, Wilhelm Rontgen, discovered electromagnetic waves of high frequency commonly called X rays today.

By the end of the 1800s, a significant number of the fundamental equations, laws, and relationships had been established and various fields of study including electronics power generation, and calculating equipment started to develop in earnest. Since 1900s, the improvements in theory and practice of electric circuits have not stopped.

SYSTEMS OF UNITS

In the past, the systems of units most commonly used were the English and metric, as outlined by Table 1.1 Note that while the English system is based on a single standard, the metric is subdivided into two interrelated standards: the MKS and CGS. Fundamental quantities of these systems are compared in Table 1.1 along with their abbreviations. The MKS and CGS system draw their named from the units of measurement used with each system; the MKS system uses Meters, Kilograms, and Seconds, while the CGS system uses Centimeters, Grams, and Seconds. SI is the abbreviation of international system of units given in Table 1.1.

	Metric		
English	MKS	CGS	SI
Length: Yard (yd) (0.914 m)	Meter (m) (39.37 in.) (100 cm)	Centimeter (cm) (2.54 cm = 1 in.)	Meter (m)
Mass: Slug (14.6 kg)	Kilogram (kg) (1000 g)	Gram (g)	Kilogram (kg)
Force: Pound (lb) (4.45 N)	Newton (N) (100,000 dynes)	Dyne	Newton (N)
Temperature: Fahrenheit (°F) $\left(=\frac{9}{5}°C+32\right)$	Celsius or Centigrade (°C) $\left(=\frac{5}{9}(°F-32)\right)$	Centigrade (°C)	Kelvin (K) K = 273.15 + °C
Energy: Foot-pound (ft-lb) (1.356 joules)	Newton-meter (N-m) or Joule (J) (0.7378 ft-lb)	Dyne-centimeter or Erg (1 joule = 10^7 ergs)	Joule (J)
Time: Second (s)	Second (s)	Second (s)	Second (s)

S.I. Prefixes :

An SI prefix is a name attached to the beginning of an SI unit to form either a decimal multiple or submultiple of the SI unit. For example, kilo is a prefix corresponding to one thousand. So, a kilometer is 1000 m.

The SI prefixes have symbols just as the SI units do. Following is a list of the SI prefixes and symbols along with the corresponding powers of 10.

Multiplier	*Prefix*	*Symbol*
10^{18}	exa	E
10^{15}	peta	P
10^{12}	tera	T
10^{9}	giga	G
10^{6}	mega	M
10^{3}	kilo	k
10^{2}	hecto	h
10^{1}	deka	da
10^{-1}	deci	d
10^{-2}	centi	c
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	n
10^{-12}	pico	p
10^{-15}	femto	f
10^{-18}	atto	a

For circuit analysis, only some of these prefixes are important : mega, kilo, Millie, micro, nano, and Pico.

Chapter Two

CURRENT, VOLTAGE AND POWER

Atoms And Their Structure

A basic understanding of the fundamental concepts of current and voltage requires a degree of familiarity with the atom and its structure. The simplest of all atoms is the hydrogen atom, made up of two basic particles, the proton and the electron, in the relative positions shown in Fig. 2.1(a). The nucleus of the hydrogen atom is the proton, a positively charged particle. The orbiting electron carries a negative charge that is equal in magnitude to the positive charge of the proton. In all other elements, the nucleus also contains neutrons, which are slightly heavier than protons and have no electrical charge. The helium atom, for example, has two neutrons in addition to the electrons and two protons as shown in Fig. 2.1(b). In all neutral atoms the number of electrons is equal to the number of protons. The mass of the electron is 9.11×10^{-28} g, and that of the proton and neutron is 1.672×10^{-24} g. The mass of the proton (or neutron) is therefore approximately 1836 times that of the electron. The radii of the proton, neutron, and electron are all of the order of magnitude of 2×10^{-15} m.

For the hydrogen atom, the radius of the smallest orbit followed by the electron is about 5×10^{-11} m. The radius of this orbit is approximately 25.000 times that of the basic constituents of the atom. This is approximately equivalent to a sphere the size of a dime rotating about another sphere of the same size more than a quarter of a mile away.

Different atoms will have various numbers of electrons in the concentric shells about the nucleus. The first shell, which is closest to the nucleus, can contain only two electrons. If an atom should have three electrons, the third must go to the next shell. The second shell can contain a maximum of eight electrons, the third 18, and the fourth 32, as determined by the equation $2n^2$, where n is the shell number. These shells are usually denoted by a number (n = 1, 2, 3, ...) or letter (n = k, l, m, ...).

Each shell is then broken down into subshells, where the first subshell can contain a maximum of two electrons, the second subshell six electrons, the third 10 electrons, and the fourth 14, as shown in Fig. 2.2. The subshells are usually denoted by the letters s, p, d, and f, in that order, outward from the nucleus.

It has been determined by experimentation that unlike charges attract, and like charges repel. The force of attraction or repulsion between two charged bodies Q_1 and Q_2 can be determined by Coulomb's law:

$$F \text{ (attraction or repulsion)} = \frac{kQ_1Q_2}{r^2} \qquad (2.1)$$

Where F is in Newton, k = constant = 9.0×10^9 $N.m/c^2$, Q_1 and Q_2 are the charges 1,2 coulombs (to be introduced in Section 2.2), and r is the distance in meters between the two charges. In particular, note the squared r term in the denominator resulting in rapidly decreasing levels of F for increasing values of r.

In the atom, therefore, electrons will repel each other, and protons and electrons will attract each other. Since the nucleus consists of many positive charges (protons), a strong attractive force exists for the electrons in orbits close to the nucleus [note the effects of a large charge Q and a small distance r in Eq. (2.1)]. As the distance between the nucleus and the orbital electrons increases, the binding force diminishes until it reaches its lowest level at the outermost subshell (largest r). Due to the weaker binding forces, less energy must be expended to remove an electron from an outer subshell than from an inner subshell. Also, it is generally true that electrons are more readily removed from atoms having outer subshell that are incomplete and, in addition, possess few electrons. These properties of the atom that permit the removal of electrons under certain conditions are essential if motion of charge is to be created.

Copper is the most commonly used metal in the electrical / electronics industry. An examination of its atomic structure will help identify why it has such widespread applications. The copper atom (Fig. 2.2) has one more electron than needed to complete the first three shells. This incomplete outermost subshell, possessing only one electron, and the distance between this electron and the nucleus, reveal that the twenty-ninth electron is loosely bound to the copper atom. If this twenty-ninth electron gains sufficient energy from the surrounding medium to leave its parent atom, it is called a free electron. In one cubic inch of copper at room temperature there are approximately = 1.4 X 10^{+24} free electrons. Copper also has the advantage of being able to be drawn into long thin wires (ductility) or worked into many different shapes (malleability). Other metals that exhibit the same properties as copper, but to a different degree, are silver, gold, platinum, and aluminum. Gold is used extensively in integrated circuits where the performance level and amount of material required balance the cost factor. Aluminum has found some commercial use but suffers from being more temperature sensitive (expansion and contraction) than copper.

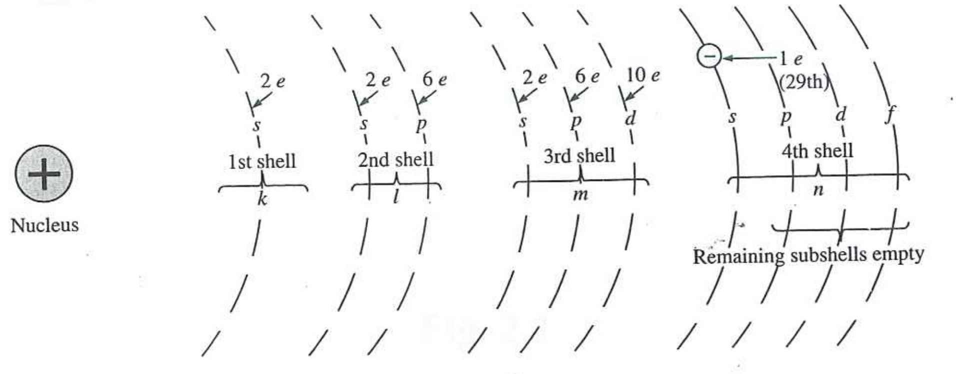


FIG. 2.2
The copper atom.

CURRENT

Consider a short length of copper wire cut with an imaginary perpendicular plane, producing the circular cross section shown in Fig. 2.3. At room temperature with no external forces applied, there exists within the copper wire the random motion of free electrons created by the thermal energy that the electrons gain from the surrounding medium. When an atom loses its free electron, it acquires a net positive charge and is referred to as a positive ion. The free electron is able to move within these positive ions and leave the general area of the parent atom, while the positive ions only oscillate in a mean fixed position. For this reason, the free electron is the charge carrier in a copper wire or in any other solid conductor of electricity.

An array of positive ions and free electrons is depicted in Fig.2.4. Within this array, the free electrons find themselves continually gaining or losing energy by virtue of their changing direction and velocity. Some of the factors responsible for this random motion include (1) the collisions with positive ions and other electrons, (2) the attractive forces for the positive ions, and (3) the force of repulsion that exists between electrons. This random motion of free electrons is such that over a period of time, the number of electrons moving to the right across the circular cross section of Fig. 2.3 is exactly equal to the number passing over to the left.

With no external forces applied, the net flow of charge in a conductor in any one direction is zero.

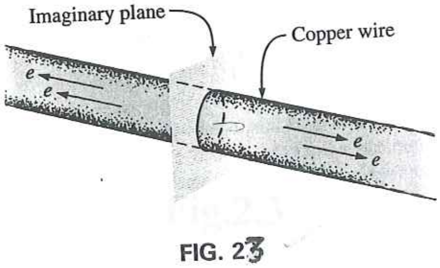


FIG. 2.3

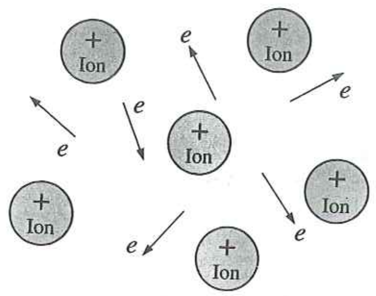


Fig.2.4

Let us now connect this copper wire between two battery terminals as shown in Fig.2.5. The battery, at the expense of chemical energy, places a positive charge on one terminal and a net negative charge on the other. The instant the wire is connected between these two terminals, the free electrons of the copper wire drift toward the positive terminal, while the positive ions will simply oscillate in a mean fixed position. The negative terminal is a supply of electrons to be drawn from when the electrons of the copper wire drift toward the positive terminal. The chemical activity of the battery will absorb the electrons at the positive terminal and maintain a steady supply of electrons at the negative terminal.

If 6.242×10^{18} electrons drift at uniform velocity through the imaginary circular cross section of Fig.2.5 in 1 second, the flow of charge, or current, is said to be 1 ampere (A)

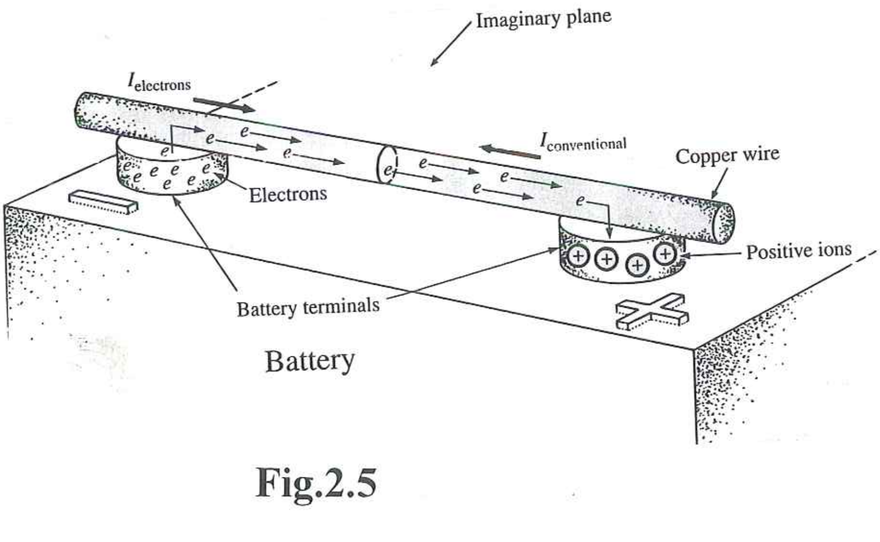


Fig.2.5

The current in amperes can now be calculated using the following equation:

$$I = \frac{Q}{t} \quad \left.\begin{array}{l} \mathbf{I = amperes\ (A)} \\ \mathbf{Q = coulombs\ (C)} \\ \mathbf{t = seconds\ (s)} \end{array}\right] \quad 2.2$$

Through algebraic manipulations, the other two quantities can be determined as follows:

$$\mathbf{Q = It\ (coulombs\ (C)} \qquad (2.3)$$

and

$$t = \frac{Q}{I} \quad \mathbf{(seconds,\ S)} \qquad (2.4)$$

Example 2.1 **The charge flowing through the imaginary surface of Fig. 2.5 is 0.16 C every 64 ms. Determine the current in amperes.**

Solution: Eq. (2.2):

$$I = \frac{Q}{t} = \frac{0.16\ \mathrm{C}}{64 \times 10^{-3}\ \mathrm{s}} = \frac{160 \times 10^{-3}\ \mathrm{C}}{64 \times 10^{-3}\ \mathrm{s}} = \mathbf{2.50\ A}$$

Example 2.2 **Determine the time required $4\ \mathrm{X}\ 10^{16}$ electrons to pass through the imaginary surface of Fig. 2.5 if the current is 5 mA.**

Solution: Determine Q:

$$4 \times 10^{16}\ \cancel{\text{electrons}}\left(\frac{1\ \mathrm{C}}{6.242 \times 10^{18}\ \cancel{\text{electrons}}}\right) = 0.641 \times 10^{-2}\ \mathrm{C}$$
$$= 0.00641\ \mathrm{C} = 6.41\ \mathrm{mC}$$

Calculate t [Eq. (2.4)]:

$$t = \frac{Q}{I} = \frac{6.41 \times 10^{-3}\ \mathrm{C}}{5 \times 10^{-3}\ \mathrm{A}} = \mathbf{1.282\ s}$$

A second glance at Fig. 2.5 will reveal that two directions of charge flow have been indicated. One is called conventional flow, and the other is called electron flow. This

text will deal only with conventional flow for a variety of reasons, including the fact that it is the most widely used at educational institutions and in industry, is employed in the design of all electronic device symbols, and is the popular choice for all major computer software packages. The flow controversy is a result of an assumption made at the time electricity was discovered that the positive charge was the moving particle in metallic conductors. Be assured that the choice of conventional flow will not create great difficulty and confusion in the chapters to follow:

2.3 VOLTAGE

The flow of charge described in the previous section is established by an external "pressure" derived from the energy that a mass has by virtue of its position: potential energy,

Energy, by definition, is the capacity to do work. If a mass (m) is raised to some height (h) above a reference plane, it has a measure of potential energy expressed in joules (J) that is determined by

$$w \text{ (potential energy)} = mgh \quad \text{(joules, J)} \qquad (2.5)$$

where g is the gravitational acceleration (9.754 m/s^2). This mass now has the "potential" to do work such as crush an object placed on the reference plane. If the weight is raised further, it has an increased measure of potential energy and can do additional work. There is an obvious difference in potential between the two heights above the reference plane.

In the battery of Fig. 2.5 the internal chemical action will establish (through an expenditure of energy) an accumulation of negative charges (electrons) on one terminal (the negative terminal) and positive charges (positive ions) on the other (the positive terminal). A "positioning" of the charges has been established that will result in a potential difference between the terminals. If a conductor is connected between the terminals of the battery, the electrons at the negative terminal have sufficient potential energy to overcome collisions with other particles, in the conductor and the repulsion from similar charges to reach the positive terminal to which they are attracted.

Charge can be raised to a higher potential level through the expenditure of energy from an external source, or it can lose potential energy as it travels through an electrons system. In any case, by definition:

A potential difference of 1 volt (V) exists between two points if 1 joule (J) of energy is exchanged in moving 1 coulomb (C) of charge between the two points.

The unit of measurement is volt was chosen to honor

Pictorially, if one joule of energy (l J) is required to move the one coulomb (l C) of charge of Fig. 2.6 from position x to position y, the potential difference or voltage between the two points is one volt (l V). If the energy required to move the l C of charge increases to 12 J due to additional opposing forces then the potential difference will increase to 12 V. Voltage is therefore an indication of how much energy is involved in moving a charge between two points in an electrical system. Conversely, the higher the voltage rating of an energy source such as a battery, the more energy available to move charge through the system. Note in the above discussion that two points are always involved when talking about voltage or potential difference. In the future, therefore, it is very important to keep in mind that a potential difference or voltage is always measured between two points in the system. Changing either point may change the potential difference between the two points under investigation.

In general, the potential difference between two points is determined by

$$V = \frac{W}{Q} \quad \text{(volts)} \tag{2.6}$$

Through algebraic manipulations, we have

$$W = QV \quad \text{(joules)} \tag{2.7}$$

And $$V = \frac{W}{Q} \quad \text{(coulombs)} \tag{2.8}$$

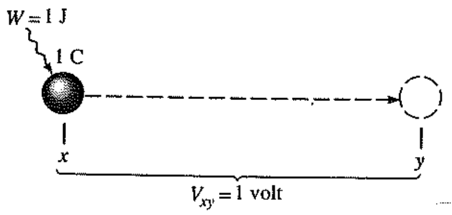


Fig.2.6
Defining the unit of measurement for voltage.

EXAMPLE: Find the potential difference between two points in an electrical system if 60 J of energy are expended by a charge of 20 C between these two points.

Solution: Eq. (2.6):

$V = 60/20 = 30V$

EXAMPLE: Determine the energy expended moving a charge of 50 μC potential difference of 6 V.

Solution: Eq. (2.7):

$W = QV = 0.3\ mJ$

Notation plays a very important role in the in the analysis of electrical and electronic systems. To distinguish between sources of voltage (batteries and the like) and losses in potential across dissipative elements, the following notation will be used:

E for voltage sources (volts)
V for voltage drops (volts)

An occasional source of confusion is the terminology applied to this subject matter. Terms commonly encountered include potential, potential difference, voltage, voltage difference (drop or rise), and electromotive force. As noted in the description above, some are used interchangeably.

In summary, the applied potential difference (in volts) of a voltage source in an electric circuit is the "pressure" to set the system in motion and "cause" the flow of charge or current through the electrical system. A mechanical analogy of the applied voltage is the pressure applied to the water in a main. The resulting flow of water through the system is likened to the flow of charge through an electric circuit. Without the applied pressure from the spigot, the water will simply sit in the hose, just as the electrons of copper wire do not have a general direction without an applied voltage.

2.4 FIXED (dc) SUPPLIES

The terminology dc employed in the heading of this section is an abbreviation for direct current, which encompasses the various electrical systems in which there is unidirectional ("one direction") flow of charge.

De Voltage Sources

Since the dc voltage source is the more familiar of the two types of supplies, it will be examined first. The symbol used for dc voltage supplies appears in Fig.2.7. The relative lengths of the bars indicate the terminals they represent.

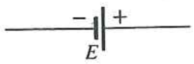


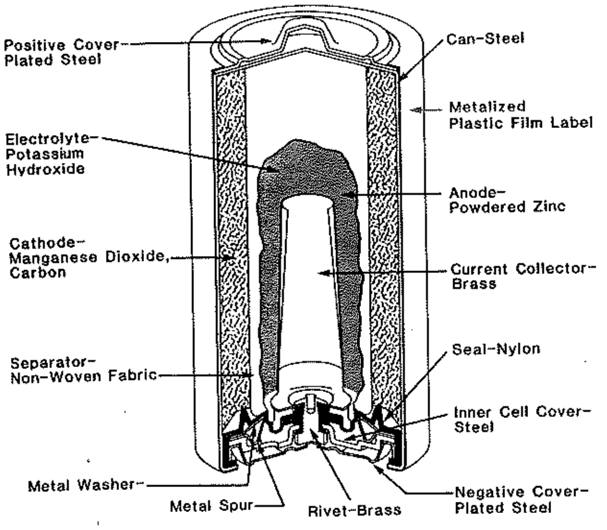
Fig.2.7

Dc voltage sources can be divided into three broad categories:
(1) batteries (chemical action)
(2) generators (electromechanically),
(3) power supplies (rectification).

Batteries For the layperson, the battery is the most common of the dc sources. By definition, a battery (derived from the expression "battery of cells") consists of a combination of two or more similar cells. A cell being the fundamental source of electrical energy developed through the conversion of chemical or solar energy. All cells can be divided into the primary or secondary types. The secondary is rechargeable, whereas the primary is not. That is, the chemical reaction of the secondary cell can be reversed to restore its capacity. The two most common rechargeable batteries are the lead-acid unit (used primarily in automobiles) and the nickel-cadmium battery (used in calculators, tools, photoflash units, shavers, and so on). The obvious advantage of the rechargeable unit is the reduced costs associated with not having to continually replace discharged primary cells.

All the cells appearing in this chapter except the solar cell, which absorbs energy from incident light in the form of photons, establish a potential difference at the expense of chemical energy. In addition, each has a positive and a negative electrode and an electrolyte to complete the circuit between electrodes within the battery. The electrolyte is the contact element and the source of ions for conduction between the terminals.

The popular alkaline primary battery employs a powdered zinc anode (+), a potassium (alkali metal) hydroxide electrolyte, and a manganese dioxide, carbon cathode (-) as shown in Fig.2.8 (a). In particular, note in Fig.2.8 (b) that the larger the cylindrical unit the higher the current capacity. The lantern is designed primarily for long-term use.



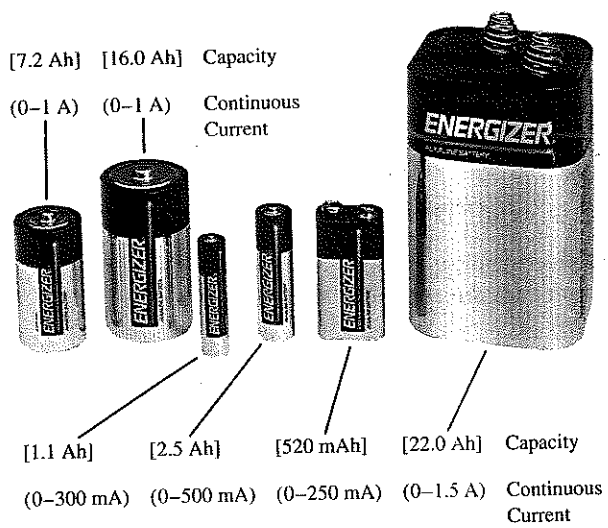


Fig. 2.8

For the secondary lead-acid unit appearing in Fig.2.9 the electrolyte is sulfuric acid and the electrodes are spongy lead (Pb) and lead peroxide (Pbo_2). When a load is applied to the battery terminals, there is a transfer of electrons from the spongy lead electrode to the lead peroxide electrode through the load. This transfer of electrons will continue until the battery is completely discharge. The discharge time is determined by how diluted the acid has become and how heavy the coating of lead sulfate is on each plate. The state of discharge of a lead storage cell can be determined by measuring the specific gravity of electrolyte with a hydrometer. The specific gravity of a substance is defined to be the ratio of the weight of a given volume of the substance to the weight of an equal volume of water at 4°C. For fully charged batteries, the specific gravity shout be somewhere between 1.28 and 1.30. When the specific gravity drops to about 1.1, the battery should be recharged.

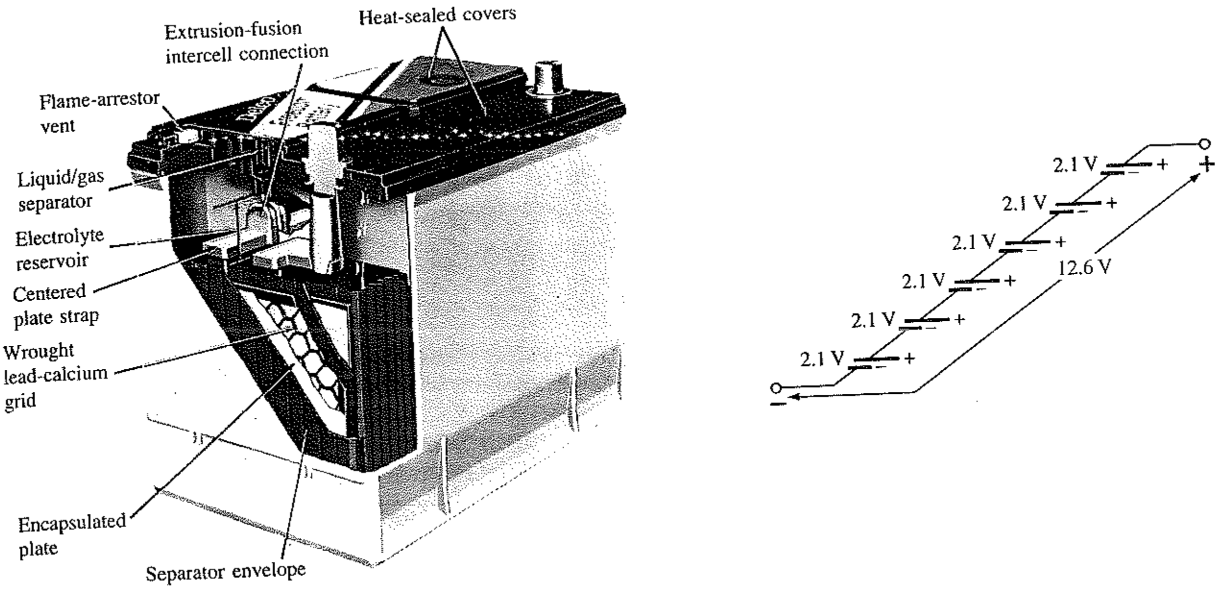


Fig.2.9

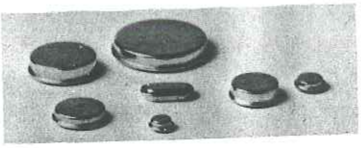
Since the lead storage cell is a secondary cell, it can be recharged at any point during the discharge phase simply by applying and external dc source across the cell that will pass current through the cell in a direction opposite to that in which the cell supplied current to the load. This will remove the lead sulfate from the plates and restore the concentration of sulfuric acid.

The output of lead storage cell over most of the discharge phase is about 2.1 V. In the commercial lead storage batteries used in the automobile, the 12.6 V can be produced by six cells in series, as shown in Fig. 2.9. The use of a grid made from a wrought lead-calcium alloy strip rather than the lead-antimony cast grid commonly used has results in maintenance-free batteries such as that appearing in the same figure. The lead-antimony structure was susceptible to corrosion, overcharge, gassing, water usage, and self- discharge. Improved design with the lead-calcium grid has either eliminated or substantially reduced most of these problems.

The nickel-cadmium battery is rechargeable battery that has been receiving enormous interest and development in recent years. A number of such batteries manufactured by the Eveready Battery Corporation appear in Fig.2.10. In the fully charged condition the positive electrode is nickel hydroxide [$Ni(OH)_2$]; the negative electrode, metallic cadmium (Cd); and the electrolyte, potassium hydroxide (KOH). The oxidation (increased oxygen content) of the negative electrode occurring simultaneously with the reduction of the positive electrode provides the required electrical energy. The advantage of such cells is that the active materials go through a change in oxidation state necessary to establish the required ion level without a change in the physical sate. This establishes an excellent recovery mechanism for the recharging phase.

1.2 V 4 Ah | 1.2 V 1.2 Ah | 7.2 V 100 mAh | 1.2 V 500 mAh | 1.2 V 180 mAh

(a)



Eveready® BH 500 cell
1.2 V, 500 mAh
App: Where vertical height is severe limitation

(b)

Fig.2.10

The life (hours) of any battery is given by the following equation:

$$Life(hours) = \frac{amper - hour\ \ rating(Ah)}{amper\ \text{drawn}\ (A)} \qquad \textbf{(2.9)}$$

Problems:

1-A charge of 16 C flows through a surface every 64S, determine the current in amperes.

2-Deteremine the time required for 2 X 10^{16} electrons to pass through a surface if the current is 2.5 mA.

3-Find the potential difference between two points in an electrical system if 160 J are expended by a charge of 120C between these two points.

4-If a current of 40A exists for 1 min how many coulombs of charge have passed through the wire?

5-Will a fuse rated at l A "blow" if 86C pass through it in 1.2 min?

6-What would you prefer? a: A piaster for ever electron that passes through a wire in 0.01 μ s at a current of 2 mA OR b: A pound for every electron that passes through a wire in 1.5 ns if the current is 100 μ A.

7-What is the voltage between 2 points if 96 mJ of energy are required to move 50 X 10^{18} electrons between the 2 points?

8-If the potential difference between 2 points is 42V, how much work is required to bring 6C from one point to the other?

9-If a conductor with a current of 200 m.A passing through it converts 40 J of electrical energy into heat in 30S, what is the potential drop across the conductor?

10-Charge is flowing through a conductor at the rate of 420 c/min. If 742 J of electrical energy are converted to heat in 30 sec, what is the potential drop across the conductor?

11-The potential difference between 2 points in an electric circuit is 24 V. If 0.4 J energy were dissipated in a period of 5 ms, what would the current be between the two points?

12-A portable TV using 12 V, 3-Ah battery cooperate for a period of about 5.5 h. What is the average current drawn during this? What is the energy expended by the battery in Joules?

17-What is the cost of using the following at 8 piasteres /kWhr? 110 W stereo set for 4h, 1200-W projector for 20 min, 60-W tape recorder for 1.5 hr and 150 W color television set for 3h 45 min .

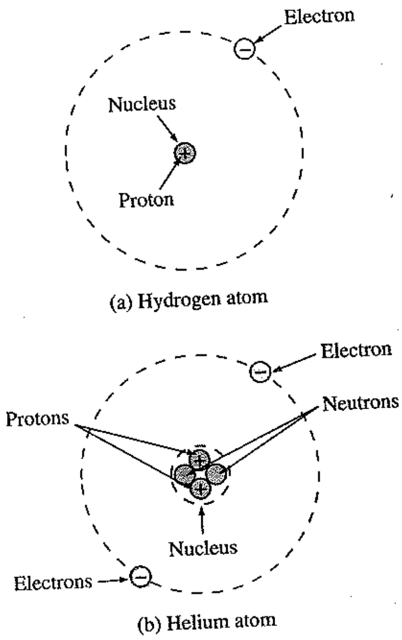


FIG. 2.1
The hydrogen and helium atoms.

Chapter Three

RESISTANCE, OHM'S LAW AND KIRCHOFF's LAWS

INTRODUCTION

The flow of charge through any material encounters an opposing force similar in many respects to mechanical friction. This opposition, due to the collisions between electrons and between electrons and other atoms in the material, which converts electrical energy into heat, is called the resistance of the material. The unit of measurement of resistance is the ohm, for which the symbol is Ω, the capital Greek letter omega. The circuit symbol for resistance appears in Fig. 3.2 with the graphic abbreviation for resistance (R).

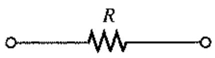


FIG. 3.1
Resistance symbol and notation.

The resistance of any material with a uniform cross-sectional areas is determined by the following four factors:
1-Material
2-Length
3-Cross-sectional area
4-Temperature

The chosen material, with its unique molecular structure, will react differentially to pressures to establish current through its core. Conductors that permit a generous flow of charge with little external pressure will have low resistance levels, while insulators will have high resistance characteristics.

As one might expect, the longer the path the charge must pass through, the higher the resistance level, whereas the larger the area (and therefore available room), the lower the resistance. Resistance is thus directly proportional to length and inversely proportional to area.

As the temperature of most conductors increases, the increased motion of the particles with the molecular structure makes it increasingly difficult for the "free" carriers to pass through, and the resistance level increases.

At fixed temperature of 20°C (room temperature), the resistance is related to the other three factors by

$$R = \rho \frac{\ell}{A} \text{ (ohms, } \Omega) \tag{3.1}$$

where ρ (Greek letter rho) is a characteristics of the material called the resistively, ℓ is the length of sample, and A is the cross-sectional area of the sample.

3.2 RESISTANCE ; CIRCULAR WIRES

For a circular wire, the quantities in Eq. (3.1) are defined by Fig, 3.2. For two wires of the same physical size at the same temperature, as shown in Fig. 3.3(a),

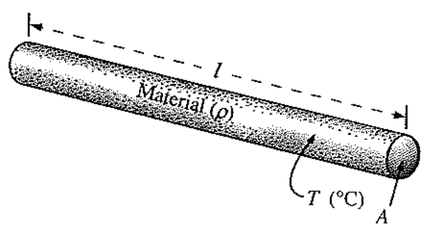


FIG. 3.2
Factors affecting the resistance of a conductor.

The higher the resistively, the more the resistance.

As indicated in Fig.3.3(b),

The longer the length of a conductor, the more the resistance.

Figure 3.3(c) reveals for remaining similar determining variables that the smaller the area of a conductor, the more the resistance.

Finally, Fig.3.3(d) states for metallic wires of identical construction and material

The higher the temperature of a conductor, the more the resistance.

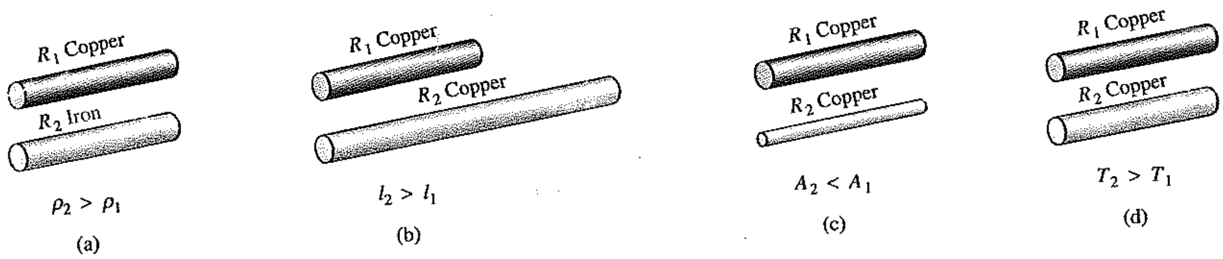


FIG. 3.3

Cases in which $R_2 > R_1$. For each case, all remaining parameters that control the resistance level are the same.

Ohm's Law:

The relation between the voltage across resistor is proportional to the current following in it, i.e.,

$V \alpha I$ or $V = IR$

where R is the resistance of the resistor. This relation is known as Ohm's law. It can take different forms such as :

$$R = V/I \quad \text{and} \quad I = V/R \tag{3.1}$$

The SI unit of the resistance is the ohm with symbol Ω. Ohm's law means that the relation between voltage and current is linear as shown in Fig. 3.4. However, there are nonlinear elements whose relation is nonlinear as shown in Fig. 3.5.

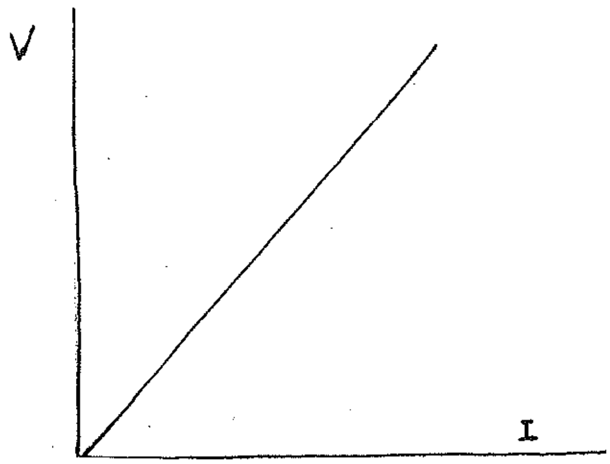


Figure 3.4

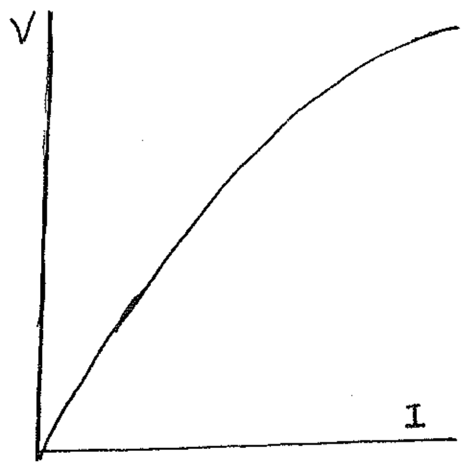


Figure 3.5

In this book, only linear elements of resistor are considered. Another form of Ohm's law is:

$$I = GV \tag{3.2}$$

In which G is the constant of proportionality. This G is the quantity symbol for conductance. The SI unit of conductance is the siemens with symbol S.

Of course G = I/R and also has the unit of mho ℧. The greater the conductance, the greater the current for a given voltage.

Table 3-1 gives resistivities at 20°C for some materials arranged in order of their values. Note the wide range of resistivities. A good conductor has a resistivity close to 10^{-8} Ω.m.

Table 3-1

Material	*Resistivity (ohm-meters)*
Silver	1.64×10^{-8}
Copper, annealed	1.72×10^{-8}
Copper, hard-drawn	1.78×10^{-8}
Gold	2.45×10^{-8}
Aluminum	2.83×10^{-8}
Constantan	49×10^{-8}
Nichrome	100×10^{-8}
Germanium	0.45
Silicon	2500
Water	5000
Paper	10^{10}
Mica	5×10^{11}
Pyrex glass	10^{12}
Quartz	10^{17}

Materials with resistivities greater than 10^{10} Ω.m are insulators.

Materials with resistivities in the range 10^{-4} to 10^{7} Ω.m are classed as semiconductors.

Example

What is the resistance of annealed copper 10 m length and 1 mm in diameter. Compare this value with that obtained if the wire is made of Aluminum.

Solution :

For annealed copper $$R = \frac{1.72 \text{ x } 10^{-8} \text{ x } 10}{\Pi(0.5)^2 \text{ x } 10^{-6}} = 0.2195\,\Omega$$

For Aluminum $$R = \frac{2.83 \text{ x } 10^{-8} \text{ x} 10}{\Pi(0.5)^2 \text{ x} 10^{-6}} = 0.3612\,\Omega$$

$P = VI = I^2 R = V^2/R$

Obviously, copper has less resistance, but it is more expensive than aluminum.

The resistance of metallic conductors increases with rise in temperature. If R_o is the resistance of a conductor at 0^oC, R_t at t^oC and t is the rise in temperature, then :

$$R_t = R_o (1 + \alpha_o t) \qquad (3\text{-}5)$$

where α_0 is the temperature coefficient of resistance at 0^oC and is defined as :

The increase in resistance per ohm for a rise in temperature of 1^oC from 0^oC .

It is found that the value of temperature-coefficient is not the same at all temperatures even for a given material.

Its value depends upon the initial temperature on which the increment in resistance is based.

If α_0 **= tempt. coeff. at 0°C**

α_1 **= tempt. coeff. at** t_1 **°C**

α_2 **= tempt. coeff. at** t_2 **°C**

Then
$$\alpha_1 = \frac{\alpha_0}{1 + \alpha_0 t_1} \qquad \textbf{(3-6)}$$

and
$$\alpha_2 = \frac{\alpha_0}{1 + \alpha_0 t_2} \qquad \textbf{(3-7)}$$

The relation between α_1 **and** α_2 **is given by :**

$$\alpha_2 = \frac{1}{1/\alpha_1 + (t_2 - t_1)} \qquad \textbf{(3-8)}$$

The resistivity, like resistance, also increases with temperature and the increase is given by the relation,

$$\rho_t = \rho_0(1 + \alpha_0 t) \qquad \textbf{(3-9)}$$

Example

Calculate the resistance of a 100 m length of wire having a uniform cross-sectional area of 0.1 mm^2 if the wire is made of manganese having a resistivity of 50 μΩ-cm.

Solution :

Formula used : $R = \rho \frac{\ell}{a}$

Here ρ = **50 x 10^{-6} Ω-cm**

ℓ = **100 m = 10,000 cm = 10^4 cm**

a = 0.1 mm^2 = 10^{-3} cm^2

R = 50 x 10^{-6} x 10^4/10^{-3} = 500 Ω

Example 3-3 :

An aluminum wire 750 cm long is connected in parallel with a copper wire 600 cm long. when a current of 5A is passed through the combination, it is found that the current in the aluminum wire is 3A. The diameter of the aluminum wire is 1 mm. Determine the diameter of the copper wire. Resistivity of copper is 1.7 μΩ-cm, aluminum 2.8 μΩ-cm.

Solution :

Let R_1 and R_2 be the resistances of the aluminum and copper wire respectively.

Current through aluminum wire, $I_1 = 3A$

Current through copper wire, $I_2 = 2A$

Since currents are inversely proportional to resistances,

$$\therefore \qquad \frac{R_1}{R_2} = \frac{I_2}{I_1} = \frac{2}{3}$$

Now $\qquad R_1 = \rho_1 \ell_1/a_1$

and $\qquad R_2 = \rho_2 \ell_2/a_2$

$$\therefore \qquad \frac{R_1}{R_2} = \frac{\ell_1}{\ell_2} \times \frac{a_2}{a_1} \times \frac{\rho_1}{\rho_2}$$

or $$\frac{2}{3} = \frac{750}{600} \times \frac{a_2}{a_1} \times \frac{2.8}{1.7}$$

$$\frac{a_2}{a_1} = \frac{2}{3} \times \frac{600}{750} \times \frac{1.7}{2.8} = \frac{34}{105}$$

or $$\frac{\Pi d^2/4}{\Pi \times 1^2/4} = \frac{34}{105}$$

$$\therefore \qquad d = \sqrt{34/105} = 0.569 \text{ mm}$$

Example

A specimen of copper wire has a specific resistance of 1.7×10^{-6} ohm-cm at $0^{\circ}C$ and has temperature coefficient of 1/254.5 at $20^{\circ}C$. Find the specific resistance and temperature coefficient at $70^{\circ}C$.

Solution :

Given : $\rho_0 = 1.7 \times 10^{-6}$ **ohm-cm**

$\alpha_{20} = 1/254.5$

Now $$\alpha_{20} = \frac{\alpha_0}{1 + \alpha_0 \times 20}$$

$$\therefore \quad \frac{1}{254.5} = \frac{\alpha_0}{1 + 20\alpha_0} \qquad \therefore \quad \alpha_0 = 1/234.5$$

Now

$$\alpha_{70} = \frac{\alpha_0}{1 + 70\alpha_0}$$

$$= \frac{1/234.5}{(1 + 70/234.5)} = 1/304.5$$

$$\rho_{70} = \rho_0(1 + 70\alpha_0)$$

$$= 1.7 \times 10^{-6}\left(1 + \frac{70}{234.5}\right)$$

$$= 2.208 \times 10^{-6} \text{ ohm-cm}$$

Example

The specific resistance of platinum at 0^oC is 10.3 $\mu\Omega$-cm. How long must a wire of No. 32 S.W.G. (diameter = 0.0274 cm) platinum be to have a resistance of 4Ω at 0^oC. What will be the resistance of the wire at 100^oC if the temperature coefficient of platinum is 0.0038 per ^oC ?

Solution :

Formula used $$R_0 = \rho_0 \frac{\ell}{a}$$

Here $R_0 = 4\Omega$; $\rho_0 = 10.3 \times 10^{-6}\Omega\text{-cm}$

$$\text{diameter} = 0.0274 \text{ cm} = 2.74 \times 10^{-2} \text{ cm}$$

$$a = \Pi(2.74 \times 10^{-2})^2/4 = \Pi \times 1.37^2 \times 10^{-4} \text{ cm}^2$$

$$\ell = aR_0/\rho_0$$

$$= \frac{\Pi \times 1.37^2 \times 10^{-4} \times 4}{10.3 \times 10^{-6}} = 229.1 \text{ cm}$$

Now $$R_t = R_0(1 + \alpha t)$$

$$= 4(1 + 100 \times 0.0038)$$

$$= 5.52\ \Omega$$

Example

Find the resistance of a cable 9.660 m long and of cross-section 64.5 mm^2 at a mean temperature of 35°C , assuming that the resistance of 1 m of copper of 1 mm^2 area is 1/58 Ω at 20°C and the temperature coefficient is 1/234.5 $°C^{-1}$ at 0°C.

Solution :

$$\alpha_{20} = \frac{\alpha_0}{1 + 20\alpha_0}$$

$$= \frac{1/234.5}{1 + (20/234.5)} = \frac{1}{254.5}\ {}^{o}C^{-1}$$

Now $\quad R_{35} = R_{20}\,[1 + \alpha_{20}(35 - 20)]$

The resistance of a cable 1 m long and 1 mm^2 cross-section at 20°C is 1/58 Ω. Hence, its resistance at 35°C is :

$$R_{35} = \frac{1}{58}\left[1 + \frac{1}{254.5}(35 - 20)\right]$$

$$= 1/55\ \Omega$$

The resistance of a cable 9.660 m long and cross-section 64.5 mm^2 at 35 °C is :

$$= \frac{1}{55} \times \frac{9.660}{64.5} = 2.72\ \Omega \times 10^{-3}$$

Example :

A coil of copper wire has a resistance of 90 Ω at 20°C and is connected to a 230-V supply. By how much must the voltage be increased in order to maintain the current constant if the temperature of the coil rises to 60°C ? Take the temperature coefficient of resistance of copper as 0.00428 from 0°C.

Solution :

$$\alpha_0 = 0.00428\ °C^{-1}$$

Current at $20°C = 230/90 = 23/9A$

$$\alpha_{20} = \frac{\alpha_0}{1 + 20\alpha_0} = \frac{0.00428}{1 + 0.0856}$$

$$= 0.00428/1.0856 = 0.00394\ °C^{-1}$$

Now $R_{60} = R_{20}\ [1 + \alpha_{20}(60 - 20]$

$$= 90(1 + 40 \times 0.00394)$$

$$= 104.18\ \Omega$$

If the current is to be maintained constant at its previous value, then voltage required

$$= \frac{23}{9} x\,104.18 = 266.24 \text{ V}$$

Voltage increase = 266.24 - 230 = 36.24 V

Resistor Color Codes :

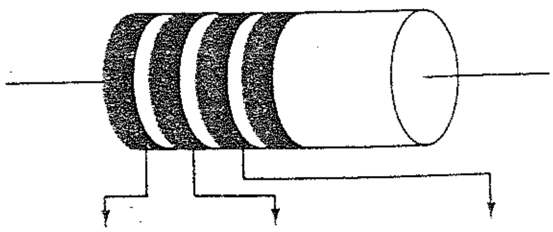
Resistors are either fixed or variable. We will consider fixed resistors. The most common fixed resistor is the carbon-composition resistor. It is a composition of carbon granules and an insulating powder mixed and bound together in the shape of a rod. Wire leads are embedded in the ends and an insulating coating covers the rod to make a cylindrically shaped package. Color bands around the resistor specify the resistance.

The first and second bands represent the first and second digits, respectively. The third band determines the power of 10 multiplier for the first two digits, actually the number of zeros that follow the second digit or a multiplying factor if gold or silver.

The fourth band is the manufacturer's tolerance which is an indication of the precision by which the resistor is made. If the fourth band is omitted the tolerance is assumed to be ± 20% gold and silver correspond to 5% & 10%. The fifth band is a reliability factor which gives the percentage of failure per 1000 hrs of use.

Table 3-2 gives Carbon Composition Resistor Color Codes. It is to be noted that these resistors are low-wattage.

Table 3-2 : Carbon-Composition Resistor Color Code



Color	*First Band Digit*	*Second Band Digit*	*Third Band Multiplier*
Black	0	0	$10^0 = 1$
Brown	1	1	$10^1 = 10$
Red	2	2	$10^2 = 100$
Orange	3	3	$10^3 = 1000$
Yellow	4	4	$10^4 = 10\ 000$
Green	5	5	$10^5 = 100\ 000$
Blue	6	6	$10^6 = 1\ 000\ 000$
Violet	7	7	$10^7 = 10\ 000\ 000$
Gray	8	8	$10^8 = 100\ 000\ 000$
White	9	9	$10^9 = 1\ 000\ 000\ 000$
Gold			0.10
Silver			0.010

Example **:**

A resistor has band colors in the order of yellow, violet, brown, and silver. Find the resistance.

Solution :

To find the resistance, we just match the colors with the digits.

yellow	violet	brown	silver
↓	↓	↓	↓
4	7	10^1	10%

The resistance is 470 Ω with a tolerance of 10%.

Example **:**

A resistor has band colors in the order of blue, gray, yellow, and gold. What is the resistance.

Solution :

We match the first three colors with digits from table 3-2 : blue (6), gray (8), and yellow (4). The corresponding value is 68×10^4. Of course, the gold band indicates a tolerance of 5%. So, the resistance is 680 kΩ with a tolerance of 5%.

Variable Resistors :

Variable resistors have a terminal resistance that can be varied by any method. Most of these resistors have three terminals. If this device is used as a variable resistor, it is usually referred as a rheostat. On the other hand, if it is used for controlling potential levels, it is commonly called a potentiometer. A variable resistor is shown in Fig. 3-6. The resistance Rac is always fixed at the full rated value, while Rab or Rbc can be varied from a minimum of zero ohms to a maximum value equal to the full rated value. In all cases, the following relation is hold

$$R_{ac} = R_{ab} + R_{bc} \tag{3 - 10}$$

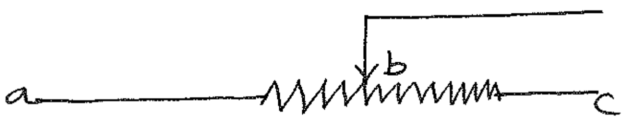


Figure 3.6

DC Voltage and Current Sources :

An ideal dc voltage source produces a constant voltage independent of the current flow through it. The current can be 1 A or 10^6 A, and in either direction, and the ideal dc voltage source still produces the same constant voltage.

Several circuit diagram symbols are used for ideal dc voltage sources. A battery symbol has pairs of long and short parallel lines as in Fig. 3-7 (a) and (b).

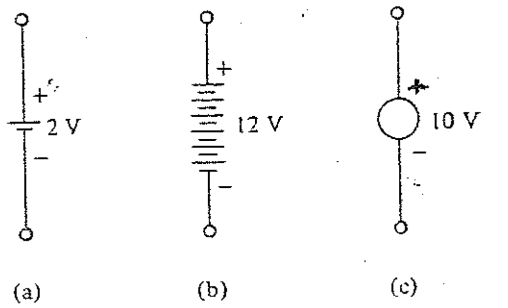


Figure 3-7

The source could be a generator or a power supply, although for these we would most likely use the symbol in Fig. 3-7 (c).

Actually, there are no ideal voltage sources. No practical voltage source produces a voltage independent of the current flow through it. A practical voltage source may, however, perform like an ideal voltage source in series with a resistor. (Components in series carry the same current.) Figure 3-8 shows this. This series component is called the internal resistance—not resistor—of the voltage source. When current flows from the source, the terminal voltage drops to less than the ideal source voltage because of the voltage drop across this internal resistance,

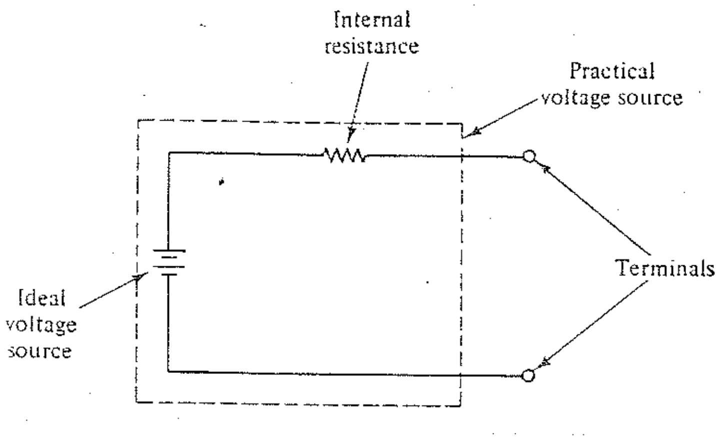


Figure 3-8

As mentioned, a voltage source has a relation between its voltage and current. Figure 3-9 (a) shows this relation for an ideal voltage source of 12 V. As shown by the horizontal line, the voltage source produces 12 V for any current, positive or negative. Figure 3-9 (b) shows the voltage-current relation for a practical 12-V source. The graph is a straight line with a negative slope, which means that the voltage decreases for increasing current.

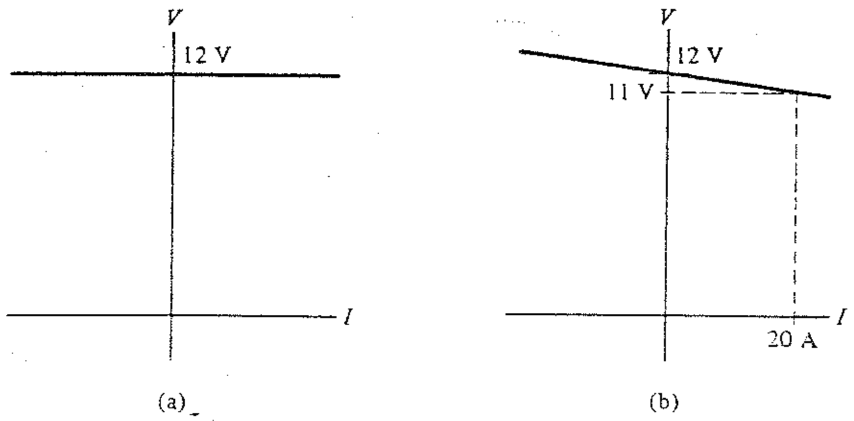


Figure 3-9 (a) , (b)

So much for dc voltage sources. The other kind of dc source is the dc current source. An ideal dc current source delivers a certain number of amperes irrespective of the voltage across it. The voltage can be 1 V, 10^6 V, or — 1000 V, and the dc ideal current source still delivers the same constant current.

Different symbols are used for ideal current sources. Two of these symbols are shown in Fig. 3-10.

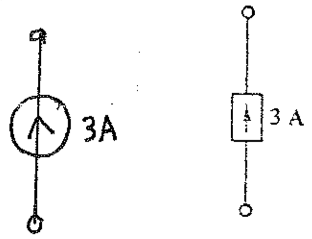


Figure 3-10

As might be expected from our ideal voltage source discussion, ideal current sources do not exist. However, many practical current sources act like an ideal current source in parallel with a resistor, as illustrated in Fig. 3-11. (Two components are in parallel when they have the same voltage across them).

This parallel resistor is called the internal resistance of the current source.

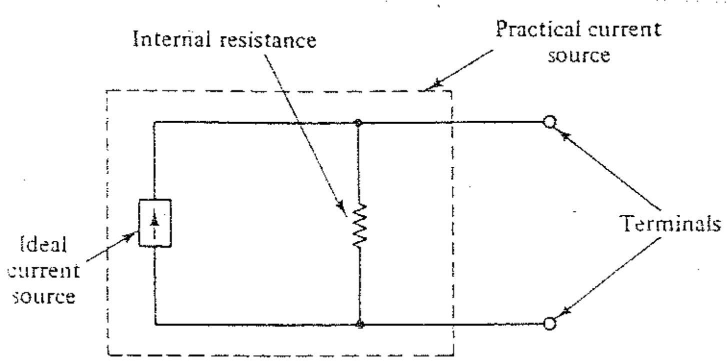


Figure 3-11

Topology of Electric Circuits :

The general characteristics of any electric circuit can be known if the geometrical properties are known. Any electric circuit consists of active elements (e.g. battery) or passive elements (e.g. resistor).

The following definitions are considered.

1- Node :

The first definition we will consider is that of a node. Nodes are the places in circuits at which components are connected together.

In Fig. 3-9a, there are five nodes (1, 2, 3, 4, 5).

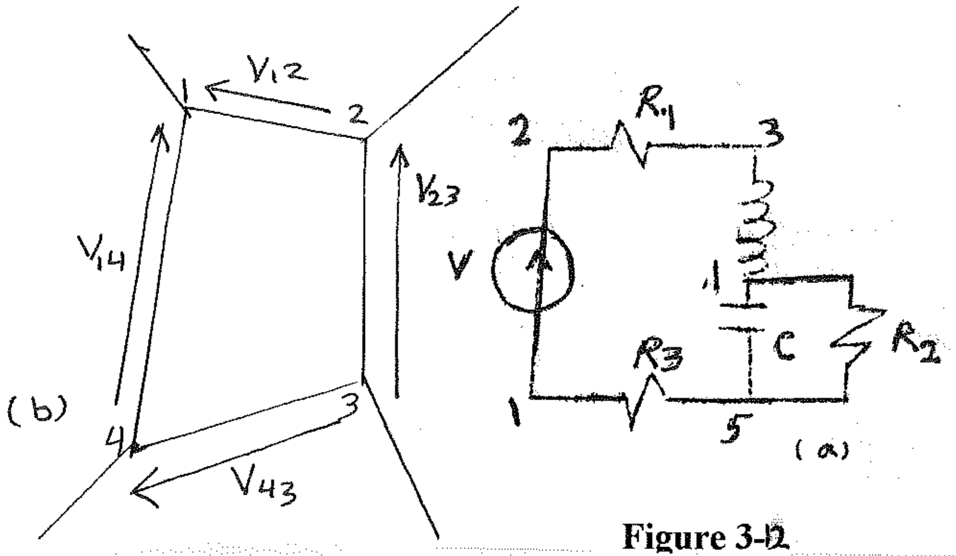


Figure 3-12

2- Branch :

Another circuit definition is that of a branch. A circuit branch is a single component, other than a short circuit, between two nodes. The circuit of Fig. 3-12a has six branches—four resistor branches one condenser and one voltage source branch. Sometimes we will relax this branch definition when a node connects just two branches and consider these two components to be a single branch, especially if the two branches contain the same type of component. We can do this in Fig. 3-12a for the three left branches that contain resistors. We can combine these resistors into a single equivalent resistor. In doing this we are in effect considering these branches to be a single branch consisting of a voltage source and one equivalent resistance. In this case, there are two nodes and three branches.

3- Loop and Mesh :

As explained above almost all circuits have closed paths. These closed paths have more common names : loops and meshes. A loop is any closed path. A mesh is a loop that does not have a closed path in its interior. In Fig. 3-12a there are three loops and two meshes.

The number of independent loops ℓ is given by

$$\ell = b - n + 1 \qquad (3\text{-}11)$$

where b is the number of branches in the circuit

n is the number of circuit nodes

In Fig. 3-12a $b = 3$, $n = 2$ $\therefore\ \ell = 2$

Kirchoff's Current Law :

Kirchoff's Current Law, abbreviated KCL, has three equivalent versions : At any instant in a circuit :

(1) The algebraic sum of the currents leaving a closed surface is zero.

(2) The algebraic sum of the currents entering a closed surface is zero.

(3) The algebraic sum of the currents entering a closed surface equals the algebraic sum of those leaving.

In our circuit applications the closed surfaces of interest are almost always those enclosing nodes. So, we might prefer to use nodes instead of closed surfaces in these three equivalent KCL versions.

As shown in Fig. 3-13, the following equation can be written :

$$I_1 + I_2 + I_4 - I_3 - I_5 = 0 \qquad (3\text{-}12)$$

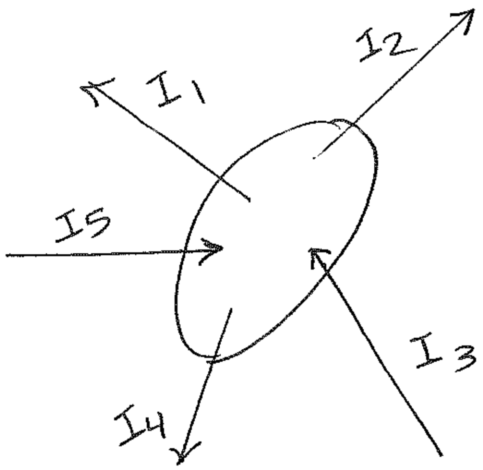


Figure 3-13

In equation (3-12) the currents leaving the closed surface are considered positive, while those entering the closed surface are negative.

If I_1 , I_2 , I_3, I_4 are given by 8 , 9 , 7 , 4 A respectively then $I_5 = I_1 + I_2 + I_4 - I_3$ or $I_5 = 14$ A

Kirchoff's Voltage Law :

Kirchoff's voltage law, abbreviated KVL, has three equivalent versions : At any instant around a loop, in either a clockwise or counterclockwise direction :

(1) The algebraic sum of the voltage drops is zero.

(2) The algebraic sum of the voltage rises is zero.

(3) The algebraic sum of the voltage drops equals the algebraic sum of the voltage rises.

The word algebraic just means that in adding, we include the signs of the voltage drops and rises. Remember that a voltage rise is a negative voltage drop and a voltage drop is a negative voltage rise.

Incidentally, the validity of KVL does not depend on having a circuit of wires and components. KVL applies as well to a closed path in space. But our interest is in its application to circuit analysis.

The circuit shown in Fig. 3-9 is taken as an example. Applying KVL in loop 12341, the following equation can be obtained :

$$V_{12} + V_{23} - V_{43} - V_{14} = 0$$

Another application of KVL is shown in Fig. 3-10 (a) and (b)

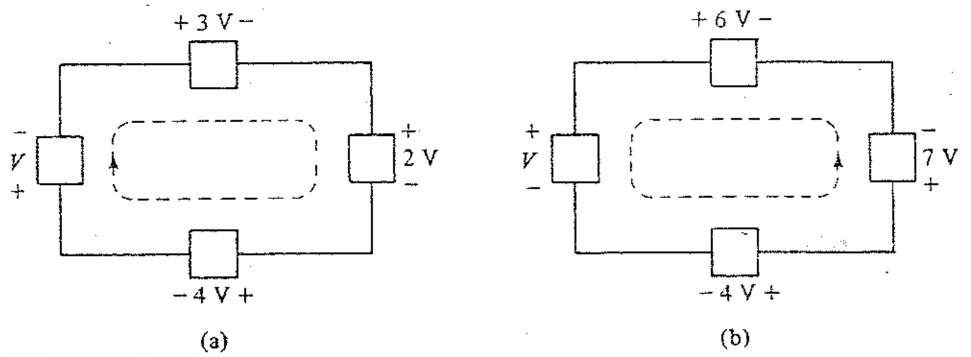


Figure 3-12

Assume that we want the unknown voltage V in each circuit. For the circuit of Fig. 3-12(a) we will arbitrarily use KVL version 1 and algebraically sum the voltage drops around the loop in a clockwise direction. In doing this we must remember that a voltage drop is from positive to negative. We can start at any point. Starting at the upper left-hand corner we have $3 + 2 + 4 + V = 0$, or $V = -9V$. We would get the same answer by going around the loop counterclockwise. It does not matter which way we go. For the circuit of Fig. 3-12(b) we will arbitrarily use KVL version 2 and sum the voltage rises around the loop in the indicated counterclockwise direction. Again, it does not matter where we start. Starting at the upper right-hand corner we have $6 - V + 4 - 7 = 0$ or $V = 3V$. In the KVL equation the unknown voltage V and the 7 V are negative because they are voltage drops and we are summing voltage rises. Going around clockwise gives the same answer.

EXAMPLE Determine the unknown voltages for the networks of Figs. 3.13 and 3.14 using Kirchhoff's voltage law.

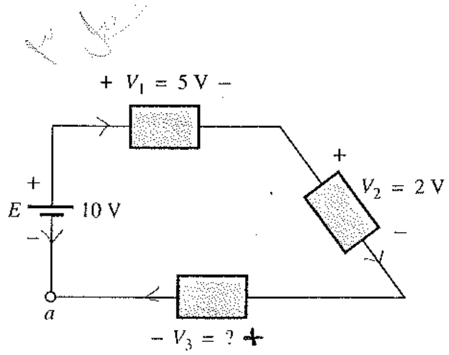


FIG. 3.13

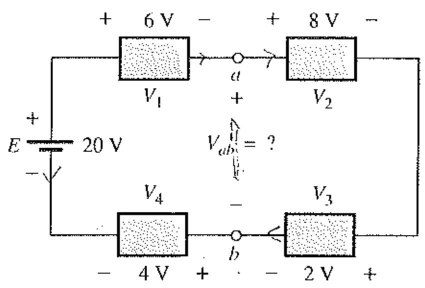


FIG. 3.14

Solution: In each case, note that we do not need to know whether there is a voltage source or dissipative element in each container. Once the magnitude and polarity of the voltage for each element are known, Kirchhoff's voltage law can be applied.

For Fig. 3.13, applying Kirchhoff's voltage law in the clockwise direction starting at point a will result in

$$E - V_1 - V_2 - V_3 = 0$$

and

$$V_3 = E - V_1 - V_2 = 10 - 5\text{ V} - 2\text{ V} = 10\text{ V} - 7\text{ V} = \mathbf{3\text{ V}}$$

In the case of Fig. 3.14, the voltage to be determined is not across a single element but between two points in the network. There are two routes to follow toward a solution.

First, Kirchhoff's voltage law can be applied in the clockwise direction around a closed loop including the voltage source E. That is,

$$+E - V_1 - V_{ab} - V_4 = 0$$

and

$$V_{ab} = E - V_1 - V_4 = 20\text{ V} - 6\text{ V} - 4\text{ V} = 20\text{ V} - 10\text{ V} = \mathbf{10\text{ V}}$$

The other alternative is to apply the law in a clockwise direction around a closed path that includes only V_2 and V_3. That is,

$$+V_{ab} - V_2 - V_3 = 0$$

and $$V_{ab} = V_2 + V_3 = 8\text{ V} + 2\text{ V} = \mathbf{10\text{ V}}$$

Note that both routes result in the same solution.

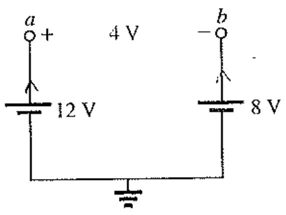


FIG. 3.15

It is important to realize that a difference in potential can exist between any two points in a network or even between adjoining networks. In other words, there is no requirement that the two points be connected by current-carrying elements. For instance, in Fig. 3.15 there is a difference in potential of 4 V between the positive terminals of the supplies even though the supplies are not joined by a current-carrying element.

EXAMPLE Find V_1 and V_2 for the network of Fig. 5.16.

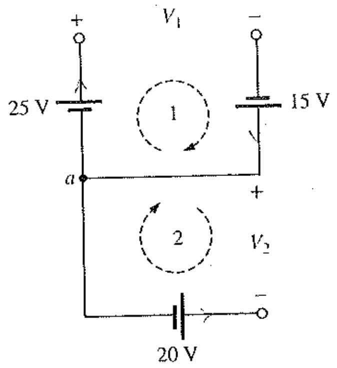


FIG. 5.16

Solution: For path 1, starting at point a in a clockwise direction:

$$+25\text{ V} - V_1 + 15\text{ V} = 0$$

and

$$V_1 = \mathbf{40\ V}$$

For path 2, starting at point a in a clockwise direction:

$$-V_2 - 20\text{ V} = 0$$

and

$$V_2 = \mathbf{-20\ V}$$

The minus sign simply indicates that the actual polarities of the potential difference are opposite the assumed polarity indicated in Fig. 5.16.

EXAMPLE For the circuit of Fig. 5.17:

a. Find R_T.
b. Find I.
c. Find V_1 and V_2.
d. Find the power to the 4-Ω and 6-Ω resistors.
e. Find the power delivered by the battery, and compare it to that dissipated by the 4-Ω and 6-Ω resistors combined.
f. Verify Kirchhoff's voltage law (clockwise direction).

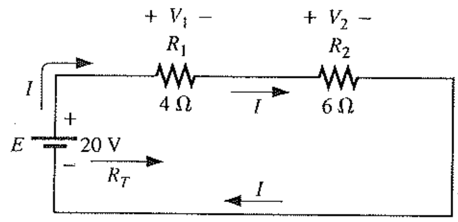


FIG. 5.17

Solutions:

a. $R_T = R_1 + R_2 = 4\ \Omega + 6\ \Omega = \mathbf{10\ \Omega}$

b. $I = \dfrac{E}{R_T} = \dfrac{20\text{ V}}{10\ \Omega} = \mathbf{2\ A}$

c. $V_1 = IR_1 = (2\text{ A})(4\ \Omega) = \mathbf{8\ V}$
$V_2 = IR_2 = (2\text{ A})(6\ \Omega) = \mathbf{12\ V}$

d. $P_{4\Omega} = \dfrac{V_1^2}{R_1} = \dfrac{(8\text{ V})^2}{4} = \dfrac{64}{4} = \mathbf{16\ W}$

$P_{6\Omega} = I^2R_2 = (2\text{ A})^2(6\ \Omega) = (4)(6) = \mathbf{24\ W}$

e. $P_E = EI = (20\text{ V})(2\text{ A}) = \mathbf{40\ W}$
$P_E = P_{4\Omega} + P_{6\Omega}$
$40\text{ W} = 16\text{ W} + 24\text{ W}$
$40\text{ W} = 40\text{ W}$ (checks)

f. $\Sigma_{\circlearrowright} V = +E - V_1 - V_2 = 0$
$E = V_1 + V_2$
$20\text{ V} = 8\text{ V} + 12\text{ V}$
$20\text{ V} = 20\text{ V}$ (checks)

EXAMPLE For the circuit of Fig. 5.18:

a. Determine V_2 using Kirchhoff's voltage law.
b. Determine I.
c. Find R_1 and R_3.

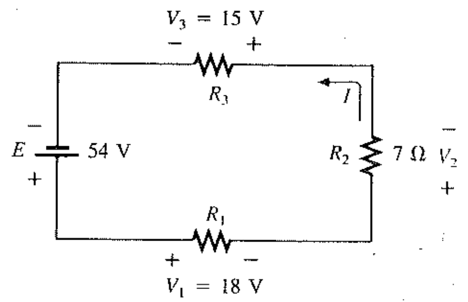


FIG. 5.18

Solutions:

a. Kirchhoff's voltage law (clockwise direction):

$$-E + V_3 + V_2 + V_1 = 0$$

or

$$E = V_1 + V_2 + V_3$$

and $V_2 = E - V_1 - V_3 = 54\text{ V} - 18\text{ V} - 15\text{ V} = \mathbf{21\ V}$

b. $I = \dfrac{V_2}{R_2} = \dfrac{21\text{ V}}{7\ \Omega} = \mathbf{3\ A}$

c. $R_1 = \dfrac{V_1}{I} = \dfrac{18\text{ V}}{3\text{ A}} = \mathbf{6\ \Omega}$

$R_3 = \dfrac{V_3}{I} = \dfrac{15\text{ V}}{3\text{ A}} = \mathbf{5\ \Omega}$

EXAMPLE Determine the currents I_3 and I_4 of Fig. 3.19 using Kirchhoff's current law.

Solution: We must first work with junction a, since the only unknown is I_3. At junction b there are two unknowns and both cannot be determined from one application of the law.

At a:

$$\Sigma I_{\text{entering}} = \Sigma I_{\text{leaving}}$$
$$I_1 + I_2 = I_3$$
$$2\text{ A} + 3\text{ A} = I_3$$
$$I_3 = \mathbf{5\text{ A}}$$

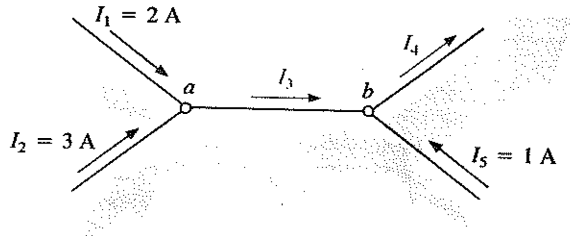


FIG. 3.19

At b:

$$\Sigma I_{\text{entering}} = \Sigma I_{\text{leaving}}$$
$$I_3 + I_5 = I_4$$
$$5\text{ A} + 1\text{ A} = I_4$$
$$I_4 = \mathbf{6\text{ A}}$$

EXAMPLE Determine I_1, I_3, I_4, and I_5 for the network of Fig. 3-20

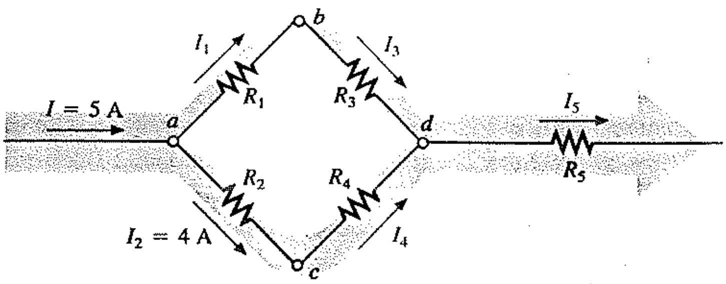


FIG. 3.20

Solution: At a:

$$\Sigma I_{\text{entering}} = \Sigma I_{\text{leaving}}$$
$$I = I_1 + I_2$$
$$5\text{ A} = I_1 + 4\text{ A}$$

Subtracting 4 A from both sides gives

$$5\text{ A} - 4\text{ A} = I_1 + 4\cancel{\text{A}} - 4\cancel{\text{A}}$$
$$I_1 = 5\text{ A} - 4\text{ A} = \mathbf{1\text{ A}}$$

At b:

$$\Sigma I_{\text{entering}} = \Sigma I_{\text{leaving}}$$
$$I_1 = I_3 = \mathbf{1\text{ A}}$$

as it should, since R_1 and R_3 are in series and the current is the same in series elements.

At c:

$$I_2 = I_4 = \mathbf{4\text{ A}}$$

for the same reasons given for junction b.

At d:

$$\Sigma I_{\text{entering}} = \Sigma I_{\text{leaving}}$$
$$I_3 + I_4 = I_5$$
$$1\text{ A} + 4\text{ A} = I_5$$
$$I_5 = \mathbf{5\text{ A}}$$

If we enclose the entire network, we find that the current entering is $I = 5$ A; the net current leaving from the far right is $I_5 = 5$ A. The two must be equal, since the net current entering any system must equal that leaving.

EXAMPLE Determine the currents I_3 and I_5 of Fig. 3.21 through applications of Kirchhoff's current law.

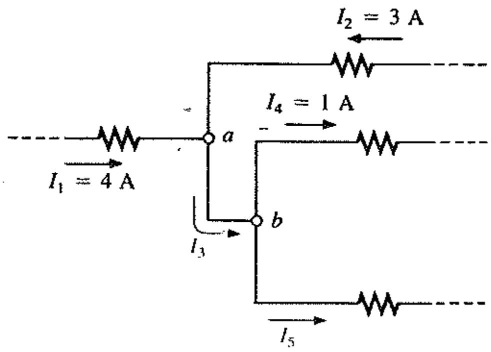


FIG. 3.21

Solution: Note that since node b has two unknown quantities and node a only one, we must first apply Kirchhoff's current law to node a. The result can then be applied to node b. For node a,

$$I_1 + I_2 = I_3$$
$$4\text{ A} + 3\text{ A} = I_3$$

and
$$I_3 = \mathbf{7\text{ A}}$$

For node b,

$$I_3 = I_4 + I_5$$
$$7\text{ A} = 1\text{ A} + I_5$$

and
$$I_5 = 7\text{ A} - 1\text{ A} = \mathbf{6\text{ A}}$$

EXAMPLE Find the magnitude and direction of the currents I_3, I_4, I_6, and I_7 for the network of Fig. 3.22. Even though the elements are not in series or parallel, Kirchhoff's current law can be applied to determine all the unknown currents.

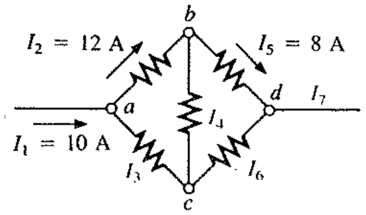


FIG. 3.22

Solution: Considering the overall system, we know that the current entering must equal that leaving. Therefore,

$$I_7 = I_1 = \mathbf{10\text{ A}}$$

Since 10 A are entering node a and 12 A are leaving, I_3 must be supplying current to the node. Applying Kirchhoff's current law at node a,

$$I_1 + I_3 = I_2$$
$$10\text{ A} + I_3 = 12\text{ A}$$

and
$$I_3 = 12\text{ A} - 10\text{ A} = \mathbf{2\text{ A}}$$

At node b, since 12 A are entering and 8 A are leaving, I_4 must be leaving. Therefore,

$$I_2 = I_4 + I_5$$
$$12\text{ A} = I_4 + 8\text{ A}$$

and
$$I_4 = 12\text{ A} - 8\text{ A} = \mathbf{4\text{ A}}$$

Problems :

1- A wire wound resistor is to be made with a resistance of 20 Ω. Some wire is available with a cross-sectional area of .002 in^2 and a resistivity of 16×10^{-6} Ω in. Calculate the length of wire required, and the power loss in the resistor when a P.d of 10V is applied to it.

2- An aluminum wire 10 meters long and 2 mm in diameter is connected in parallel with a copper wire 6 meters long. A total current of 2 A is passed through the combination and it is found that the current through the aluminum wire is 1.25 A. Calculate the diameter of the copper wire.

3- A conductor has a cross-section of 15 mm^2 and a specific resistance of 7.6 μΩ.cm at 0°C. Get the resistance of the conductor at 50°C per Km if the temperature coefficient at 0°C is .005 /°C.

4- It is found that resistance of a coil of wire increases from 50Ω to 58Ω when temperature rises from 15 °C to 55°C. Calculate the temperature coefficient at 0 °C of the conductor material.

5- It is found that the resistance of a coil of wire increases from 100Ω at 21.5°C to 125Ω at 85.5°C. Estimate the resistance temperature coefficient of the conductor material at 0°C.

6- Find the current flowing at the instant of switching on a 100-W metal filament lamp on to a 200 V circuit, given that the incandescent filament temperature is 200°C and the resistance temperature coefficient at the room temperature of 20°C is 0.005 $^{\circ}C^{-1}$.

7- What is the voltage drop across a 5-Ω resistor carrying 4 A ?

8- What is the current drawn by a 10-Ω resistor with 100V across it?

9- What is the resistance of an ammeter that indicates 10 A while having 0.10 V across it ?

10- What is the resistance of a voltmeter that reads 200 V when drawing 60 μA ?

11- What is the conductance of a 2.7-kΩ resistor ?

12- What is the resistance corresponding to 50 mS ?

13-What is the resistance of an electric heater that consumes 1900 W when connected to 115-V lines ?

14- What is the resistance range for 2% , 620-Ω resistors ?

15- A 0.25-A current flows through a 5%, 15-Ω resistor. What range must the resistor voltage be in ?

16- A 5% , 20-kΩ resistor has a voltage of 125 V. What range must the resistor current be in ?

17- Find the minimum resistance value and tolerance of a carbon-composition resistor having band colors in the order of red, violet, black, and missing.

18- A 12-V battery has a 0.05-Ω internal resistance. What is the battery terminal voltage when the battery delivers 40 A ?

19- A battery that produces 12 V on open circuit produces 10 V when delivering 10 A. What is the model for this battery.

20- A 5-A current source has 80-kΩ internal resistance. What is the voltage across this source when it delivers 4.2 A ?

21- A short circuit across a current source draws 10 A. But when the current source terminal voltage is 50 V, the external current is 8 A. What is the model for this current source ?

22- A short-circuit across a current source draws 15 A. And when the current source has an open circuit across it, its terminal voltage is 300 V. What is the model for this current source.

23- A voltmeter having internal resistance of 20- kΩ is connected to a series resistance R and an ideal voltage source of 230 V. The reading of the voltmeter is 160 V. Get R .

24- Using Ohm's and Kirchoff's Laws, find the current in all the branches of Fig. 3-23.

25- In Fig. 3-24, gets R if the power dissipated in resistance 12 Ω is 36 W.

26- Find the potential of the point Y in Fig. 3-25.

27- Find the current in all branches of the network shown in Figure 3-26.

28- For the circuit shown in Fig. 3-27, calculate : -

(a) the current in the 100-ohm resistor.

(b) the potential difference across the 20-ohm resistor.

(c) the power dissipated in the 50-ohm resistor.

29- Two batteries P and Q are connected in parallel to a resistance of 2.5 ohms. Battery P has an e.m.f. of 110 volts and an internal resistance of 0.25 ohm, and Q has 100 volts and 0.20 ohm. Find the value and direction of the current in each branch.

30- In the circuit shown in Fig. 3-28, find the value of the resistance R required to limit the current in the 6-volt battery to 5 amperes.

31.A portion of residential service to a home is shown in Fig.3.29

a.Calculate the supply current. Will the circuit breaker trip2.

b.What is the total resistance of the network ?

c.Determine the power supplied by the 120-V source. How does it compare to the total power of the load?

32. Determine the currents I_1 and I_s for the networks of Fig. 3.30.

33. Determine the power delivered by the dc battery in Fig. 3.31.

34. For the network of Fig.3.32. a.Find the source current I_s.

b. Calculate the power dissipated by the 4- Ω resistor.

c. Find the current I_2.

35. For the network of Fig. 3.33 a. Get O.C voltage V_L.

b.Place a short circuit across the output terminals and determine the current through the short circuit.

c.If the 2.2-k Ω resistor is short circuited, what is the V_1?.

d.Repeat part (b) with the 4.7-kΩ resistor replaced by an open circuit.

36. For the network of Fig. 3.34 determine

a. The short-circuit currents I_1 and I_2.

b. The voltages V_1 and V_2. c. The source current I_s.

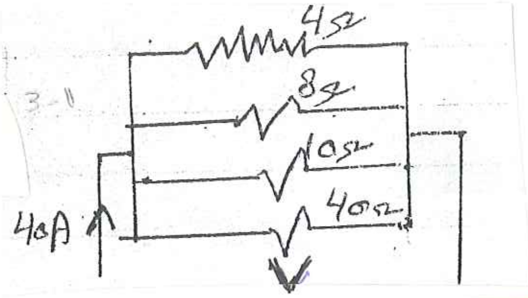
37. Based on the measurements of Fig. 3.35, determine whether the network is operating correctly. If not, try to determine why not.

38. Referring to the network of Fig. 3.36, determine is V_a = 8.8 V the correct reading for the given configuration? If not, change the 1 k Ω to obtain the correct reading.

39.a.The voltage V_a for the network of Fig. 3.37 is –1 V. If it suddenly jumped to 20 V, what could have happened to the circuit structure? Localize the problem area.

b. If the voltage V_a is 6 V rather than – 1 V, try to explain what is wrong about the network construction. The voltages 20 V, - 4 V and the 1 k Ω resistance are fixed.

Figure 3-23



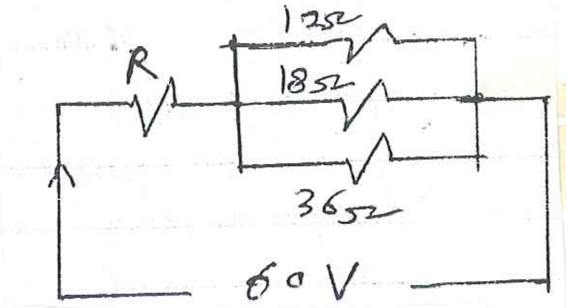


Figure 3-24

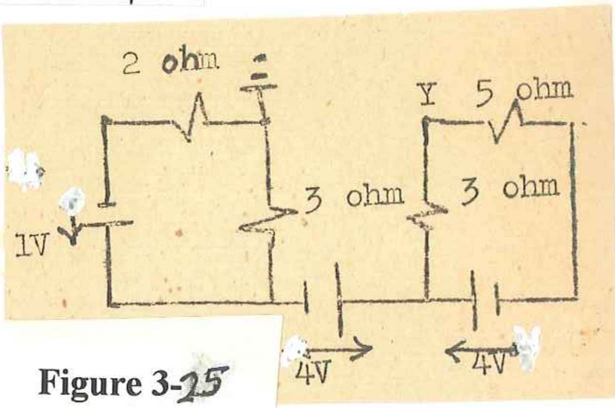


Figure 3-25

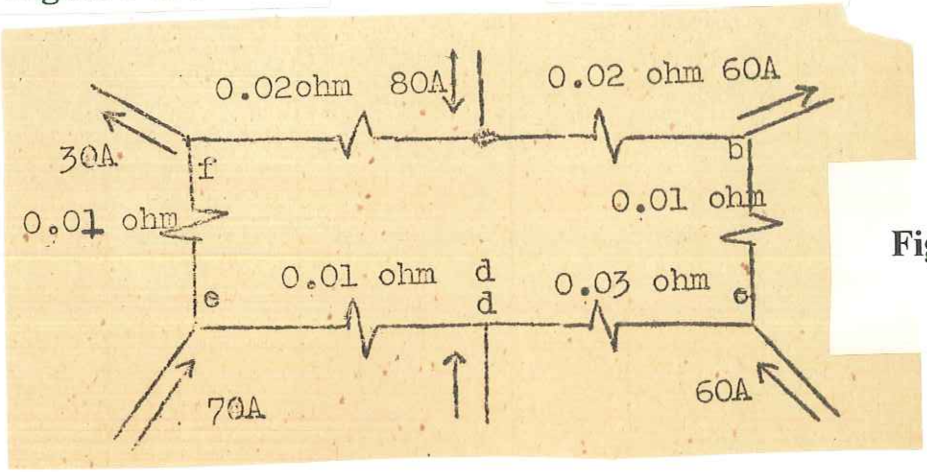


Figure 3-26

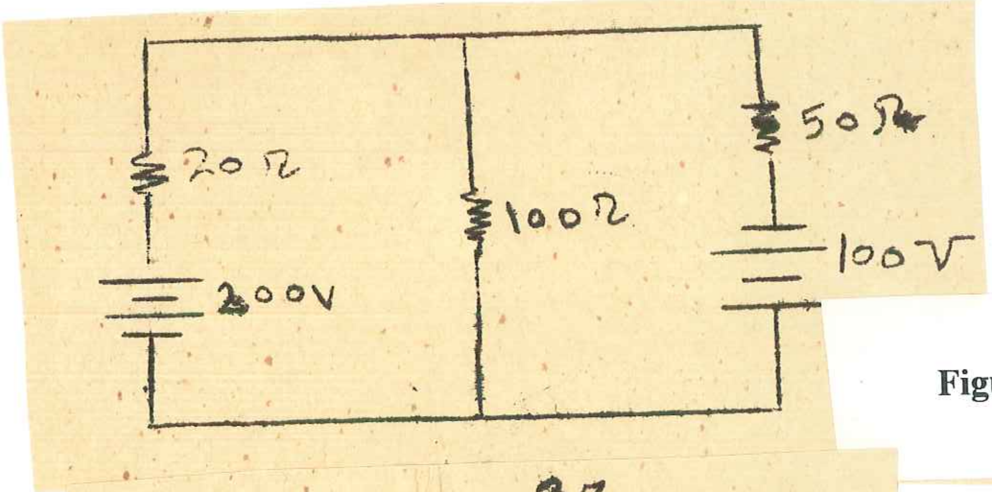


Figure 3-27

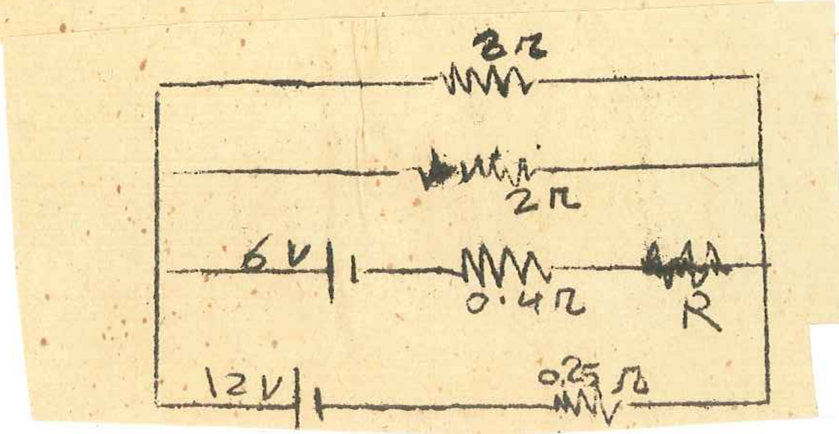


Figure 3-28

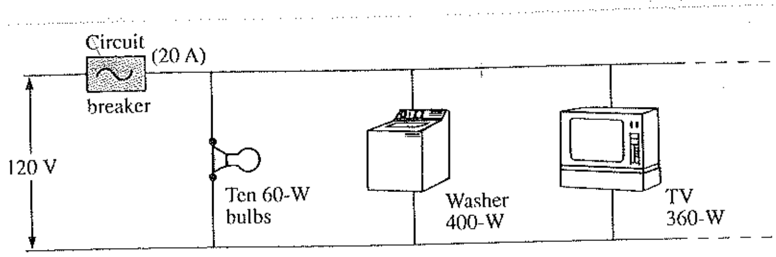


FIG. 3-29

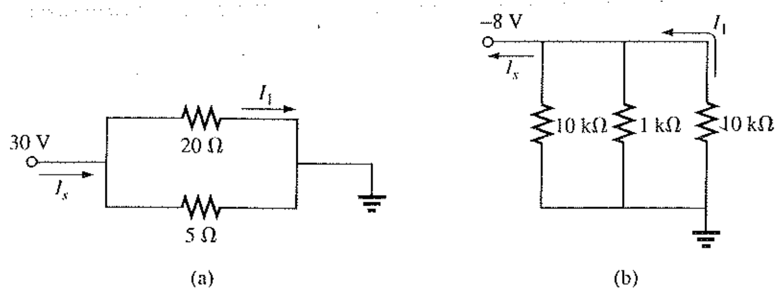


FIG. 3-30

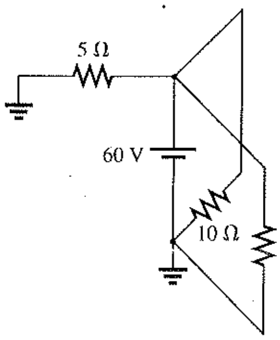


FIG. 3-31

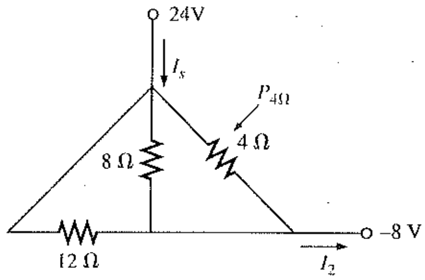


FIG. 3-32

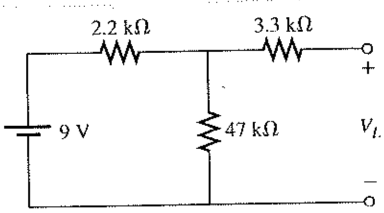


FIG. 3-33

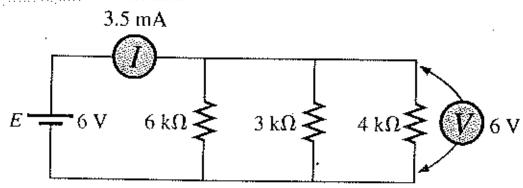


FIG. 3-35

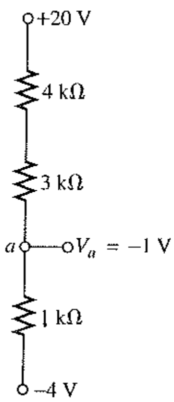


FIG. 3-37

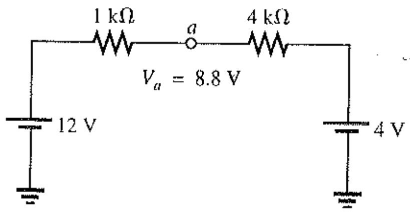


FIG. 3-36

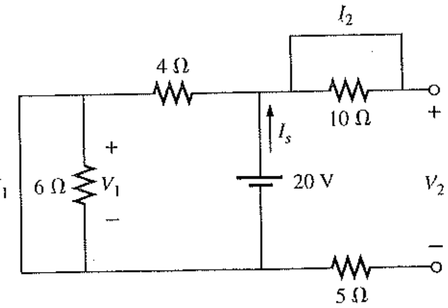


FIG. 3-34

Chapter Four

SOME METHODS FOR SOLUTION OF ELECTRIC CIRCUITS

At first, there are two rules that must be mentioned :

1- The first rule is that any branch in the circuit having zero voltage drop can be removed or replaced by a short circuit without any change in the operation of the remaining circuit.

2- The second rule is that any two nodes having the same potential can be connected directly without any change in the operation of the remaining circuit.

Step by Step Simplification :

The electric circuits can be simplified by sequential elemination of some nodes or branches.

Sometimes, the whole circuit can be simplified to one loop.

Source Equivalence :

Solution of some electric circuits requires that all the sources are voltage circuits or current circuits. Voltage sources and current sources are considered equivalent if each source is replaced by the other source without changing the conditions (current or volt in the external circuit). A voltage source is shown in Fig. 4-1. It consists of an e.m.f. of V_o in series with internal resistance R_o . R_L is the load resistance. This voltage source is equivalent to a current source shown in Fig. 4-2. I_o equals V_o / R_o and R_o is the previously mentioned internal resistance of voltage source. The condition of equivalence is satisfied here.

In Fig. 4-1, the current in R_L equals :

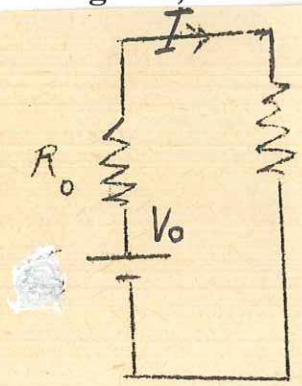


Figure 4-1

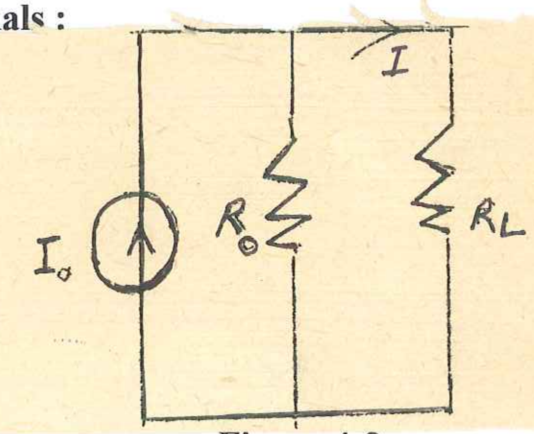


Figure 4-2

$$I = V_o/(R_o + R_L) \qquad (4\text{-}1)$$

In Fig. 4-2, the current in R_L equals : $I = I_oR_o/(R_o + R_L)$

but $I_oR_o = V_o$

$$\therefore \quad I = V_o/(R_o + R_L) \qquad (4\text{-}2)$$

Equations (4-1) and (4-2) are the same.

Combination of Elements :

1- Elements Connected in Series :

Two elements are considered to be in series if they have a common point. This common point has no connection to any branch carrying current. Consider the actual voltage sources having induced e.m.f's. V_1 , V_2 , . . . V_n and their internal resistances are R_1 , R_1 , ... R_n respectively. They are connected in series as shown in Fig. 4-8. They can be replaced by an equivalent element having e.m.f. of V_o and internal resistance R_o given by :

$$V_o = V_1 + V_2 - V_3 + \ldots\ldots V_n$$

or
$$V_o = \sum_{m=1}^{n} V_m \tag{4-3}$$

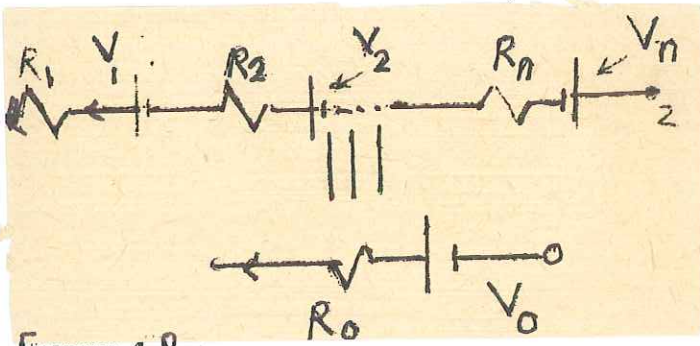


Figure 4-8

It is to be noticed that the current in all the series elements is the same. Consider the case of series passive elements R_1 , R_2 , ... R_N . We can conclude that the equivalent resistance of any number of resistors in series equals the sum of the individual resistances :

$$R_r = R_1 + R_2 + R_3 + \ldots + R_N \tag{4-4}$$

As A special case, if all N resistors have the same resistance R, then,

$$R_r = R + R + R + \ldots + R = NR \qquad (4\text{-}5)$$

Example :

What is the total resistance of 4-Ω , 6-Ω , 7-Ω , 10-Ω and 3-Ω resistors in series ?

Solution :

Because the resistors are in series, we need only add their resistances to find the total resistance: $R_r = 4 + 6 + 7 + 10 + 3 = 30\ \Omega$.

Example :

In the circuit of Fig. 4-9, find the current and all the unknown voltages.

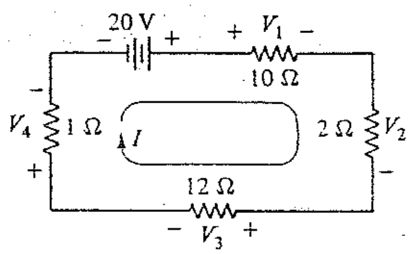


Figure 4-9

Solution :

The best approach here is to find the total resistance and divide it into the applied voltage to get the current I. The total resistance is $10 + 2 + 12 + 1 = 25\ \Omega$. This divided into the applied voltage gives the current: $I = 20/25 = 0.8$ A. Now that we have the current,we can use

Ohm's law to find the unknown voltages: $V_1 = 10I = 8V$, $V_2 = 2I = 1.6V$, $V_3 = 12I = 9.6$ V, and $V_4 = I = 0.8$ V. As a check we will sum these voltages to make certain their sum equals the applied source voltage: $8 + 1.6 + 9.6 + 0.8 = 20$ V, which checks, as by KVL it must unless we make an error. It is good practice, and really essential, to check results whenever possible. It is also important to check each step in an analysis before going to the next step.

Now let us change the single loop quite a bit and see if we can still apply basic principles in analyzing a circuit.

Example :

Find the current and the unknown voltages in the circuit of Fig. 4-10.

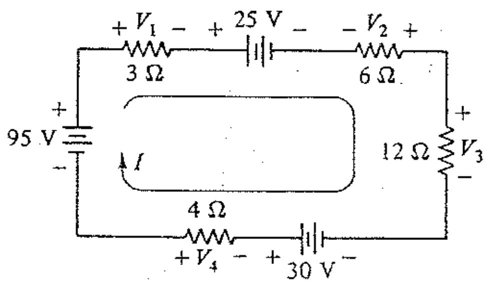


Figure 4-10.

Solution :

This is a series circuit because all components carry the same current. So, to find the current we sum the resistance : $R_r = 3 + 6 + 12 + 4 = 25\ \Omega$. Next, we sum the voltage rises from sources in the direction of I : $V_r = 95 - 25 + 30 = 100$ V. The 25 V is negative because it is a voltage drop and we are summing voltage rises. Finally, we divide this total voltage by the total resistance to get the current :

$$I = \frac{V_r}{R_r} = \frac{100}{25} = 4\ A$$

Then from Ohm's law,

$V_1 = 3I = 12$ V $V_2 = -6I = -24$ V $V_3 = 12I = 48$ V $V_4 = -4I = -16$ V

V_2 and V_4 are negative because the arrow for I enters their negatively referenced terminals (non associated references). Checking, 12 - (- 24) + 48 - (- 16) = 100 V, the net voltage applied.

Interchanging series elements does not affect the total resistance, current or power to each element.

Voltage Divider Rule :

In a series circuit the voltage across the resistive elements will be divided as the magnitude of the resistance levels. Consider the circuit shown in Fig. 4-11. The total resistance R_T equals $(R_1 + R_2)$. The voltages V_1, V_2 can be given by :

$$V_1 = E\ R_1/R_T \tag{4-6}$$

and

$$V_2 = E\ R_2/R_T \tag{4-7}$$

If there is n series resistances, then

$$R_T = R_1 + R_2 + \ldots + R_n \tag{4-8}$$

and the voltage drops across resistance R_X equals :

$$V_x = R_x\ E/R_T \qquad (4\text{-}9)$$

where E is the impressed voltages.

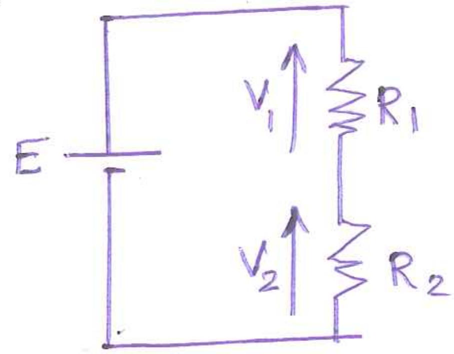


Figure 4-11

In words, the voltage divider rule states that : the voltage across a resistor in a series circuit is equal to the value of this resistor times the total impressed voltage across the series elements divided by the total resistance of the series elements.

Example :

Using the voltage divider rule, get V_1 and V_3 in the circuit of Fig. 4-12.

Solution : $V_1 = R_1\ E / R_T = 2\text{-k}\Omega \times 45 / 15\text{-k}\Omega = 6\ V$

$V_3 = R_3\ E / R_T = 8\text{-k}\Omega \times 45 / 15\text{-k}\Omega = 24\ V$

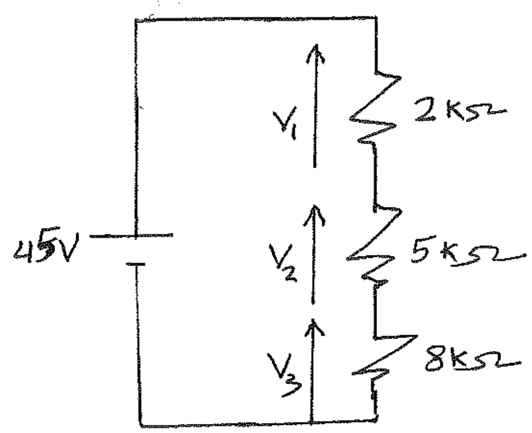


Figure 4-12

<u>Elements Connected in Parallel</u> :

Two elements, branches or networks are in parallel if they have two points in common. Consider the active elements shown in Fig. 4-13, the equivalent element can be given by an e.m.f. V_o given by :

$$ {}_o = \sum_{m=1}^{n} V_m G_m / \sum_{m=1}^{n} G_m \tag{4-10}$$

and a resistance R_o given by :

$$R_o = 1 / \sum_{m=1}^{n} G_m \tag{4-11}$$

where $G_m = 1 / R_m$

This result is known as Milliman's theorem.

We can generalize and conclude that any number of resistors in parallel have a combined or equivalent conductance of total conductance equal to the sum of the conductances of all the parallel resistors :

$$G_r = G_1 + G_2 + \ldots\ldots + G_N \tag{4-12}$$

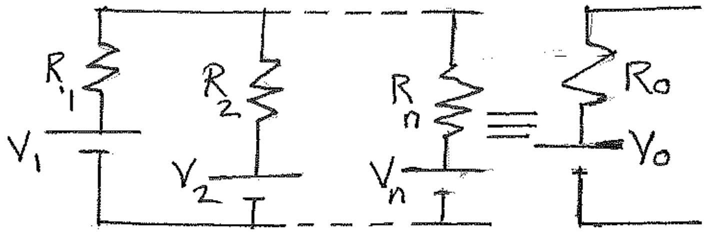


Figure 4-13

If all N conductances have the same value G_r = NG. On inversion this becomes R_r = R/N : the net resistance of the parallel combination of resistors with the same resistance is this resistance divided by the number of resistors.

An important consequence of this conductance equation is that the net resistance of parallel resistors is always less that the least resistance of the resistors in the parallel combination. This fact, which is often a convenient check, follows from the total conductance being greater than any individual conductance. So, the inverse of the total conductance is less than the inverse of any individual conductance. These inverses are, of course, the resistance. If one resistor has a resistance that is much less than any of the others, R_T is only slightly less than this least resistance.

Example :

What is the total or input conductances and resistance of parallel resistors having of 1 Ω, 0.5 Ω, 0.25 Ω, 0.2 Ω, and 0.125 Ω ?

Solution :

Because the resistors are in parallel, we need only add the individual conductances to get the total, equivalent, or input conductance (the terms are synonymous) : $G_r = 1 + 2 + 4 + 5 + 8 = 20$ S. The corresponding resistance is the inverse of G_r : $R_T = 1/120 = 0.05\ \Omega$. Notice that R_T is less than the least resistance, $0.125\ \Omega$, of any of the parallel resistors.

Ladder Network Analysis :

Although we can use mesh or nodal analysis on a ladder network with a single voltage source at one end, as in Fig. 4-13́, sometimes voltage division is easier. For this we combine resistances starting from the end opposite the input until the network has just a single equivalent resistor between the V_1 node and ground. Then we find V_1 by voltage division. Next, we use V_1 and voltage division to find V_2. Finally, we find V_3 from V_2, again by voltage division.

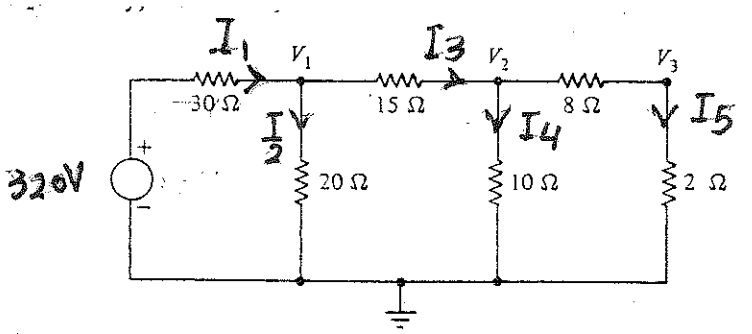


Figure 4-13́

The method always begins at the end opposite the input. There, in Fig. 4-13 an 8-Ω resistor and a 2-Ω resistor in series have a total resistance of 10-Ω. This 10-Ω is in parallel with a 10-Ω resistor connected between the V_2 node and ground. The net resistance of this parallel combination is $10||10 = 5$-Ω. And the network reduces to that of Fig. 4-14(a). This 5-Ω is in series with 15 Ω, producing 20 Ω in parallel with the 20-Ω resistor between the V_1 node and ground.

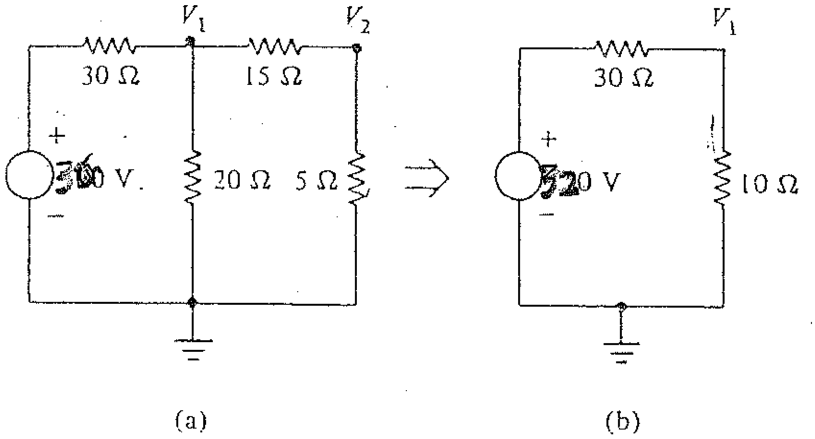


Figure 4-14

They combine to $20||20 = 10$-Ω as in Fig. 4-14(b).

With the network reduced to a voltage source in series with two resistors, we can apply voltage division to Fig. 4-14(b) to get V_1.

$$V_1 = \frac{10}{30 + 10} \times 320 = 80 \text{ V}$$

With V_1 now known, we can apply voltage division to Fig. 6-14(a) to find V_2.

$$V_2 = \frac{5}{15 + 5} V_1 = 20 \text{ V}$$

And from the original circuit, we can now get V_3 .

$$V_3 = \frac{2}{8+2} V_2 = 4 \text{ V}$$

With all node voltages known, we can find any current of interest, as should be evident.

Another solution can be obtained by assuming the current, at the far end, equal unity i.e., $I_5 = 1$ A. By Ohm's Law, then $V_2 = 10$ V , $I_4 = 1$ A. Using Kirchoff's Current Law, then $I_3 = 2$ A. We can conclude that $V_1 = 40$ V, $I_2 = 2$ A & $I_1 = 4$ A. The supply voltage thus equals 160 V, but the actual supply voltage equals 320 V. Accordingly all currents and node voltages must be multiplied by 320/160 i.e., by 2. Thus the actual voltages and currents are given by $I_5 = 2$ A , $V_2 = 20$ V , $I_4 = 2$ A , $I_3 = 4$ A , $V_1 = 80$ V , $I_2 = 4$ A and $I_1 = 8$ A.

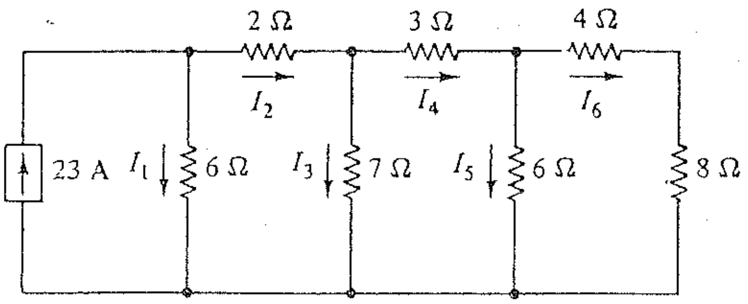


Figure 4-15

Now consider a ladder network energized by a current source as in Fig. 4-15. We could make a source conversion to get the network

in the form of Fig. 4-13, and then use voltage division as before. But an alternative procedure is to leave the network as shown and use current division instead of voltage division. With current division, we reduce the network as in voltage division until there are just two resistors, but they will be in parallel instead of in series. Then we use current division repeatedly, away from the source.

We will use this approach on the circuit of Fig. 4-15. At the right end the 4-Ω resistor in series with the 8-Ω resistor produces 12-Ω in parallel with the 6-Ω resistor. The two resistances combine to

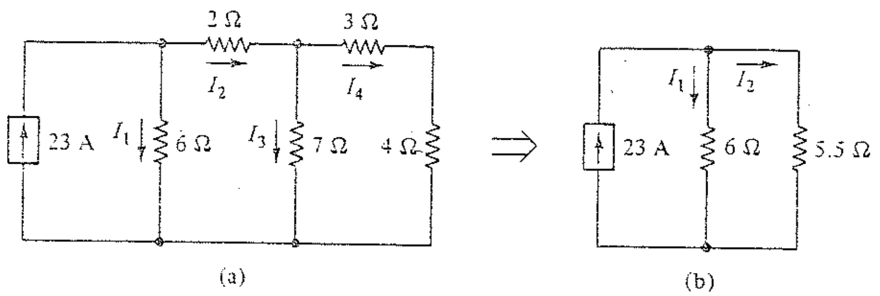


Figure 4-16

6||20 = 4-Ω as in Fig. 4-16(a). Next, the 3-Ω and 4-Ω resistances add to 7-Ω, which combines with the 7 Ω of the original circuit to produce 7||7 = 3.5-Ω. And this 3.5-Ω is in series with 2-Ω, giving 5.5 Ω as in Fig. 4-16(b). We can use current division now that the circuit is reduced to two parallel resistors.

From current division on Fig. 4-16(b),

$$I_1 = \frac{5.5}{6 + 5.5} \times 23 = 11 \text{ A}$$

Leaving $I_2 = 23 - 11 = 12$ A. With I_2 known, we can use current division on Fig. 4-16(a) :

$$I_3 = \frac{7}{7+7} I_2 = \frac{12}{2} = 6 \text{ A}$$

making $I_4 = 12 - 6 = 6$ A. Finally, with I_4 known we can use current division on the original circuit :

$$I_5 = \frac{12}{6+12} I_4 = \frac{2}{3} \times 6 = 4 \text{ A}$$

So, $I_6 = 6 - 4 = 2$ A.

Y-Δ And Δ-Y Conversions :

Figure 4-17(a) shows a Y (wye) resistor network and Fig. 4-17(b) a Δ (delta) resistor network. These networks have other names. Because the Y network can be arranged in the shape of a T, it is also called a T (tee) network when so arranged. Also, because the Δ network can be arranged in the shape of a Π, it is also called a Π(pi) network.

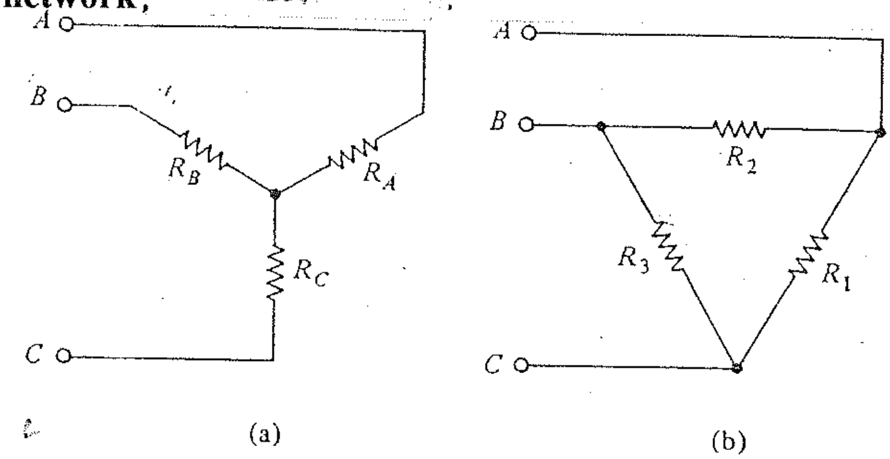


Figure 4-17

Conversion is possible from a Y to an equivalent Δ and also from a Δ to an equivalent Y. Here, equivalent means equivalent only for voltages and currents external to the Y and Δ. Internally to the Y or Δ, the voltages and currents change.

Although these conversions are never necessary, they are convenient for some circuits, especially three-phase circuits, which are a special case of ac circuits that we will study later. In some dc circuits these conversions are also helpful in combining resistors to reduce the circuits. For example, if as in Fig. 4-18, a circuit has series resistors in lines to a Δ, the line resistors are neither in seeries nor in parallel with the Δ resistors. But with conversion of the Δ to a Y, the Y resistors becomes in series with the line resistors and can be combined with them to reduce the circuit.

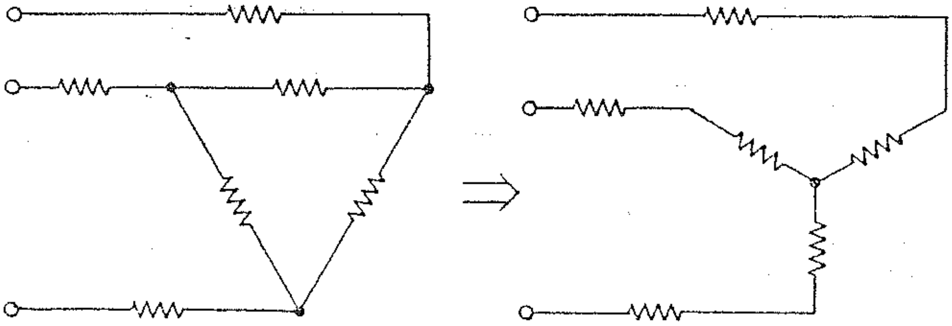


Figure 4-18

In Fig. 4-17 for a Δ-to-Y conversion the formulas are :

$$R_A = \frac{R_1 R_2}{R_1+R_2+R_3} \quad R_B = \frac{R_2 R_3}{R_1+R_2+R_3} \quad R_C = \frac{R_1 R_3}{R_1+R_2+R_3} \tag{4-13}$$

And for converting from a Y to Δ they are :

$$R_1 = \frac{R_A R_B + R_A R_C + R_B R_C}{R_B} \qquad R_2 = \frac{R_A R_B + R_A R_C + R_B R_C}{R_C}$$

$$R_3 = \frac{R_A R_B + R_A R_C + R_B R_C}{R_A} \tag{4-14}$$

Notice that in the Δ-to-Y conversion formulas, the denominators are the same : $R_1 + R_2 + R_3$, the sum of Δ resistances. In the Y-to-Δ conversion formulas it is the numerators that are the same : $R_A R_B + R_A R_C + R_B R_C$, the sum of the different products of the Y resistances taken two at a time. So, the denominators of the Δ-to-Y formulas and the numerators of the Y-to-Δ formulas are easy to remember.

For the other parts of the formula, drawing the Y inside the Δ as in Fig. 4-19 is a good memory aid. For each Y resistor in the Δ-to-Y conversion formulas, the two resistance in the numerator product are those of the two Δ resistors in Fig. 4-7 immediately adjacent to the Y resistor of interest : R_1 and R_2 are adjacent to R_A and R_2 and R_3 are adjacent to R_B and R_1 and R_3 are adjacent to R_C. As mentioned, the denominators are the same : $R_1 + R_2 + R_3$.

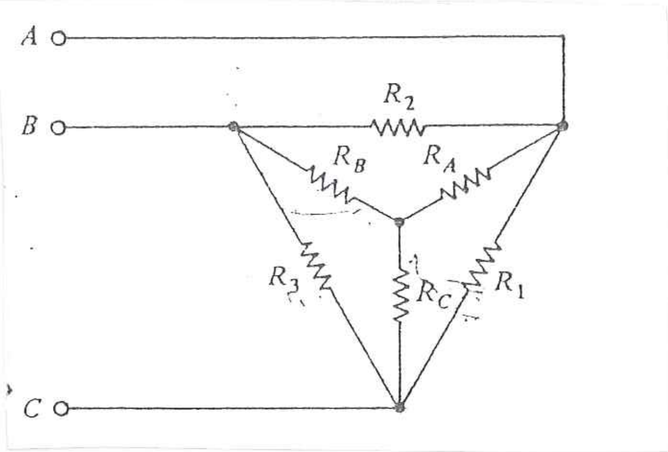


Figure 4-19

For each Δ resistor in the Y-to-Δ conversion formulas, the single Y resistance in each denominator is that of the Y resistor in Fig. 6-7 opposite the Δ resistor of interest. In Fig. 6-7, R_B is opposite R_1, R_C is opposite R_2, and R_A is opposite R_3. As mentioned, the numerators are the same : $R_AR_B + R_AR_C + R_BR_C$.

Frequently, all three resistors in a Y have the same resistance, say R_Y. In this case the resistors of the equivalent Δ also have the same resistance. It is $3R_Y$, as the conversion formulas give. If the original circuit has a Δ in which all three resistors have the same resistance, say R_Δ, then the reesistors of the equivalent Y each have a resistance of $R_\Delta/3$, as the conversion formulas give.

<u>Example</u> :

Use Δ and Y conversion formulas in analyzing the circuit of Fig. 4-20(a).

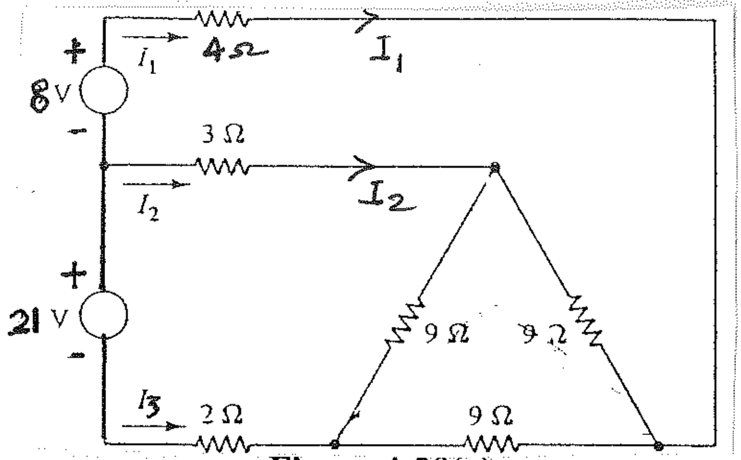


Figure 4-20(a)

***Solution* :**

Here, mesh analysis is easy enough to use, resulting in just three mesh equations. On the other hand, it may be easier to convert the Δ to a Y to reduce the circuit to only two meshes. We will do this. Because the resistances are equal, the equivalent Y resistances are each one-third of the Δ resistances, or 3 Ω. Fig. 4-20(b) shows the network with the equivalent Y replacing the Δ. The resistances combine to $4 + 3 = 7\ \Omega$, and $2 + 3 = 5\ \Omega$, as shown in Fig. 4-20(b).

Using Kirchoff's Voltage Law for loops 1 and 2 shown in Fig. 4-20(b), we get for loop 1 :

$$8 - 7\,I_1 + 6\,I_2 = 0 \qquad (4\text{-}15)$$

and for loop 2 :

$$21 - 6\,I_2 + 5\,I_3 = 0 \qquad (4\text{-}16)$$

Applying Kirchoff's current law at node b we get :

$$I_1 + I_2 + I_3 = 0 \qquad (4\text{-}17)$$

Solving equations (4-15), (4-16) & (4-17), we get :

$$I_1 = 2\text{A}\ ,\ I_2 = 1\text{ A} \ \text{ and } \ I_3 = -3\text{ A}$$

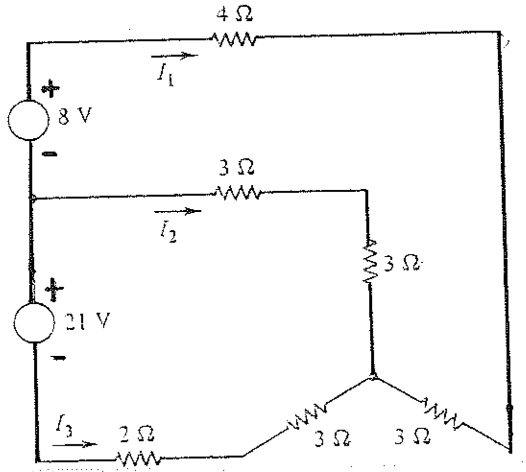


Figure 4-20(b)

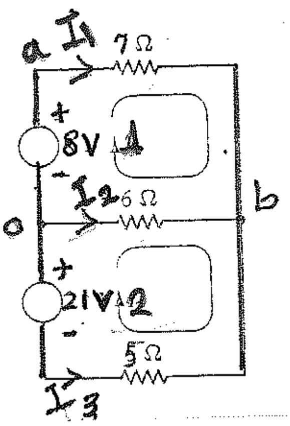


Figure 4-20(c)

***Example* :**

Find the indicated currents in the circuit of Fig. 4-21.

***Solution* :**

The circuit is reducible, containing a Y or 6 Ω resistors inside a Δ of 9 Ω resistors. We will convert this Y-Δ combination to a single Y that we can combine with the series resistors to make a simple two-mesh circuit. The Y of 6 Ω resistors converts to a ý Δ of 18 Ω resistors, which resistors are in parallel with the 9 Ω resitors of the

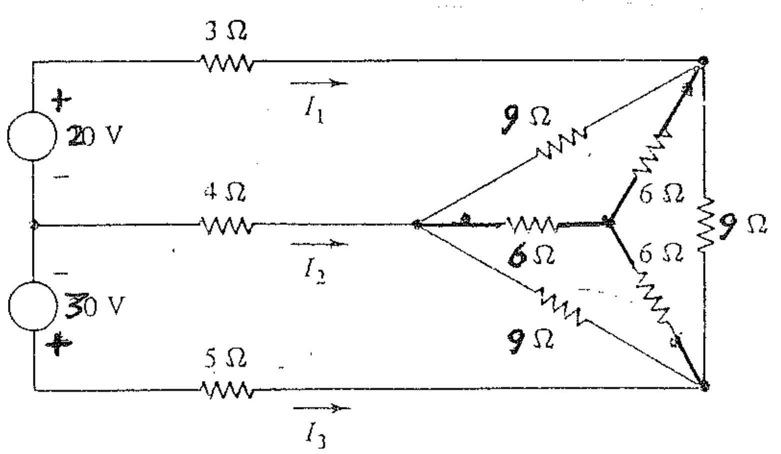


Figure 4-21

original Δ, as shown in Fig. 4-22(a). The parallel resitors combine to a single Δ of 6 Ω resistors as in Fig. 4-22(b). And this Δ converts to the Y of 2 Ω resistors of Fig. 4-22(c). Finally, these resistors combine with the series resistors to form the simple two-mesh circuit of Fig. 4-22(d).

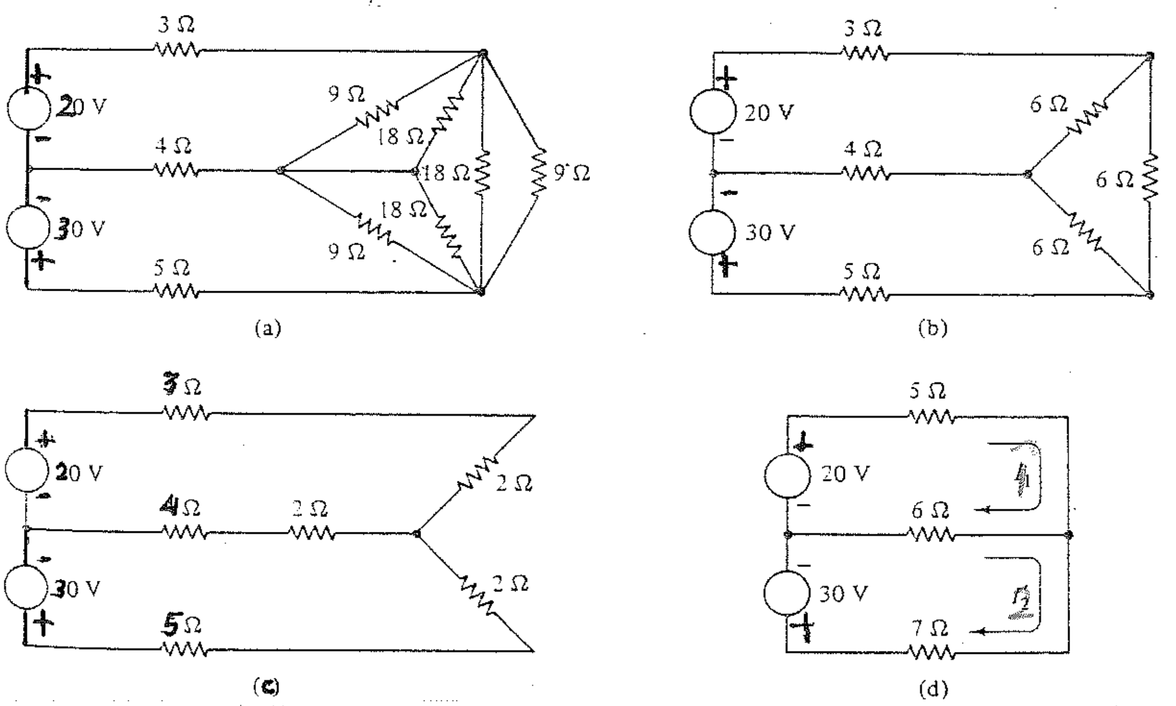


Figure 4-22

Applying Kirchoff's voltage and current laws as in the previous example we get :

$I_1 = 0.748$, $I_2 = -2.71$ and $I_3 = 1.962$ A

Bridge Network Analysis :

A bridge network has two joined Δ's or, depending on point of view, two joined Y's , with a shared branch as illustrated in Fig. 4-23 (a). Although the bridge network usually appears in this form, those

of Fig. 4-23 (b) and (c) are also common. Clearly, these three networks are electrically the same.

Δ-Y conversions are often helpful in analyzing a bridge network.

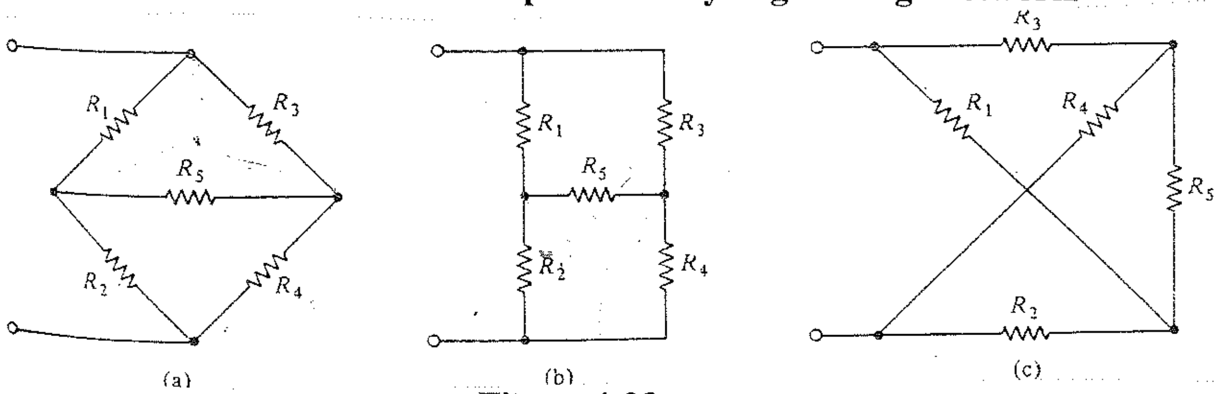


Figure 4-23

Example :

Find I in the circuit of Fig. 4-24.

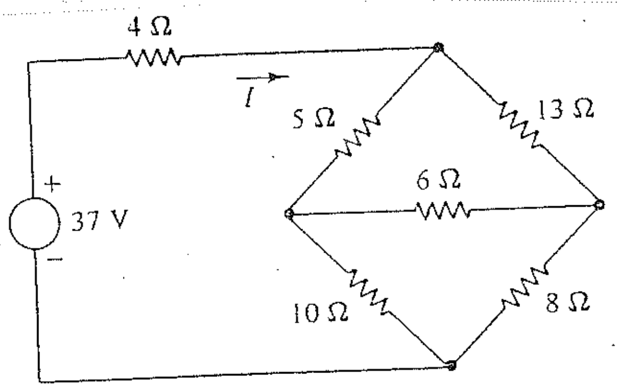


Figure 4-24.

Solution :

Analysis of this network, as it stands, requires either three mesh or three nodal equations. But the number of equations needed reduces to just one with a Δ-to-Y conversion followed by combining resistors in series and parallel. Figure 4-25(a) has the graphic aid for converting the top Δ to a Y. From it,

$$R_1 = \frac{5 \times 13}{5 + 6 + 13} = 2.7\ \Omega \qquad R_2 = \frac{5 \times 6}{5 + 6 + 13} = 1.25\ \Omega$$

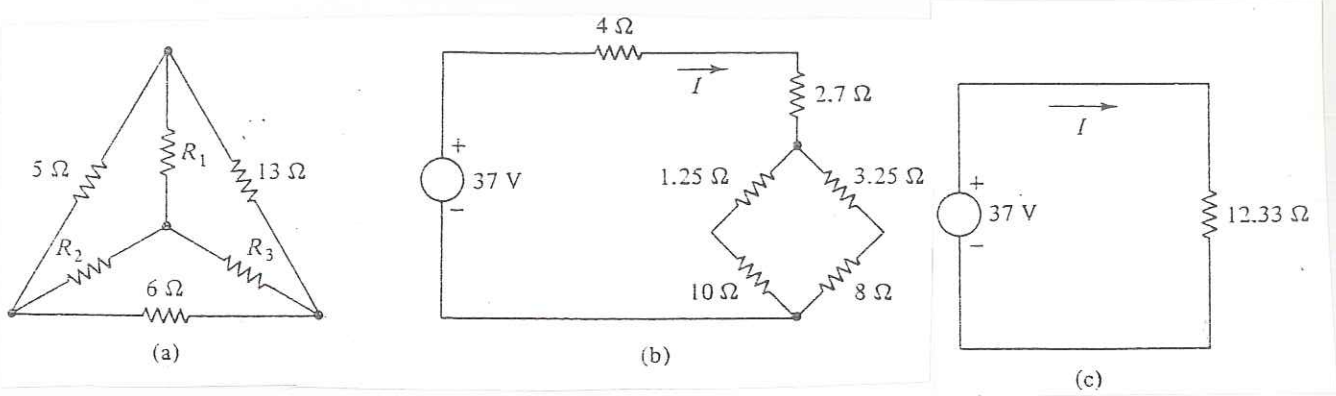


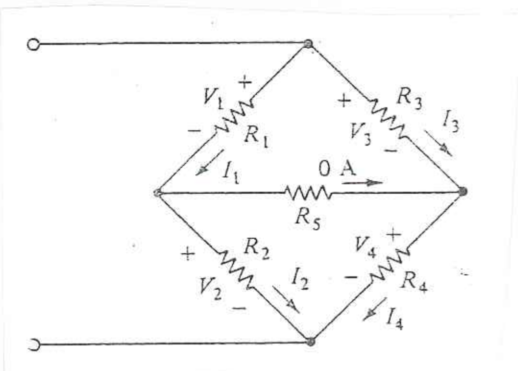
Figure 4-25.

$$R_3 = \frac{6 \times 13}{5 + 6 + 13} = 3.25\ \Omega$$

With the conversion the network reduces to the series-parallel network of Fig. 4-25 (b) in which the 1.25-Ω Y resistor is in series with the original 10-Ω resistor and the 3.25-Ω Y resistor is in series with the original 8-Ω resistor —these two arrangements being in parallel. All the resistors, being either in series or in parallel, combine to a single 12.33-Ω resistor as in Fig. 4-25 (c). From it, I = 37/12.33 = 3 A.

If the bridge center branch carries no current, the bridge is said to be balanced. For balance, the resistances of the four other branch resistors must have a specific relationship. To find this resistor relationship needed for balance, consider the bridge of Fig. 4-26.

Figure 4-26



If the center branch carries no current, then $I_2 = I_1$ and $I_4 = I_3$. Also, the voltage drop is zero across the center branch resistor R_5. It follows that from KVL around the top Δ, V_1 must equal V_3 or, what amounts to the same thing, $I_1 R_1 = I_3 R_3$. Likewise, V_2 equals V_4, or $I_1R_2 = I_3R_4$. Taking the ratio of these two equations eliminates the currents,

$$\frac{I_1 R_1}{I_1 R_2} = \frac{I_3 R_3}{I_3 R_4}$$

$$\frac{R_1}{R_2} = \frac{R_3}{R_4} \qquad (4\text{-}18)$$

Equivalently,

$$\frac{R_1}{R_3} = \frac{R_2}{R_4} \qquad (4\text{-}19)$$

either of which is the resistance relationship required for bridge balance.

Notice that R_5, the resistance of the center branch resistor, does not appear in this relationship. Because the resistance of R_5 has no effect on the balance, this resistance can be an open circuit or a short circuit or anything in between.

<u>Example</u> **:**

We must R be to balance the bridge in circuit of Fig. 4-27(a) ? Also, in the bridge in the circuit of Fig. 4-27(b), select R_1 and R_2 to balance the bridge.

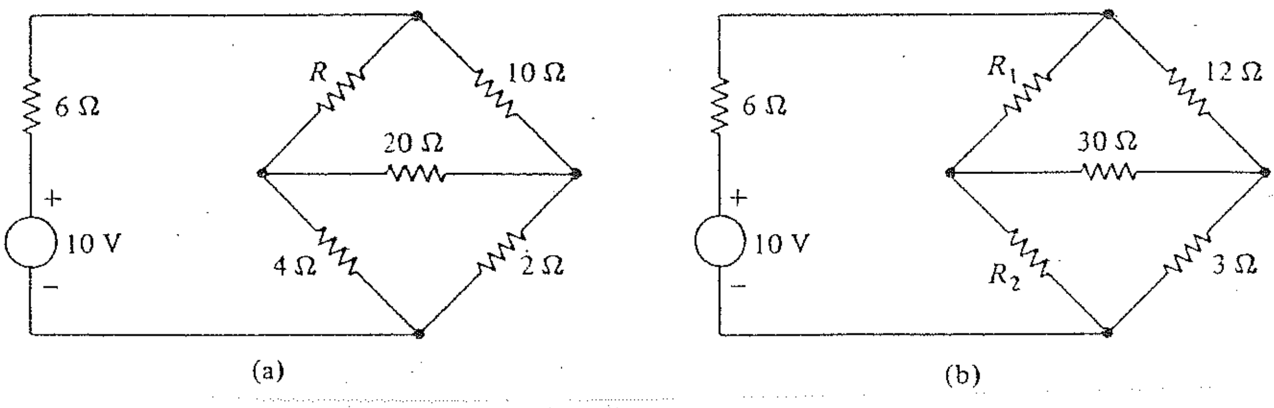


Figure 4-27

Solution :

The balance equation for the bridge of Fig. 4-27(a) is :

$$\frac{R}{4} = \frac{10}{2}$$

From which R = 20 Ω. For the bridge of Fig. 4-27(b) the balance equation is :

$$\frac{R_1}{R_2} = \frac{12}{3} = 4$$

Because this balance equation has two unknowns, we can select any resistance for either R_1 or R_2 and then use this equation to find the other resistance. As an illustration, for a selection of $R_2 = 6\ \Omega$, th equation gives :

$$R_1 = 4R_2 = 4 \times 6 = 24\ \Omega$$

For $R_1 = 600\ \Omega$.

$$R_2 = \frac{R_1}{4} = \frac{600}{4} = 150\ \Omega$$

So, this second bridge circuit has more than one solution. In fact, it has an infinite number of them.

The balanced bridge is the basis of an accurate resistance measurer, the Wheat stone bridge, the circuit of which is in Fig. 4-28.

A commercial Wheat stone bridge has one or more standard resistors R_1, R_2,

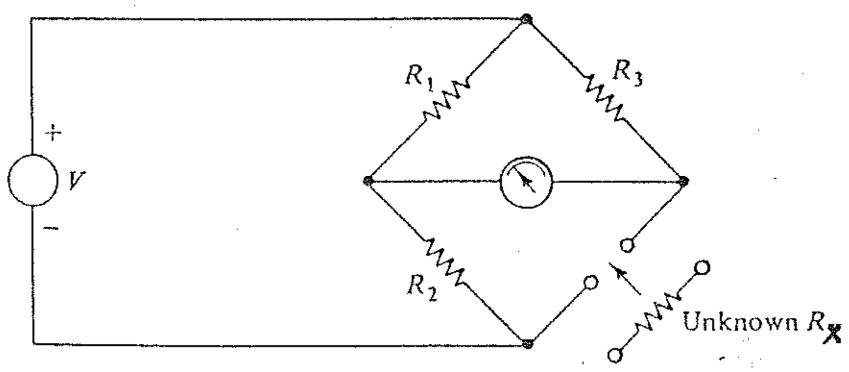


Figure 4-28

and R_3 and a sensitive current indicator, such as a galvanometer, in the center branch. The remaining branch has two terminals for connection to the resistor R_x having the unknown resistance. Operation merely requires connecting in the resistor R_x and adjusting dials to vary the resistances of the standard resistors until the galvanometer reads zero. Then the bridge is in balance, so

$$\frac{R_1}{R_2} = \frac{R_3}{R_x}$$

or $$R_x = \frac{R_2 R_3}{R_1} \qquad (4\text{-}20)$$

The value of R_x can then be read directly from the dials.

Loop Current Method :

In the loop current method, we suppose fictitious currents circulating in the selected independent loops. These currents are known as the loop currents whose number equals number of independent loops ℓ . As previously mentioned in equation (3-11).

$$\ell = b - n + 1 \tag{3-11}$$

The loop current method equations are less than the equations needed in case of using Kirchoff's laws by (n - 1). The branch currents equal the algebric sum of the loop-currents that pass through these branches. The following example shows the loop current method.

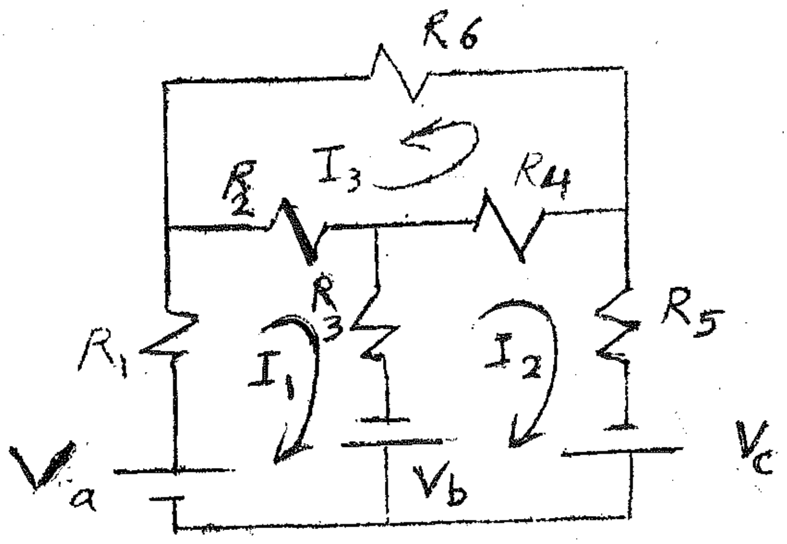


Figure 4-29

Consider the circuit shown in Figure 4-29 : $b = 6$, $n = 4$ and $\ell = 6 - 4 + 1 = 3$.

The independent loops and the fictitious currents are shown in Figure 4-29. Applying Kirchoff's voltage law for loops 1 , 2 and 3 , we get :

<u>For loop (1)</u> :

$$I_1 R_1 + (I_1 + I_3) R_2 + (I_1 - I_2) R_3 = V_a + V_b$$

Or $$I_1 (R_1 + R_2 + R_3) - I_2 R_3 + I_3 R_2 = V_a + V_b \qquad (4\text{-}21)$$

<u>For loop (2)</u> :

$$I_2 R_5 + (I_2 - I_1) R_3 + (I_2 + I_3) R_4 = V_c - V_b$$

Or $$- I_1 R_3 + I_2 (R_3 + R_4 + R_5) + I_3 R_4 = V_c - V_b \qquad (4\text{-}22)$$

<u>For loop (3)</u> :

$$I_3 R_6 + (I_1 + I_3) R_2 + (I_2 + I_3) R_4 = 0$$

Or $$I_1 R_2 + I_2 R_4 + I_3 (R_2 + R_4 + R_6) = 0 \qquad (4\text{-}23)$$

Equations (4-21), (4-22) and (4-23), shows that the loop current passing through considered loop is multiplied by the sum of all

resistance found in the loop. Thus sum is known as the self resistance R_{kk} for loop K. In our case :

$$\left.\begin{aligned} R_{11} &= R_1 + R_2 + R_3 \\ R_{22} &= R_3 + R_4 + R_5 \\ R_{33} &= R_2 + R_4 + R_6 \end{aligned}\right] \qquad (4\text{-}24)$$

The currents in the other loops are multiplied by the common resistances between the considered loop and the other loops in which the other currents pass. In our case :

$$R_{21} = R_{12} = -\,R_3 \quad , \quad R_{32} = R_{23} = R_4 \quad , \quad R_{13} = R_{31} = -\,R_2 \qquad (4\text{-}25)$$

Note that R_{12} is negative as I_1 and I_2 pas in R_3 in opposite direction. Similarly it can be explained why R_{31} is negative. In general, R_{1k} is positive if I_1 and I_k pass in the same direction in R_{1k} and vice versa. Equations (4-21), (4-22) and (4-23) can be putt in the following form :

$$I_1 R_{11} + I_2 R_{12} + I_3 R_{13} = V_{11} \qquad (4\text{-}26)$$

$$I_1 R_{21} + I_2 R_{22} + I_3 R_{23} = V_{22} \qquad (4\text{-}27)$$

$$I_1 R_{31} + I_2 R_{32} + I_3 R_{33} = V_{33} \qquad (4\text{-}28)$$

The self and mutual resistance are given by equations (4-24) and (4-25).

V_{11} , V_{22} , V_{33} , are the sum of the e.m.f's. we met on going around loops 1, 2 and 3 respectively.

The e.m.f's are taken positive if the loop currents are going out of the source positive terminal and vice versa. In general for any circuit having ℓ independent loops, the loop current equations are given by :

$$\left.\begin{array}{l} I_1 R_{11} + I_2 R_{12} + \cdots + I_k R_{1k} + \cdots + I_\ell R_{1\ell} = V_{11} \\ I_1 R_{21} + I_2 R_{22} + \cdots + I_k R_{2k} + \cdots + I_\ell R_{2\ell} = V_{22} \\ \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\quad \vdots \\ I_1 R_{k1} + I_2 R_{k2} + \cdots + I_k R_{kk} + \cdots + I_\ell R_{k\ell} = V_{kk} \\ \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\quad \vdots \\ I_1 R_{\ell 1} + I_2 R_{\ell 2} + \cdots + I_k R_{\ell k} + \cdots + I_\ell R_{\ell\ell} = V_{\ell\ell} \end{array}\right] \qquad (4\text{-}29)$$

Where R_{kk} and $R_{\ell k}$ are the self and mutual resistance respectively. Also, $R_{\ell k} = R_{k\ell}$. As previously defined, V_{kk} is the sum of e.m.f's we met on going around loop k.

***<u>Example</u>* :**

Find the currents in all branches of circuit shown in Fig. 4-30 if :

$V_1 = 24$ V , $V_2 = 96$ V , $V_3 = 48$ V ,

$R_3 = R_5 = 8\ \Omega$, $R_2 = R_4 = 16\ \Omega$.

***<u>Solution</u>* :**

Writing loop equations, we get :

<u>For loop (1)</u> :

$$V_1 + V_3 = I_1 (R_3 + R_4) + 0 - I_3 R_4$$

$$72 = 24\, I_1 - 16\, I_3 \qquad (4\text{-}30)$$

<u>For loop (2)</u> :

$$- V_1 + V_2 = 0 + (R_2 + R_5)\, I_2 - I_3 R_2$$

$$72 = 24\ I_2 - 16\ I_3 \qquad (4\text{-}31)$$

For loop (3) :

$$-\ V_2 = -\ I_1\ R_4 - I_2\ R_2 + I_3\ (R_2 + R_4)$$

$$-\ 96 = -\ 16\ I_1 - 16\ I_2 + 32\ I_3 \qquad (4\text{-}32)$$

Solving the three above equations, we get :

$I_1 = 3A$, $I_2 = 3A$, $I_3 = 0$

also,

$I_{R3} = 3A$, $I_{R4} = 3A$, $I_{R5} = 3A$, $I_{R2} = 3A$ **and** $I_{v1} = 0$

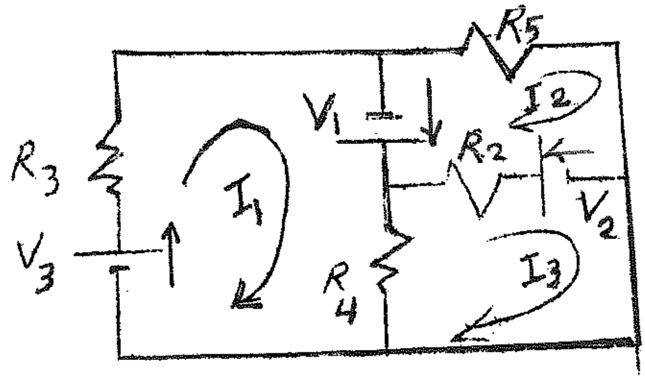


Figure 4-30

Example :

Write loop equations for finding the current I_1 in the $6\ \Omega$ resistance in the circuit of Fig. 4-31.

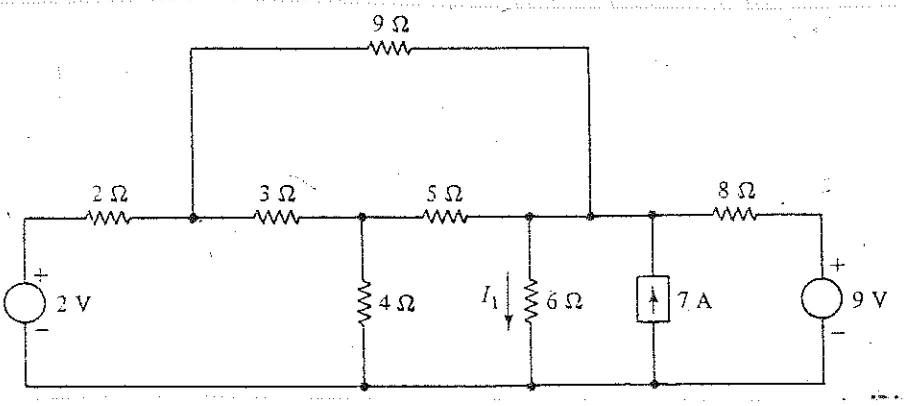


Figure 4-31

Solution :

A source conversion should not be made on the 7-A current source because this conversion involves the 6 Ω resistor through which the current is wanted, and the conversion would change the current. With the current source present, the circuit has five meshes, and that means that five loops are necessary. Of course, only one loop should go through the 7-A current source and only one through the 6 Ω resistor. Even with these constraints there are various choices for the loops. Figure 4-32 shows one suitable selection.

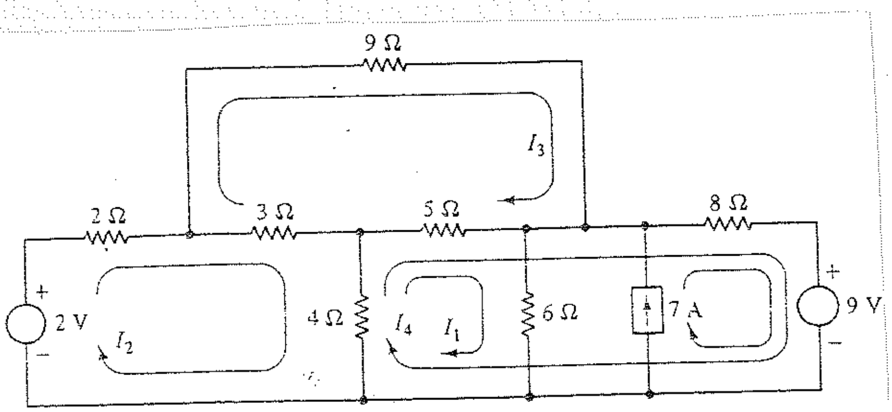


Figure 4-32

For loop 1 the self-resistance is $6 + 4 + 5 = 15\ \Omega$. Its mutual resistance with loop 2 is $4\ \Omega$, and because I_2 goes through the mutual resistor in a direction opposite to I_1, the I_2 coefficient is negative : - 4. The mutual resistance with loop 3 is $5\ \Omega$. With I_1 and I_3 having opposite directions through the $5\ \Omega$ mutual resistor, the I_3 coefficient is negative : - 5. The resistance mutual to loops 1 and 4 is a total of $9\ \Omega$. The 9 coefficient for I_4 is positive because I_1 and I_4 flow through the $5\ \Omega$ and $4\ \Omega$ resistors in the same direction. There are no voltage sources for loop 1, which means that the loop 1 KVL equation has 0 on the right side. From our consideration this equation is :

$$15I_1 - 4I_2 - 5I_3 + 9I_4 = 0$$

In like fashion, the second and third loop equations are :

$$-4I_1 + 9I_2 - 3I_3 - 4I_4 = 2$$

$$-5I_1 - 3I_2 + 17I_3 - 5I_4 = 0$$

The equation for loop 4 requires some explanation, as it has the 7-A current in it. Really through, we treat this 7-A loop current just like any other loop current, the only difference being that it is known. So, it produces constants on the left side that we must substract from both sides to get the equation into standard form. Thus, for loop 4,

$$9I_1 - 4I_2 - 5I_3 + 17I_4 + (7)(8) = -9$$

or $$9I_1 - 4I_2 - 5I_3 + 17I_4 = -65$$

Placing these four loop equations together clearly shows the same off-diagonal symmetry of the mesh equations.

$$\begin{aligned}
15I_1 - 4I_2 - 5I_3 + 9I_4 &= 0 \\
-4I_1 + 9I_2 - 3I_3 - 4I_4 &= 2 \\
-5I_1 - 3I_2 + 17I_3 - 5I_4 &= 0 \\
9I_1 - 4I_2 - 5I_3 + 17I_4 &= -65
\end{aligned}$$

Node Voltage Method :

The method is an application to Kirchoff's current law. The voltages of different nodes are determined with respect to the reference node voltage usually equal to zero. The different node voltages are assumed to be positive. If these voltages are determined, then the currents in the different branches are easily obtained using Ohm's law.

As an example consider the circuit shown in Fig. 4-33. There are three nodes 1, 2 and 3. Node 3 is taken as a reference equal to zero. Each voltage source is substituted by a current source and the resistors are described by their conductances G as shown in Figure 4-34.

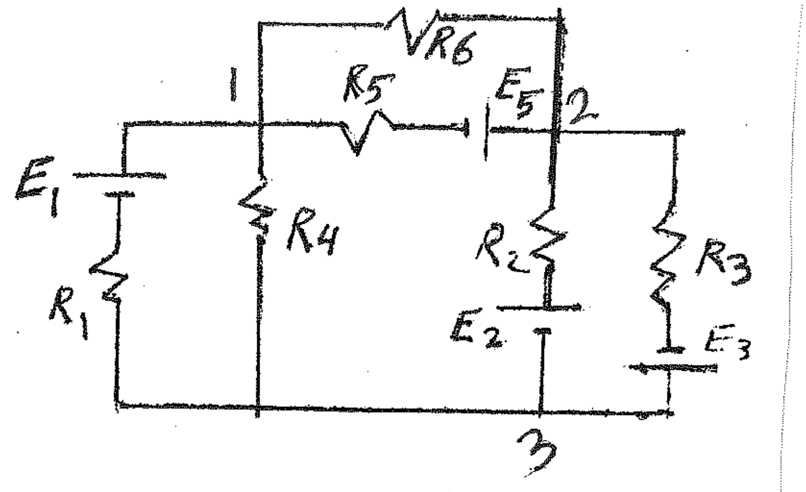


Figure 4-33

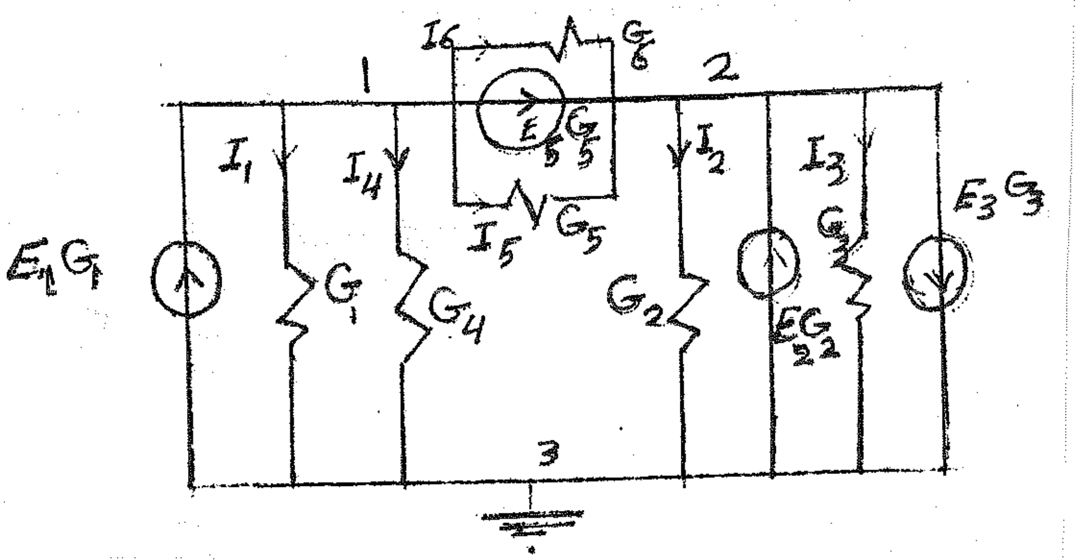


Figure 4-34

Applying Kirchoff's current law at nodes 1 and 2, we get for

<u>node 1</u> :

$$I_1 + I_4 + I_5 + I_6 = E_1 G_1 - E_5 G_5 \qquad (4\text{-}33)$$

and for <u>node 2</u> :

$$I_2 + I_3 - I_5 - I_6 = E_2 G_2 - E_3 G_3 + E_5 G_5 \qquad (4\text{-}34)$$

If V_1 and V_2 are the voltages of nodes 1 & 2 respectively, we get :

$I_1 = V_1 G_1$, $I_4 = V_1 G_4$, $I_5 = (V_1 - V_2)\, G_5$,

$$I_6 = (V_1 - V_2)\, G_6 \quad , \quad I_2 = V_2 G_2 \quad , \quad I_3 = V_2 G_3 \qquad (4\text{-}35)$$

Substituting for the currents I_1 , I_2 , . . . I_6 from equation (4-35) in equations (4-33) & (4-34), we get :

$$V_1 G_1 + V_1 G_4 + (V_1 - V_2)\, G_5 + (V_1 - V_2)\, G_6 = E_1 G_1 - E_5 G_5$$

$$V_1 (G_1 + G_4 + G_5 + G_6) - V_2 (G_5 + G_6) = E_1 G_1 - E_5 G_5$$

$$- V_1 (G_5 + G_6) + V_2 (G_2 + G_3 + G_5 + G_6) = E_5 G_5 + E_2 G_2 - E_3 G_3$$

$$\ldots\ldots\ldots\ldots (4\text{-}36)$$

In equation (4-36), the voltage of the considered node is multiplied by the sum of conductances connected to this node and the voltages of the other nodes are multiplied by the negative sum of the conductances connecting these nodes by the considered node. The algebraic sum equals all the current sources or (equivalent current sources of voltage sources) connected to the considered node. The current source entering the node is positive and vice versa.

Generally for any circuit of n nodes, there are (n - 1) equations given as follows :

$$V_1 G_{11} + V_2 G_{12} + \ldots + V_k G_{1k} + \ldots + V_{n-1} G_{1,\, n-1} = I_{11}$$

$$V_1 G_{21} + V_2 G_{22} + \ldots + V_k G_{2k} + \ldots + V_{n-1} G_{2,\, n-1} = I_{22}$$

$$V_1 G_{k1} + V_2 G_{k2} + \ldots + V_k G_{kk} + \ldots + V_{n-1} G_{k,\, n-1} = I_{kk}$$

$$V_1 G_{n-1,\, 1} + V_2 G_{n-1,\, 2} + \ldots + V_k G_{n-1,\, k} + \ldots + V_{n-1} G_{n-1,\, n-1} = I_{n-1,\, n-1}$$

$$\ldots\ldots\ldots\ldots (4\text{-}37)$$

where G_{kk} , G_{nk} are the self and mutual conductances respectively. The self conductance of a node equals the sum of conductances connected to this node. The mutual conductance between two nodes equals the negative sum of the conductances connecting the two nodes. I_{kk} equals the sum of the current sources or equivalent current sources entering the node. If I_{kk} enters the node, then it is positive and vice versa.

Example :

What are the nodal equations for the circuit of Fig. 4-35 ? Get also the current in the different branches.

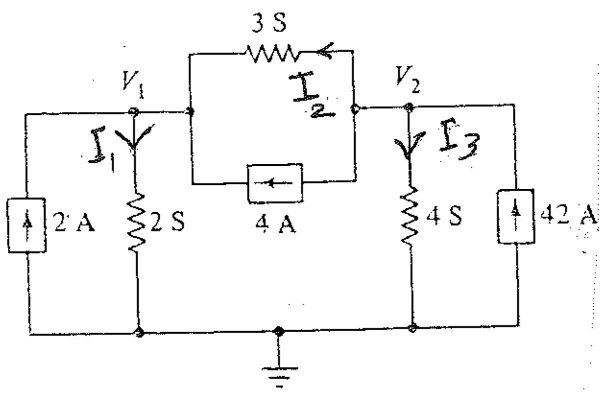


Figure 4-35

Solution :

The node 1 self-conductance is 2 + 3 = 5 S. That of node 2 is 4 + 3 = 7 S. The mutual conductance is 3 S. The current into node 1 from current sources is 4 + 2 = 6A. That into node 2 is 42 - 4 = 38A. So, the resulting nodal equations are :

$$5V_1 - 3V_2 = 6 \tag{4-38}$$

$$-3V_1 + 7V_2 = 38 \tag{4-39}$$

the solutions to which are $V_1 = 6$ V and $V_2 = 8$ V.

This is the shortcut self-conductance and mutual conductance approach. Perhaps the left sides of these equations are worth verifying by considering the individual resistor currents. At node 1 the current down through the 2-S resistor is $2V_1$. The current to the right through the 3-S resistor is $3(V_1 - V_2)$. The resulting KCL equation at node 1 is :

$$2V_1 + 3(V_1 - V_2) = 6$$

or $$5V_1 - 3V_2 = 6$$

which checks the first equation. At node 2 the current down through the 4-S resistor is $4V_2$. That to the left through the 3-S resistor is $3(V_2 - V_1)$. So, the KCL equation at node 2 is :

$$4V_2 + 3(V_2 - V_1) = 38$$

or $$-3V_1 + 7V_2 = 38$$

To get the currents, the relation $I = V\,G$ is used. Then :

$I_1 = 2\,V_1 = 12$ A, $I_3 = 4\,V_2 = 32$ A and $I_2 = 3(V_2 - V_1) = 3 \times 2 = 6$ A.

Example :

Get the current in different branches of the circuit shown in Figure 4-36 using node voltage method if :

$R_1 = 3\ \Omega$, $R_2 = R = 6\ \Omega$, $R_3 = R_4 = 2\ \Omega$,

$E_1 = 48$ V , $E_2 = 24$ V , $E_3 = E = 12$ V .

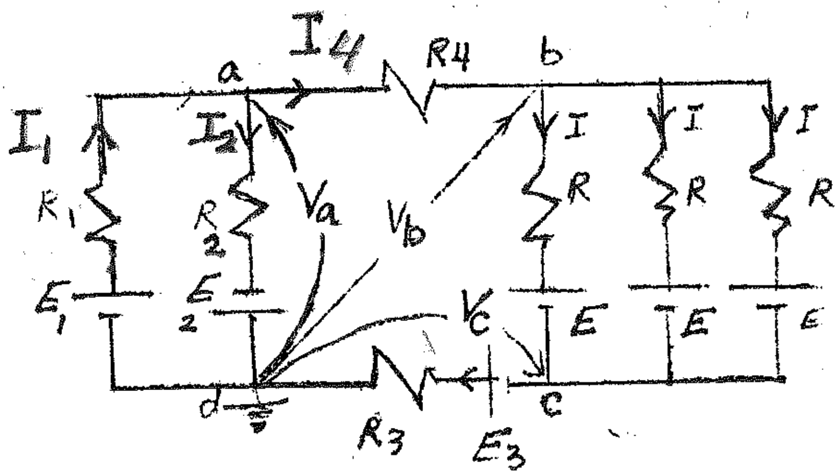


Figure 4-36

Solution :

Node voltage equations for nodes a , b & c are :

$$V_a G_{aa} + V_b G_{ab} + V_c G_{ac} = I_a \qquad (4\text{-}40)$$

$$V_a G_{ab} + V_b G_{bb} + V_c G_{bc} = I_b \qquad (4\text{-}41)$$

$$V_a G_{ac} + V_b G_{bc} + V_c G_{cc} = I_c \qquad (4\text{-}42)$$

The number of (independent nodes equals n - 1 or 4 - 1 = 3. The reference node is d node. In equations (4-40), (4-41) and (4-42), the G's and I's are given by :

$$G_{aa} = G_1 + G_2 + G_3 = \frac{1}{3} + \frac{1}{6} + \frac{1}{2} = 1 \quad ,$$

$$G_{bb} = G_4 + 3G = \frac{1}{2} + \frac{3}{6} = 1 \quad ,$$

$$G_{cc} = G_3 + 3G = \frac{1}{2} + \frac{3}{6} = 1 \quad ,$$

$$G_{ab} = G_{ba} = - G_4 = - \frac{1}{2} \quad ,$$

$$G_{ac} = G_{ca} = 0 \quad ,$$

$$G_{bc} = G_{cb} = - 3 \times \frac{1}{6} = - \frac{1}{2} \quad ,$$

$$I_a = E_1 G_1 - E_2 G_2 = \frac{48}{3} - \frac{24}{6} = 12 \quad ,$$

$$I_b = 3 E G = 3 \times \frac{12}{6} = 6 \quad ,$$

and

$$I_c = -3 E G - E_3 G_3 = -6 - \frac{12}{6} = -12 \quad ,$$

Substituting for the G's and I's in node voltage equations, we get :

$$V_a - 0.5 V_b = 12 \tag{4-43}$$

$$- 0.5 V_a + V_b - 0.5 V_c = 6 \tag{4-44}$$

$$- 0.5 V_b + V_c = - 12 \tag{4-45}$$

Solving equations (4-43), (4-44) and (4-45), we get :

$V_a = 18 \text{ V}$, $V_b = 12 \text{ V}$, $V_c = -6 \text{ V}$,

$V_{ab} = 18 - 12 = 6 \text{ V}$,

$V_{bc} = 12 + 6 = 18 \text{ V}$,

$I_1 = (E_1 - V_a) / R_1 = (48 - 18) / 3 = 10 \text{ A}$,

$I_2 = (E_2 + V_a) / R_2 = (24 + 18) / 6 = 7 \text{ A}$,

$I_3 = E_3 + V_c / R_3 = (12 - 6) / 2 = 3 \text{ A}$,

$I_4 = V_{ab} / R_4 = 6 / 2 = 3 \text{ A}$,

and

$I = (- E + V_{bc}) / R = (- 12 + 18) / 6 = 1 \text{ A}$

Dependent Sources and Circuit Analysis :

A dependent source, also called a controlled source, produces a voltage or current as a function of a voltage or current elsewhere. The circuit of Fig. 4-37 contains a current source that produces a current as a function of the current through the 12-Ω resistor. Because the source current depends on this resistor current, this source is a dependent current source. In contrast, the independent sources we have been considering produce voltages or currents independently of any voltages or currents anywhere in the circuits.

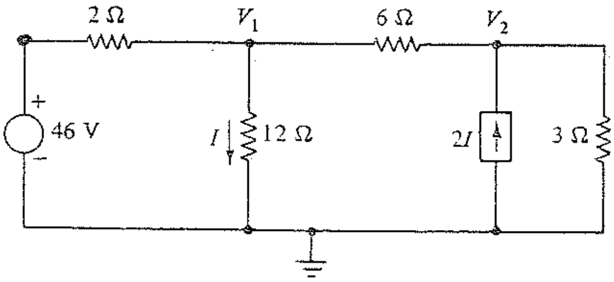


Figure 4-37

This dependent source, which produces a current that is a function of current, is a current-dependent current source. There are also voltage-dependent current sources. They produce currents as functions of voltages elsewhere in the circuits. Likewise, dependent voltage sources are either voltage-dependent or current-dependent.

Nodal and loop analyses for circuits containing dependent sources are about the same as for circuits having only independent sources. Usually, though, there are a few more equations.

Example :

Find V_2 in the circuit of Fig. 4-37.

Solution :

For this circuit no one analysis method is obviously best. So, we will arbitrarily select nodal analysis. At the V_1 node gives :

$$\left(\frac{1}{2}+\frac{1}{12}+\frac{1}{6}\right)V_1 - \left(\frac{1}{6}\right)V_2 = \frac{1}{2} x\, 46$$

which simplifies to $9V_1 - 2V_2 = 276$. At the V_2 node the equation is :

$$-\frac{1}{6}V_1 + \left(\frac{1}{6} + \frac{1}{3}\right)V_2 = 2I$$

But $I = V_1 / 12$. Substituting this in the last equation, we have :

$$-\left(\frac{1}{6}\right)V_1 + \left(\frac{1}{6} + \frac{1}{3}\right)V_2 = 2\left(\frac{V_1}{12}\right)$$

which reduces to $-2V_1 + 3V_2 = 0$. Solving the two equations :

$$9V_1 - 2V_2 = 276$$

$$-2V_1 + 3V_2 = 0$$

gives $V_2 = 24$ V.

Example :

Find the current I_0 in the circuit of Fig. 4-38, which has a current-dependent voltage source.

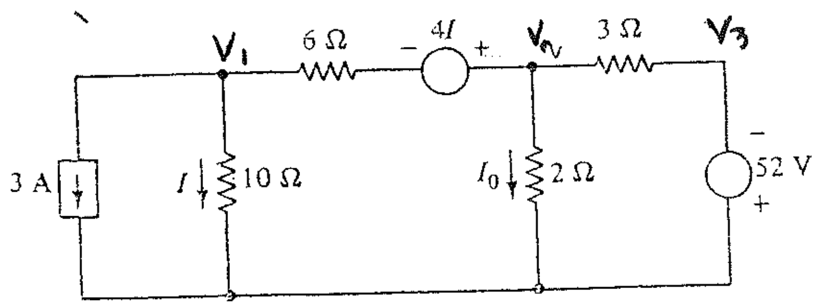


Figure 4-38

Figure 4-39 shows the loop selection dictated by the preference for a single loop current through the current source and also single loop currents through the 10-Ω and 2-Ω resistor branches.

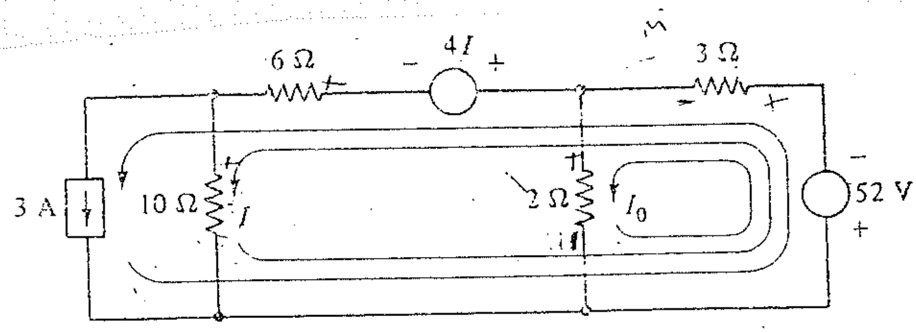


Figure 4-39

This choice for loops does not restrict the closed paths selected for KVL. To show this we will use KVL on the middle mesh, and also the right one, even through there is no loop current for the middle mesh. Around the middle mesh,

$$10I - 2I_0 + 4I + 6(I + 3) = 0$$

which simplifies to $20I - 2I_0 = -18$. Around the right mesh,

$$2I_0 + 52 + 3(I_0 + I + 3) = 0$$

which simplifies to $3I + 5I_0 = -61$. Solving the two KVL equations,

$$20I - 2\,I_0 = -18$$

$$3I + 5I_0 = -61$$

Produces $I_0 = -11$ A. Incidentally, also from these equations $I = -2$ A, the negative sign of which means that the dependent voltage source of 4I produces 8 V but of a polarity opposite to that shown by this source in Fig. 4-39.

Another Solution :

We choose three loops as shown in Fig. 4-40

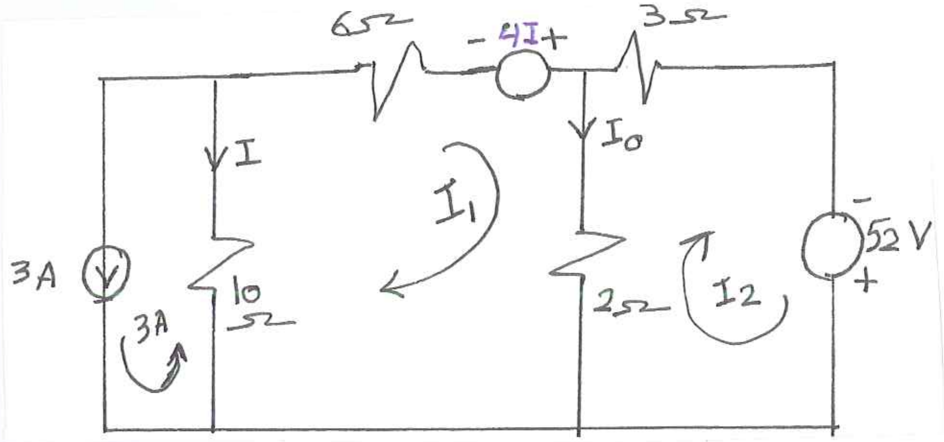


Figure 4-40

Using Kirchoff's current law, we get :

$$I = -(I_1 + 3) \qquad (4\text{-}43)$$

Loop current equations, for loops 1 & 2, are :

$$4I = 18I_1 + 3 \times 10 - 2\, I_2 \qquad (4\text{-}44)$$

$$52 = 5\, I_2 - 2\, I_1 \qquad (4\text{-}45)$$

Substituting for I from equation (4-43) in equation (4-44), it can be written as : $-42 = 22\, I_1 - 2\, I_2$

or $$-21 = 11\, I_1 - I_2 \qquad (4\text{-}46)$$

Solving equations (4-45) & (4-46), we get :

$I_1 = -1$ A , $I_2 = 10$ A and

$I_0 = (I_1 - I_2) = -11$ A.

this is the same result obtained by the first solution.

EXAMPLE Determine R_1, R_2, and R_3 for the voltage divider supply of Fig. 4.241 Can 2-W resistors be used in the design?

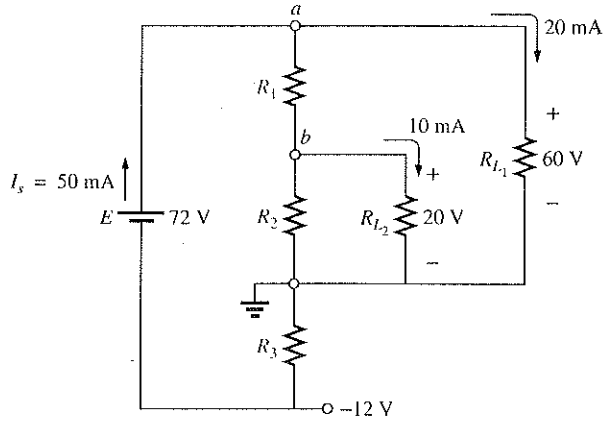


FIG. 4.241

Solution: R_3:

$$R_3 = \frac{V_{R_3}}{I_{R_3}} = \frac{V_{R_3}}{I_s} = \frac{12\text{ V}}{50\text{ mA}} = \mathbf{240\ \Omega}$$

$$P_{R_3} = (I_{R_3})^2 R_3 = (50\text{ mA})^2\, 240\ \Omega \doteq 0.6\text{ W} < 2\text{ W}$$

R_1: Applying Kirchhoff's current law to node *a:*

$$I_s - I_{R_1} - I_{L_1} = 0$$

and $\quad I_{R_1} = I_s - I_{L_1} = 50\text{ mA} - 20\text{ mA}$

$$= 30\text{ mA}$$

$$R_1 = \frac{V_{R_1}}{I_{R_1}} = \frac{V_{L_1} - V_{L_2}}{I_{R_1}} = \frac{60\text{ V} - 20\text{ V}}{30\text{ mA}} = \frac{40\text{ V}}{30\text{ mA}}$$

$$= \mathbf{1.33\ k\Omega}$$

$$P_{R_1} = (I_{R_1})^2 R_1 = (30\text{ mA})^2\, 1.33\text{ k}\Omega = 1.197\text{ W} < 2\text{ W}$$

R_2: Applying Kirchhoff's current law at node *b:*

$$I_{R_1} - I_{R_2} - I_{L_2} = 0$$

and $\quad I_{R_2} = I_{R_1} - I_{L_2} = 30\text{ mA} - 10\text{ mA}$

$$= 20\text{ mA}$$

$$R_2 = \frac{V_{R_2}}{I_{R_2}} = \frac{20\text{ V}}{20\text{ mA}} = \mathbf{1\ k\Omega}$$

$$P_{R_2} = (I_{R_2})^2 R_2 = (20\text{ mA})^2\, 1\text{ k}\Omega = 0.4\text{ W} < 2\text{ W}$$

Since P_{R_1}, P_{R_2}, and P_{R_3} are less than 2 W, 2-W resistors can be used for the design.

We will now consider a network with only one source of voltage to point out that mesh analysis can be used to advantage in other than multisource networks.

EXAMPLE 8.18 Find the current through the 10-Ω resistor of the network of Fig. 4.42

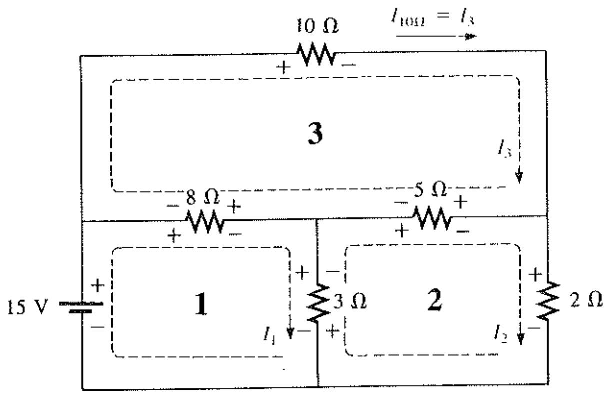


FIG. 4.42

Solution:

$$
\begin{aligned}
I_1&: \quad (8\,\Omega + 3\,\Omega)I_1 - (8\,\Omega)I_3 - (3\,\Omega)I_2 = 15\text{ V}\\
I_2&: \quad (3\,\Omega + 5\,\Omega + 2\,\Omega)I_2 - (3\,\Omega)I_1 - (5\,\Omega)I_3 = 0\\
I_3&: \quad (8\,\Omega + 10\,\Omega + 5\,\Omega)I_3 - (8\,\Omega)I_1 - (5\,\Omega)I_2 = 0
\end{aligned}
$$

$$
\begin{aligned}
11I_1 - 8I_3 - 3I_2 &= 15\\
10I_2 - 3I_1 - 5I_3 &= 0\\
23I_3 - 8I_1 - 5I_2 &= 0
\end{aligned}
$$

or

$$
\begin{aligned}
11I_1 - 3I_2 - 8I_3 &= 15\\
-3I_1 + 10I_2 - 5I_3 &= 0\\
-8I_1 - 5I_2 + 23I_3 &= 0
\end{aligned}
$$

and

$$
I_3 = I_{10\Omega} = \frac{\begin{vmatrix} 11 & -3 & 15 \\ -3 & 10 & 0 \\ -8 & -5 & 0 \end{vmatrix}}{\begin{vmatrix} 11 & -3 & -8 \\ -3 & 10 & -5 \\ -8 & -5 & 23 \end{vmatrix}} = 1.220\text{ A}
$$

The next example has only one source applied to a ladder network.

EXAMPLE Write the nodal equations and find the voltage across the 2-Ω resistor for the network of Fig. 4.43.

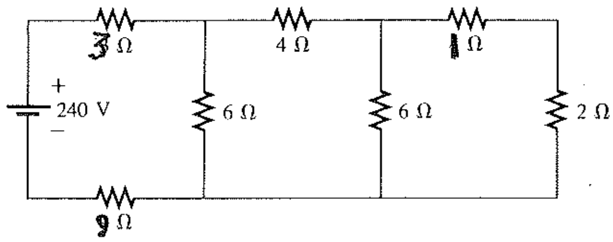


FIG. 4.43

Solution: The nodal voltages are chosen as shown in Fig. 4.44.

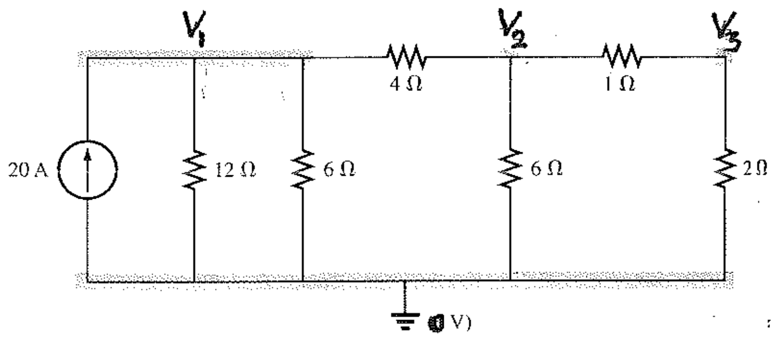


FIG. 4.44

$$V_1:\quad \left(\frac{1}{12\,\Omega}+\frac{1}{6\,\Omega}+\frac{1}{4\,\Omega}\right)V_1-\left(\frac{1}{4\,\Omega}\right)V_2+\quad 0 \quad = 20\text{ V}$$

$$V_2:\quad \left(\frac{1}{4\,\Omega}+\frac{1}{6\,\Omega}+\frac{1}{1\,\Omega}\right)V_2-\left(\frac{1}{4\,\Omega}\right)V_1-\left(\frac{1}{1\,\Omega}\right)V_3=0$$

$$V_3:\quad \left(\frac{1}{1}+\frac{1}{2\,\Omega}\right)V_3-\left(\frac{1}{1\,\Omega}\right)V_2+\quad 0 \quad =0$$

and

$$\begin{aligned} 0.5V_1 \;-0.25V_2 \quad &+0=20\\ -0.25V_1 \;+\frac{17}{12}V_2 \quad &-1V_3=0\\ 0 \quad -1V_2 \quad &+1.5V_3=0 \end{aligned}$$

Note the symmetry present about the major axis. Application of determinants reveals that

$$V_3 = V_{2\Omega} = \mathbf{10.667\ V}$$

PROBLEMS

1- Convert the voltage source of Fig. 4-41 to an equivalent current source for :

(a) $V = 10\ V$, $R = 2\ \Omega$

(b) $V = 50\ V$, $R = 10\ \Omega$

2- Convert the current source of Fig. 4-42 to an equivalent voltage for :

(a) $I = 10\ A$, $R = 2\ \Omega$

(b) $I = 4\ A$, $R = 2\ k\Omega$

(c) $I = -5\ mA$, $R = 20\ k\Omega$

3- Find the total resistance and current I for each circuit of Figure 4-43.

4- For the circuit of Fig. 4-44, the total resistance is given. Find the unknown resistances and the current I for each circuit.

5- Determine the current I and its direction for each network of Fig. 4-45.

6- Find the unknown voltage source, resistor and the direction of current for the networks of Fig. 4-46.

7- In the networks of Fig. 4-47 get V_1 & V_2 using Kirchoff's voltage law.

8- Determine I & V_1 for the network of Fig. 4-48.

9- For the circuit of Fig. 4-49, get V_1 , V_2 , V_3 , I and R_T. Also, get the power dissipated by each resistor and check power balance. If the resistors are available with wattage ratings of 0.5 , 1 and 2 W, what minimum wattage rating can be used for each resistor in the circuit.

10- For the circuit of Fig. 4-50, find the unknown quantities.

11- **Eight lights are connected in series as shown in Fig. 4-51 :**

(a) If the set is connected to a 120 V source, what is the current of the source if each lamp has an internal resistance is 15 Ω.

(b) If one lamp burns out, what is the effect on the remaining lamps.

12- **Using the voltage divider rule, find V_{ab} for the circuits of Figure 4-52.**

13- **Find the unknown resistance using the voltage divider rule for the circuit of Fig. 4-53.**

14- **In the circuit of Fig. 4-54, $R_2 = 2\ R_1$, find the unknown quantities.**

15- **Determine V_a & V_b for the networks of Fig. 4-55.**

16- **For the network of Fig. 4-56, get V_a , V_b , V_c & V_d .**

17- **Determine R_1, R_2 & R_3 of Fig. 4-57 if $R_2 = 5\ R_1$ and $R_3 = 0.5\ R_1$.**

18- Eight lamps are connected in parallel as shown in Fig. 4-58.

(a) If the set is connected to a 120 V source, what is the current of the source if each lamp has internal resistance of 1.8 KΩ.

(c) If one lamp burns out, what is the effect on the remaining lamps.

(d) Compare the parallel arrangement of Fig. 4-58 with the series arrangement of Fig. 4-51. What are the relative advantages of the parallel system as compared to the series arrangement.

19- Calculate the effective resistive of the circuit and the voltage drop across each resistance when a P. D. of 60 V is applied between points A and B in Fig. 4-59.

20- A resistance of 10 Ω is connected in series with two resistances each of 15 Ω arranged in parallel. What resistance must be shunted across this series-parallel combination so that the total current taken by the circuit be 1.5 A with 20 V applied.

21- Three resistances R , 2 R and 3 R are connected in delta. Determine the resistances for an equivalent star connection.

22- In Fig. 4-60 , 20 V are applied across terminals AB. Determine R.

23- In the circuit given in Fig. 4-61, get the current through branch cd.

24- Determine the input resistance between A & B for the circuits shown in Fig. 4-62 (A, B), 4-63 (A , B) and 4-64 (A , B).

25- In the ladder network shown in Fig. 4-65, find the currents in all branches by two different methods.

26- In the circuit shown in Fig. 4-66, find the currents in all branches and the voltage E.

27- To determine the resistance R in circuit of Fig. 4-67, we use a voltmeter and ammeter (taking into consideration their resistances) connected as shown. When switch is in position (1) I_{1A} 1.6 A and U_{1V} = 2.35 V. When switch is in position (2) I_{2A} = 1.25 A and U_{2V} = 1 V. Calculate the resistance R. Determine also r_A and r_V.

28- For the circuit in Fig. 4-68, find :

(a) the current in the 40-ohm resistance, if the applied voltage U = 310 V.

(b) the voltage U, if the current in the 8-ohm resistance is 1 A.

29- The circuit given in Fig. 4-69 is supplied from a voltage source U_o = 75 V, from one side and a current source I_o = 15 A. From the other side, find currents in all branches of the network.

30- Find the currents in all branches and the voltage of the source for the circuit given in Fig. 4-70, if the current in R_5 is 4 A.

31- Determine the values of U_1 and U_2 in the circuit shown in Fig. 4-71, so that each source will supply the same power.

$U_{ab} = 600$ V , $I_{ab} = 50$ A

$R_1 = 2.2$ ohm , $R_2 = 1.1$ ohm

$R_3 = 0.3$ ohm , $R_4 = 0.5$ ohm

32- For the given circuit in Fig. 4-72, find r_o and U.

33- In the circuit shown in Fig. 4-73, calculate the current I by means of the method of "step by step" simplification. All the resistances are equal and have the value of 6 Ω. U = 180 V.

34- Calculate the the currents in all branches of the given circuit of Fig. 4-74, using the direct application of Kirchoff's laws.

35- Using loop-current method determine the currents in all branches of the network of Fig. 4-75.

$R_1 = 18$ ohms $R_2 = 18$ ohms $R_3 = 10$ ohms

$R_4 = 36$ ohms $R_5 = 10$ ohms $U_1 = 90$ volts

$U_2 = 45$ volts

36- In the network of Fig. 4-76, determine the currents in all branches by the help of the loop-current method.

$R_1 = 10$ ohms $\quad R_2 = 15$ ohms $\quad R_3 = 5$ ohms

$R_4 = 20$ ohms $\quad R_5 = 10$ ohms $\quad R_6 = 20$ ohms

$R_7 = 5$ ohms $\quad U_1 = 100$ volt $\quad U_2 = 30$ volt

$U_3 = 40$ volt

37- For the network shown in Fig . 4-77, get the current in the 132 ohm resistor and the P. d. across the 20 ohm resistor using milliman's theorem.

38- A circuit is made up as shown in Fig. 4-78, get the current in 12 V battery. Check the answer by milliman's theorem.

39- In Fig. 4-79, get I.

40- In the circuit shown in Fig. 4-80, get I.

41- Find the voltage V_0 across the 4 Ω resistor in the circuit of Fig. 4-81.

42- Find the voltage V_0 in the circuit of Fig. 4-82.

43- Use branch analysis to solve for the indicated branch currents in Fig. 4-83.

44- Repeat Prob. 43 using mesh analysis.

45- Repeat Prob. 43 with all resistances doubled.

46- Repeat Prob. 45 Using mesh analysis.

47- Repeat Prob. 43 with the 3-Ω resistor changed to 2 Ω and 12 V source changed to 8 V.

48- Repeat Prob. 47 Using mesh analysis.

49- Use mesh analysis to find the indicated mesh currents in Fig. 4-84

50- Repeat Prob. 49 with all resistances doubled.

51- **Repeat Prob. 49 with the 3 Ω resistor changed to 8 Ω and the 4 Ω resistor changed to 9 Ω.**

52- **Show a circuit corresponding to the mesh equations $5\ I_1 - 2\ I_2 = 10$ and $-2\ I_1 + 7\ I_2 = -\ 6$. Also, find the currents.**

53- **Show a circuit corresponding to the mesh equations $10\ I_1 - 3I_2 = -7$ and $3I_1 + 15\ I_2 = 10$. Also, find the currents.**

54- **Show a circuit corresponding to the mesh equations $14\ I_1 - 8I_2 = 13$ and $-\ 8\ I_1 + 20\ I_2 = -\ 11$. Also, find the currents.**

55- **Show a three-mesh circuit corresponding to the following description. Meshes 1, 2 and 3 have self-resistances of 10 Ω, 16 Ω, and 13 Ω, respectively. The mutual resistances are 2 Ω for meshes 1 and 2, 3 Ω for meshes 1 and 3, and 4 Ω for meshes 2 and 3. The total source voltages are 3 V aiding for mesh 1 , 4 V opposing for mesh 2, and 6 V opposing for mesh 3. Also, find the currents.**

56- Show a three-mesh circuit corresponding to the following description. Meshes 1, 2 and 3 have self-resistances of 20 Ω, 30 Ω, and 40 Ω, respectively. The mutual resistances are 4 Ω for meshes 2 and 3. The total source voltages are 10 V aiding for mesh 1, 12 V opposing for mesh 2, and 16 V aiding for mesh 3. Also, find the currents.

57- Find the currents and show a circuit corresponding to the mesh equations :

$$21\,I_1 - 6\,I_2 - 8\,I_3 = 10$$

$$-6\,I_1 + 30\,I_2 - 12\,I_3 = -6$$

$$-8\,I_1 - 12\,I_2 + 40\,I_3 = -12$$

58- In Fig. 4-83 find I_3 by using loop analysis with I_3 being the only loop current through the 2 Ω resistor.

59- In Fig. 4-84 use loop analysis to find the current out of the negative terminal of the 6 V source. Have only one loop current through this source.

60- Repeat Prob. 59 for the current out of the positive terminal of the 10 V source.

61- Repeat Prob. 59 for the current out of the positive terminal of the 4 V source.

62- Use nodal analysis to find V_1 and V_2 in the circuit of Fig. 4-85.

63- Repeat Prob. 62 with the 5 A source changed to 23 A and the 29 A source changed to 45 A.

64- Repeat Prob. 62 with the 5 A source changed to 34 A and 29 A source changed to 62 A.

65- Repeat Prob. 62 using mesh analysis.

66- Repeat Prob. 63 using mesh analysis.

67- Repeat Prob. 64 using mesh analysis.

68- Use nodal analysis to find V_1 and V_2 and V_3 in the circuit of Fig. 4-86.

69- Repeat Prob. 4-68 with the current sources changed from 4 A to 29 A, 11 to 34 A, and from 12 A to 6 A.

70- Repeat Prob. 68 with the current sources changed from 4 A to 22 A, 11 A to – 28 A, and from 12 to 31 A.

71- Repeat Prob. 68 using mesh or loop analysis.

72- Repeat Prob. 69 using mesh or loop analysis.

73- Repeat Prob. 70 using mesh or loop analysis.

74- Find the voltages and show a circuit corresponding to the nodal equations :

$$10\,V_1 - 6\,V_2 - 2\,V_3 = 4$$

$$-6\,V_1 + 14\,V_2 - 5\,V_3 = 2$$

$$-2\,V_1 - 5\,V_2 + 9\,V_3 = 6$$

75- Repeat Prob. 74 for :

$$15\, V_1 - 4\, V_2 - 3\, V_3 = -3$$

$$-4\, V_1 + 11\, V_2 - 7\, V_3 = 8$$

$$-3\, V_1 - 7\, V_2 + 14\, V_3 = -7$$

76- Show a circuit corresponding to the following nodal equations :

$$20\, V_1 - 5\, V_2 - 4\, V_3 - 3\, V_4 = 4$$

$$-5\, V_1 + 15\, V_2 - 4\, V_3 - 5\, V_4 = -6$$

$$-4\, V_1 - 4\, V_2 + 25\, V_3 - 10\, V_4 = 8$$

$$-3\, V_1 - 5\, V_2 - 10\, V_3 + 18\, V_4 = 0$$

77- Two parallel-connected 12 V batteries energize a 15 Ω load. One battery has a 0.05 Ω internal resistance and the other a 0.1 Ω internal resistance. What analysis method is best for finding the load voltage ? Find this voltage and use it to find the battery currents.

78- Find the nodal equations of Fig. 4-87.

79- Two ideal voltage sources deliver the power into the circuit of Fig. 4-88. Determine the magnitude and direction of the current flowing through the resistor R_7 using the node equations :

$R_1 = 5$ ohm	$R_2 = R_3 = 3$ ohm	$R_4 = 6$ ohm
$R_5 = R_6 = 8$ ohm	$R_7 = 2$ ohm	$R_8 = 4$ ohm
$V_1 = 60$ volt	$V_2 = 39$ volt	

80- Solve the circuit of Fig. 4-89, and check the power balance :

$R_1 = 10$ ohm	$R_2 = 20$ ohm	$V_3 = 40$ volt
$R_4 = 3.3$ ohm	$R_5 = 20$ ohm	
$V_1 = 80$ volt	$V_2 = 100$ volt	$R_3 = 3$ ohm

81- In the network of Fig. 4-90 calculate the current in each branch.

82- In the network shown in Fig. 4-91, $V = 10$ volt, find the current in each branch. Check also the power balance.

70. Find the voltage V_2 and the current I_1 for the network of Fig. 4.96.

71.a. Convert the voltage sources of Fig. 4-97 to current sources.

b. Find the voltage V_{ab} and the polarity of points a and b.

c. Find the magnitude and direction of the current I.

72. For the network of Fig. 4.98:

a. Convert the voltage source to a current source.
b. Reduce the network to a single current source and determine the voltage V_1.
c. Using the results of part (b), determine V_2.
d. Calculate the current I_2.

73. For the network of Fig. 4.99:

a. Write the equations necessary to solve for the branch currents.
b. Solve for the branch current through the resistor R_3.

74. Using mesh analysis, determine the current through the 5-Ω resistor for each network of Fig.4.100. Then determine the voltage V_a.

75. a. Write the nodal equations for the networks of Fig.4.101.

b. Solve for the nodal voltages.

c. Determine the magnitude and polarity of the voltage across each resistor.

76. Determine the current through the source resistor R_s of each network of Fig. 4.102 using either mesh or nodal analysis. Discuss why you chose one method over the other.

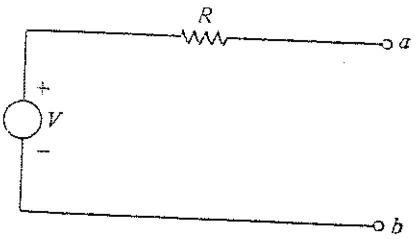


Figure 4-41

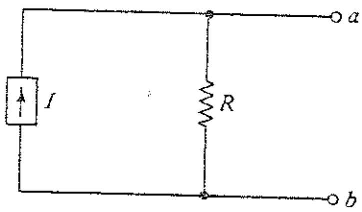


Figure 4-42

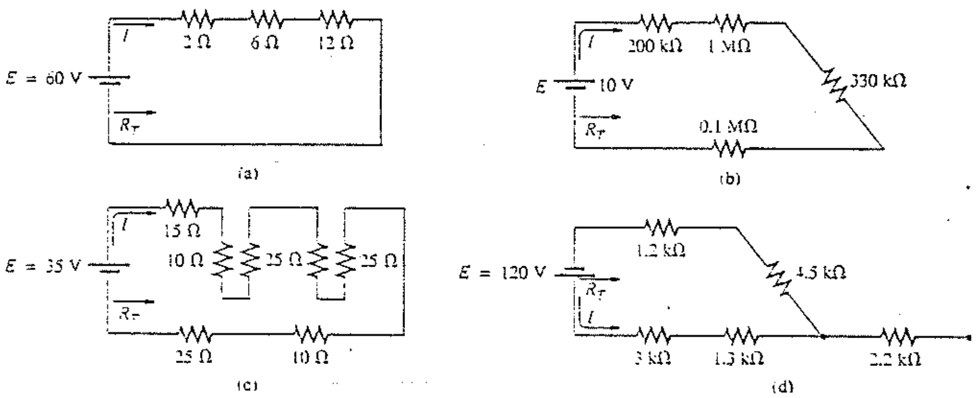


Figure 4-43

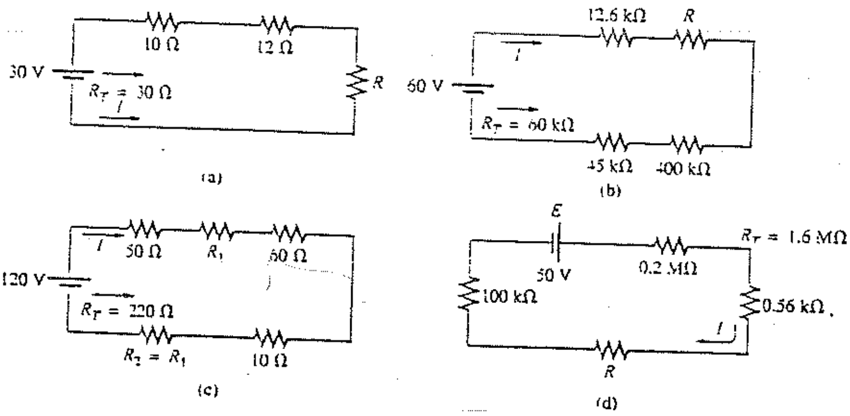


Figure 4-44

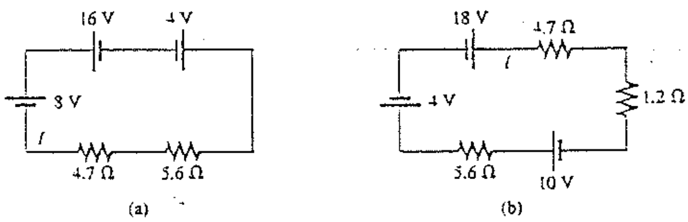


Figure 4-45

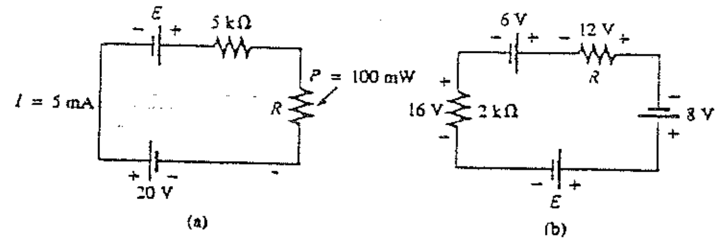


Figure 4-46

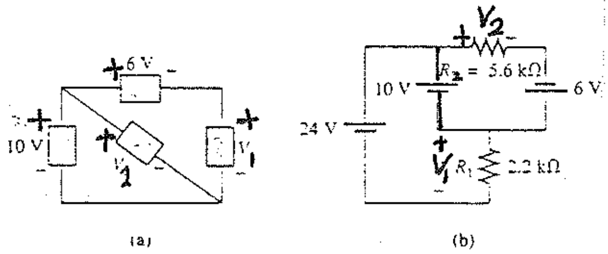


Figure 4-47

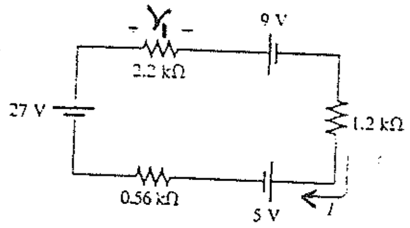
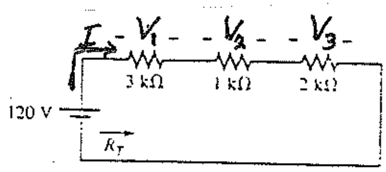


Figure 4-48



← **Figure 4-49**

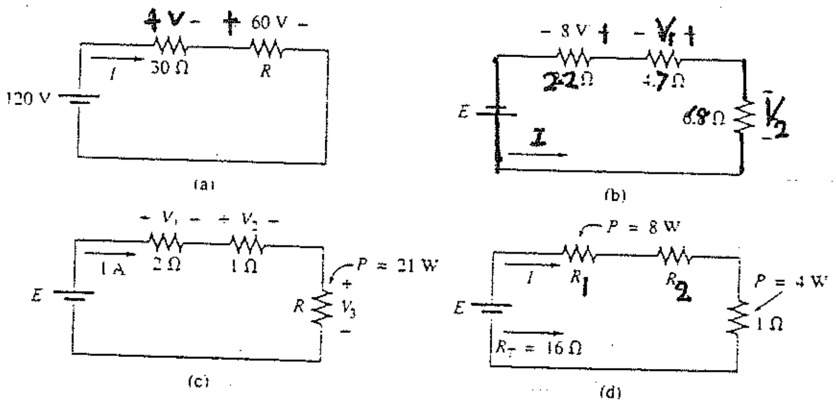


Figure 4-50

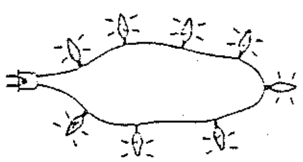


Figure 4-51

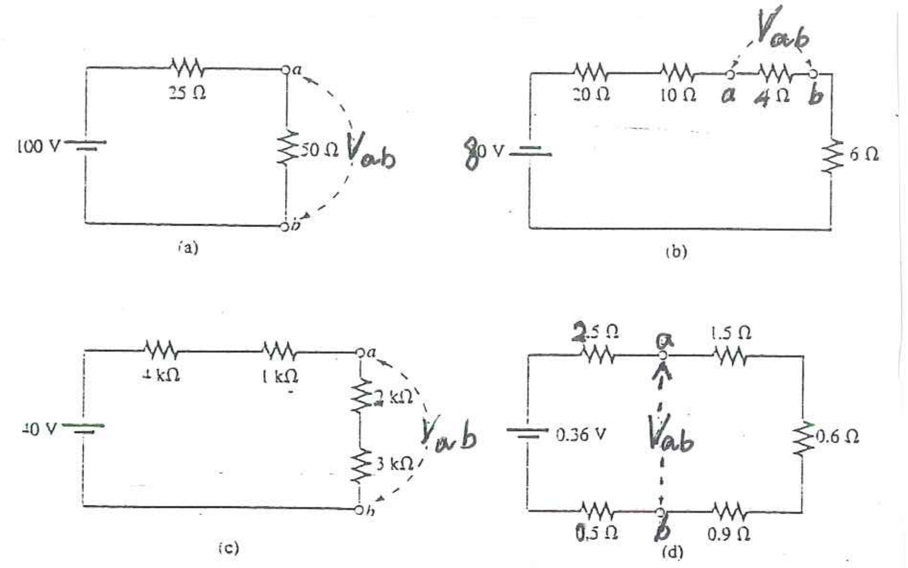


Figure 4-52

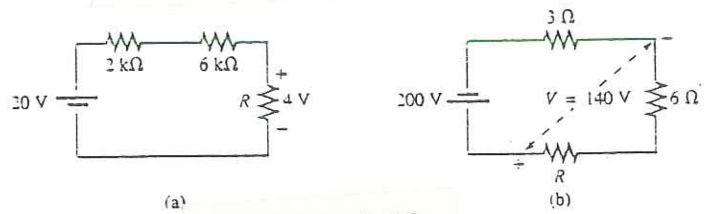


Figure 4-53

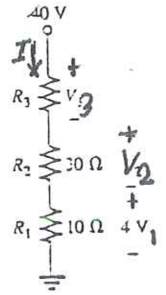


Figure 4-54

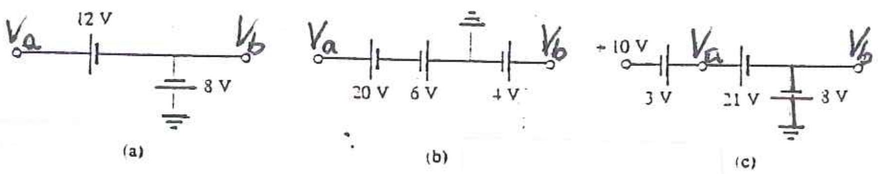


Figure 4-55

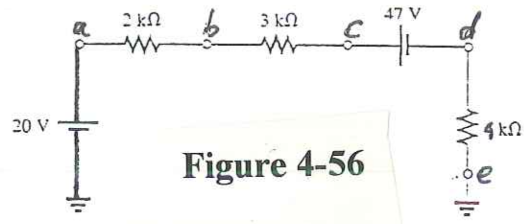
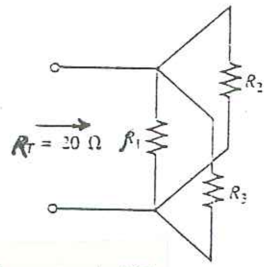


Figure 4-56



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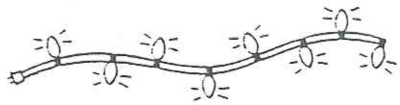

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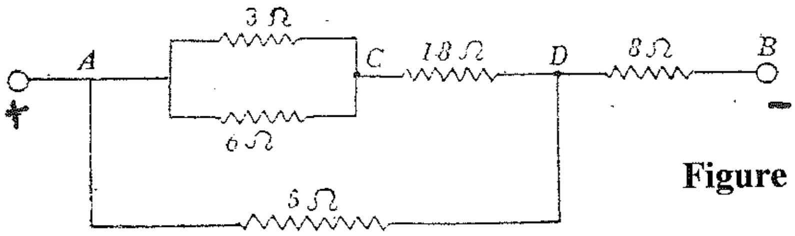


Figure 4-59

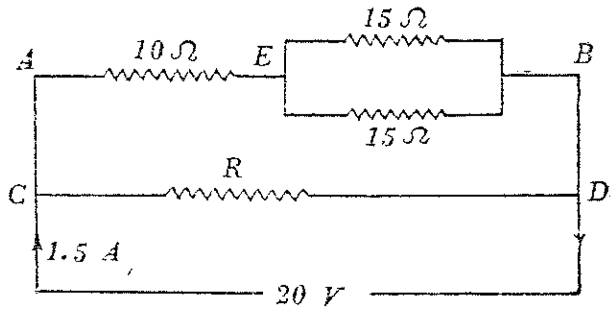
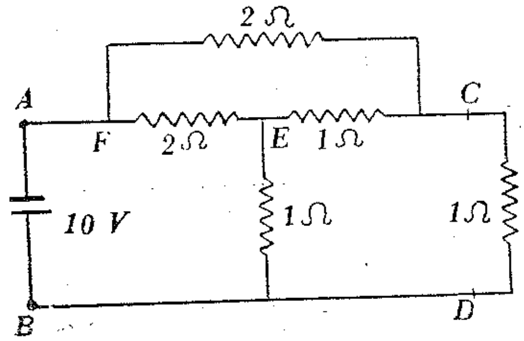


Figure 4-60



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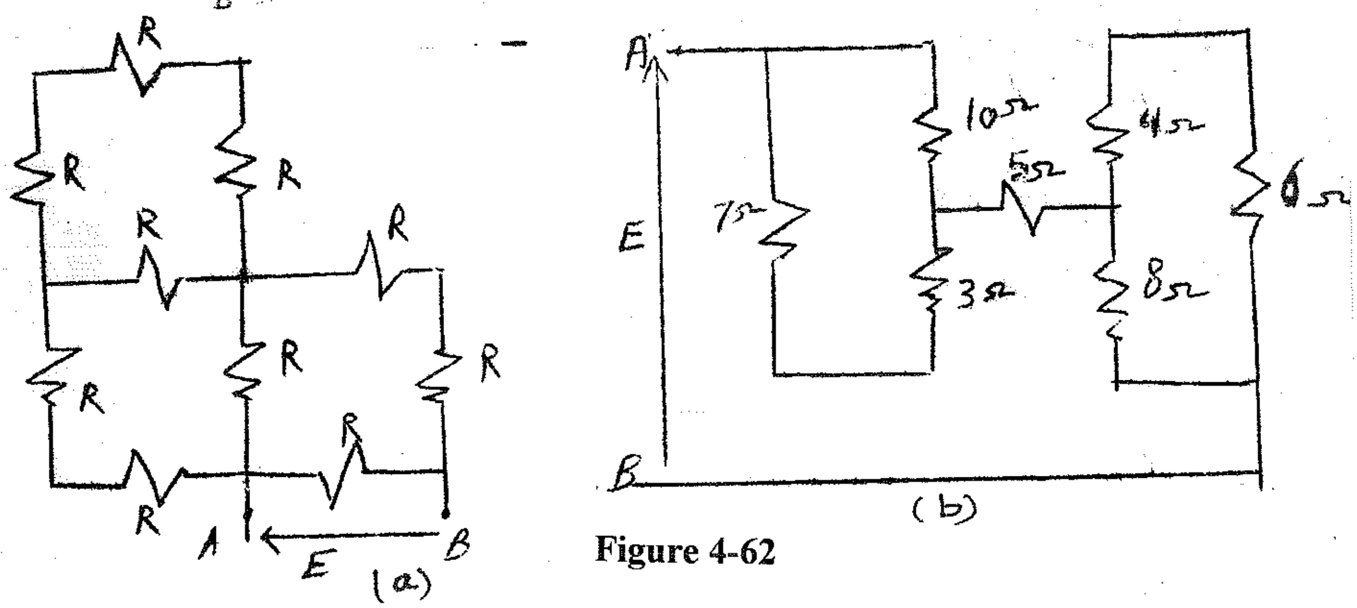

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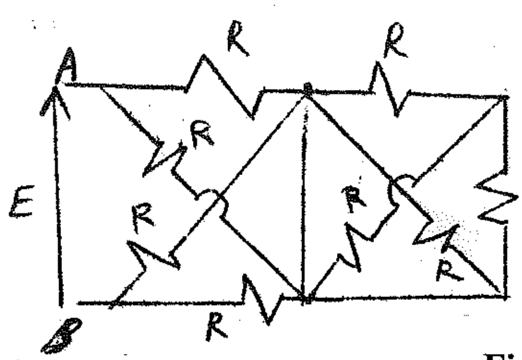

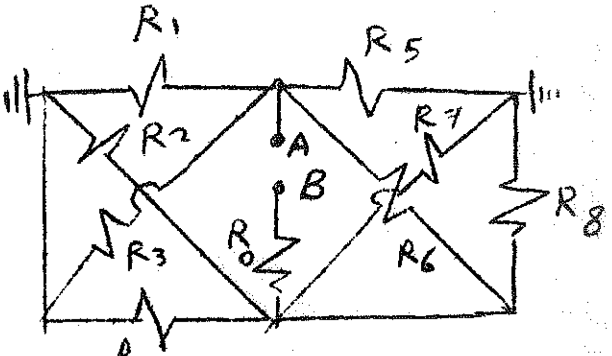

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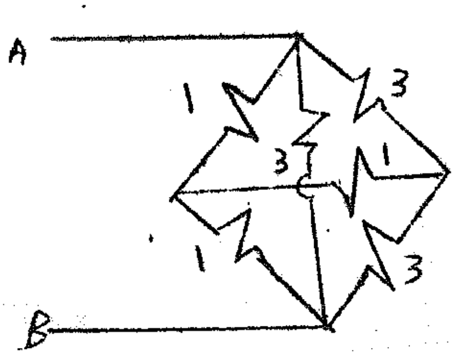

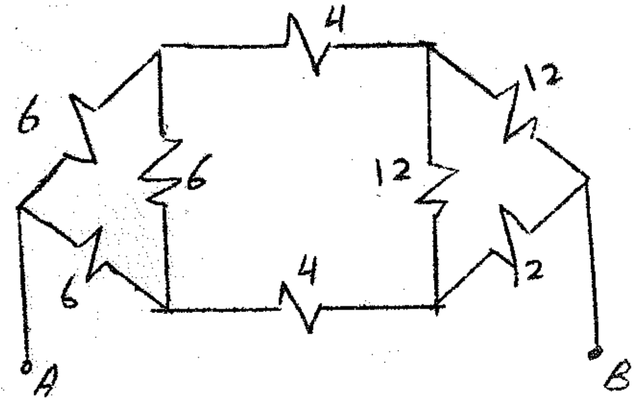

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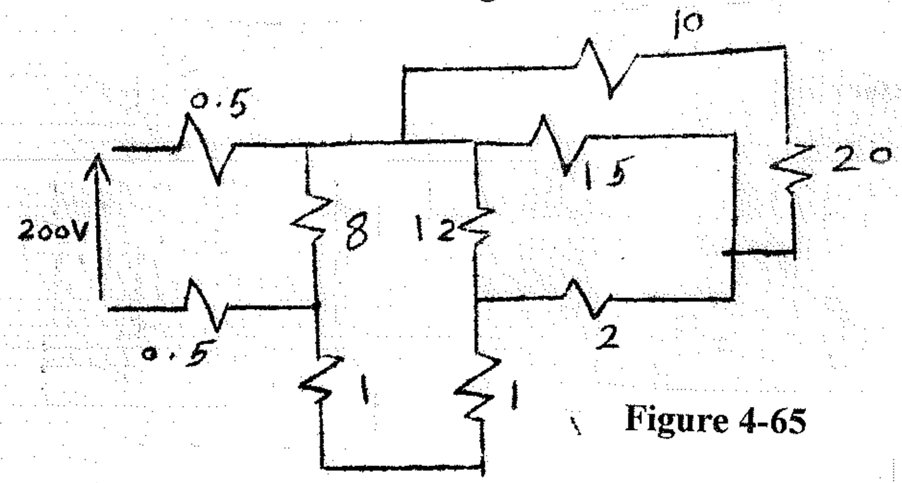

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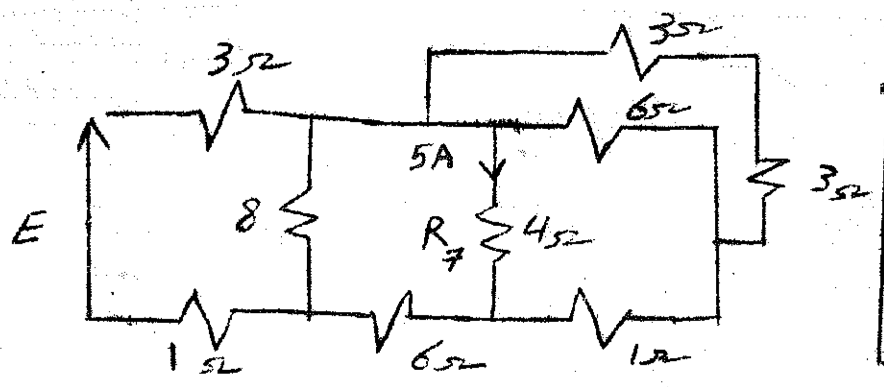


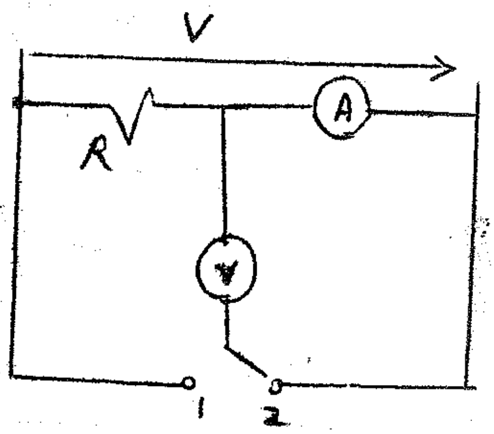


Figure 4-66

Figure 4-67

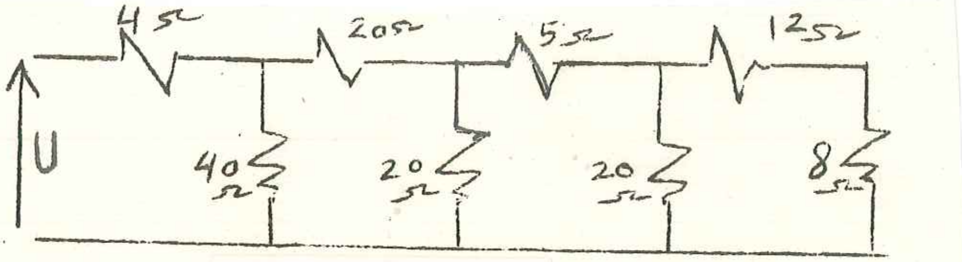
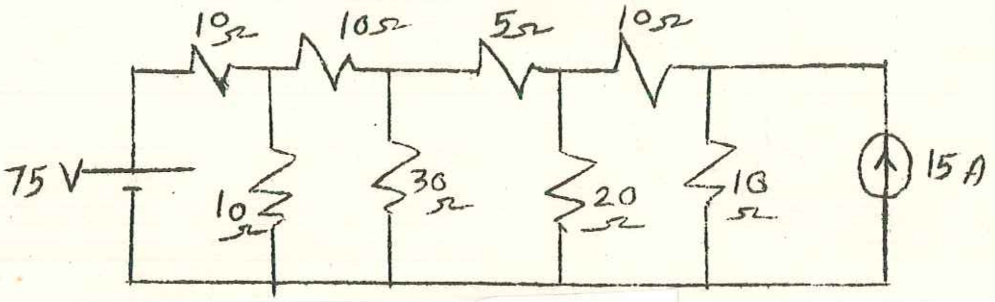


Figure 4-68



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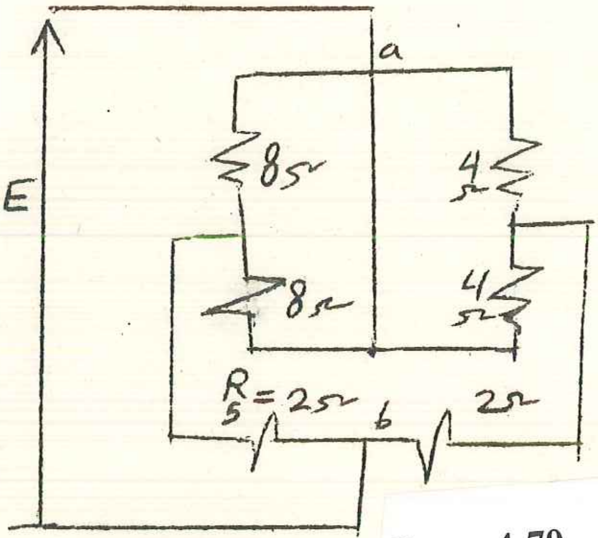
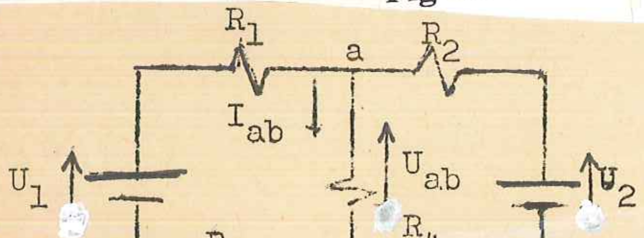


Figure 4-70



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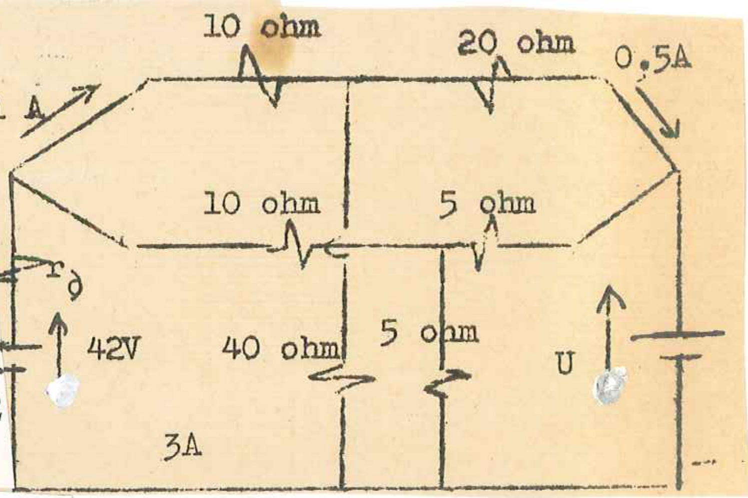

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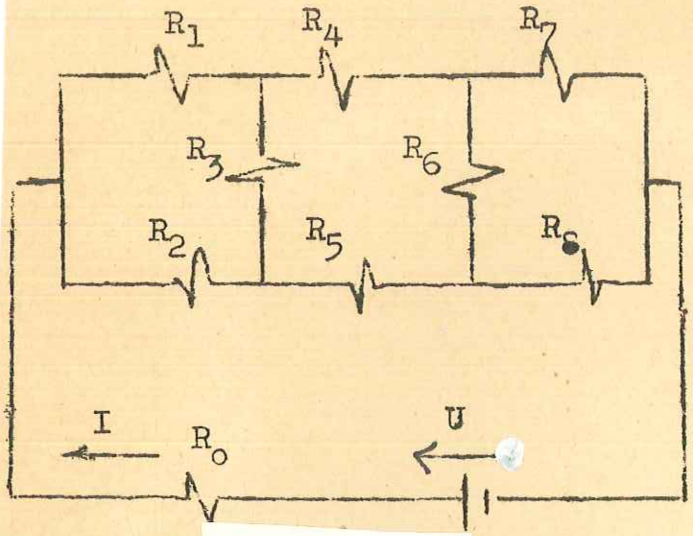

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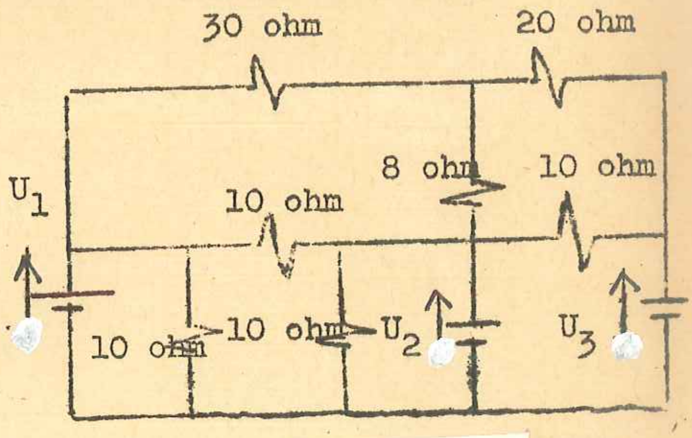

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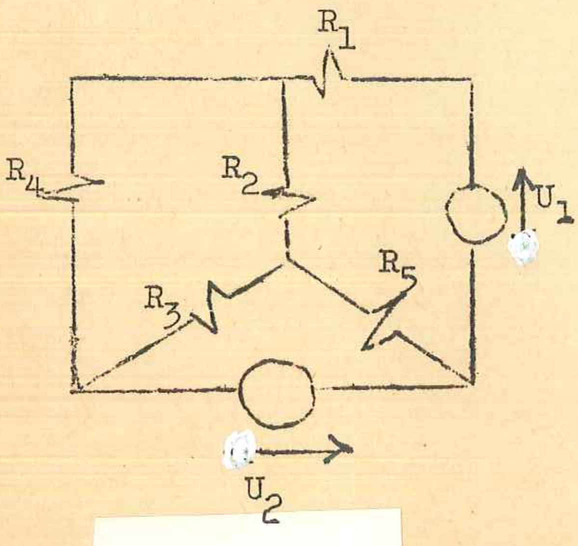

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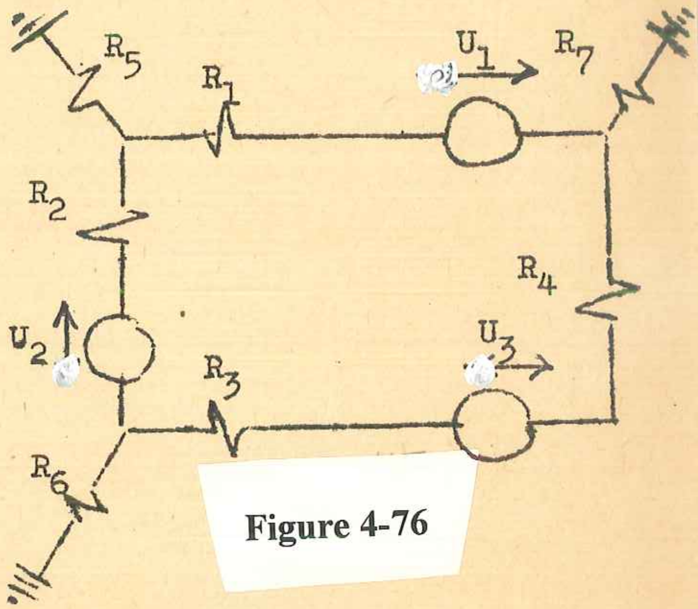

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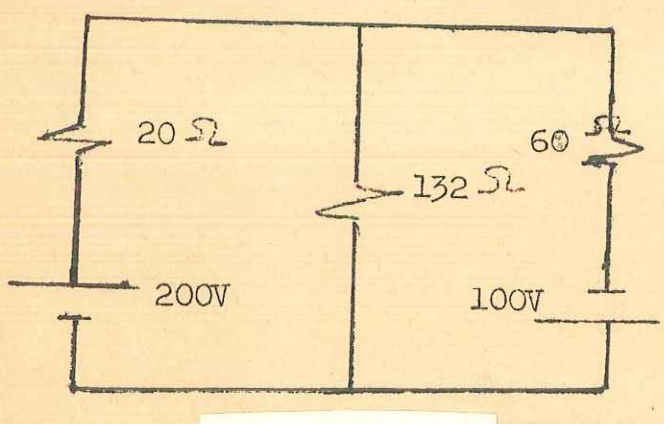

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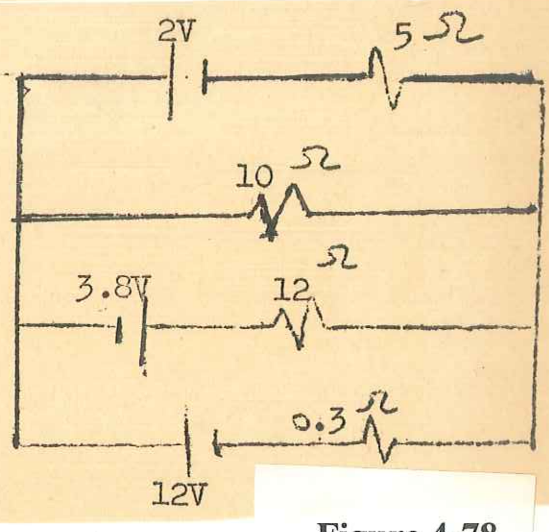

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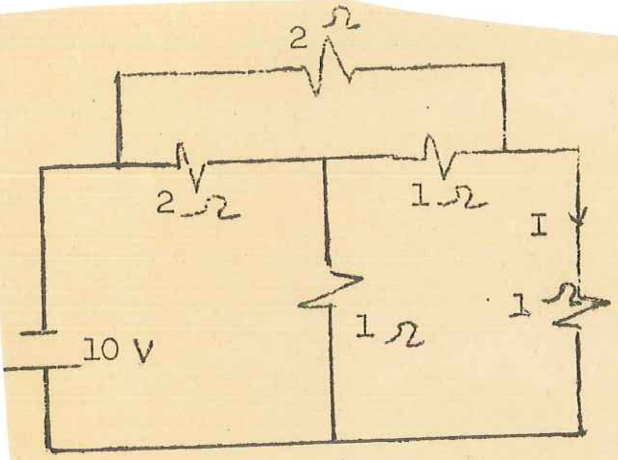


Figure 4-79

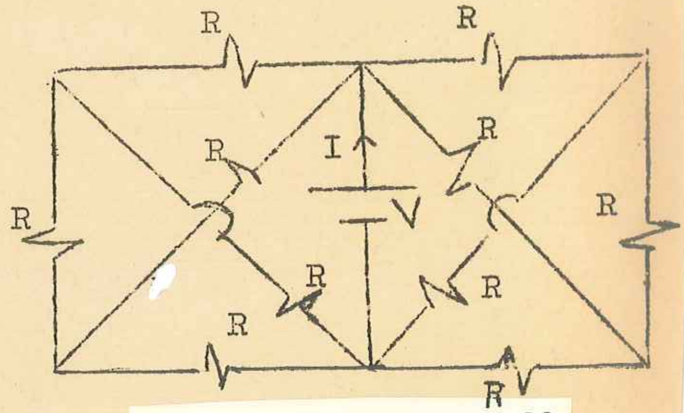


Figure 4-80

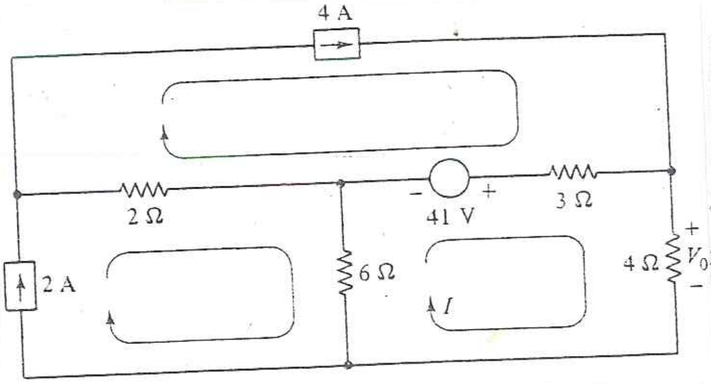
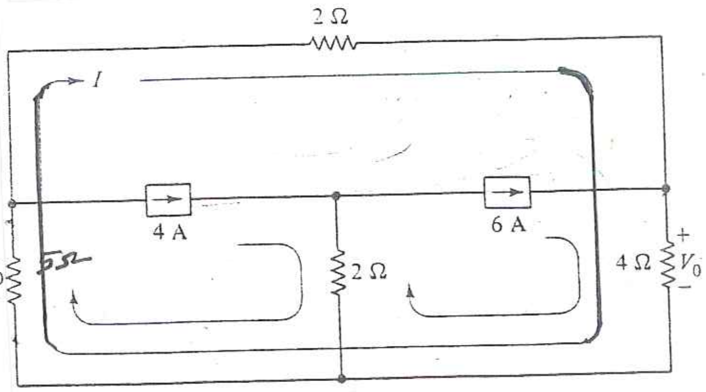


Figure 4-81



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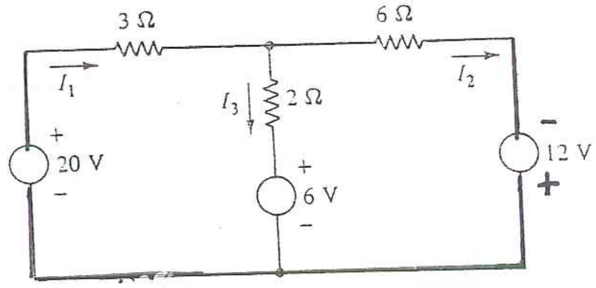

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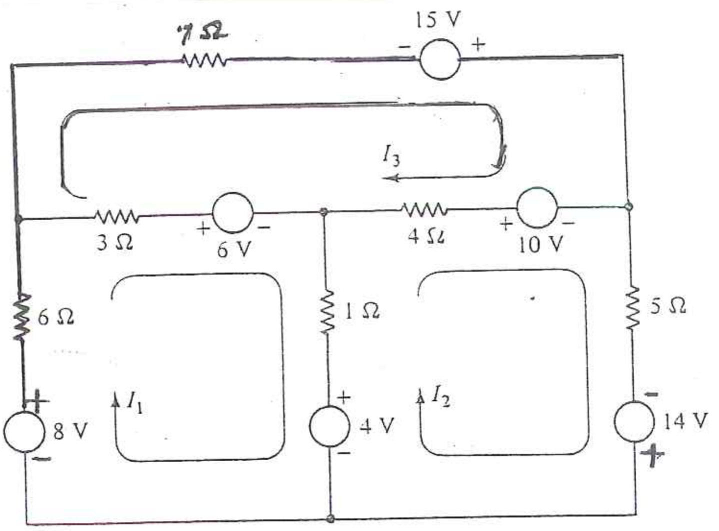

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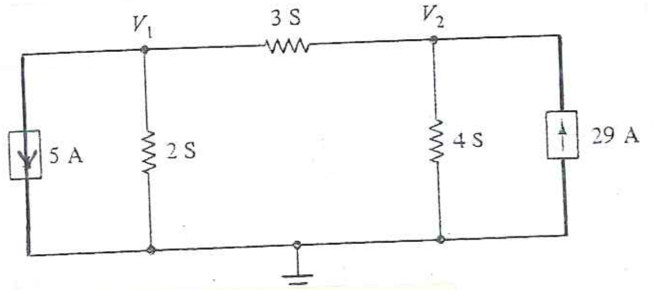

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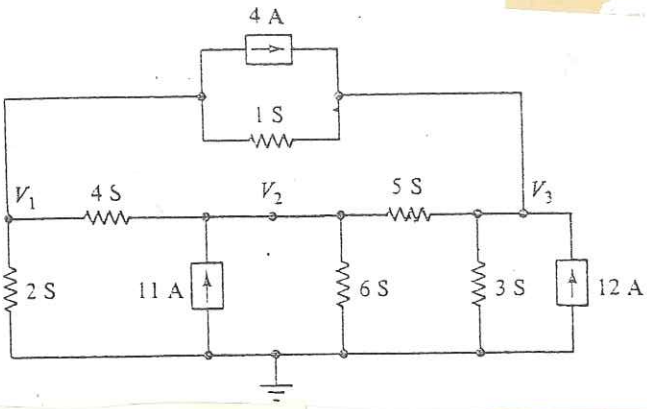
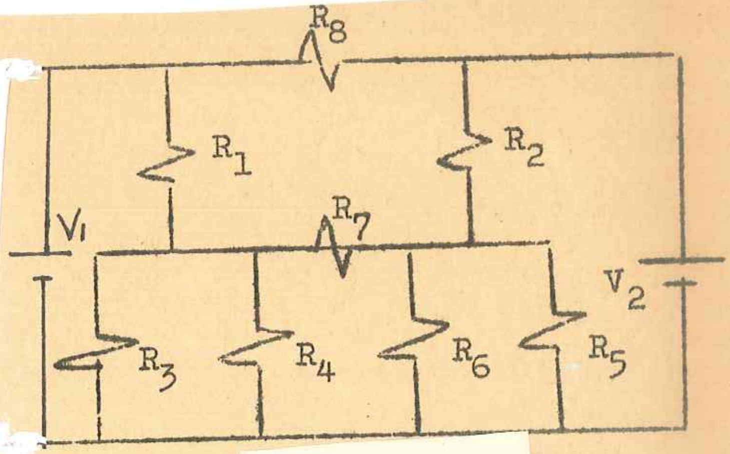


Figure 4-86



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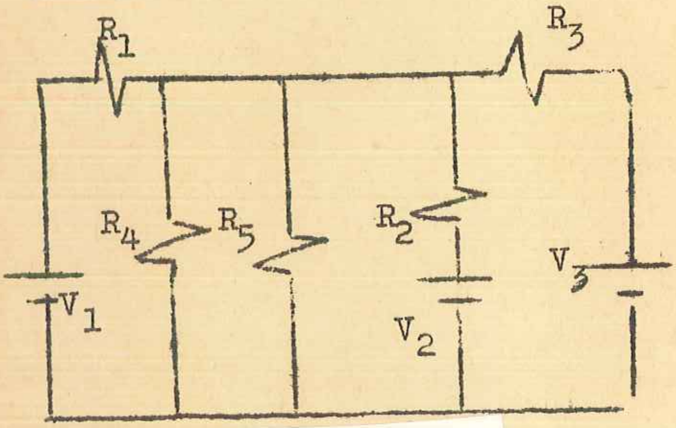


Figure 4-89

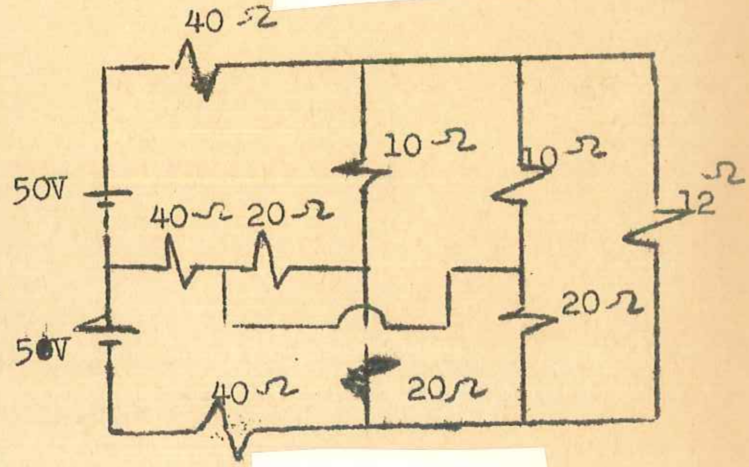


Figure 4-90

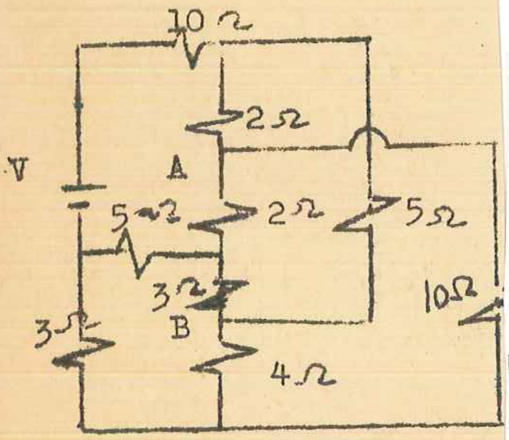


Figure 4-91

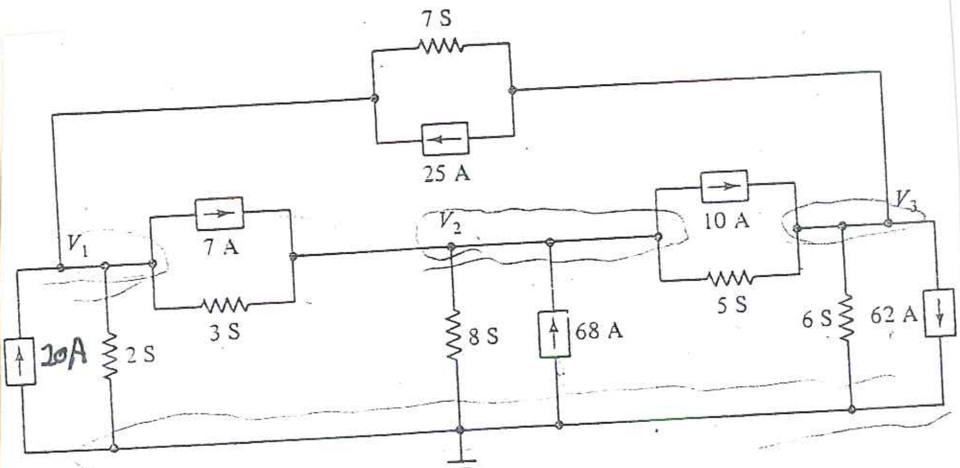


Figure 4-87

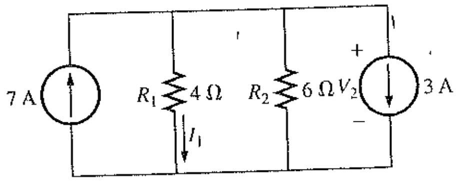


FIG. 4.96

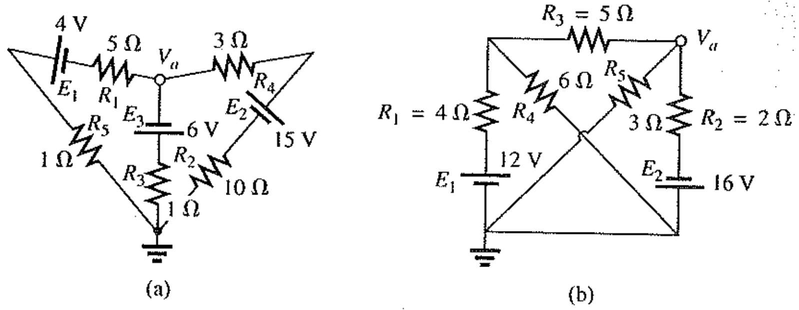


FIG. 4.100

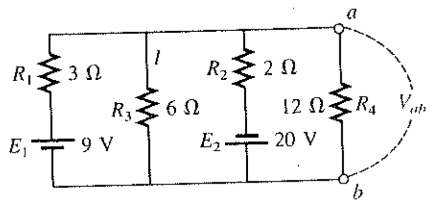


FIG. 4.97

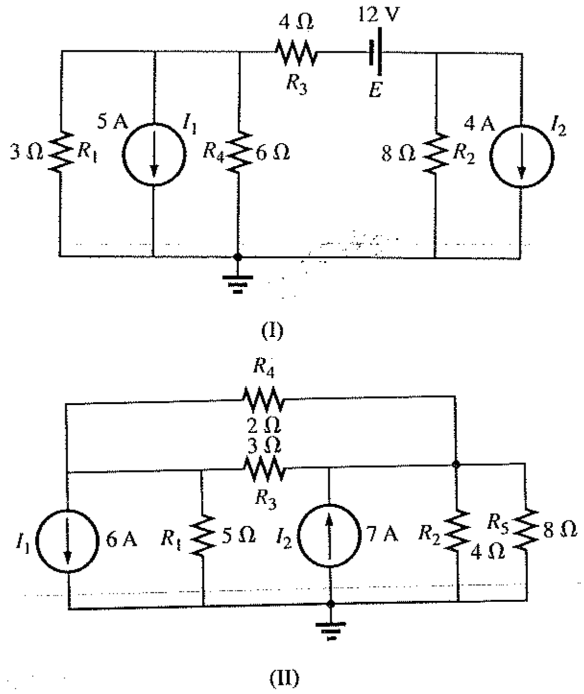


FIG. 4.101

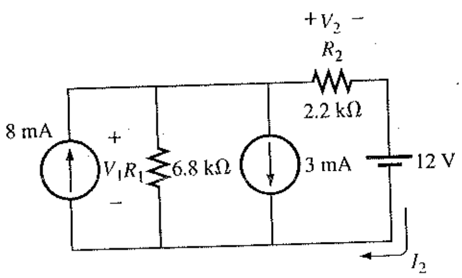


FIG. 4.98

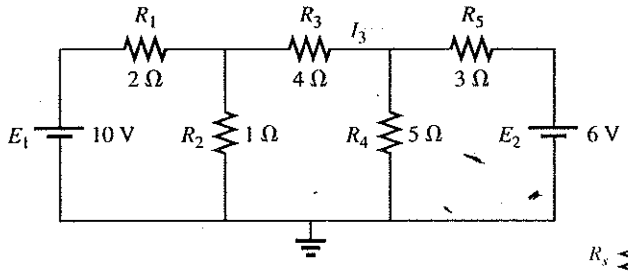


FIG. 4.99

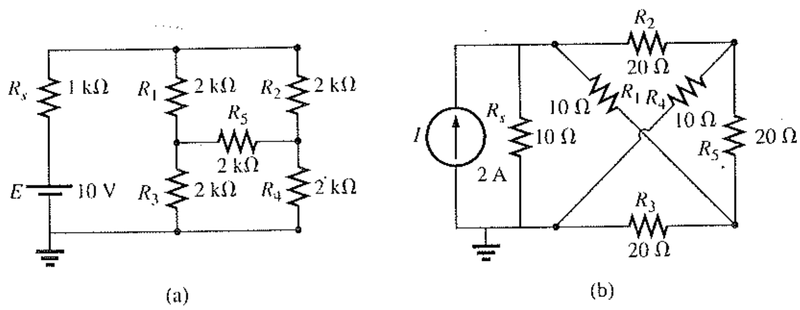


FIG. 4.102

Chapter Five

NETWORK THEOREMS

INTRODUCTION

This chapter will introduce the important fundamental theorems of network analysis. Included are the *superposition, Thevenin's, Norton's, maximum power transfer, substitution, Millman's,* and *reciprocity* theorems. We will consider a number of areas of application for each. A thorough understanding of each theorem is important because a number will be applied repeatedly in the material to follow.

SUPERPOSITION THEOREM

The superposition theorem, like the methods of the last chapter, can be used to find the solution to networks with two or more sources that are not in series or parallel. The most obvious advantage of this method is that it does not require the use of a mathematical technique such as determinants to find the required voltages or currents. Instead, each source is treated independently, and the algebraic sum is found to determine a particular unknown quantity of the network.

The superposition theorem states the following:

The current through, or voltage across, an element in a linear bilateral network is equal to the algebraic sum of the currents or voltages produced independently by each source.

When applying the theorem it is possible to consider the effects of two sources at the same time and reduce the number of networks that have to be analyzed, but, in general,

$$\boxed{\begin{array}{c}\text{Number of networks}\\ \text{to be analyzed}\end{array} = \begin{array}{c}\text{Number of}\\ \text{independent sources}\end{array}} \qquad (5.1)$$

To consider the effects of each source independently requires that sources be removed and replaced without affecting the final result. To remove a voltage source when applying this theorem, the difference in potential between the terminals of the voltage source must be set to zero (short circuited); removing a current source requires that its terminals be opened (open circuit). Any internal resistance or conductance associated with the displaced sources is not eliminated but must still be considered.

Figure 5.1 reviews the various substitutions required when removing an ideal source, and Fig. 9.2 reviews the substitutions with practical sources that have an internal resistance.

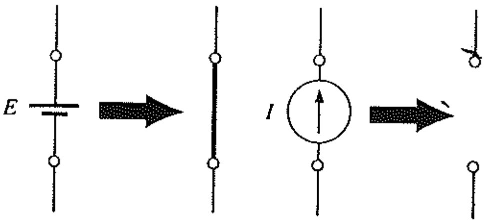


FIG. 5.1
Removing the effects of ideal sources.

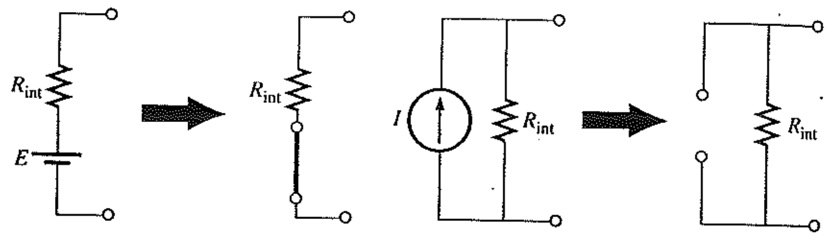


FIG. 5.2
Removing the effects of practical sources.

The total current through any portion of the network is equal to the algebraic sum of the currents produced independently by each source. That is, for a two-source network, if the current produced by one source is in one direction, while that produced by the other is in the opposite direction through the same resistor, *the resulting current is the difference of the two and has the direction of the larger.* If the individual currents are in the same direction, *the resulting current is the sum of two in the direction of either current.* This rule holds true for the voltage across a portion of a network as determined by polarities, and it can be extended to networks with any number of sources.

The superposition principle is not applicable to power effects since the power loss in a resistor varies as the square (nonlinear) of the current or voltage. For this reason, the power to an element cannot be calculated until the total current through (or voltage across) the element has been determined by superposition.

EXAMPLE Determine I_1 for the network of Fig. 5.3.

Solution: Setting $E = 0$ V for the network of Fig. 5.3 results in the network of Fig. 5.4(a), where a short-circuit equivalent has replaced the 30-V source.

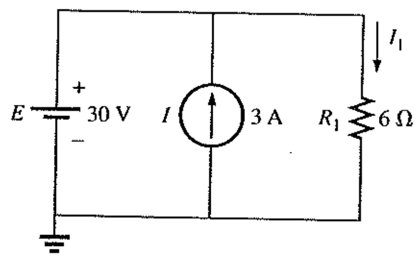
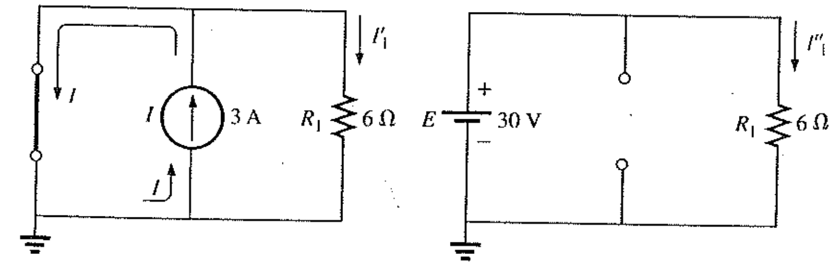


FIG. 5.3



(a) (b)

FIG. 5.4

As shown in Fig. 9.4(a), the source current will choose the short-circuit path, and $I'_1 = 0$ A. If we applied the current divider rule,

$$I'_1 = \frac{R_{sc} I}{R_{sc} + R_1} = \frac{(0\,\Omega)I}{0\,\Omega + 6\,\Omega} = 0\text{ A}$$

Setting I to zero amperes will result in the network of Fig. 9.4(b) with the current source replaced by an open circuit. Applying Ohm's law,

$$I''_1 = \frac{E}{R_1} = \frac{30\text{ V}}{6\,\Omega} = 5\text{ A}$$

Since I'_1 and I''_1 have the same defined direction in Figs. 9.4(a) and (b), the current I_1 is the sum of the two, and

$$I_1 = I'_1 + I''_1 = 0\text{ A} + 5\text{ A} = \mathbf{5\text{ A}}$$

Note in this case that the current source has no effect on the current through the 6-Ω resistor since the voltage across the resistor must be fixed at 30 V because they are parallel elements.

EXAMPLE Using superposition, determine the current through the 4-Ω resistor of Fig. 9.5. Note that this is a two-source network of the type considered in Chapter 8.

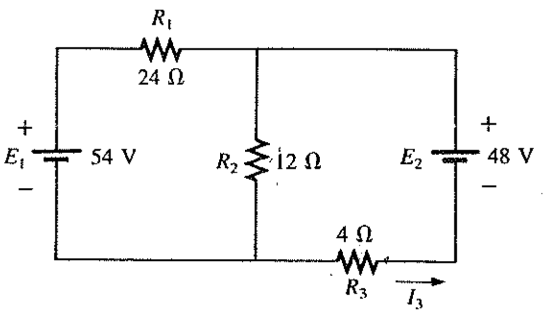


FIG. 9.5

Solution: *Considering the effects of a 54-V source (Fig. 9.6):*

$$R_T = R_1 + R_2 \| R_3 = 24\,\Omega + 12\,\Omega \| 4\,\Omega = 24\,\Omega + 3\,\Omega = 27\,\Omega$$

$$I = \frac{E_1}{R_T} = \frac{54\text{ V}}{27\,\Omega} = 2\text{ A}$$

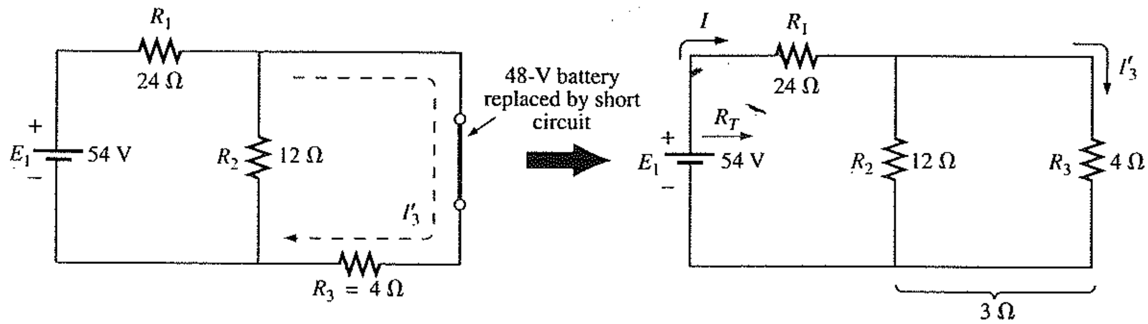


FIG. 9.6

By the current divider rule,

$$I'_3 = \frac{R_2 I}{R_2 + R_3} = \frac{(12\ \Omega)(2\ \text{A})}{12\ \Omega + 4\ \Omega} = \frac{24\ \Omega}{16} = 1.5\ \text{A}$$

Considering the effects of the 48-V *source* (*Fig.* 9.7):

$$R_T = R_3 + R_1 \parallel R_2 = 4\ \Omega + 24\ \Omega \parallel 12\ \Omega = 4\ \Omega + 8\ \Omega = 12\ \Omega$$

$$I''_3 = \frac{E_2}{R_T} = \frac{48\ \text{V}}{12\ \Omega} = 4\ \text{A}$$

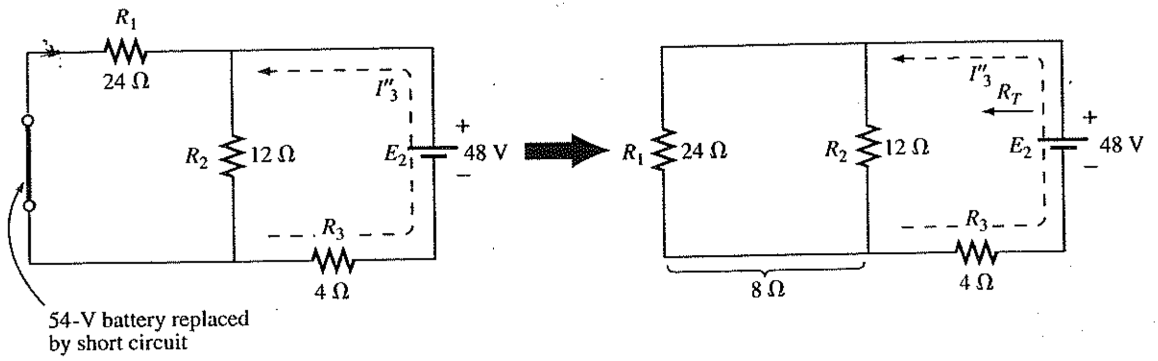


FIG. 5.7

The total current through the 4-Ω resistor (Fig. 5.8) is

$$I_3 = I''_3 - I'_3 = 4\ \text{A} - 1.5\ \text{A} = \mathbf{2.5\ A} \qquad (\text{direction of } I''_3)$$

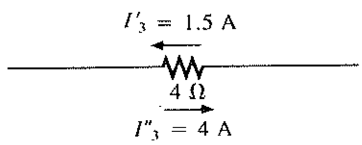


FIG. 5.8

EXAMPLE Using superposition, find the current through the 6-Ω resistor of the network of Fig. 5.9.

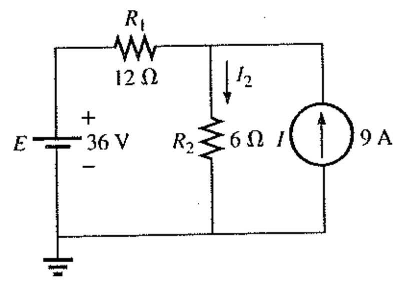


FIG. 5.9

Solution: *Considering the effect of the 36-V source (Fig. 5.10):*

$$I'_2 = \frac{E}{R_T} = \frac{E}{R_1 + R_2} = \frac{36\text{ V}}{12\ \Omega + 6\ \Omega} = 2\text{ A}$$

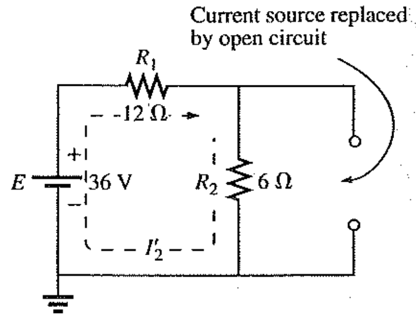


FIG. 5.10

Considering the effect of the 9-A source (Fig. 5.11):

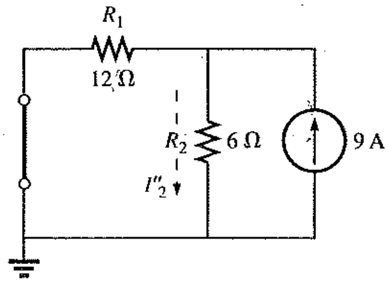


FIG. 5.11

Applying the current divider rule,

$$I''_2 = \frac{R_1 I}{R_1 + R_2} = \frac{(12\ \Omega)(9\text{ A})}{12\ \Omega + 6\ \Omega} = \frac{108\text{ A}}{18} = 6\text{ A}$$

The total current through the 6-Ω resistor (Fig. 5.12) is

$$I_2 = I'_2 + I''_2 = 2\text{ A} + 6\text{ A} = \mathbf{8\text{ A}}$$

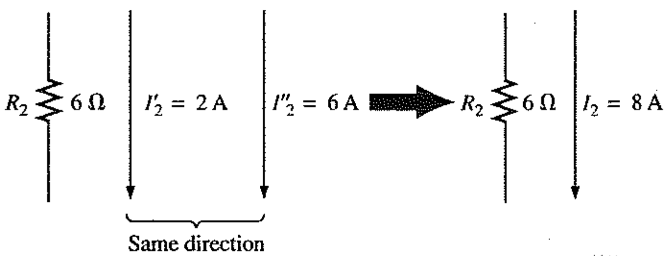


FIG. 5.12

The power to the 6-Ω resistor is

$$\text{Power} = I^2R = (8\text{ A})^2(6\ \Omega) = \mathbf{384\text{ W}}$$

The calculated power to the 6-Ω resistor due to each source, *misusing* the principle of superposition, is

$$P_1 = (I'_2)^2R = (2\text{ A})^2(6\ \Omega) = 24\text{ W}$$
$$P_2 = (I''_2)^2R = (6\text{ A})^2(6\ \Omega) = 216\text{ W}$$
$$P_1 + P_2 = 240\text{ W} \neq 384\text{ W}$$

This results because 2 + 6 = 8, but

$$(2)^2 + (6)^2 \neq (8)^2$$

As mentioned previously, the superposition principle is not applicable to power effects, since power is proportional to the square of the current or voltage (I^2R or V^2/R).

Figure 5.13 is a plot of the power delivered to the 6-Ω resistor versus current.

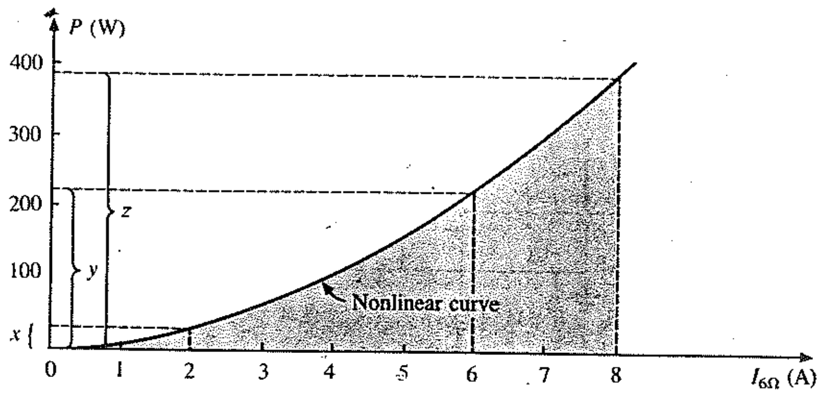


FIG. 5.13

Obviously, $x + y \neq z$, or 24 W + 216 W ≠ 384 W, and superposition does not hold. However, for a linear relationship, such as that between the voltage and current of the fixed-type 6-Ω resistor, superposition can be applied, as demonstrated by the graph of Fig. 5.14, where $a + b = c$, or 2 A + 6 A = 8 A.

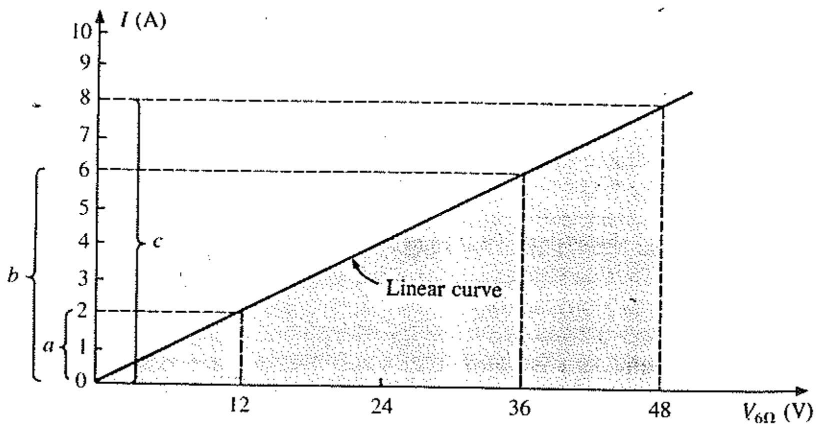


FIG. 5.14

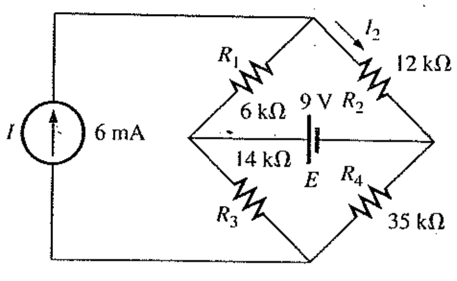


FIG. 5.15

EXAMPLE 9.4 Using the principle of superposition, find the current I_2 through the 12-kΩ resistor of Fig. 5.15.

Solution: *Considering the effect of the* 6-mA *current source (Fig.* 5.16):

Current divider rule:

$$I'_2 = \frac{R_1 I}{R_1 + R_2} = \frac{(6\ \text{k}\Omega)(6\ \text{mA})}{6\ \text{k}\Omega + 12\ \text{k}\Omega} = 2\ \text{mA}$$

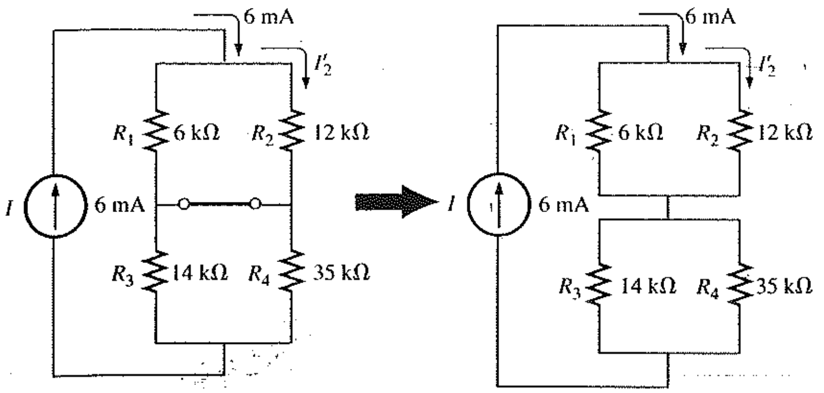


FIG. 9.16

Considering the effect of the 9-V *voltage source (Fig.* 9.17):

$$I''_2 = \frac{E}{R_1 + R_2} = \frac{9 \text{ V}}{6 \text{ k}\Omega + 12 \text{ k}\Omega} = 0.5 \text{ mA}$$

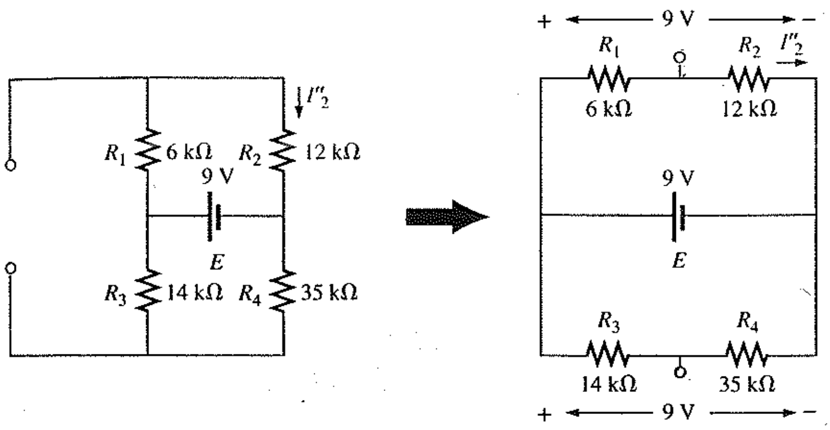


FIG. 9.17

Since I'_2 and I''_2 have the same direction through R_2, the desired current is the sum of the two:

$$\begin{aligned} I_2 &= I'_2 + I''_2 \\ &= 2 \text{ mA} + 0.5 \text{ mA} \\ &= \mathbf{2.5 \text{ mA}} \end{aligned}$$

EXAMPLE 9.5 Find the current through the 2-Ω resistor of the network of Fig. 9.18. The presence of three sources will result in three different networks to be analyzed.

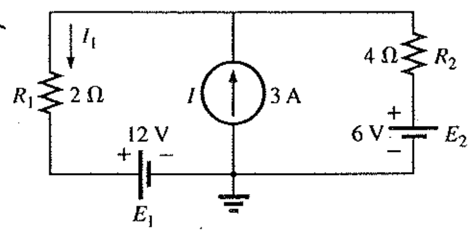


FIG. 9.18

Solution *Considering the effect of the* 12-V *source (Fig.* 9.19):

$$I'_1 = \frac{E_1}{R_1 + R_2} = \frac{12 \text{ V}}{2 \,\Omega + 4 \,\Omega} = \frac{12 \text{ V}}{6 \,\Omega} = 2 \text{ A}$$

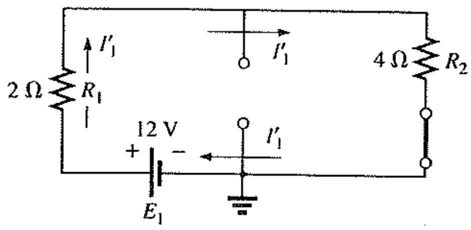


FIG. 5.19

Considering the effect of the 6-V *source (Fig.* 5.20):

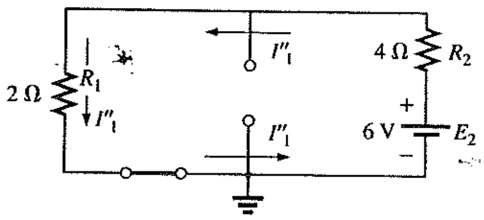


FIG. 5.20

$$I''_1 = \frac{E_2}{R_1 + R_2} = \frac{6\text{ V}}{2\ \Omega + 4\ \Omega} = \frac{6\text{ V}}{6\ \Omega} = 1\text{ A}$$

Considering the effect of the 3-A *source (Fig.* 9.21):

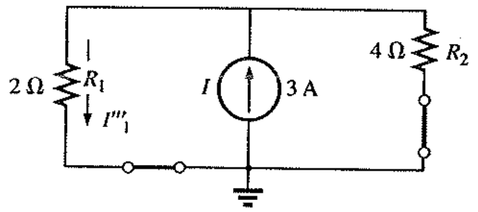


FIG. 5.21

Applying the current divider rule,

$$I'''_1 = \frac{R_2 I}{R_1 + R_2} = \frac{(4\ \Omega)(3\text{ A})}{2\ \Omega + 4\ \Omega} = \frac{12\text{ A}}{6} = 2\text{ A}$$

The total current through the 2-Ω resistor appears in Fig. 5.22, and

$$\begin{aligned} I_1 &= \overbrace{I''_1 + I'''_1}^{\substack{\text{Same direction}\\ \text{as } I_1 \text{ in Fig. 9.18}}} \overset{\substack{\text{Opposite direction}\\ \text{to } I_1 \text{ in Fig. 9.18}}}{-} I'_1 \\ &= 1\text{ A} + 2\text{ A} - 2\text{ A} = \mathbf{1\ A} \end{aligned}$$

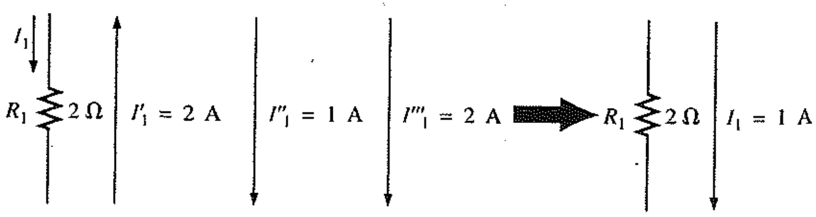


FIG. 5.22

Example :

Use superposition to find the current I in the circuit of Fig. 5-23

Solution :

The current I has two components, I_v from the 7 V source and I_c from the 1.5 A source : $I = I_v + I_c$. To find I_v we kill the current source as illustrated in Fig. 5-24(a). Even with the current source killed, the presence of the dependent source prevents us from finding I_v by some simple procedure such as voltage or current division. But mesh analysis is convenient. The two mesh equations are :

$$9I' - 4I_v = 7$$

$$-4I' + 12I_v = 7V_4$$

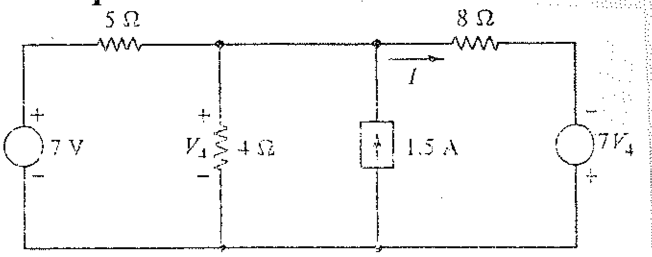


Figure 5-23

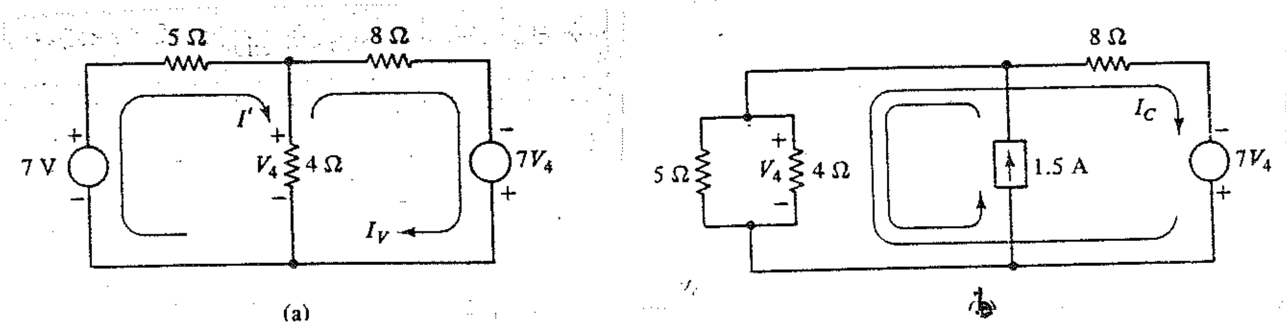


Figure 5-24

In the second equation we need to substitute for V_4. Clearly, $V_4 = 4(I' - I_v)$. Substituting this and simplifying, we get :

$$9I' - 4I_v = 7$$

$$-32I' + 40I_v = 0$$

from which $I_v = 0.966$ A.

To find I_c we kill the independent voltage source as illustrated in Fig. 5-24 (b). This replacing of the independent voltage source with a short circuit puts the 4 Ω and 5 Ω resistors in parallel for a net resistance of 2.22 Ω. With these resistances combined, the outside loop has a KVL equation of :

$$(2.22 + 8)I_c - (2.22)(1.5) = 7\ V_4$$

But $V_4 = (1.5 - I_c)(2.22)$. With this substitution the equation becomes :

$$10.22\ I_c - 3.33 = (7)(1.5 - I_c)(2.22)$$

which simplifies to $25.76I_c = 26.24$ and $I_c = 1.034$ A. This added to I_v gives the total current I :

$$I = I_v + I_c = 0.966 + 1.034 = 2\ A$$

Thevenin's and Norton's Theorems :

Thevenin's and Norton's theorems are probably the most important network theorems. With these theorems we can often reduce networks to make them easier to analyze.

To apply these theorems we separate the network of interest into part A and part B, as shown in Fig. 5-25(a), with two wires joining the two parts. One part must be linear and bilateral, but the other part need not be. It can anything.

M. L. Thevenin's, a French engineer, first published a statement of his theorem in 1883. He said that the linear, bilateral portion, say part A, of a network can be replaced by s specific voltage source in series with a specific resistor, as in Fig. 5-25(b), without any changes in

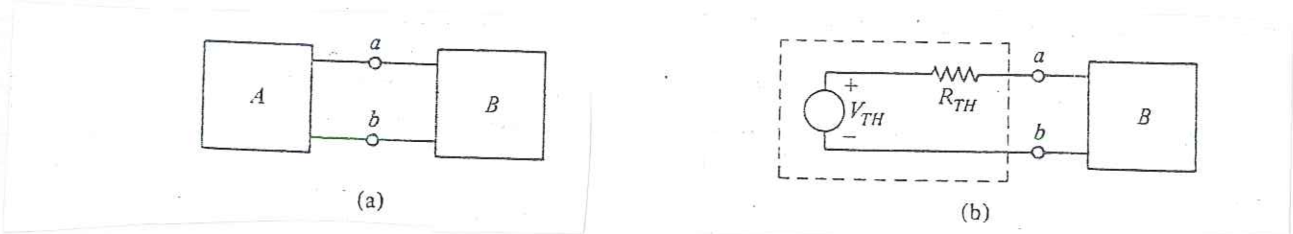


Figure 5-25

the other part, part B. In honor of Thevenin, the voltage source is called the Thevenin voltage source and it has the symbol V_{TH}. And the resistance is called the Thevenin resistance. Its symbol is R_{TH}.

The voltage for the Thevenin source is the open-circuit voltage of part A. That is, it is the voltage between terminals a and b with part B replaced by an open circuit. Another way of considering it is as if the wires are cut at terminals a and b and a voltmeter connected to measure the voltage across these terminals. This open-circuit voltage is almost always different from the voltage at the a-b terminals with part B connected. The two voltages usually differ in magnitude and may even differ in polarity. Of course, the polarity of the source must agree with that of the open-circuit voltage. For example, if at the open circuit, terminal a is positive with respect to terminal b, the Thevenin voltage source must have its positive terminal toward terminal a.

The series resistance is the resistance of part A at terminals a and b with all independent sources killed. Put another way, with the wires cut at a and b, Thevenin's resistance is the resistance measured by an ohmmeter connected to terminals a and b with all independent voltage sources in part A replaced by short circuits and all independent current sources replaced by open circuits. For some networks in which part B is just a load, this Thevenin resistance is called the output resistance of part A, or the input resistance between a and b.

Thevenin's theorem guarantees only that the voltages and currents in part B do not change when part A is replaced by its Thevenin equivalent network. The voltages and currents in the Thevenin circuit itself usually differ considerably from those in part A except at terminals a and b, where they are the same.

Example :

Find the Thevenin equivalents of the network portions of Fig. 5-20

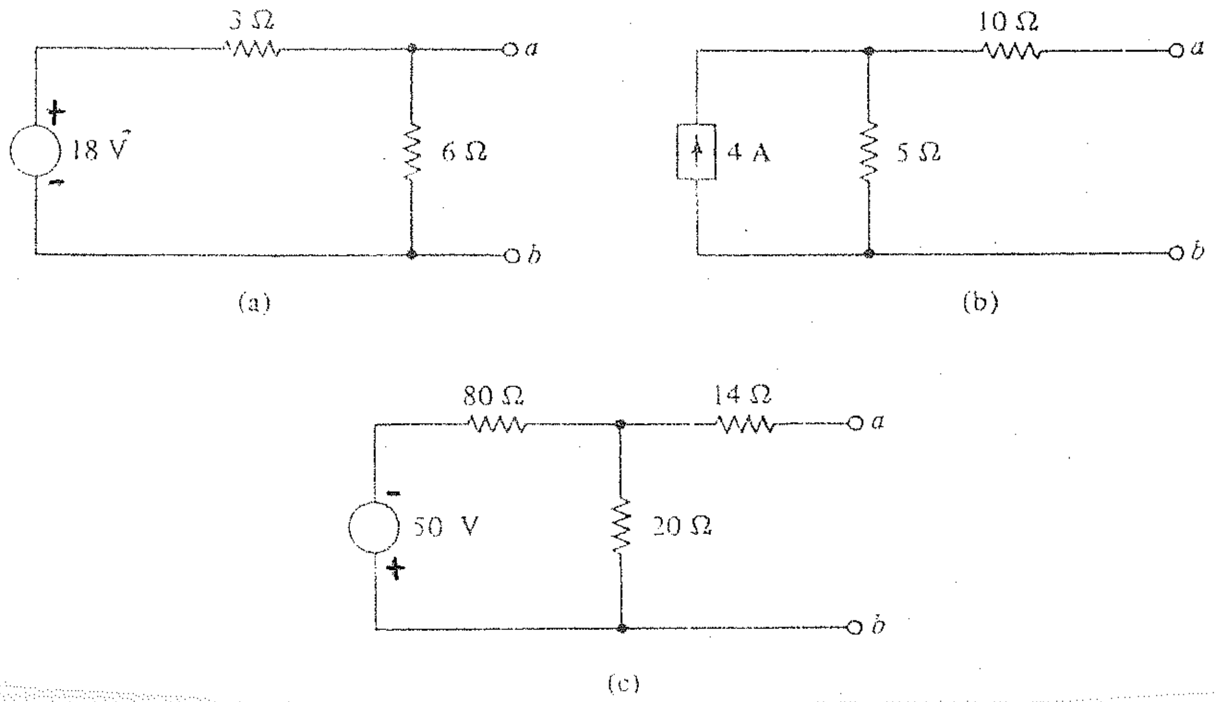


Figure 5-26

Solution :

Each network portion in Fig. 5-26 corresponds to the linear, bilateral part A of Fig. 5-25(a). There is no showing of part B because it has no effect on the Thevenin equivalent networks. As should be apparent, the interconnecting wires are considered to be cut at terminals a and b. One other general matter : although what are shown are just parts of networks, we will call them networks because it is conventional to do so, just as it is conventional to speak of their Thevenin equivalent circuits or networks.

For the network of Fig. 5-26(a), the open-circuit voltage at terminals a and b is just the 6 Ω resistor voltage drop, which by voltage division is :

$$V_{TH} = \frac{6}{6+3} \text{x}\,18 = 12\text{ V}$$

with terminal a positive with respect to terminal b. To find Thevenin's resistance, we replace the 18 V source with a short circuit as in Fig. 5-27(a) and then find the resistance at terminals a and b. This short circuit places the 3 Ω resistor in parallel with the 6 Ω

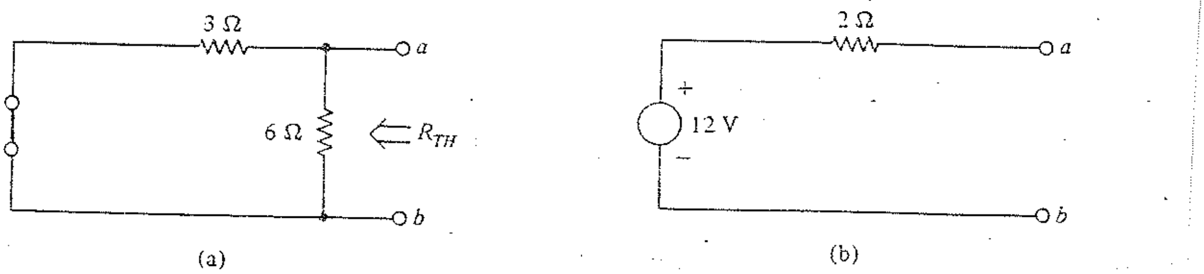


Figure 5-27

resistor, giving $R_{TH} = 3||6 = 2\ \Omega$. Figure 5-27(b) shows the Thevenin equivalent network.

For the network of Fig. 5-26(b), the open-circuit of Thevenin voltage is the same as the voltage across the 5 Ω resistor, which voltage is 20 V–the 4 A of the source times the 5 Ω of the resistor. Because of the zero current flow through it, the 10 Ω resistor has no voltage across it to aid or to oppose the voltage across the 5 Ω resistor.

To find Thevenin's resistance, we replace the current source with an open circuit as shown in Fig. 5-18(a). The 5 Ω resistor then becomes in series with the 10 Ω resistor to make $R_{TH} = 5 + 10 = 15\ \Omega$. The Thevenin equivalent is thus as illustrated in Fig. 5-18(b).

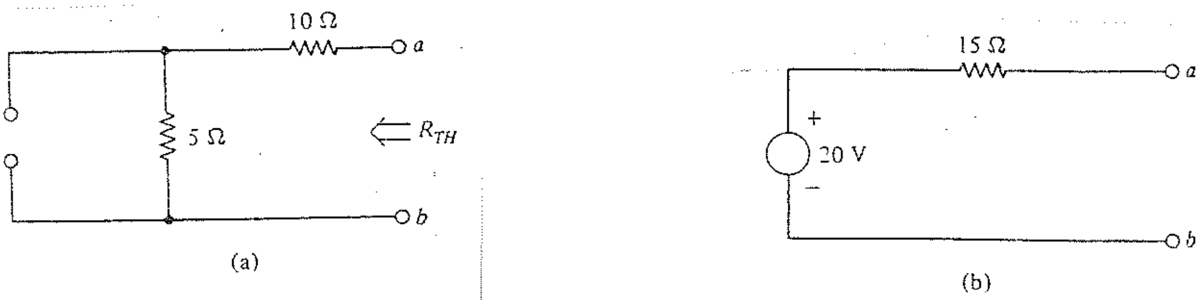


Figure 5-18

Finally, the Thevenin equivalent for the network of Fig. 5-16(c) has Thevenin voltage that from voltage division is :

$$V_{TH} = \frac{20}{20+80} \text{ x } 50 = 10 \text{ V}$$

with terminal b positive with respect to terminal a. The 14 Ω resitor does not enter into this computation because it has zero current flow through it and so zero voltage drop across it. The Thevenin resistance is the resistance at terminals a and b with the voltage

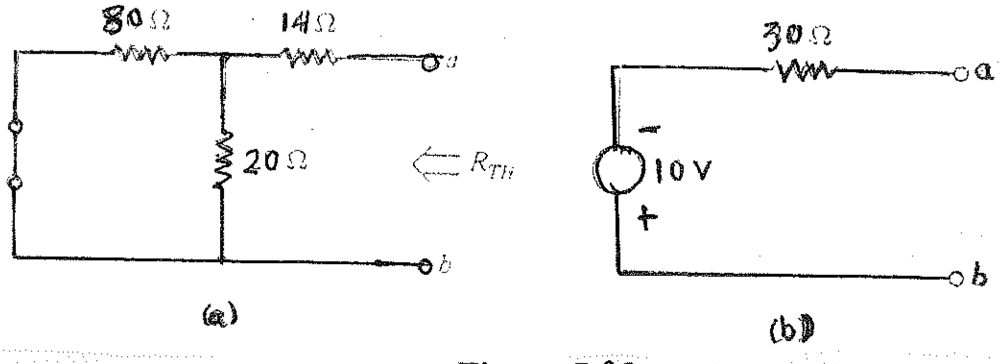


Figure 5-19

source replaced by a short circuit, as in Fig. 5-29(a). This short circuit places the 80 Ω resistor in parallel with the 20 Ω resistor, which parallel combination is in series with the 14 Ω resistor. The result is a Thevenin resistance of :

$$R_{TH} = 14 + 80 \,||\, 20 = 14 + \frac{80 \times 20}{80 + 20} = 30\,\Omega$$

Figure 5-29(b) has the Thevenin equivalent.

Another popular way to find R_{TH} is from the short-circuit I_{SC} that would flow in a short circuit placed across terminals a and b as in

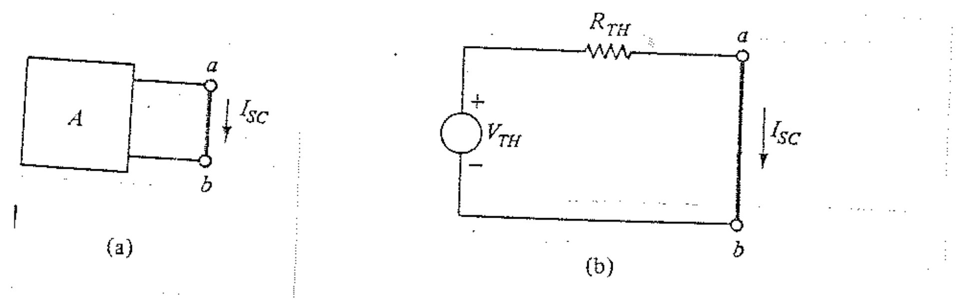


Figure 5-30

Fig. 5-30(a). This short-circuit current from terminal a to terminal b is related to the Thevenin voltage and resistance as can be seen from Fig. 5-30(b). Specifically, V_{TH}, R_{TH}, and I_{SC} have the relation $V_{TH} = I_{SC} R_{TH}$ or, alternatively, $R_{TH} = V_{TH} / I_{SC}$.

What this means is that R_{TH} is the ratio of the open-circuit voltage at terminals a and b, and the short-circuit current between them.

Example :

Use V_{TH} and I_{SC} to find R_{TH} for the Thevenin equivalent of the network of Fig. 5-31.

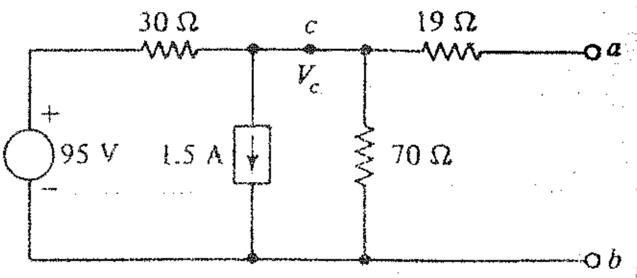


Figure 5-31

Solution :

Because of the open circuit at terminals a and b, no current flows through the 19 Ω resistor, so it has 0 V across it. Consequently, the voltage at node c is the same as that at node a, which means that the voltage from node c to node b is the open-circuit or Thevenin voltage. A single KCL equation at node c gives this voltage :

$$\frac{V_c}{70}+1.5+\frac{V_c-95}{30}=0$$

From which $V_{TH} = V_c = 35$ V.

Now we need the short-circuit current I_{SC} as illustrated in Figure

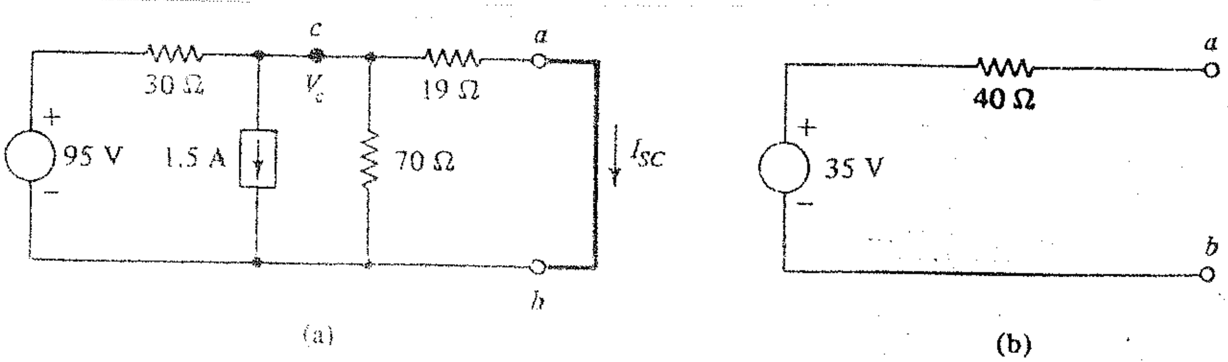


Figure 5-32

5-32 (a). Nodal analysis is probably the best method for finding it. A single KCL equation at node c will give V_c and with V_c known, we can find I_{SC} from $I_{SC} = V_c/19$. Do not make the mistake of thinking that this V_c is the same as that just calculated. The one we just found is for an open-circuit load. But the one we want now is for a short-circuit load, and so we should not expect V_c to remain the same. For the calculation of this different.

Applying KCL at node c results in :

$$\frac{V_c - 95}{30} + 1.5 + \frac{V_c}{70} + \frac{V_c}{19} = 0$$

the solution to which is $V_c = 16.625$ V. So,

$$I_{SC} = \frac{16.625}{19} = 0.875 \text{ A}$$

and

$$R_{TH} = \frac{V_{TH}}{I_{SC}} = \frac{35}{0.875} = 40\,\Omega$$

Figure 5-32(b) shows the Thevenin equivalent.

For a circuit like that of Fig. 5-31, using V_{TH} and I_{SC} to find R_{TH} is too much work. We only did it for an exercise. Actually, for this circuit we can find R_{TH} easier by killing the sources and finding the input resistance at terminals a and b by combining resistances. With the sources killed, the 30 Ω and 70 Ω resistors are in

parallel, producing a net resistance of 30 || 70 = 21 Ω. This parallel combination is in series with the 19 Ω resistor, so R_{TH} =21 + 19 = 40Ω.

Selecting the best method for finding R_{TH} is a matter of judgment. Certainly, though, if killing the sources produces a series-parallel resistive network between terminals a and b, the I_{SC} approach needs usually more work.

The Norton Equivalent :

The Norton equivalent circuit consists of an independent current source in parallel with the Norton equivalent resistance. We can derive it from the Thevenin equivalent circuit simply by making a source transformation. Thus the Norton current equals the short-circuit current at the terminals of interest, and the Norton resistance is identical to the Thevenin resistance.

Fig. 5-32(b), shows the Norton equivalent network.

Notice that the Norton current $I_N = V_{TH} / R_{TH}$ is just I_{SC} the short-circuit current at terminals a and b but oppositely directed. If the short-circuit current is down, I_N is up, and vice versa. This I_N must be oppositely directed to produce a short-circuit current in the same direction as I_{SC}. After all, the Norton network is equivalent to the original network for all external loads, including short circuits.

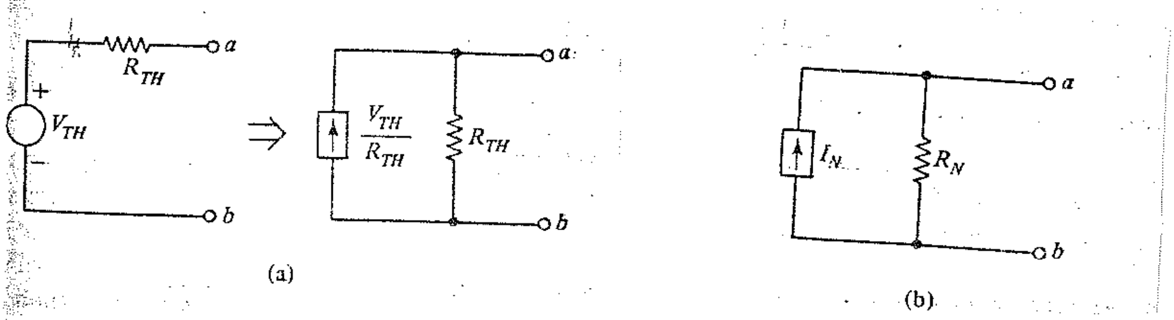


Figure 5-32

Example :

In the circuit shown in Fig. 5-33, get the current in the 8 Ω resistor.

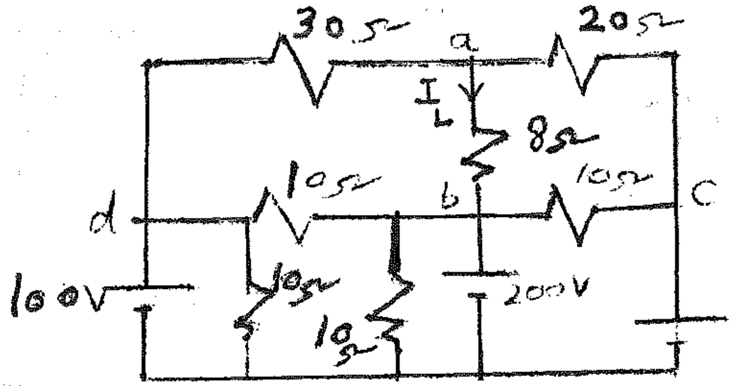


Figure 5-33

Solution :

To get R_{TH} or R_e, we kill all the active sources and get the input resistance between a & b. As a result all the 10 Ω resistances are short circuited and the circuit will be as shown in Fig. 5-34.

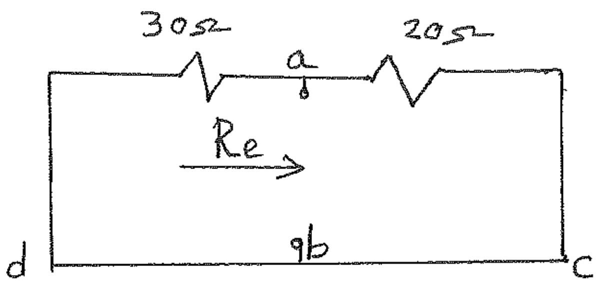


Figure 5-34

In Fig. 5-34, the resistors 30 Ω and 20 Ω are in parallel.

So, R_e is given by :

$$R_e = (20 \times 30) / (20 + 30) = 12\ \Omega$$

As previously mentioned Re is the same for Thevenin's and Norton's theorems :

a) **Solution by Thevenin's theorem** :

We calculate the open circuit voltage at terminals a & b as shown in Fig. 5-20.

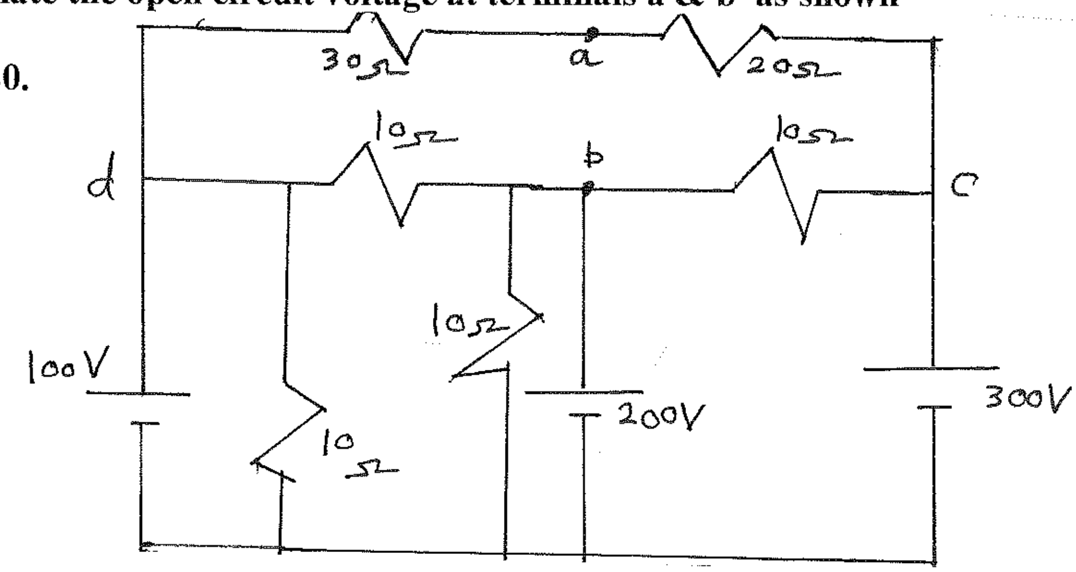


Figure 5-35

Then we get : $E_e = E_{cb} - E_{ca}$

$$E_{cb} = 300 - 200 = 100\ V$$

$$E_{ca} = \frac{300 - 100}{30 + 20} \times 20 = 80\ V$$

$\therefore \qquad \mathbf{E_e = 100 - 80 = 20\ V}$

In the circuit of Fig. 5-26, we get : $I_L = \dfrac{E_e}{R_e + R_1} = 20/(12+8) = 1\,A$

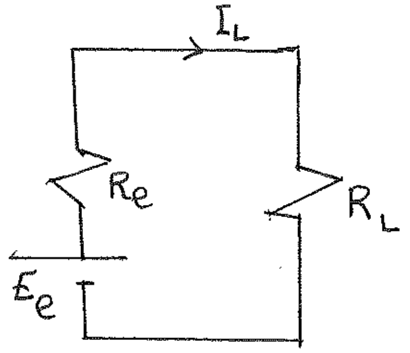


Figure 5-26

To get I_N , the terminals a & b are short circuited, then the short circuit (s – c) current I_e is given by the circuit shown in Fig. 5-27.

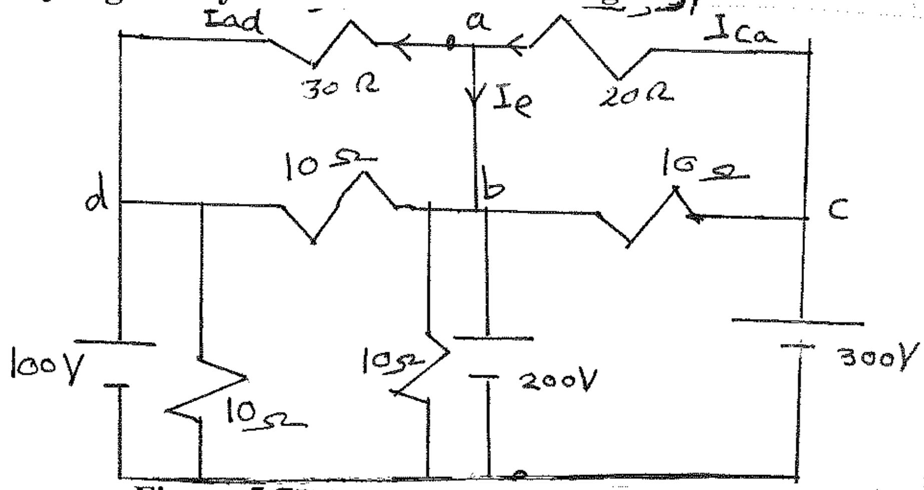


Figure 5-27

$\mathbf{I_e = I_{ca} - I_{ad}}$ **,** $\qquad \mathbf{I_{ca} = (300 - 200) / 20 = 5\ A}$ **,**

$\mathbf{I_{ad} = (200 - 100) / 30 = 3.33\ A,}$ $\qquad$ **and** $\mathbf{I_e = 1.67\ A}$

In the equivalent Norton's circuit shown in Fig. 5-28, we get :

$$I_L = I_e \frac{R_e}{R_e + R_1} = 1\ A$$

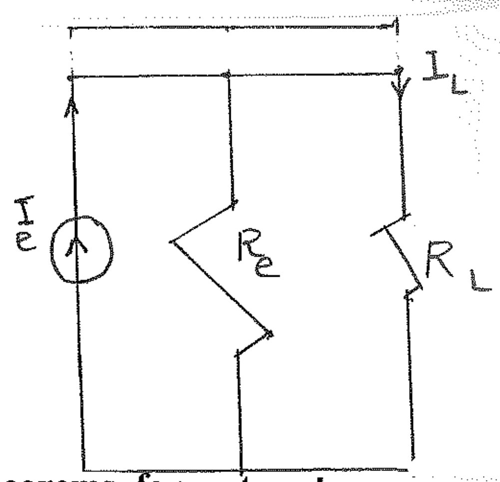


Figure 5-38

We can use Thevinin's and Norton's theorems for networks having dependent sources. In doing this, however, we do not usually kill the dependent sources in finding the Thevenin resistance. In general, we do not kill the dependent sources if the controlling branch for the dependent source is in the part of the network we are finding the equivalent of. Almost always it is.

Example :

Find the Thevenin resistance for the network of Fig. 5-39.

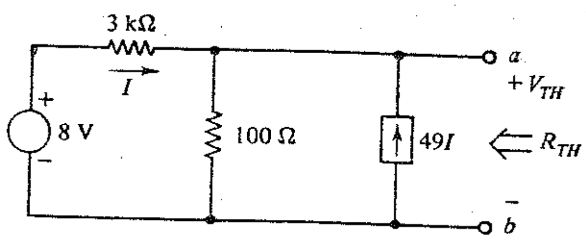


Figure 5-39

Solution :

We will find the Thevenin resistance by two ways : first from V_{TH} / I_{SC}, and second from the $V_T - I_T$ input resistance method with the independent voltage source killed. For the first way, KCL at the top right node gives :

$$\frac{V_{TH}}{100} + \frac{V_{TH} - 8}{3000} = 49\,I$$

But $$I = \frac{-V_{TH} + 8}{3000}$$

With this substitution the equation becomes

$$\frac{V_{TH}}{100} + \frac{V_{TH} - 8}{3000} = 49\left(\frac{V_{TH} - 8}{3000}\right)$$

If we clear the denominator and rearrange, we get $80V_{TH} = 400$,from which $V_{TH} = 5$ V.

Next, we short terminals a and b, as in Fig. 5-40 to find I_{SC}. Because the short circuit places 0 V across the 100 Ω resistor, no current flows through this resistor. So, $I_{SC} = I + 49\,I = 50\,I$. From the outside loop, clearly $I = 8/3000$. This substituted in makes :

$$I_{SC} = 50\left(\frac{8}{3000}\right) = \frac{400}{3000} = 0.133 \text{ A}$$

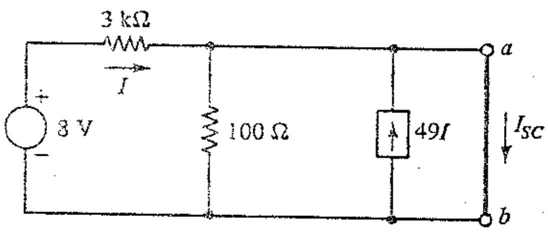


Figure 5-40

With V_{TH} and I_{SC} now known, we can find the Thevenin resistance:

$$R_{TH} = \frac{V_{TH}}{I_{SC}} = \frac{5}{0.133} = 37.5\ \Omega$$

The other way of finding R_{TH} is form the $V_T - I_T$ method applied to the network with the independent source killed. We apply a test

voltage V_T , calculate the resulting source current I_T , and then find R_{TH} from $R_{TH} = V_T / I_T$. Figure 5-41 shows the circuit with the independent source killed and the V_T source applied. The source current I_T equals the sum of the currents flowing in the three circuit branches :

$$I_T = -49I + \frac{V_T}{100} + \frac{V_T}{3000}$$

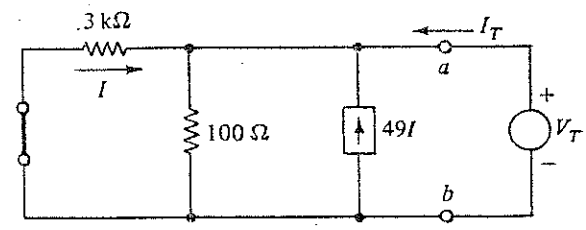


Figure 5-41

The test voltage V_T is across the 3 kΩ resistor, making :

$$I = -V_T/3000. \text{ So,}$$

$$I_T = -49\left(\frac{-V_T}{3000}\right) + \frac{V_T}{100} + \frac{V_T}{3000} = \frac{80}{3000}V_T$$

and $$R_{TH} = \frac{V_T}{I_T} = \frac{3000}{80} = 37.5\ \Omega$$

which checks with the value of R_{TH} from the first method.

Example :

Consider the circuit shown in Fig. 5-42, find the equivalent Thevenin.

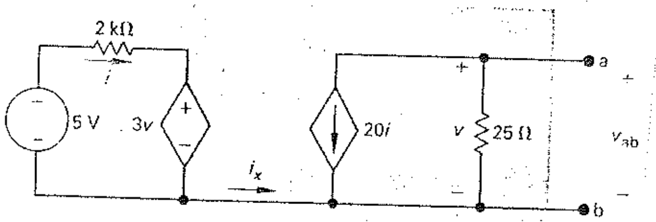


Figure 5-42

Solution **:**

The first step in analyzing the circuit in Fig. 5-42 is to recognize that the current labeled i_s must be zero. (Note the absence of a return path for i_s to enter the left-hand portion of the circuit). The open circuit, or Thevenin, voltage will be the voltage across the 25 Ω resistor. With $i_s = 0$,

$$V_{TH} = v_{ab} = (-20i)(25) = 500\ I \qquad (5\text{-}1)$$

The current i is :

$$i = \frac{5 - 3v}{2000} = \frac{5 - 3V_{TH}}{2000} \qquad (5\text{-}2)$$

In writing Eqn. (5-2), we recognize that the Thevenin voltage is identical to the control voltage. When we substitute Eqn. (5-2) into Eqn. (5-1), we obtain :

$$V_{TH} = -5\ V \qquad (5\text{-}3)$$

To calculate the short-circuit, we place a short circuit across a, b. When the terminals a, b are shorted together, the control voltage v is reduced to zero. Therefore, with the short in place, the circuit

shown in Fig. 5-42 becomes that shown in Fig. 5-43. With the short circuit shunting the 25 Ω resistor, all the current form the dependent current source appears in the short, so

$$i_{SC} = -20\,i \qquad (5\text{-}4)$$

As the voltage controlling the dependent voltage source has been reduced to zero, the current controlling the dependent current source is :

$$i = \frac{5}{2000} = 2.5 \text{ mA.} \qquad (5\text{-}5)$$

Substituting Eqn. (5-5) to Eqn. (5-4) yields a short-circuit current of

$$i_{SC} = -20(2.5) = -50 \text{ mA} \qquad (5\text{-}6)$$

From Eqns. (5-3) and (5-6),

$$R_{TH} = \frac{V_{TH}}{I_{SC}} = \frac{-5}{-50} \times 10^3 = 100\ \Omega \qquad (5\text{-}7)$$

Figure 5-44 illustrates the Thevenin equivalent circuit for the circuit shown in Fig. 5-42. Note that the reference polarity marks on the Thevenin voltage source in Fig. 5-44 agree with Eqn. (5-3).

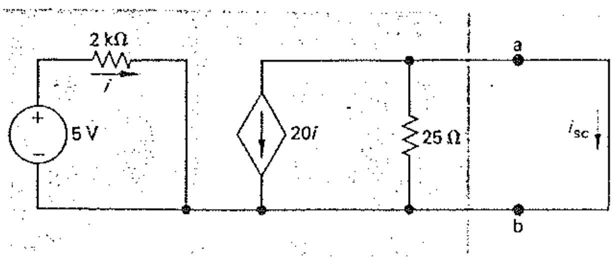


Figure 5-43

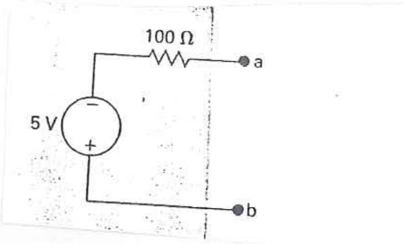


Figure 5-44

We now consider the alternative technique for finding the Thevenin resistance R_{TH}. We first deactivate the independent voltage source from the circuit and then excite the circuit from the terminals a , b with either a test voltage source or a test current source. If we apply a test voltage source we will know the voltage of the dependent voltage source and hence the controlling current i. Therefore we optain for the test voltage source. Figure 5-45 shows the circuit for computing the Thevenin resistance. The externally applied test voltage source is denoted v_T and the current that it delivers to the circuit is labeled I_T . To find the Thevenin resistance, we simply solve the circuit shown in Fig. 5-45 for the ratio of the voltage to the current at the test source; that is , $R_{TH} = v_T / i_T$. From Fig. 5-45:

$$I_T = \frac{v_T}{25} + 20i \quad , \tag{5-8}$$

$$i = \frac{-3v_T}{2} \ mA. \tag{5-9}$$

We then substitute Eqn. (5-9) into Eqn. (5-8) and solve the resulting equation for the ratio v_T / i_T :

$$I_T = \frac{v_T}{25} - \frac{60\, v_T}{2000} \quad ,$$

$$\frac{I_T}{v_T} = \frac{1}{25} - \frac{6}{200} = \frac{50}{5000} = \frac{1}{100} \qquad \textbf{(5-10)}$$

From Eqn. (5-10) ,

$$R_{TH} = \frac{v_T}{I_T} = 100\ \Omega \qquad \textbf{(5-11)}$$

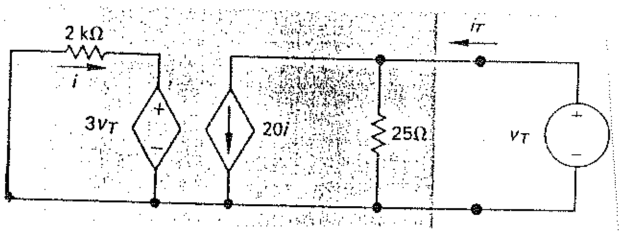


Figure 5-45

<u>Maximum Power Transfer Theorem</u> :

The maximum power transfer theorem states that :

A resistive load receives maximum power from a linear bilateral dc network if the load resistance of the network equals Thevenin resistance of the network as "seen" by the load.

As background for understanding this theorem, consider a resistor load R_L , as in Fig. 5.31, energized by a dc network with indicated Thevenin equivalent.

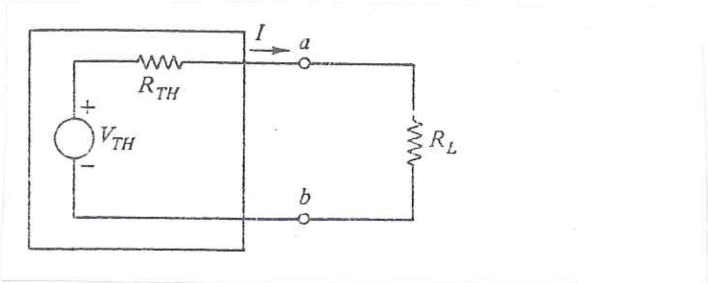


Figure 5-31

The problem is to determine the value of R_L that permits maximum power delivery to R_L. Derivation of R_L requires expressing the power dissipated in R_L as a function of the three circuit parameters V_{TH}, R_{TH} and R_L. Thus :

$$p = I^2\, R_L = \left(\frac{V_{TH}}{R_{TH} + R_L}\right)^2 R_L \qquad \textbf{(5-12)}$$

Next, we recognize that for a given circuit, V_{TH} and R_{TH} will be fixed. Therefore the power dissipated is a function of the single variable R_L. To find the value of R_L that maximizes the power, we use elementary calculus; that is, we solve for the value of R_L then dp/dR_L equals zero :

$$\frac{dp}{dR_L} = V_{TH}^2 \left[\frac{(R_{TH} + R_L)^2 - R_L .2\,(R_{TH} + R_L)}{(R_{TH} + R_L)^4}\right] , \qquad \textbf{(5-13)}$$

and the derivative is zero when

$$(R_{TH} + R_L)^2 = 2\, R_L (R_{TH} + R_L) \qquad \textbf{(5-14)}$$

Solving Eqn. (5.14) yields

$$R_L = R_{TH} \qquad \textbf{(5-15)}$$

Thus maximum power transfer occurs when the load resistance R_L equals the Thevenin resistance R_{TH} . To find the maximum power delivered to R_L , we simply substitute Eqn. (5-15) into Eqn. (5-12) :

$$P_{\max} = \frac{V_{TH}^2 R_L}{(2R_L)^2} = \frac{V_{TH}^2}{4R_L} \qquad \textbf{(5-16)}$$

The power output of the source at maximum power condition equals :

$$P_1 = V^2_{TH} / (R_{TH} + R_L) = V^2_{TH} / 2\, R_L \qquad \textbf{(5-17)}$$

The efficiency in this case equals :

$$\xi = (P_{max} / P_1) \times 100 = 50\ \% \qquad \textbf{(5-18)}$$

Circuit analysis plays an important role in the analysis of systems designed to transfer power from a source to a load. We discuss power transfer in terms of two basic types of systems. One emphasizes the efficiency of the power transfer, and the other emphasizes the amount of the power transfer. Power utility systems are a good example of the first type because they are concerned with the generation, transmission, and distribution of large quantities of electric power. Communication and instrumentation systems are good examples of the second type because they are designed to transmit information via electric signals. In the transmission of

information, or data, via electric signals, the power available at the transmitter or detector is limited. Thus transmitting as much of this power as possible to the receiver, or load, becomes desirable. In such applications the amount of power being transferred is small, so the efficiency of transfer is not a primary concern.

Example :

a) For the circuit shown in Fig. 5-32, find the value of R_L that results in maximum power being transferred to R_L.

b) Calculate the maximum power that can be delivered to R_L.

c) When R_L is adjusted for maximum power transfer, what percentage of power delivered by the 360 V source reaches R_L ?

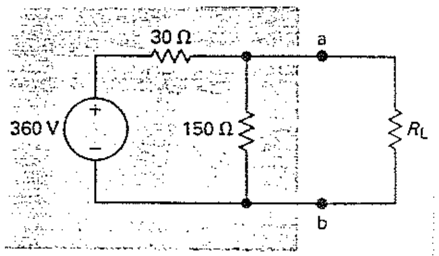


Figure 5-32

Solution :

a) The Thevenin voltage for the circuit to the left of the terminals a, b is :

$$V_{TH} = \frac{360}{180} \text{ x } 150 = 300 \text{ V}$$

The Thevenin resistance is :

$$R_{TH} = \frac{(150)(30)}{180} = 25 \ \Omega$$

Replacing the circuit to the left of the terminals a , b with its Thevenin equivalent gives the circuit shown in Fig. 5-33 which indicated that R_L must equal 25 Ω for maximum power transfer.

b) The maximum power that can be delivered to R_L is :

$$P_{max} = \left(\frac{300}{50}\right)^2 (25) = 900 \text{ W}$$

c) When R_L equals 25 Ω, the voltage v_{ab} is :

$$v_{ab} = \left(\frac{300}{50}\right) (25) = 150 \text{ V}$$

From Fig. 5-32 when v_{ab} equals 150 V, the current in the voltage source in the direction of the voltage rise across the source is :

$$i_s = \frac{360-150}{30} = \frac{210}{30} = 7 \text{ A.}$$

Therefore the source is delivering 2520 W to the circuit, or

$$P_s = I_s(360)$$

$$= 2520 \text{ W.}$$

The percentage of the source power delivered to the load is :

$$\frac{900}{2520} \text{ x } 100 = 35.71 \text{ \%}$$

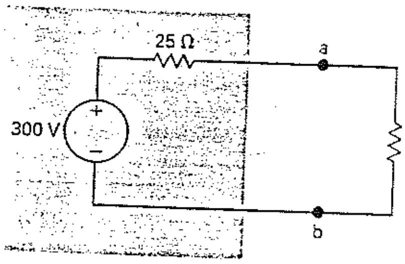


Figure 5-33

Example :

For the network of Fig. 5-34, find the resistance of R_L for maximum power dissipation in R_L. Also find this power.

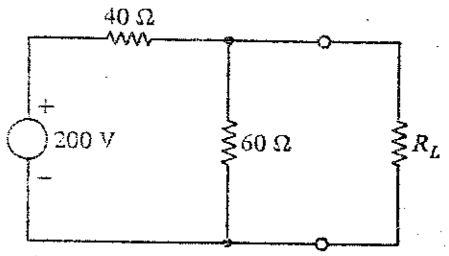


Figure 5-34

Solution :

The R_L resistance for maximum power transfer is the Thevenin resistance of the network connected to R_L. Clearly, this Thevenin resistance is $R_{TH} = 40 \parallel 60 = 24\ \Omega$ for maximum power transfer. To find this power we could return to the original circuit. Often, though, it is easier to use the Thevenin equivalent. For the circuit of Fig. 5-34 with an open circuit replacing R_L, the Thevenin voltage is, by voltage division,

$$V_{TH} = \frac{60}{60+40} \times 200 = 120\ \text{V}$$

With the Thevenin replacement, the circuit reduces to a 120 V source in series with two 24 Ω resistors. By voltage division, then, half of V_{TH}, or 60 V, drops across R_L. So, $P_{max} = \frac{60^2}{24} = 150\ \text{W}$

***Example* :**

What resistance of R_L in the circuit of Fig. 5-35, causes maximum power dissipation in R_L, and what is this power ?

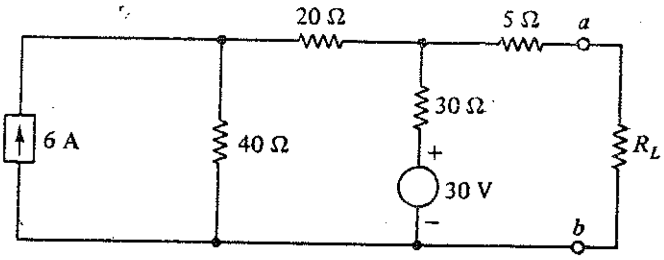


Figure 5 - 35

***Solution* :**

Again the procedure is to find the Thevenin resistance of the network connected to R_L. Also, here because of the complexity of this circuit, it is a good idea to also find V_{TH} to help in finding the power.

Killing the sources results in the network of Fig. 5-36. For this series-parallel network

$$R_{TH} = 5 + 30 \parallel (20 + 40) = 25\ \Omega$$

and so $R_L = 25\ \Omega$ for maximum power transfer.

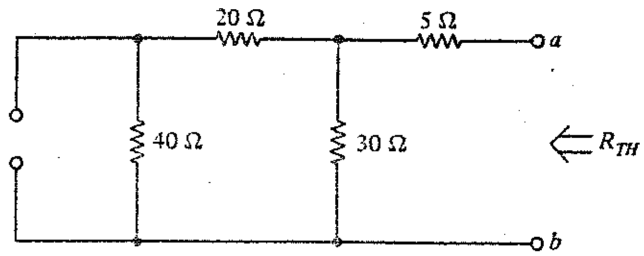


Figure 5 - 36

Now find V_{TH} . It is the voltage drop across the branch with the 30 Ω resistor and 30 V source in the network of Fig. 5-37. This is the original network of Fig. 5-35 with R_L replaced by an open circuit. One mesh equation gives the current in this branch.

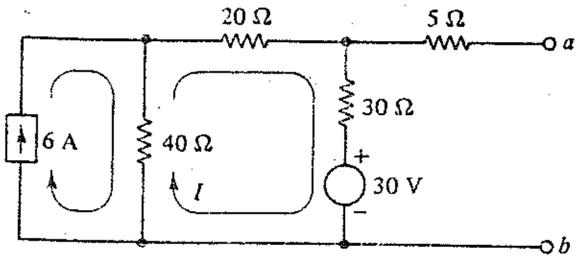


Figure 5 - 37

From this branch current we can easily find the branch voltage V_{TH}. The single mesh equation is :

$$(40 + 20 + 30)\, I = 6(40) - 30$$

from which $I = \frac{7}{3}$ A. **Then,**

$$V_{TH} = \frac{7}{3} \times 30 + 30 = 100 \text{ V}$$

And the network reduces to a series circuit of a 100 V source and two 25 Ω resistors. By voltage division, half of this voltage, 50 V, is across R_L. So,

$$P_{max} = \frac{50^2}{25} = 100 \text{ W}$$

Substitution Theorem :

The substitution theorem has to do with replacing a network branch with another branch without affecting any voltages or currents in the network. According to this theorem, the only requirement is for the substitute branch to have the same voltage as the original branch when carrying the same current. For example, consider the circuit shown in Fig. 5-38(a). It can be easily shown that the current in branch ab equals 2 A.

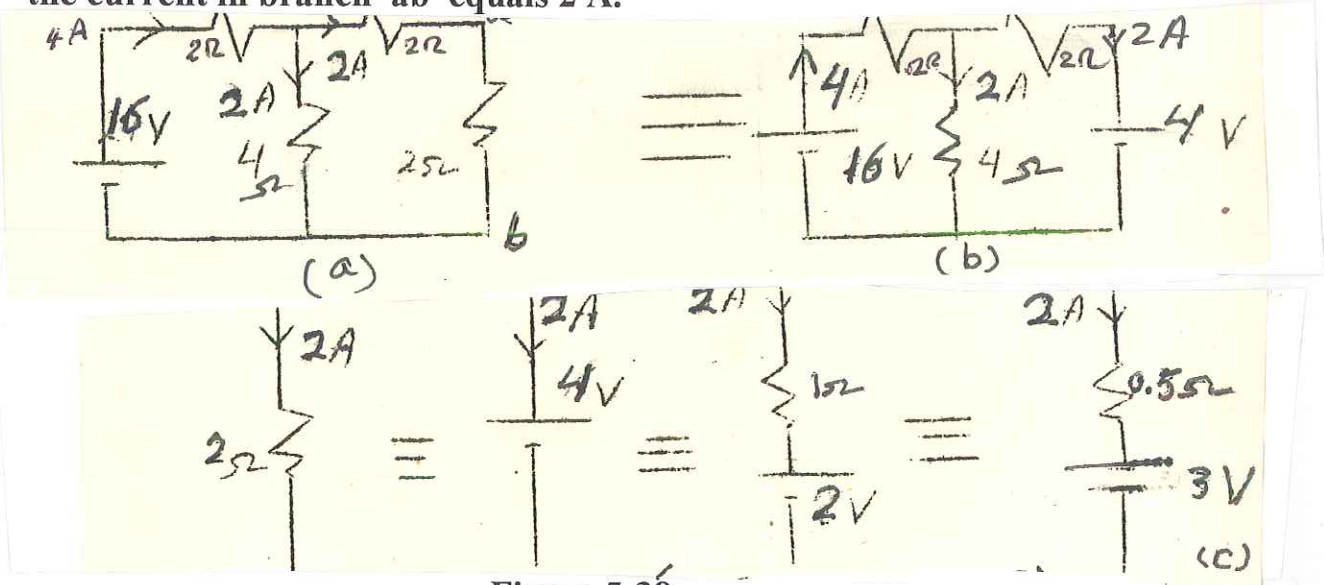


Figure 5-38

The original branch had 4V across it when carrying 2A, as shown in Fig. 5-38(b), a 4V voltage source would also do because it has 4V across it when carrying 2A or any other current. Likewise, a 2 A current source would do. There are other possibilities. A branch with a 1 Ω resistor in series with a 2 V voltage source has 4 V when carrying 2 A. Or the branch could be a 0.5 Ω resistor in series with a 3 V source. The possibilities are endless. Some of them are shown in Fig. 5-38(c).

Reciprocity Theorem :

The reciprocity theorem applies only to a linear, bilateral network with a single independent source and no dependent sources. For a network with a voltage source, the theorem states that if we put an ammeter in any branch and note the reading, and if we then interchange the voltage source and ammeter, the ammeter will still read the same. The theorem applies to only these two branches. Currents in other branches will probably change when the ammeter and voltage source are interchanged.

Example :

In the circuit shown in Fig. 5-39, show the validity of the reciprocity theorem.

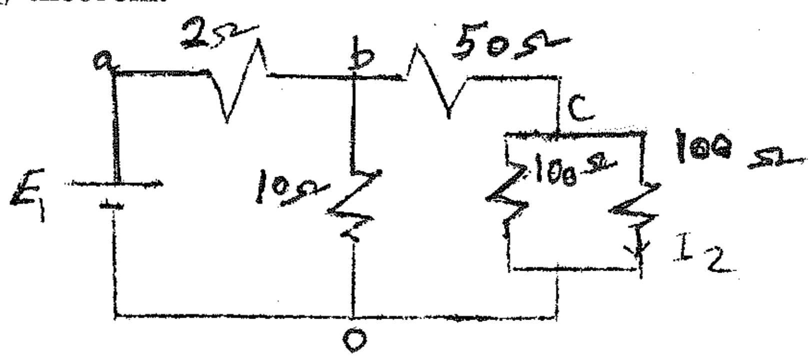


Figure 5-39

Solution :

In the circuit shown in Fig. 5-39, we get :

$E_{co} = 100\ I_2$, $E_{bo} = 200\ I_2$, $I_{bc} = 2\ I_2$,

$I_{bo} = 20\ I_2$, $I_{ab} = 22\ I_2$,

$E_{ao} = 200\ I_2 + 22 \times 2\ I_2 = 244\ I_2$

$E_1 / I_2 = 244$

Replacing the source E_1 in series with 100 Ω (in which the current I_2 was measured) and measuring the current I_1 in the branch ao, we will get the circuit shown in Fig. 5-40. We can write :

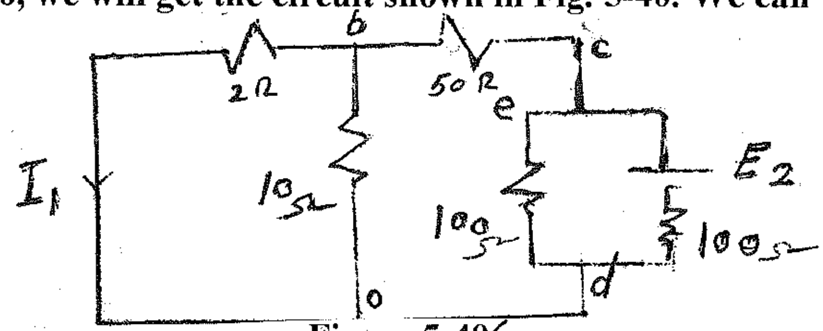


Figure 5-40

$E_{bo} = 2\ I_1$, $I_{bo} = 0.2\ I_1$,

$I_{cb} = 1.2\ I_1$, $E_{co} = 50 \times 1.2\ I_1 + 2\ I_1 = 62\ I_1$

$I_{ceo} = 6.2\ I_1 / 100 = 0.62\ I_1$

$I_{dc} = 1.2\ I_1 + 0.62\ I_1 = 1.82\ I_1$

$E_2 = E_{cd} = 62\ I_1 + 1.82\ I_1 \times 100 = 244\ I_1$

$$E_2 / I_1 = 244 \qquad (5\text{-}20)$$

Equations (5-19) and (5-20), show the validity of reciprocity theorem.

PROBLEMS

1) Find the Thevenin and Norton equivalents for the network of Fig. 5-41 for :

a) $V_s = 18$ V , $R_1 = 3\ \Omega$, $R_2 = 6\ \Omega$

b) $V_s = 100$ V , $R_1 = 5\ \Omega$, $R_2 = 20\ \Omega$

c) $V_s = 50$ V , $R_1 = 10\ \Omega$, $R_2 = 15\ \Omega$

2) Repeat Problem 1 (a) , with an added 24 Ω resistor between terminals a and b.

3) Find the Thevenin equivalent of a battery that has an 8.8 V terminal voltage when delivering 2 A and 8.6 V terminal voltage when delivering 4 A.

4) For the network of Fig. 5-42 use either Thevenin's or Norton's theorem to find the resistor that draws 2 A when connected across terminals a and b.

5) For the circuit of Fig. 5-43 find the resistances of R that produce currents through R of

a) 0.0 A

b) 0.1 A

c) 0.2 A

d) 0.3 A

6) For the circuit of Fig. 5-41 with $R_1 = 80\ \Omega$, $R_2 = 20\ \Omega$ and $V_s = 200$ V, find the resistance of the resistor that absorbs maximum power when connected across terminals a and b. Also find this power.

7) Use superposition in the circuit of Fig. 5-44 to find V. Then use superposition to find I.

8) Check the power balance in Fig. 5-45 using loop-current method and check the answer using superposition theorem R = 30 ohm.

9) Check the power balance in Fig. 5-46.

10) Find I in Fig. 5-47. Check the answer using Thevenin's theorem.

11) In Fig. 5-48. Find the value of the current in the 10 Ω resistor using :

a) The superposition theorem.

b) Millimans theorem.

c) Thevenin's theorem.

12) Determine the current which will pas in the 4 Ω resistor shown in Fig. 5-49.

13) In the circuit of Fig. 5-50. Determine the current in the resistance R_7 and the power dissipated in it, using :

a) Superposition theorem.

b) Thevenin's theorem.

c) Norton's theorem.

d) Node voltage method.

$R_1 = R_3 = R_6 = 5\ \Omega$, $R_2 = 2.5\ \Omega$, $R_7 = 1.25\ \Omega$

$R_4 = R_5 = 10\ \Omega$, $U_1 = 40\ V$, $U_2 = 50\ V$

14) Calculate all the current in branches of the circuit of Fig. 5-51 using super-position theorem and check the current through R_2 and through the source U_2 , by means of Thevenin's theorem.

$R_1 = 20\ \Omega$, $R_2 = 15\ \Omega$, $R_3 = 5\ \Omega$, $R_4 = 10\ \Omega$

$U_1 = 70\ V$ $U_2 = 90\ V$

15) In the circuit of Fig. 5-52. Find the value of the voltage U of the battery and the power delivered by it.

16) In the circuit of Fig. 5-53, find I using Thevenin's theorem.

17) Get the value of R that gives maximum power transfer in Figure 5-54 (a and b).

18) Using superposition theorem, get I in Fig. 5-55.

19) Find the value of R_L that causes maximum power tranfer in the circuit shown in Fig. 5-56. Find also the maximum power transfer.

20) Find the current and power in R to get max. power transfer to it for the circuit shown in Fig. 5-57.

21) Use the principle of superposition to find v_o in the circuit shown in Fig. 5-58.

22) a) Use the principle of superposition to find the voltage v in the circuit shown in Fig. 5-59.

b) Find the power dissipated in the 40 Ω resistor.

23) Use the prinicple of superposition to find the voltage v in the circuit shown in Fig. 5-60.

24) Find the Thevenin equivalent circuit with respect to the terminals a , b for the circuit shown in Fig. 5-61.

25) Find the Norton equivalent circuit with respect to the terminals a, b for the circuit shown in Fig. 5-62.

26) A voltmeter with an internal resistance of 100 kΩ is used to measure the voltage v_{AB} in the circuit shown in Fig. 5-63. Find the voltmeter reading.

27) Find the Thevenin equivalent with respect to the terminals a , b for the circuit in Fig. 5-64.

28) Find the Thevenin equivalent with respect to the terminals a , b for the circuit in Fig. 5-65.

29) a) Find the Thevenin equivalent with respect to the terminals a , b for the circuit in Fig. 5-66 by finding the open-circuit voltage and the short-circuit current.

b) Solve the Thevenin resistance by removing the independent sources. Compare your result to the Thevenin resistance found in part (a).

30) Find the Norton equivalent with respect to the terminals a , b in the circuit in Fig. 5-67.

31) Determine i_o and v_o in the circuit shown in Fig. 5-68 when R_o is 0, 1, 3, 5, 10, 15, 25, 40, 55, 70, 85 and 95 Ω.

32) A voltmeter with a resistance of 85.5 kΩ is used to measure the voltage v_{ab} in the circuit in Fig. 5-69.

a) What is the voltmeter reading ?

b) What is the percent error in the voltmeter reading if percent error is defined as [(measured – actual) / actual] x 100 ?

33) Determine the Thevenin equivalent with respect to the terminals a , b for the circuit shown in Fig. 5-70.

34) Find the Thevenin equivalent with respect to the terminals a , b for the circuit shown in Fig. 5-71.

35) When a voltmeter is used to measure the voltage v_e in Fig. 5-72 it reads 5.5 V.

a) What is the resistance of the voltmeter ?

b) What is the percent error in the voltage measurement ?

36) When an ammeter is used to measure the current i_ϕ in the circuit shown in Fig. 5-73 it reads 6 A.

a) What is the resistance of the ammeter ?

b) What is the percent error in the current measurement ?

37) Find the Thevenin equivalent with respect to the terminals a , b for the circuit shown in Fig. 5-74.

38) Find the Thevenin equivalent with respect to the terminals a , b for the circuit shown in Fig. 5-75.

39) A Thevenin equivalent can also be determined from measurements made at the pair of terminals of interest. Assume the following measurements were made at the terminals a , b in the circuit in Fig. 5-76.

- When a 20 Ω resistor is connected to the terminals a , b the voltage v_{ab} is measured and found to be 100 V.

- When a 50 Ω resistor is connected to the terminals a , b the voltage is measured and found to be 200 V.

- Find the Thevenin equivalent of the network with respect to the terminals a , b.

40) An automobile battery, when connected to a car radio, provides 12.5 V to the radio. When connected to a set of headlights it provides 11.8 V to the headlights. Assume the radio can be modeled as a 6 Ω resistor and the headlights can be modeled as a 0.75 Ω resistor.

What are the Thevenin and Norton equivalents for the battery ?

41) Calculate the power delivered to each resistor in problem (19). Plot the power delivered versus the resistance R_L. At what value of R_L is the power maximum ?

42) The variable resistor in the circuit in Fig. 5-77 is adjusted for maximum power transfer to R_o.

a) Find the value of R_o.

b) Find the maximum power that can be delivered in R_o.

43) The variable resistor in the circuit in Fig. 5-78 is adjusted for maximum power transfer to R_o.

What percentage of the total power developed in the circuit is delivered to R_o?

44) The variable resistor (R_L) in the circuit in Fig. 5-79 is adjusted for maximum power transfer to R_L.

a) Find the numerical value of R_L.

b) Find the maximum power transferred to R_L.

44)The variable resistor (R_o) in the circuit in Fig. 5-80 is adjusted for maximum power transfer to R_o.

a) Find the value of R_o.

c) Find the maximum power that can be delivered to R_o.

45) What percentage of the total power developed in the circuit in Fig. 5-80 is delivered to R_0.

46) The variable resistor (R_0) in the circuit in Fig. 5-81 is adjusted until it absorbs maximum power from the circuit. Find :

a) the value of R_0.

b) the maximum power ; and

c) the percent of the total power developed in the circuit that is delivered to R_0.

48) The variable resistor (R_0) in the circuit in Fig. 5-82 is adjusted for maximum power transfer to R_0.

What percent of the total power developed in the circuit is delivered to R_0?

49) A variable resistor R_0 is connected across the terminals a , b in the circuit in Fig. 5-71. The variable resistor is adjusted until maximum power transferred to R_0. Find :

a) the value of R_0.

b) the maximum power delivered to R_0, and :

c) the percentage of the total power developed in the circuit that is delivered to R_0.

50) a) Find the value of the variable resistor R_o in the circuit in Fig. 5-83 that will result in maximum power dissipation in the 8 Ω resistor.

b) What is the maximum power that can be delivered to the 8 Ω resistor.

(Hint : Hasty conclusions could be hazardous to your career.)

51) Use superposition to solve for i_o and v_o in the circuit in Fig. 5-84.

52) Use the principle of superposition to find the current i_o in the circuit shown in Fig. 5-85.

53) Use the principle of superposition to find v_o in the circuit in Figure 5-86.

54) Use the principle of superposition to find the voltage v_o in the circuit in Fig. 5-87.

55) a) In the circuit in Fig. 5-88, before the 5 mA current source is attached to the terminals a ,b the current i_o was calculated and found to be 3.5 mA. Use superposition to find the value of i_o after the current source is attached.

b) Verify your solution by finding i_o when all three sources are acting simultaneously.

56) Use the principle of superposition to find v_o in the circuit shown in Fig. 5-89.

57) Laboratory measurements on a dc voltage source yield a terminal voltage of 75 V with no load connected to the source and 60 V when loaded with a 20 Ω resistor.

a) What is the Thevenin equivalent with respect to the terminals of the dc voltage source ?

b) Show that the Thevenin resistance of the source is given by the expression :

$$R_{TH} = \left(\frac{V_{TH}}{V_0} - 1 \right) R_L \ ,$$

where :

V_{TH} = the Thevenin voltage,

V_o = the terminal voltage corresponding to the load resistance R_L .

58) An automobile generator is operating in parallel with a battery to energize a 2 Ω load. What is the load current if the open-circuit voltages are 15 V for the generator and 13 V for the battery and if the internal resistances are 0.2 Ω for the generator and 0.05 Ω for the battery ?

59) Three batteries operating in parallel energize a 10 Ω load. What is the load current if the open-circuit battery voltages and internal resistances are, respectively, 26 V and 0.2 Ω, 25 V and 0.1 Ω, and 24 V and 0.05 Ω ?

60) For the circuit of Fig. 5-44 what resistor can replace the current source without producing changes in any voltages or currents ?

61) For the circuit of Fig. 5-44 what current source can replace the 18 Ω resistor without producing changes in any voltages or currents ?

62) In the circuit of Fig. 5-43, replace the resistor R by a zero resistance ammeter and find the ammeter reading. Then interchange the 30 V source and the ammeter and find the ammeter reading again to verify the reciprocity theorem.

63) Using the superposition theorem, find the current through each resistor of the network of Fig. 5-90. Does superposition apply to power effects ? Explain.

64) Using the superposition, find I for each of the networks of Figure 5-91

65) Using the superposition, find the current through R_1 for each network of Figure 5-92.

66) Find V_2 for the network of Fig. 5-93 using superposition.

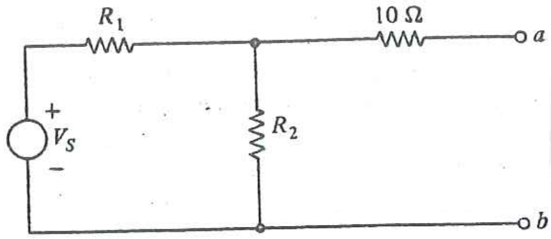
67) a) Find the Thevenin equivalent circuit for the network external to the resistor R in each of the networks of Fig. 5-94.

b) Find the power delivered to R when R is 2 Ω and 100 Ω.

68) Find the Thevenin equivalent circuit for the network external to the resistor R in each of the networks of Fig. 5-95.

69) Find the Thevenin equivalent circuit for the network external to the resistor R in each of the networks of Fig. 5-96.

70. Using superposition, find the current I through the 10-Ω resistor for each of the networks of Fig. 5.97.
71. Using superposition, find the current I through R_1 for each network of Fig. 5.98.
72. Find the Thevenin equivalent circuit for the network external to the resistor R in each of the networks of Fig. 5.99.
73. Determine the Thevenin equivalent circuit for the network external to the resistor R in both networks of Fig. 5.100.
74. Find the Norton equivalent circuit for the portions of the networks of Fig. 5.101 external to branch a –b by getting the s-c current and No resistance.
75. a. For the network of Fig. 5-102 determine the value of R for maximum power to R.
    b. Determine the maximum power to R.
    c. Plot curve of power to R versus R for equal to ¼ , ½ , ¾, 1, 1 ¼, 1 ½, 1 ¾, and 2 times the value obtained in part(a).
76. Find the resistance R_1 of Fig.5-103 such that the resistor R_4 will receive maximum power. Think !
77. Using the substitution theorem, draw three equivalent branches for the branch a-b of the network of Fig.5-104.
78. Repeat Problem 77for the network of Fig. 5-105 Be careful !
79. a. For the network of Fig. 5.106 (a) determine the current I.
    b. Repeat part (a) for the network of Fig . 5.106(b).
    c. Is the reciprocity theorem satisfied ?



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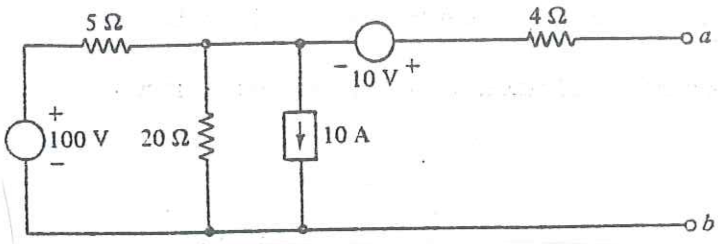

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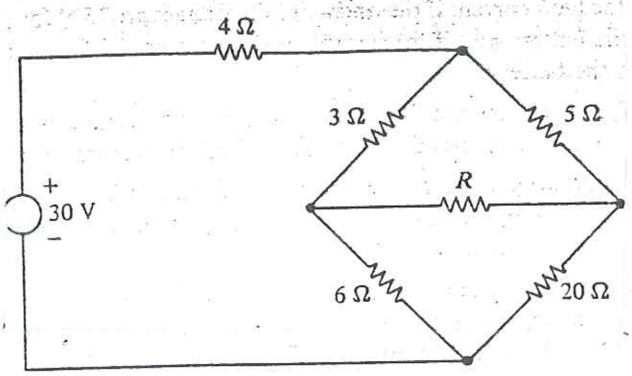

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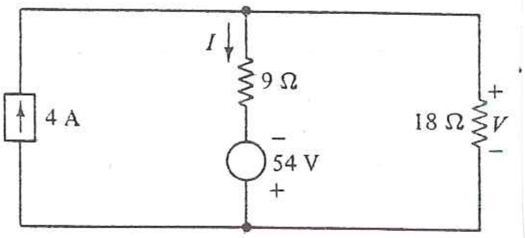

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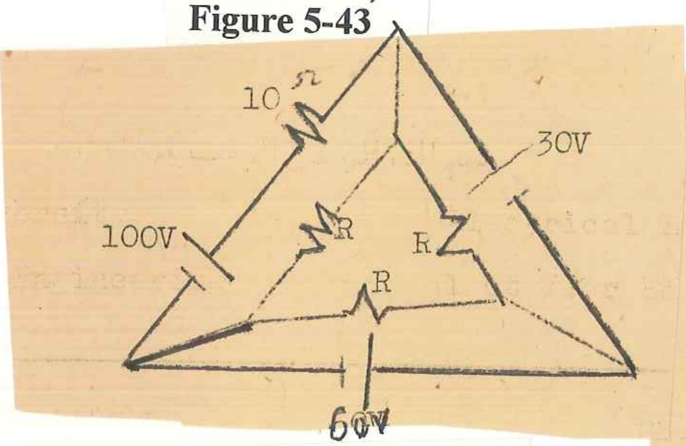

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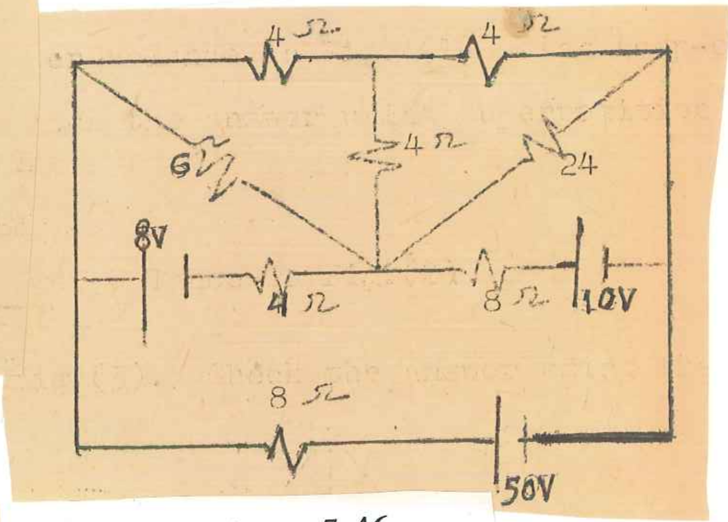

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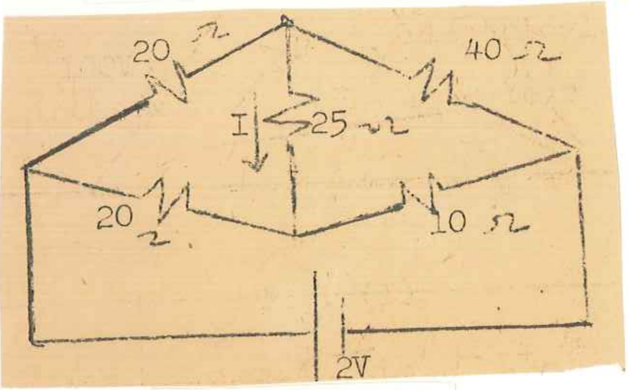

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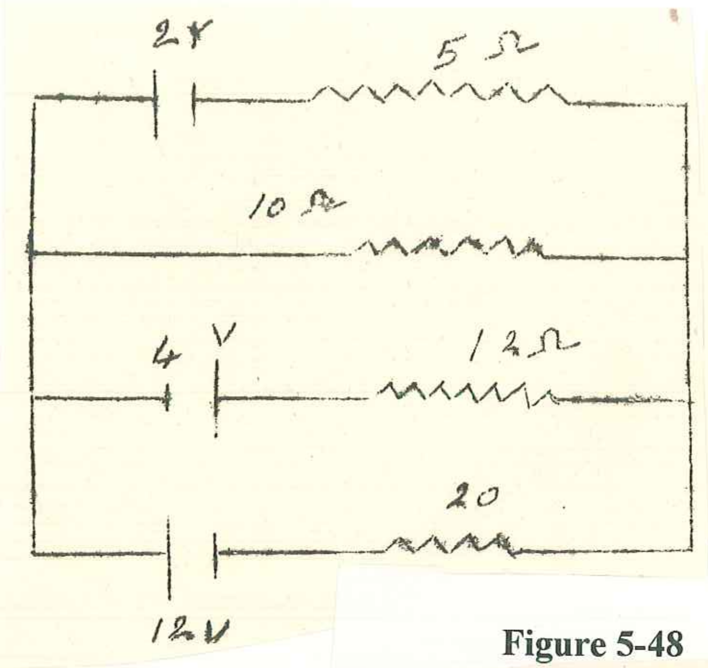

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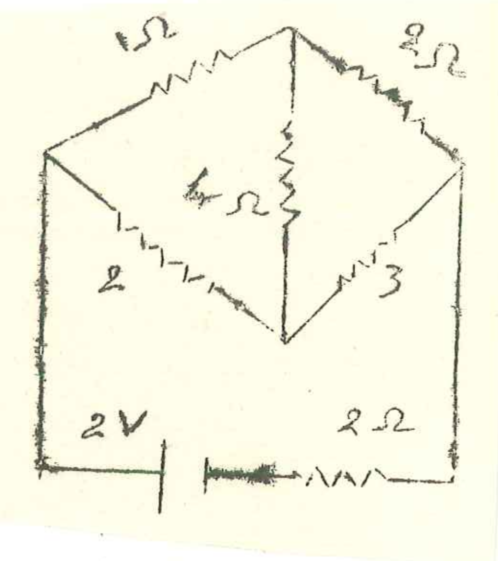

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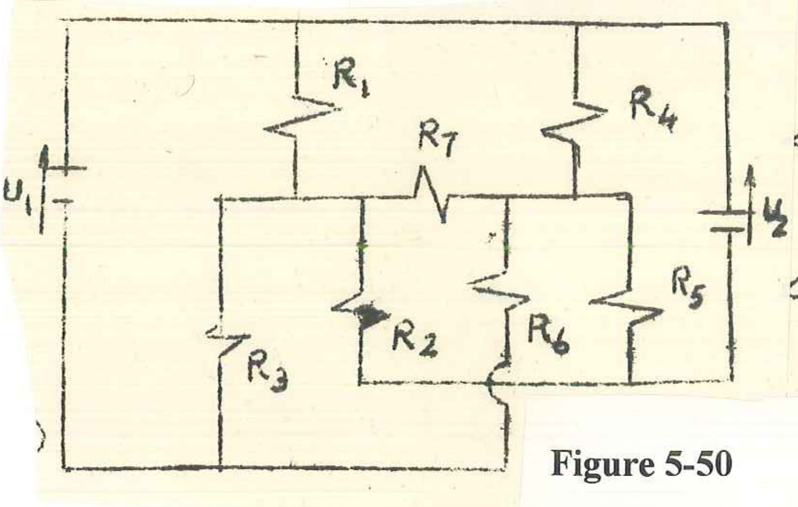

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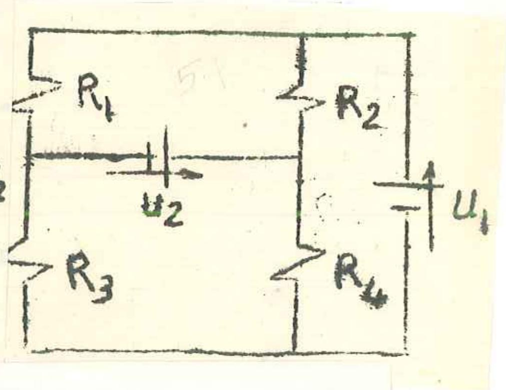

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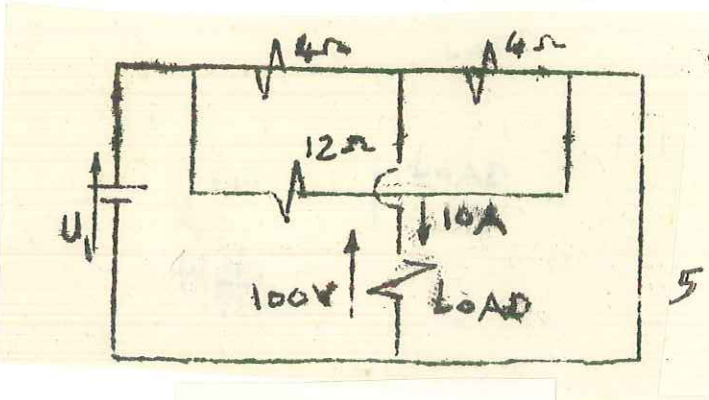

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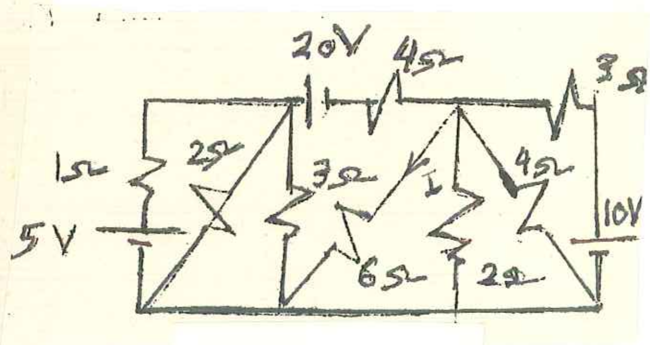

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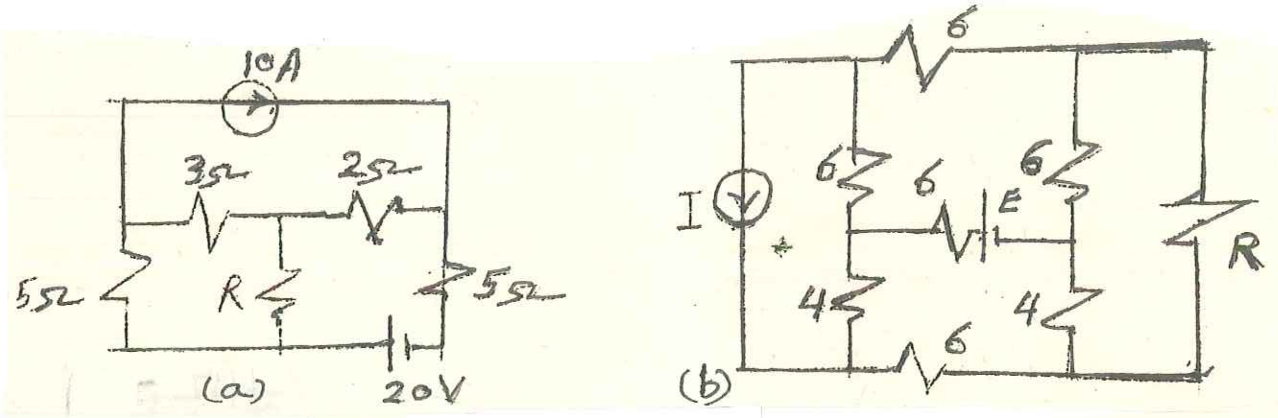

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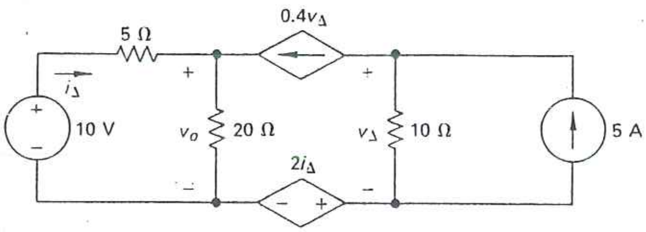

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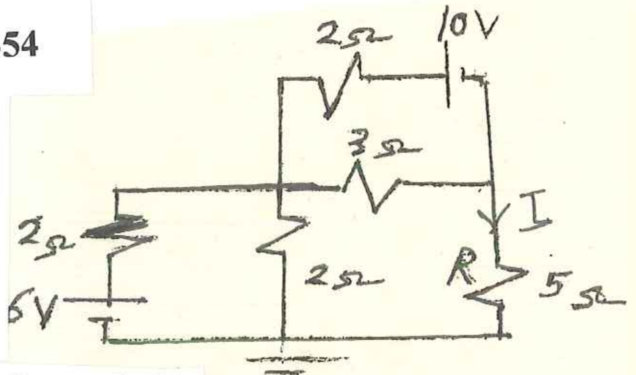

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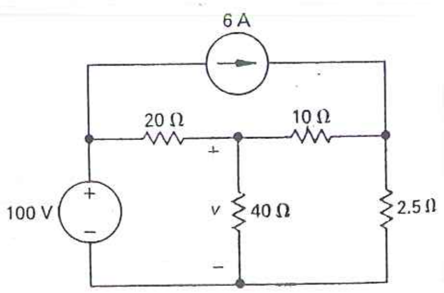

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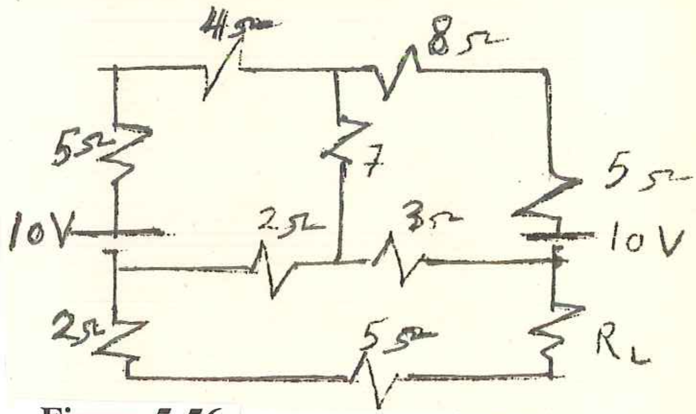

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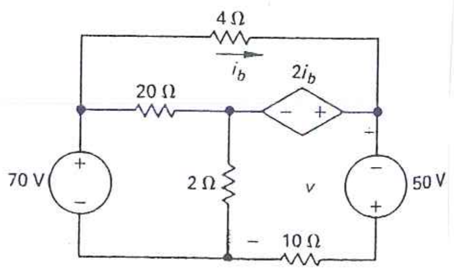

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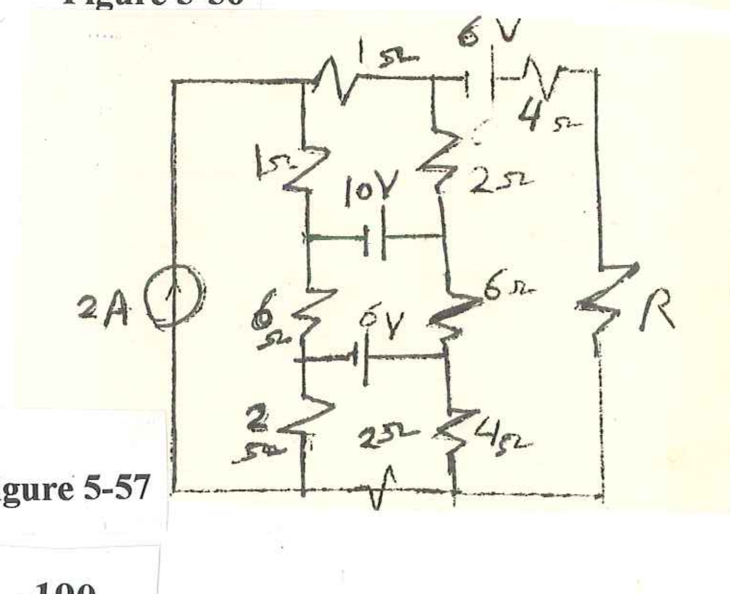

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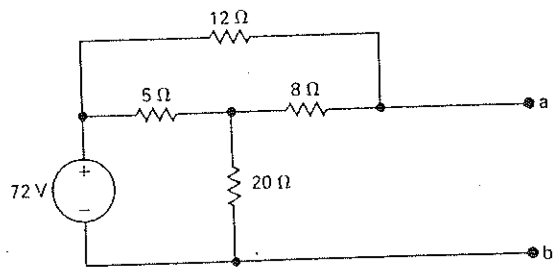

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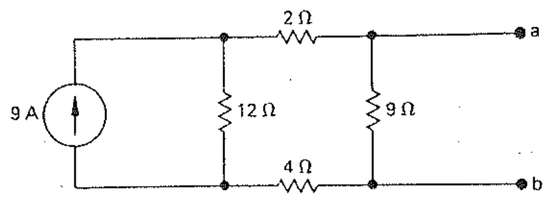

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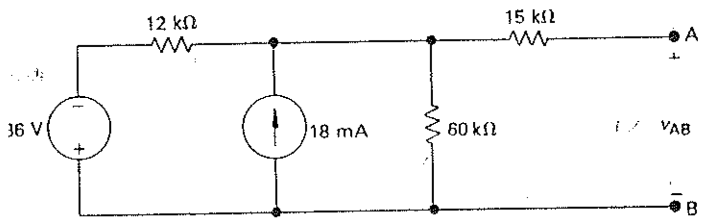

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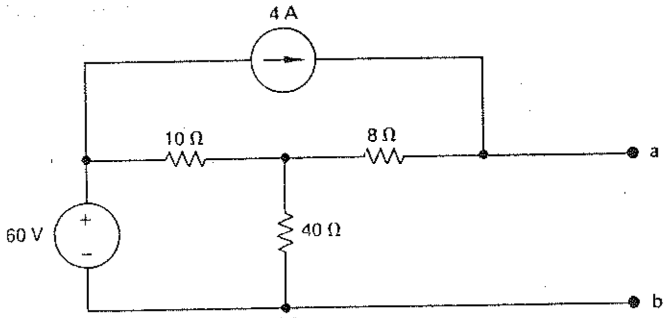

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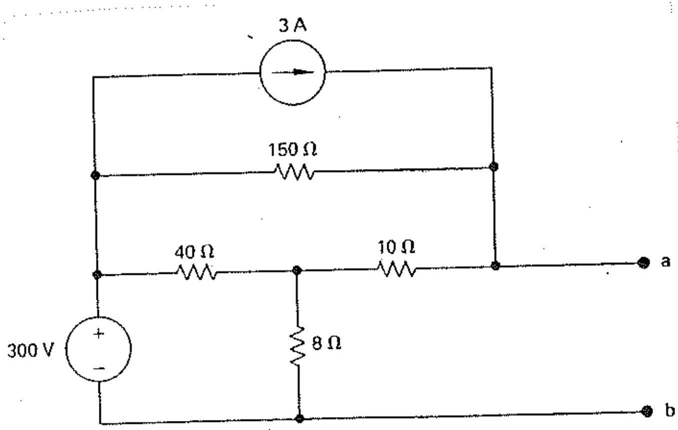

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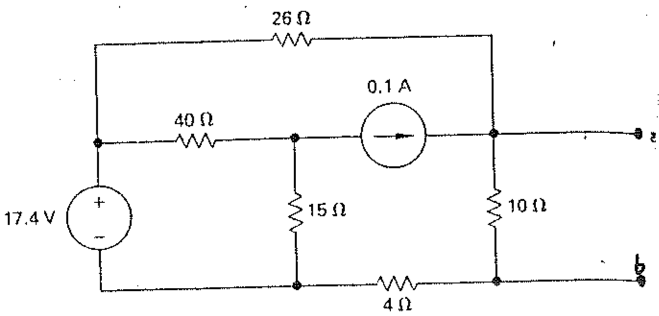

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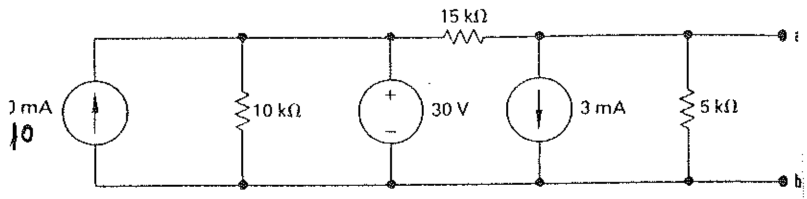

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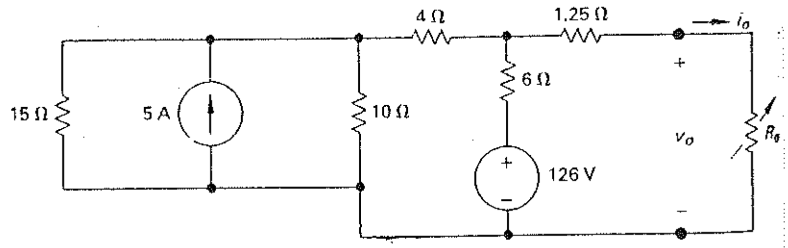

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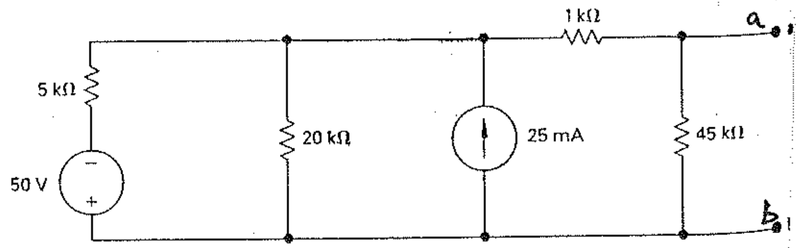

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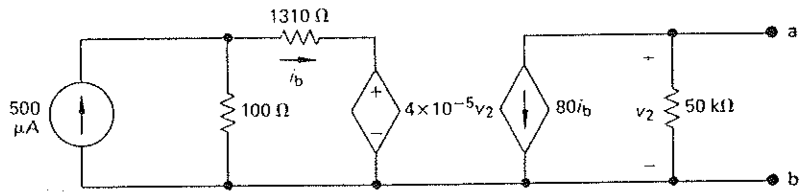

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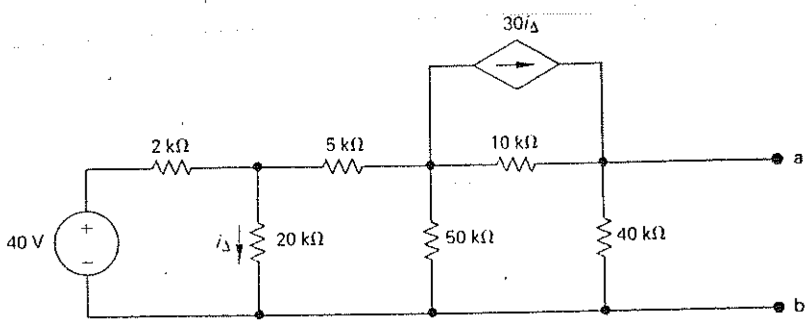

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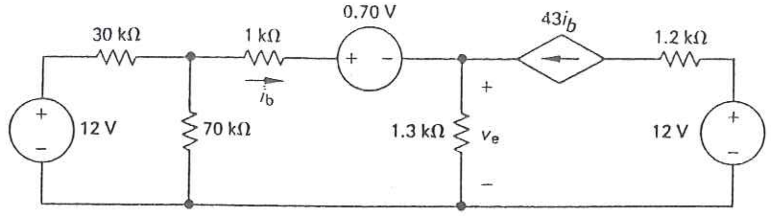

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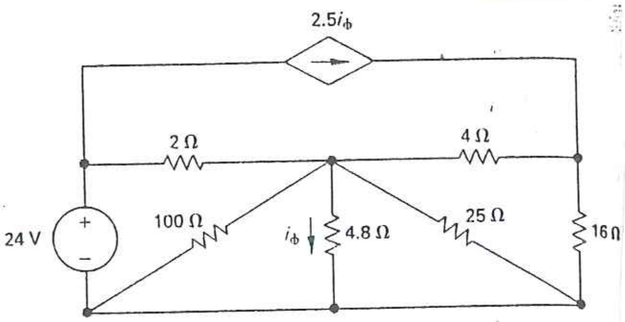

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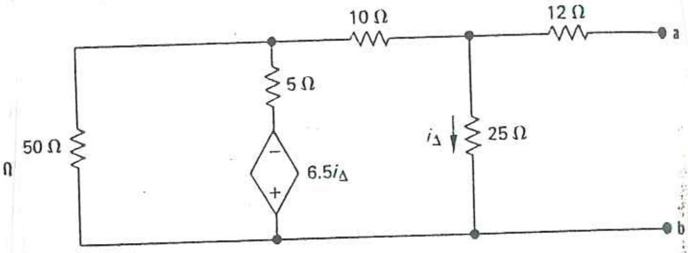

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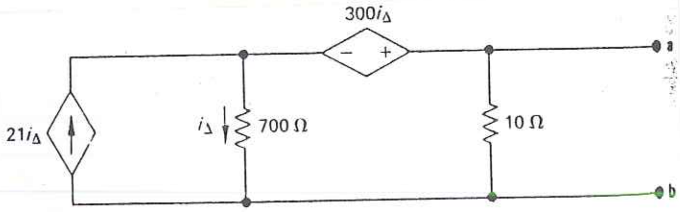

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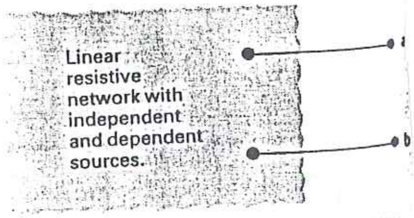

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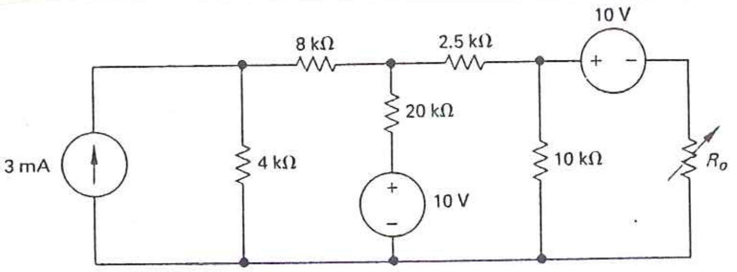

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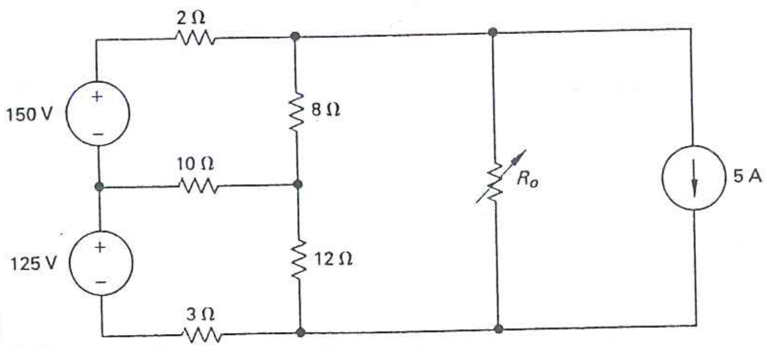

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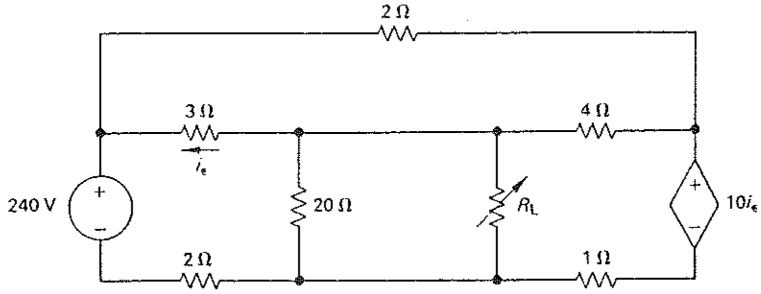

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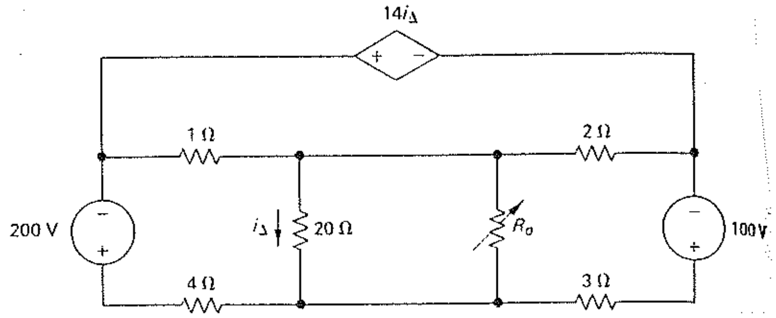
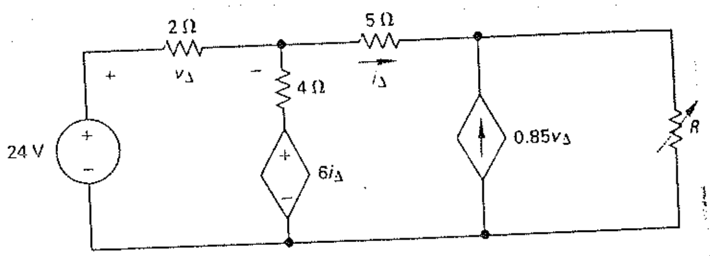


Figure 5-80



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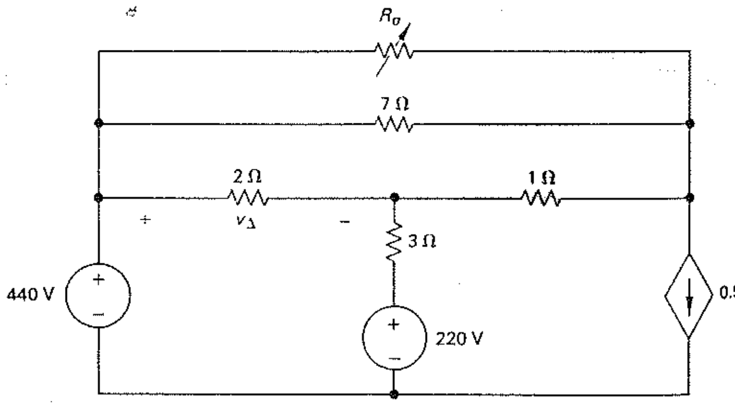

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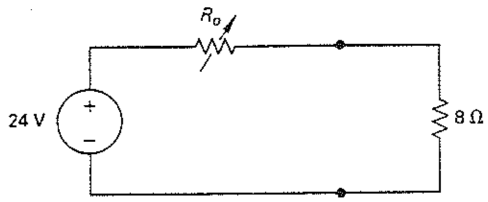

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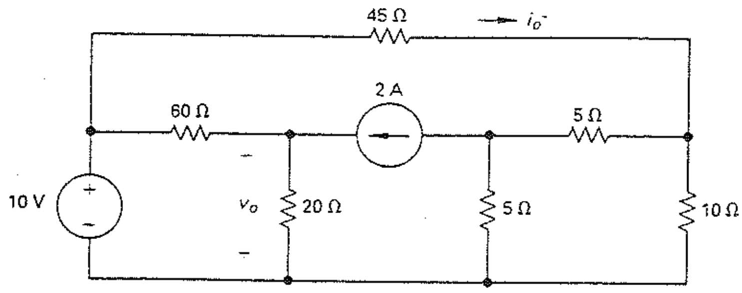

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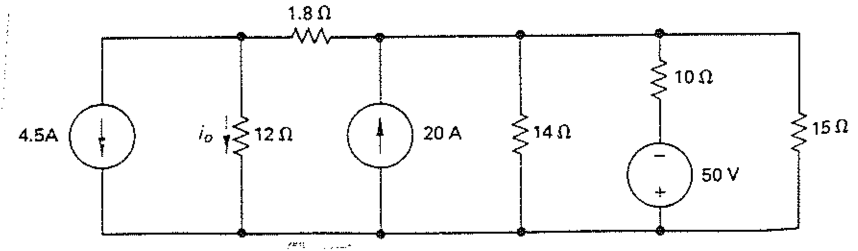

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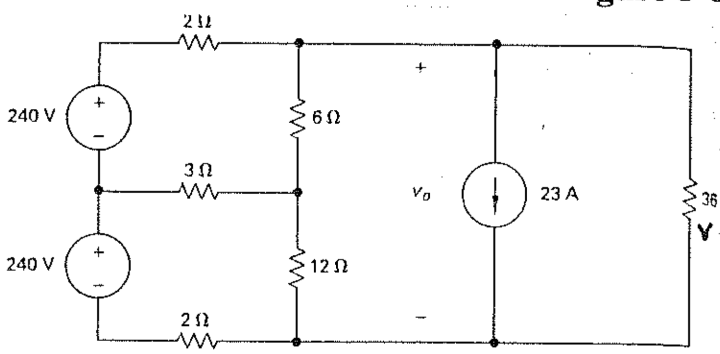

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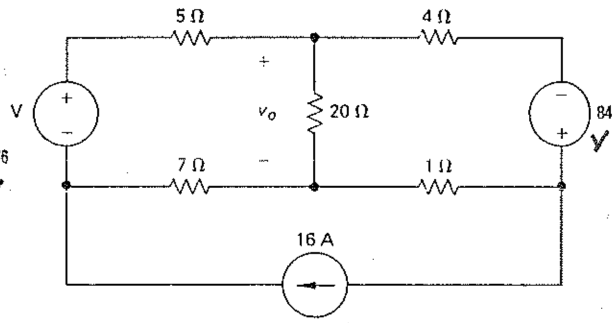

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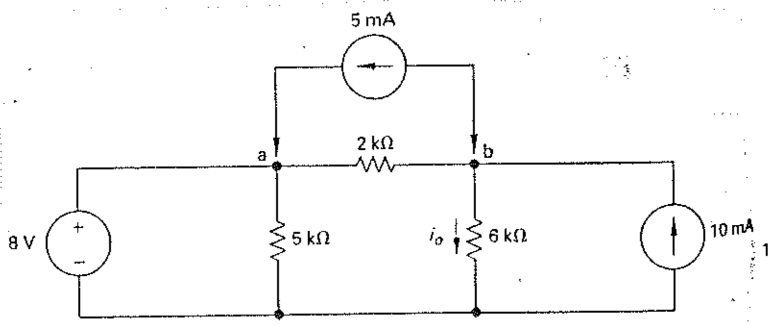

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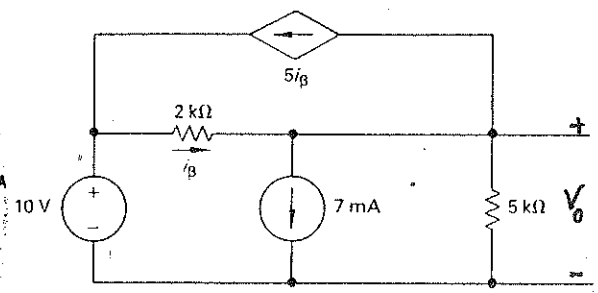

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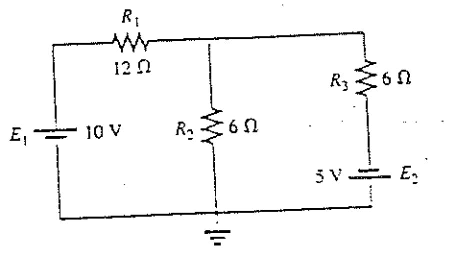

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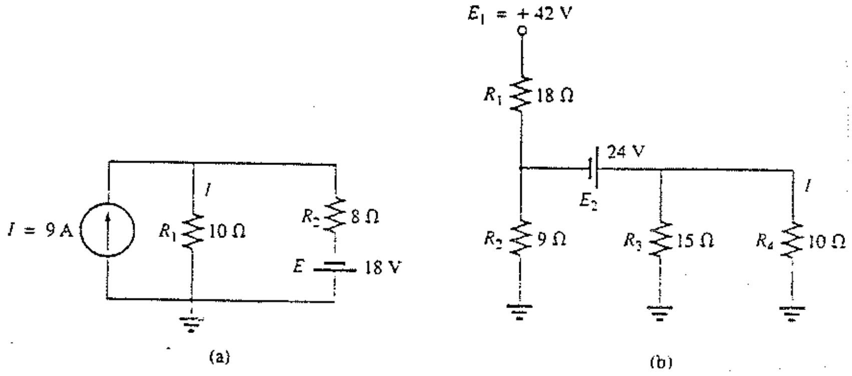

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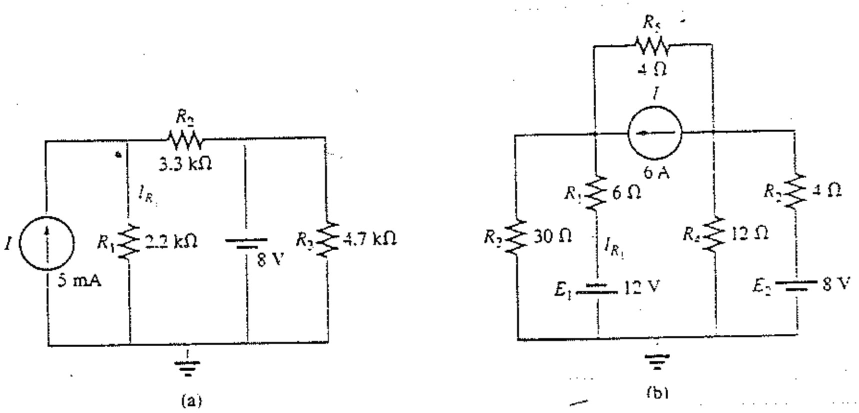

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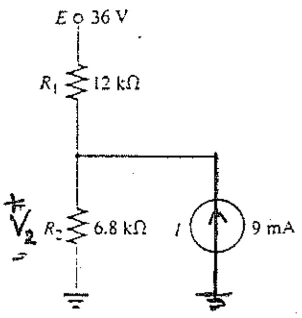

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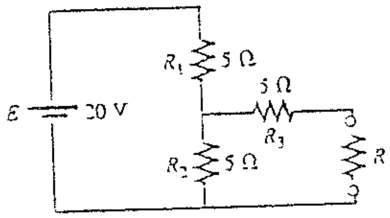

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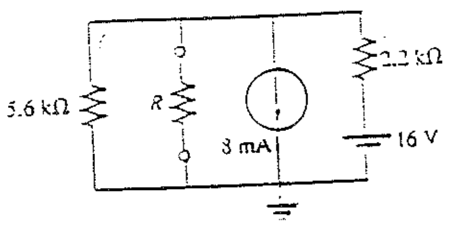
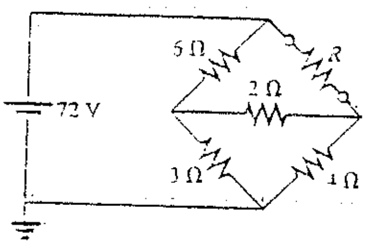


Figure 5-95



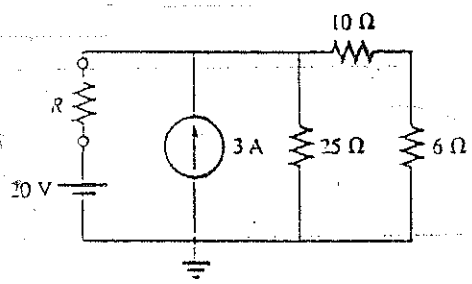


Figure 5-96

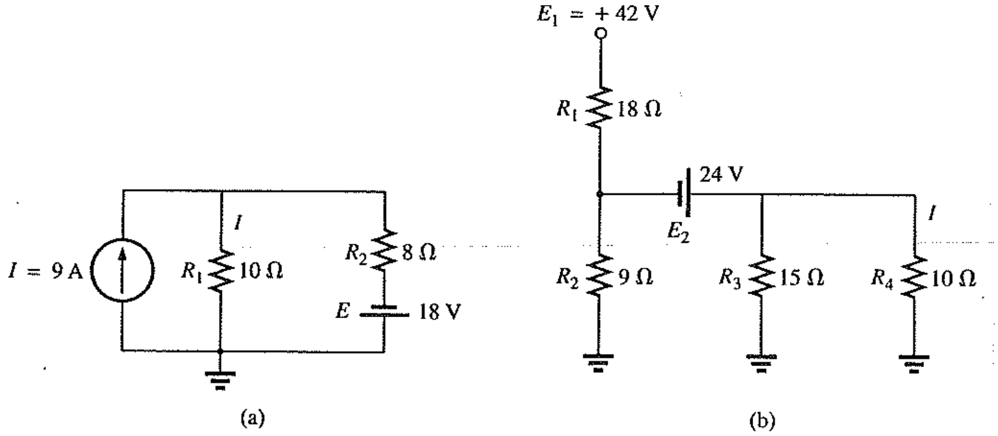


FIG. 5.97

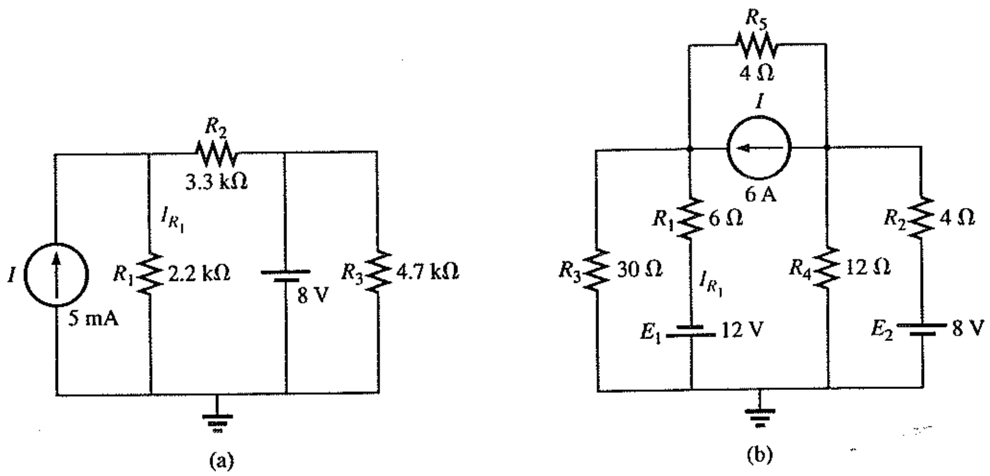


FIG. 5.98

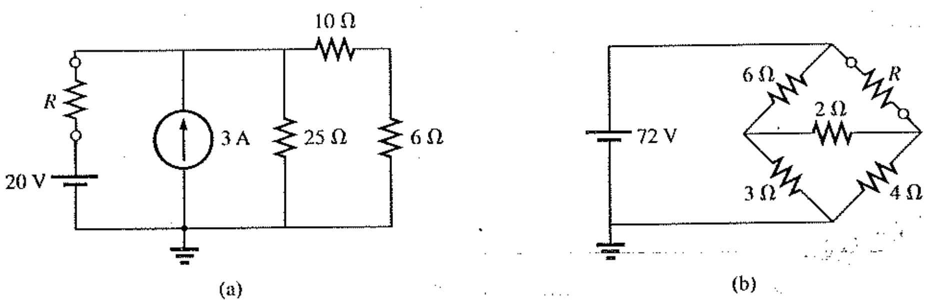


FIG. 5.99

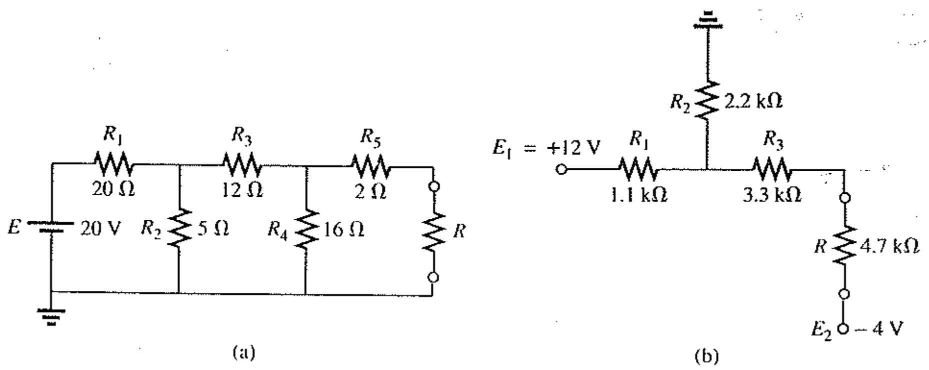


FIG. 5.100

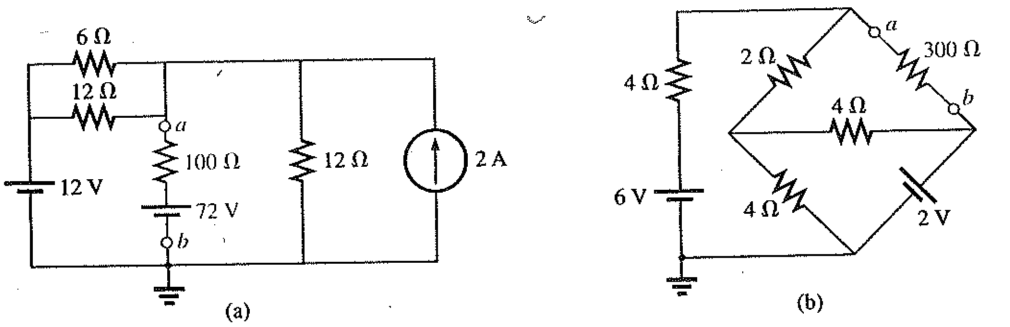


FIG. 5.101

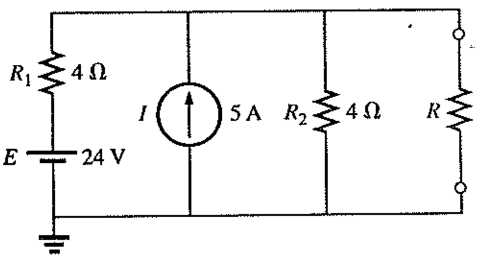


FIG. 5.102

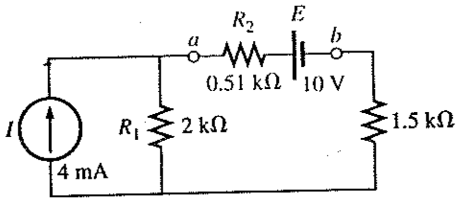


FIG. 5.104

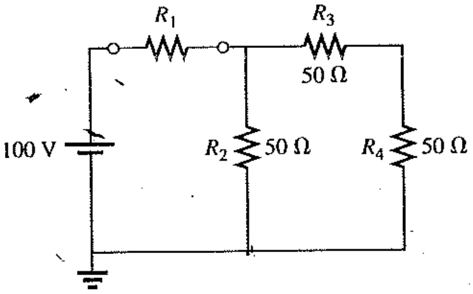


FIG. 5.103

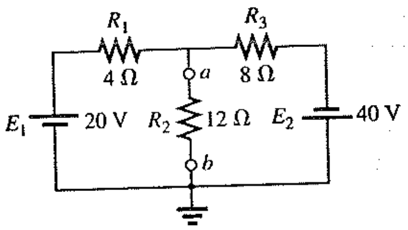


FIG. 5.105

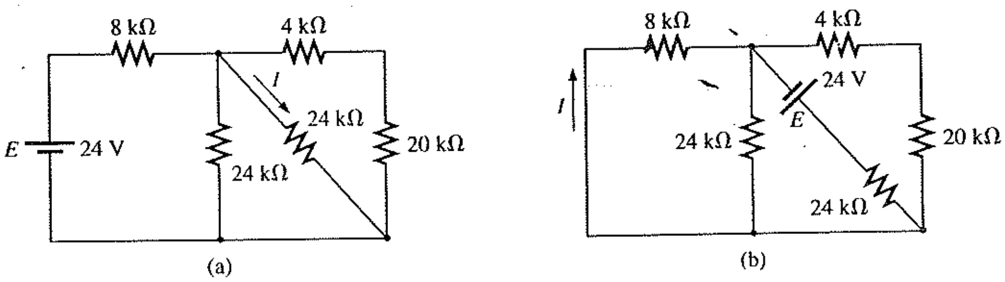


FIG. 5.106

Chapter Six

SINUSOIDAL ALTERNATING CURRENT AND VOLTAGE

Introduction :

Up to this point of our study, the independent voltage and current sources have all been DC. Now we begin the study of the analysis of networks having ac sources. The term AC is, of course, just an abbreviation for alternating current.

The word alternating usually refers only to a periodic current or voltage. And periodic means that the current or voltage varies with time such that it repeats itself after a fixed time called the period. Figure 6-1 shows some periodic waves that may be either voltages or currents. Strictly speaking, each of these waves has no beginning and no end. Actually, of course, every practical electrical voltage or current has a beginning and an end.

In Fig. 6-1 each horizontal axis (abscissa) is in time. Along each abscissa, the indicated period T is the shortest time in which a wave begins to repeat. The inverse of the period is the wave frequency, which is measured in the SI unit of hertz with unit symbol Hz. The hertz replaces the old unit of cycles per second. (A cycle is the segment of a wave occurring during one period). The quantity symbol for frequency is f. So,

$$f = \frac{1}{T}$$

In Fig. 6-1(a) is the most popular periodic wave, the sine wave, which is the principal wave for this and most of the remaining chapters. Figure 6-1(b) is a triangular wave, Fig. 6-1(c) is a sawtooth wave, Fig. 6-1(d) a square wave, and Fig. 6-1(e) a cosine wave superimposed on dc. These are examples of periodic waves, as mentioned. And all but the sawtooth wave of Fig. 6-1(c) and the wave of Fig. 6-1(e) are also alternating, as will be explained. Electronic oscillators can be purchased for generating these waves.

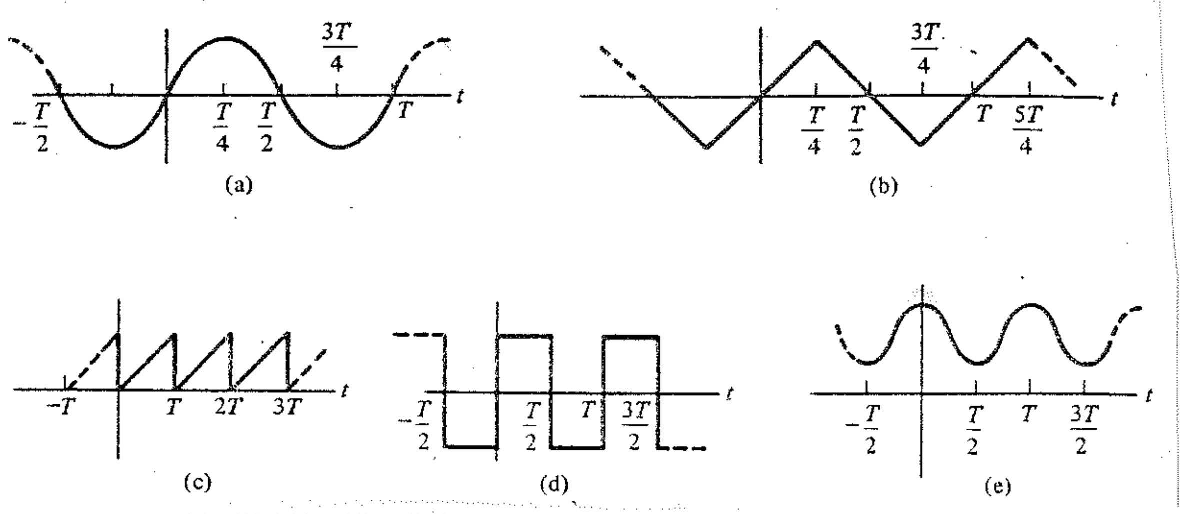


Figure 6-1

From Fig. 6-1 do not receive the impression that the period is always measured from the time that a wave goes through zero, which naturally cannot be true for the wave of Fig. 6-1(e) because it is never zero. Also, the starting time for measuring a period does not have to be t = 0 s. This starting time can be any time. The period, then, is the time between this starting time and the instant of time at which the wave first repeats—a period is the duration of a cycle.

An alternating current is a periodic current that varies with time such that during part of each period the current flows in one direction, and then for the remaining time reverses and flows in the opposite direction. This reversal is indicated in Figs. 6-1 (a), (b) and

(d) by the waves being positive or above the abscissa for a portion of a cycle and negative or below the abscissa for the remainder of the cycle.

As mentioned, the term AC is an abbreviation for alternating current. Actually, it is more than that. By common usage AC means sinusoidal. So, AC voltage means sinusoidal voltage and ac current means sinusoidal current. Other alternating waves are simply designated by name, such as square wave, triangular wave, and so on.

We will devote all of this chapter and most of the remaining chapters to the analysis of AC circuits—circuits with sinusoidal voltage and current sources. Why all this emphasis on circuits having this one type of source ? What is so important about sinusoids.

One reason for the importance of sinusoidal analysis is that almost all electrical power, at least large amounts of power, is generated sinusoidally. AC voltage generators, also called alternators, generate these sinusoidal voltages. There is some distortion of these voltages on transmission. But even so, the voltages at common electric outlets are nearly perfect sinusoids. Another reason for the importance of sinusoidal analysis is that all information in electrical form—every electrical signal—is a sum of sinusoids.

Sine and Cosine Waves :

Mathematically, though, a sine wave is described by $v = V_m \sin \omega t$ if it is a voltage and by $i = I_m \sin \omega t$ if it is a current. In these expressions the v and i represent instantaneous values and so are in lowercase, as is conventional. The V_m for voltage maximum and I_m for current maximum are the wave amplitudes or peak values. It is conventional to use uppercase letters for these. The peak value or amplitude, both mean the same, is the maximum value that a sinusoidal wave gets above the horizontal axis if the wave is plotted. The negative of this peak is the maximum negative value.

In $V_m \sin \omega t$ and $I_m \omega t$ is ω, the Greek lowercase letter omega. It is the quantity symbol for a frequency that is related to the SI unit of plane angle, the radian, with unit symbol rad. Specifically, ω is the radian frequency of the sinusoid in radians per second, the unit symbol for which is rad/s. This ω is related to f, the wave frequency in hertz, by :

$$\omega = 2\pi f$$

which specifies that the frequency of a wave in radians per second is 2π times the frequency in hertz. For example, 60 Hz corresponds to $2\pi \times 60 = 377$ rad/s.

Returning to our sinusoidal wave of $V_m \sin \omega t$ or $I_m \sin \omega t$, we should notice that the t in ωt produces a continuous change in the angle, causing the product ωt to increase linearly with time. Of course, ωt represents an angle increasing with time because ω has units of radians per second while t has units of seconds. So, their product has units of radians. For evaluating $\sin \omega t$, however, we usually prefer to have this angle in degrees, as we shall see.

If we knew the value of ω and of V_m and I_m, we could graph

both $v = V_m \sin \omega t$, provided that something like sin 30° or sin 45° made sense to us. So, we will look into this. As may be recalled from trigonometry, the sine is based on the right triangle as shown in Fig. 6-2. In this triangle the angle α is measured in a counterclockwise direction from the horizontal. It is important to remember to measure positive angles in a counterclockwise direction and negative angles in α clockwise direction. The three sides x, y, and h have specific names : x is the side adjacent the angle (α), y is the side opposite the angle, and h is the hypotenuse. By Pythagorean's theorem, $h = \sqrt{x^2 + y^2}$. And by definition, $\sin \alpha = y/h$, $\cos \alpha = x/h$, and $\tan \alpha = y/x$.

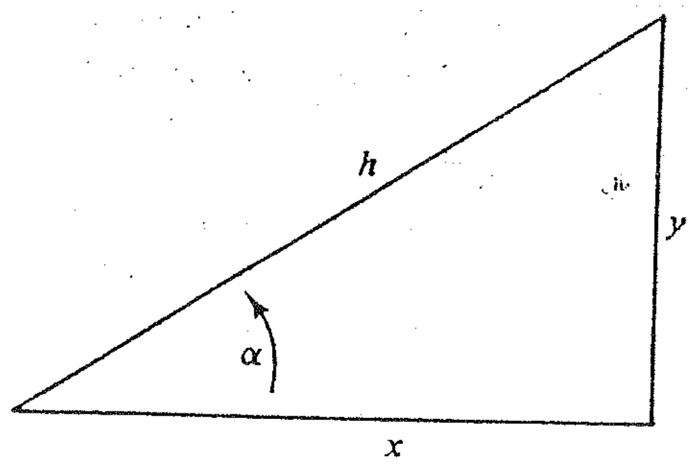


Figure 6-2

The $\sin \omega t$ in $v = V_m \sin \omega t$ and in $i = I_m \sin \omega t$ is the same sine function as in $\sin \alpha$. The only difference is that the angle ωt is not a constant as α is. Because the angle ωt varies with time, we need a different way of showing the triangle for $v = V_m \sin \omega t$ or for $i = I_m \sin \omega t$. Perhaps the best way is, as in Fig. 6-3, on a plane having horizontal and vertical axes that divide the plane into four parts, called quadrants. Here the hypotenuse is a radial line extending from the origin, and the angle ωt is the angle between the horizontal axis and the hypotenuse. As can be seen from drawing a horizontal line from the tip of the hypotenuse to the vertical axis, the height of the hypotenuse is the value of v or i corresponding to the particular angle ωt.

Although so far we have considered only the sine wave, the cosine wave of $v = V_m \sin \omega t$ or $i = I_m \sin \omega t$, illustrated in Fig. 6-4, is equally important. In fact, the sine wave and cosine wave have a common classification of sinusoid. The cosine wave has the same shape as the sine wave it can also be generated using the vertical projection of the hypotenuse in Fig. 6-3. But for this the hypotenuse line must be along the upper vertical axis at $t = 0$ s.

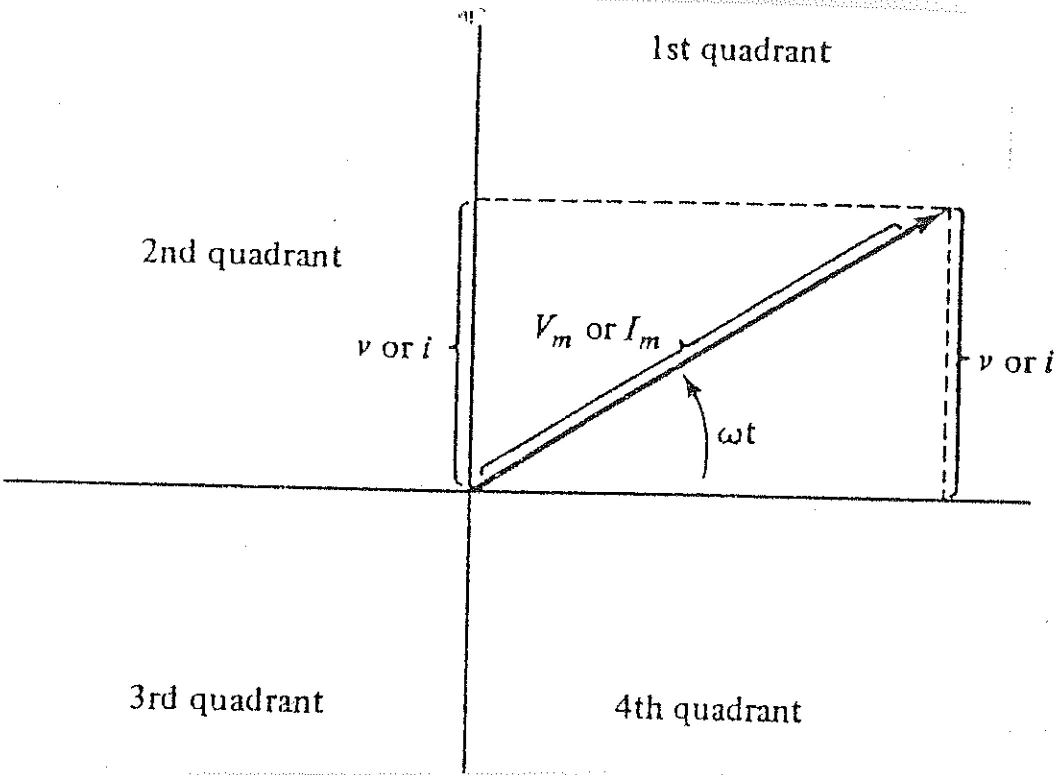


Figure 6-3

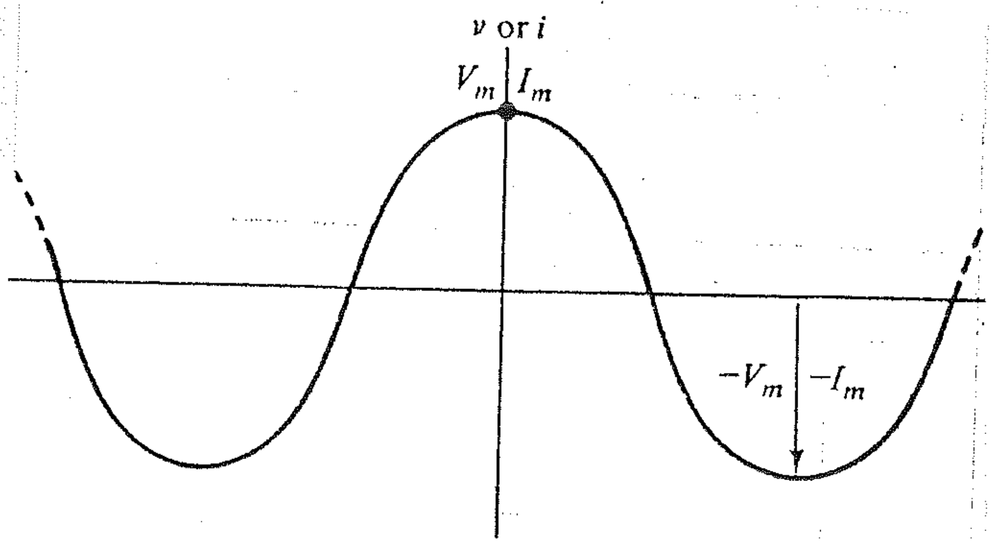


Figure 6-4

Comparing the graphs, we see that all of the cosine values occur 90° ahead of the corresponding sine values. For this reason we say

that the cosine wave leads the sine wave by 90° , or that the sine wave lags the cosine wave by 90°. Also, we can say that they have a phase angle difference of 90°. Normally, this is just called the phase difference. We will discuss this more in the next section.

<u>Phase Relations</u> :

In this section we consider phase differences in general. First, consider how the sinusoid $v_1 = 8 \sin (2\pi t + 30^\circ)$ compares with $v_2 = 8 \sin 2\pi t$. In their mathematical representations, the only difference is an added 30° in the argument of v_1 . The argument for v_1 is $2\pi t + 30^\circ$ and that for v_2 is just $2\pi t$. This 30° difference means that v_1 leads by v_2 by this 30° or , which is the same thing, v_2 lags v_1 by 30° . Also, their phase difference is 30° . We could specify this phase lead, lag, and difference in radians instead of degrees, but specifying in degrees is far more popular.

Now having compared the expressions of v_1 and v_2 , we will compare their graphs. By substituting in different values of t , we can easily get their graphs, shown superimposed in Fig. 6-5, for

comparison. Comparison is aided by having the abscissa units in degrees as illustrated, although these units could be in radians or seconds. It is important to observe here that v_1 reaches its peaks and all other values 30° ahead of or in advance of v_2. This verifies and shows that v_1 leads v_2 by 30°.

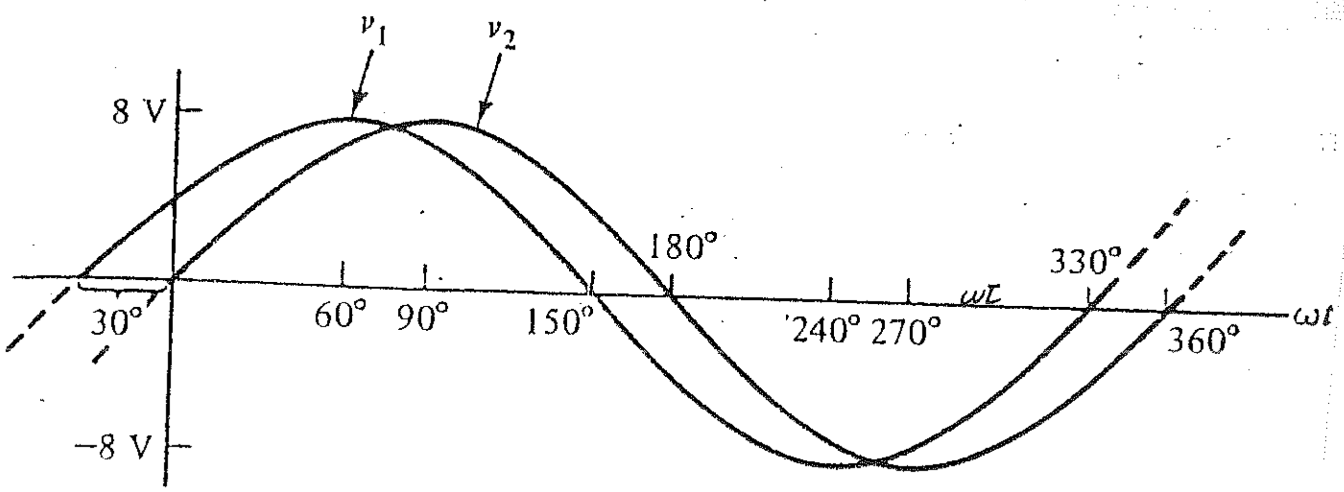


Figure 6-5

For a phase comparison to make any sense, sinusoids must have the same frequency. The reason is that the phase angles of sinusoids of different frequencies are simply not comparable because the phase differences continuously change. This sinusoids need not, however, have the same peak values. Our examples so far have had the same peak values only to make clearer the effects of phase difference.

If two sinusoids of the same frequency are zero at the same times and have positive peaks at the same times, the two sinusoids are said to be in phase. The sinusoids in Fig. 6-6(a) are in phase. In contrast, the sinusoids of Fig. 6-6(b) are just the opposite of these in-phase sinusoids. These have zeros at the same times and peaks at the same times, but the peaks are of opposite polarities. Because this corresponds to a phase difference of 180°, these sinusoids are said to be 180°, these sinusoids are said to be 180° out of phase.

The easiest way to find the phase difference between two sinusoids is to take the difference of the angles of their arguments, provided that both sinusoids are sine terms or both are cosine terms and that both have the same signs. If one sinusoid is in sine form and the other in cosine form, we should use either $\sin x = \cos(x - 90^{\circ})$ to convert the sine term to a cosine term or use $\cos x = \sin(x + 90^{\circ})$ to convert the cosine term to a sine term. And if the two sinusoids have different signs, we should preferably convert the negative sinusoid to a positive one by adding or subtracting 180° to or from the phase angle, whichever gives the smaller angle. A common error here is to move the negative sign inside the argument and change the sign of the phase angle. This is not correct! The negative sign is equivalent to a 180° phase shift, and this is different from a change in sign of the phase angle.

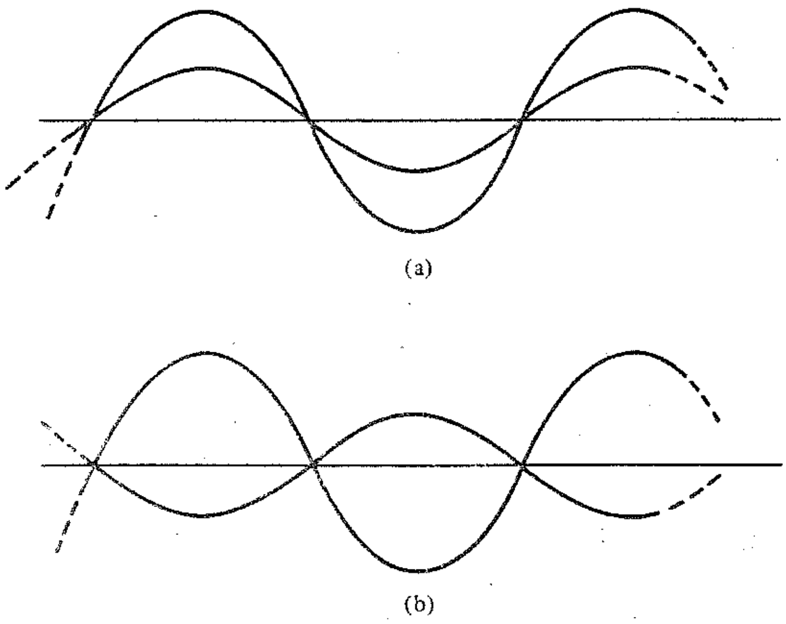


Figure 6-6

Example :

Determine the phase relations for the following :

(a) $v = 10 \sin(377t + 45^\circ)$ V and $i = 20 \sin(377t - 20^\circ)$ A

(b) $v_1 = 4 \sin(60t + 10^\circ)$ V and $v_2 = -8 \sin(60t + 95^\circ)$ V

(c) $v = 5 \cos(20t + 5^\circ)$ V and $i = 7 \sin(30t - 20^\circ)$ A

(d) $v = 5 \sin(6\pi t + 10^\circ)$ V and $i = 4 \cos(6\pi t - 15^\circ)$ A

(e) $i_1 = -6 \sin 4t$ A and $i_2 = -9 \cos(4t + 30^\circ)$ A

Solution :

(a) Because both sinusoids are of the same form and have the same sign we can get the phase difference from the difference of the phase angles. The phase difference is $45^{\circ} - (-20^{\circ}) = 65^{\circ}$, with v leading i .

(b) We need to eliminate the negative sign of v_2 by subtracting 180° from the phase angle. Subtracting is better than adding because it gives a smaller phase angle : $v_2 = -8 \sin(60t + 95^{\circ}) = 8 \sin(60t + 95^{\circ} - 180^{\circ}) = 8 \sin(60t - 85^{\circ})$. The phase difference is $10^{\circ} - (-85^{\circ}) = 95^{\circ}$ with v_1 leading v_2 by this angle.

(c) Since the radian frequency of 20 rad/s of v differs form the 30 rad/s of i, the concept of phase difference does not apply to these sinusoids.

(d) Because v and i are of different sinusoidal form, we should convert one to the form of the other. Selecting to convert i and using the trigonometric identity $\cos x = \sin(x + 90^{\circ})$, we get $i = 4 \sin(6\pi t + 75^{\circ})$. So, i leads v or v lags i by $75^{\circ} - 10^{\circ} = 65^{\circ}$.

(e) Again the sinusoids are of different form. This time, just, to be different, we will convert the sine term using $\sin x = \cos(x - 90^{\circ})$. Then, $i_1 = -6\cos(4t - 90^{\circ})$ and the phase difference is $30^{\circ} - (-90^{\circ}) = 120^{\circ}$ with i_2 leading i_1. Notice that we do not have to eliminate a negative sign because both sinusoids have negative signs.

Figure 6-7 shows the sine wave with three sets of abscissa units : seconds, radians, and degrees. Naturally, we should select just one of these. But showing all three this one time does emphasize the fact that the abscissa units can be any one of these three, whichever is

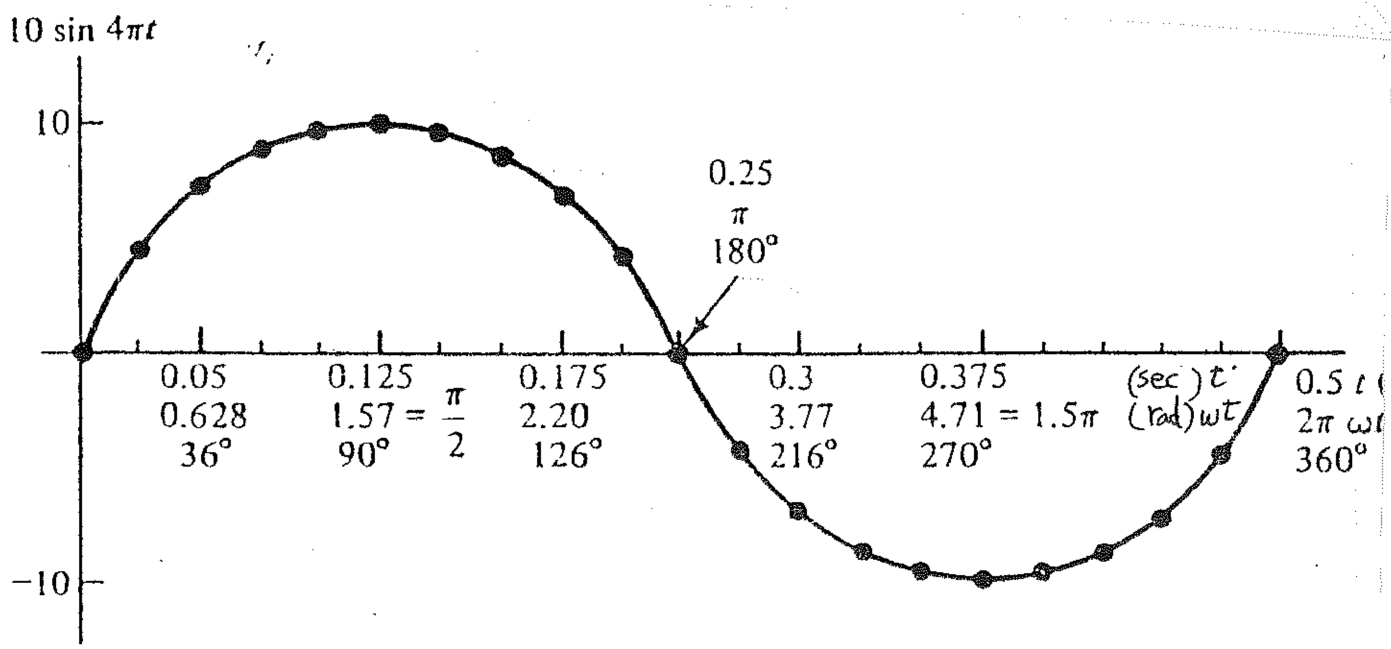


Figure 6-7

most convenient. Seconds and radians are more popular than degrees.

Sinusoidal Average Value :

The average value of a periodic wave is a quotient of area and time, with the area being that between the wave and the abscissa axis over one period and with the time being this period. The area above the abscissa is positive and that below is negative.

Example :

What is the average value of the periodic voltage v of Fig. 6-8 ?

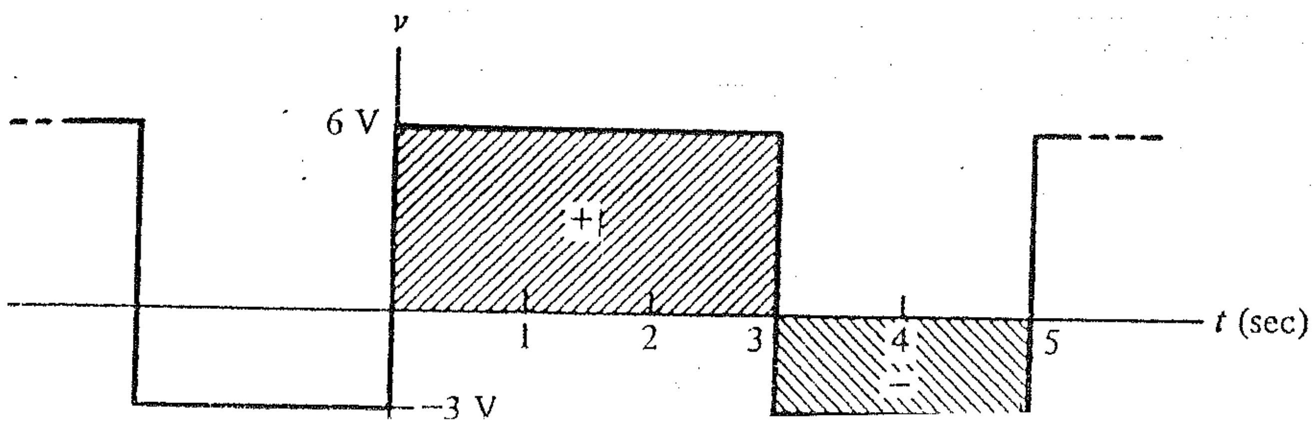


Figure 6-8

Solution :

Although we can find the average value from any period, the period from $t = 0$ s to 5 s is convenient here. The positive area from $t = 0$ s to $t = 3$ s is $6 \times 3 = 18$ V.s . The area from $t = 3$ s to $t = 5$ s, being below the abscissa, is negative: $-3 \times 2 = -6$ V.s . The total area for this period is the sum of these two areas : $18 + (-6) = 12$ V.s. This divided by the period of 5 s produces an average of

$$\frac{12}{5} = 2.4 \text{ V.}.$$

The average value of any sinusoid is zero because over a period the positive area and negative area cancel in the sum of the two areas.

Effective Value :

Although sinusoidal voltages and currents vary continuously with time, it is convenient to give them specific values based on some property. In fact, such specific values are desirable for any periodic voltage or current. But how do we select a specific value ? Average value will not do because the most important periodic function, the sinusoid, has a zero average value. Peak value may seem better, but

then a periodic waveform that is zero for almost all its period and then jumps to a large value for a short time will be classified the same as a dc wave having this high value for all time.

What scientists agreed to was a value based on the equivalent dc heating value. And they called this the effective value of the periodic voltage or current. This effective value of a periodic voltage or current equals the value of a dc voltage or current that would produce the same average power loss in a resistor that the periodic voltage or current would. We will now consider how to calculate this effective value.

The energy, dissipated by the DC current I_{eff} passing through a resistance R during a time T, is I^2_{eff} RT. This energy equals the energy dissipated by the AC current during a time T. Then :

$$I_{eff}^2 RT = \int_0^T i^2(t) R \, dt$$

or

$$I_{eff} = \sqrt{\frac{1}{T} \int_0^T i^2(t) \, dt}$$

The effective value is called the root mean square value. This is easily shown in the above equation. It is abbreviated as I_{rms} or I_{eff}.

To get I_{eff} for a sinusoidal current given by :

$$i = I_m \sin \omega t$$

Then ,

$$I_{eff} = \sqrt{\frac{1}{T} \int_0^T I_{max}^2 \sin^2 \omega t \; dt}$$

$$= \sqrt{\frac{1}{T} \int_0^T \frac{I_{max}^2}{2} (1 - \cos 2 \omega t) \; dt}$$

$$= I_{max} / \sqrt{2} = 0.707 \quad I_{max}$$

Similarly, it can be proved that the effective value of a sinusoidal voltage is given by :

$$V_{eff} = \frac{V_m}{\sqrt{2}} = 0.707 \; V_m$$

which specifies that the effective value of a sinusoidal voltage or

current is its peak value divided by the square root of 2. In this derivation, notice that R divides out. Consequently, the effective value is independent of R.

A mathematical method of finding the effective value of any periodic voltage or current, and not just sinusoids, is to :

(1) Square the periodic voltage or current.

(2) Find the average of the square over one period. Another name for this average is the mean.

(3) Take the square root of this average.

Since the voltages at electric outlets are sinusoids, sinusoidal effective values are used for the voltage specifications of electrical appliances. For example, an electric stove may require 220 V, 50 Hz, AC. This 220 V is the effective value of the voltage required, which is the voltage at the usual household electric outlets. The required frequency of 50 Hz is the Egyptian standard for electric power generation. In radians per second this frequency is $\omega = 2\pi(50) = 314$ rad/s. So, the actual voltage required for this stove is $220\sqrt{2}$ sin $(314\,t + \theta) = 311 \sin(314t + \theta)$ V. The peak value is, of course, the

effective value times the square root of 2. Whenever, a sinusoidal voltage or current is specified as a certain value, we should assume that this is the effective value.

Example :

Find the rms value of the periodic wave of Fig. 6-9.

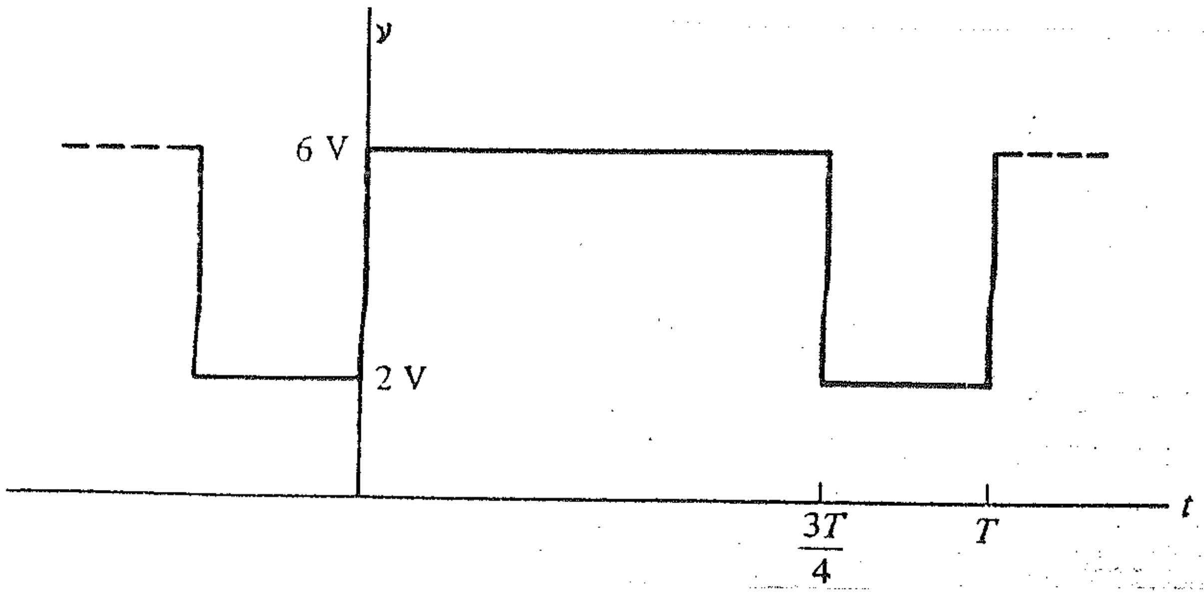


Figure 6-9

Solution :

We first square the wave as shown in Fig. 6-10. Then we find the average value of the squared wave for one period. The way to do this is to find the area under the squared wave for one period and then divide this area by the period, which is the base.

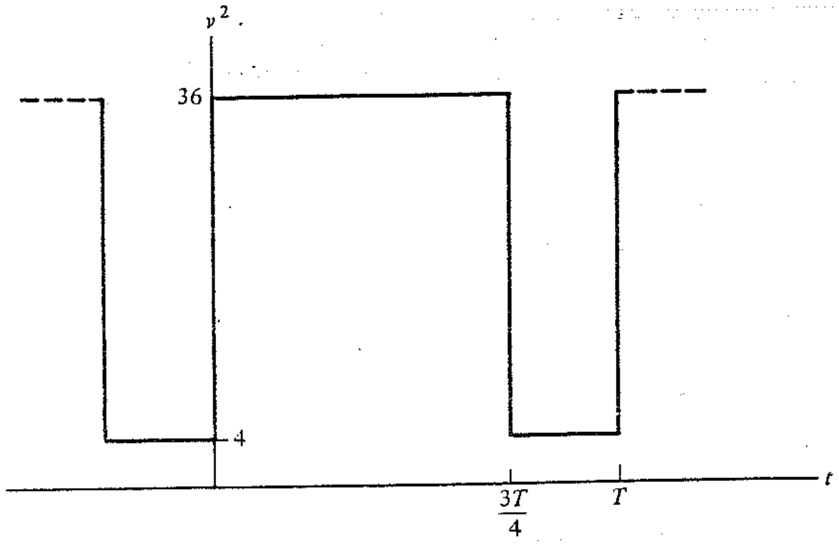


Figure 6-10

$$\textbf{area} = 36 \times \frac{3}{4}T + \frac{1 \times 4}{4}T = 28T$$

Then,

$$\textbf{average value} = \frac{area}{base} = \frac{28\,T}{T} = 28$$

As a last step we take the square root : $V_{rms} = \sqrt{28} = 5.3$ V.

Sinusoidal Currents In Ideal Elements Of Electric Circuits :

Actually Kirchoff's current, voltage law and Ohm's law apply regardless of the voltage and current. They can be dc, square wave, sawtooth, or, in particular, sinusoidal.

The instantaneous values of current and voltage are i and v respectively. At first we shall consider :

Sinusoidal Current In a Resistor :

Consider the circuit of Fig. 6-11, where :

$$i = I_m \sin(\omega t + \theta)$$

but $$v = iR$$

$$v = R\, I_m \sin(\omega t + \theta)$$

$$v = V_m \sin(\omega t + \theta)$$

where $$V_m = R\, I_m$$

The current peak equals the voltage peak divided by the resistance, and the voltage peak equals the current peak times the resistance :

$I_m = V_m/R$ and $V_m = I_m R$. Also, important is the fact that the resistor current and voltage are in phase, as shown in Fig. 6-11.

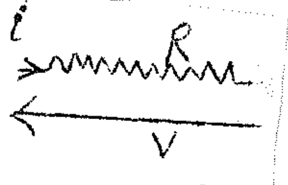
Also, dividing the above equation by $\sqrt{2}$ we get :

$$V = R\,I$$

Where V & I are the effective values of voltage and current respectively. The voltage peak is of no importance because the two curves have different scales : one is in volts and the other in amperes.

Now consider the power. The instantaneous resistor power dissipation varies with time for sinusoidal excitation because the instantaneous voltage and current vary with time, and the power is the product of these two. Specifically,

$$p = vi = [V_m \sin(\omega t + \theta)][I_m \sin(\omega t + \theta)] = V_m I_m \sin^2(\omega t + \theta)$$



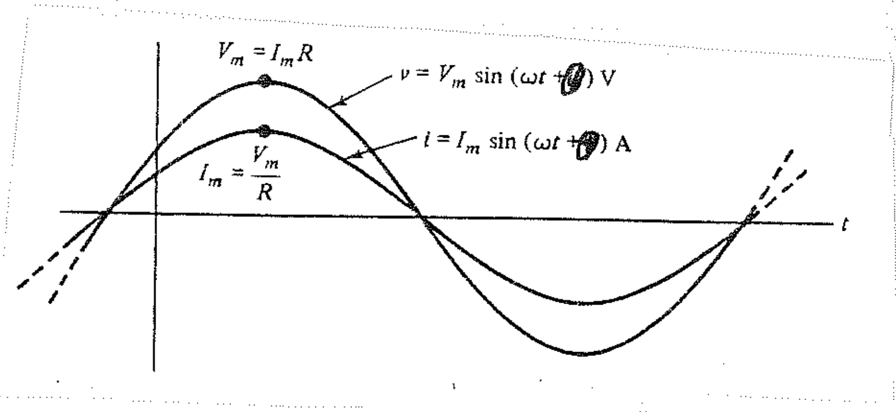


Figure 6-11(a) **Figure 6-11(b)**

From which we see that the power peak is $V_m I_m$, occurring each time $\sin(\omega t + \theta)$ is + 1 or - 1. And the power is zero when this sinusoid is zero.

We can simplify this power expression by using the trigonometric identity $\sin^2 x = (1 - \cos 2x)/2$, in which $x = \omega t + \theta$. The result is :

$$p = \frac{V_m I_m}{2} - \frac{V_m I_m}{2} \cos(2\omega t + 2\theta)$$

which is a constant plus a sinusoid of twice the frequency of voltage or current. With the sinusoidal peak of $V_m I_m/2$ of the second term just equaling the constant of the first term, the instantaneous resistor power can never be negative, since the most negative value of the sinusoid just cancels this constant. This cancellation occurs just

twice each period—at those instants at which the voltage and current are both zero. The power never being negative means that a resistor never delivers power. Instead, a resistor dissipates as heat all the energy it receives.

A plot of this instantaneous power is in Fig. 6-12. It is a sinusoid of $-V_m I_m \cos(2\omega t + 2\theta)/2$ "riding" on top of a constant of $V_m I_m/2$.

The average power supplied to a resistor cover a period is important. Inspection of Fig. 6-12, shows that $P_{av} = V_m I_m/2$ because the superimposed sinusoid is as much above this value as below it. This value is just half the power peak value. There are other ways of

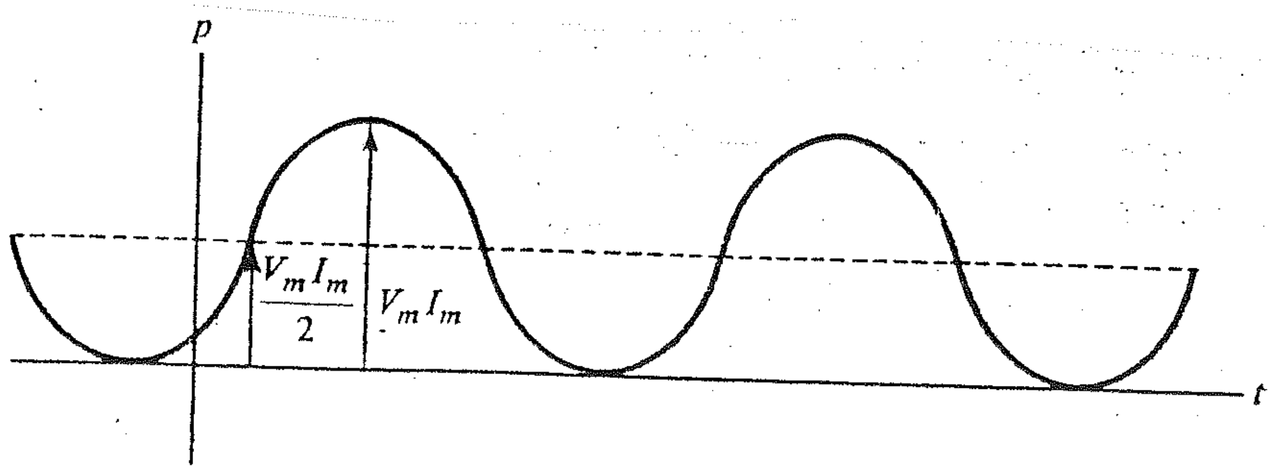


Figure 6-12

expressing this average value, which we can get from $V_m = I_mR$: $P_{av} = V_m^2/R = I_m^2R$. The average power of $V_m I_m/2$ is also apparent from the basic instantaneous resistor power equation, because the second term, the sinusoidal term has an average value of zero, leaving the first term of $V_m I_m/2$ for the average value. To repeat for emphasis, the average power absorbed by a resistor of R ohms is

$$P_{av} = \frac{V_m I_m}{2} = \frac{V_m^2}{2R} = \frac{I_m^2}{2}R$$

The above equation is often written as :

$$P = I^2 R = V^2/R$$

where P is the average power P_{av},

I is the effective value of current I_{eff},

and V is the effective value of voltage V_{eff}.

Example :

A sinusoidal voltage of 10 V peak value is impressed across an 8 Ω resistor. What is the average power dissipated in the resistor ?

Solution :

Here , $V_{eff} = V_m / \sqrt{2} = 10 / \sqrt{2}$. So,

$$P_{av} = \frac{V_{eff}^2}{R} = \frac{(10/\sqrt{2})^2}{8} = 6.25 \text{ W}$$

Example :

A 20 Ω resistor carries a current of 8 cos (377t + 30°) A. What is the average power dissipated in the resistor ?

Solution :

With the current specified, the most convenient formula is $P_{av} = I^2_{eff} R$. For this current $I_{eff} = 8 / \sqrt{2}$. So $P_{av} = (8 / \sqrt{2})^2 \times 20 = 640$ W. Notice that the frequency and phase angle of the current did not enter into this calculation.

Sinusodial Current In An Inductor :

In Fig. 6-13 a sinusoidal current source **provides current to an inductor. What is the inductor voltage ? To find this voltage we can use the basic inductor current- voltage equation $v = L\, di/dt$, which is valid regardless of the current waveshape.**

Here,

$$v = L\frac{di}{dt} = L\frac{d}{dt}\left[I_m \sin(\omega t + \theta)\right] = LI_m \frac{d}{dt}\left[\sin(\omega t + \theta)\right]$$

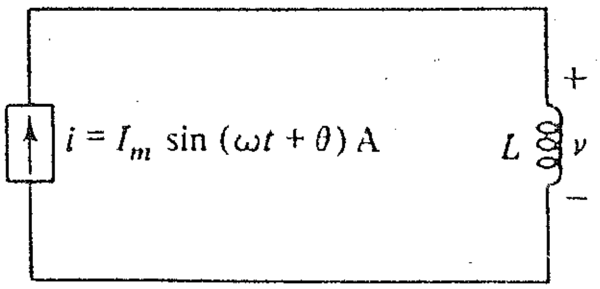


Figure 6-13

The next step requires the derivative of a sinusoidal. From calculus this is :

$$\frac{d}{dt}\left[\sin(\omega t + \theta)\right] = \omega \cos(\omega t + \theta)$$

So,

$$v = LI_m \frac{d}{dt}\left[\sin(\omega t + \theta)\right] = \omega\, LI_m \cos(\omega t + \theta)$$

What is the significance of this result ? One very important fact is that the sinusoidal inductor current produces a sinusoidal inductor voltage. Further, the sinusoidal have the same frequency. Also, because we could as well have considered the inductor voltage as being applied and the inductor current the response, it follows that a sinusoidal inductor voltage produces a sinusoidal current of thee same frequency.

The fact that a sinusoidal excitation of an inductor produces a sinusoidal response of the same frequency may not seem unexpected because it is also true of a resistor, as we have discussed. In general, for a linear resistor an excitation of a certain waveshape produces a response of the same waveshape. But for an inductor this is rare. A square-wave inductor voltage does not produce a square-wave inductor current; a sawtooth inductor voltage does not produce a sawtooth inductor current; a triangular inductor voltage does not produce a triangular inductor current, and so on. So, it is really unusual for an inductor excitation and response to have the same wave shape. In fact, a sinusoid is about the only practical wave for which this is true.

Another important fact from $v = \omega L I_m \cos(\omega t + \theta) = V_m \cos(\omega t + \theta)$ is that the peak inductor voltage is ωL times the peak inductor current : $V_m = \omega L I_m$ and $I_m = V_m/\omega L$. Compare these with the relations $V_m = RI_m$ and $I_m = V_m/R$. Clearly, an inductor has a current limiting action similar to that of a resistor, with ωL corresponding to R. Because of this correspondence, this ωL has a name : inductive reactance, and a quantity symbol, X_L .

$$X_L = \omega L$$

Dividing both sides of the equation relating V_m and I_m by $\sqrt{2}$ we get :

$$V = \omega L I$$

Where V and I are the effective values of voltage and current respectively.

Being the ratio of a voltage to a current, inductive reactance has the unit of ohm, just as does resistance.

Notice that this reactance, this current-limiting property, is not only proportional to inductance but also to frequency. The greater the sinusoidal frequency, the greater reactance. Resistance, in contrast, is not a function of frequency.

Observe from ωL that as the frequency approaches zero, the inductor approaches a short circuit ($X_L = \omega L \to 0$), in agreement with dc results. (A dc voltage or current is sometimes considered to be a sinusoid of zero frequency). At the other frequency extreme, as the frequency gets large and approaches infinity, the inductor approaches an open circuit ($X_L = \omega L \to \infty$).

The third and last important observation regarding inductor voltage is that it leads the inductor current by 90°. This is apparent from comparing $v = \omega L I_m \cos(\omega t + \theta)$ and $I = I_m \sin(\omega t + \theta)$. Both sinusoids have the same argument of $\omega t + \theta$, but the voltage is the cosine of this argument and the current is the sine of it. Of course, for the same time argument, a cosine leads a sine by 90°. This fact is important enough to repeat. For sinusoids, the inductor voltage leads the inductor current by 90° or, alternatively, the inductor

current lags the inductor voltage by 90°. Fig. 6-14 illustrates this phase difference. The dashed vertical lines are at times for which the 90° phase difference is most obvious.

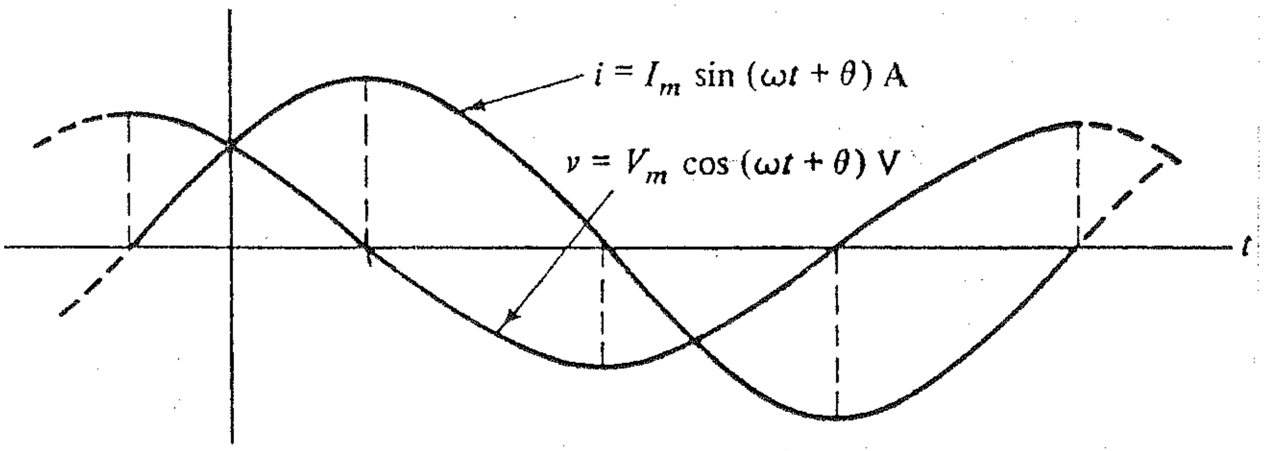


Figure 6-14

Example **:**

A voltage $v = 100 \sin(\omega t + 30^\circ)$ V is across a 2 H inductor. What is the inductor current for: (a) $\omega = 5$ rad/s and for (b) $\omega = 50$ rad/s ?

Solution **:**

(a) For $\omega = 5$ rad/s the inductor resistance is $X_L = \omega L = 5 \times 2 = 10\ \Omega$

This divided into the voltage peak gives the current peak: I_m = 100/10 A. The only other quantity needed is the phase angle. And we can get this from the fact that the current lags the voltage by 90°. So,

$$i = 10 \sin (5t + 30^\circ - 90^\circ) = 10 \sin (5t - 60^\circ) \text{ A}$$

(b) For ω = 50 rad/s the resistance is $X_L = 50 \times 2 = 100\ \Omega$, giving a current peak of $I_m = 100/100 = 1$ A. This with the 90° phase lag result in :

$$i = 1 \sin (50t + 30^\circ - 90^\circ) = \sin (50t - 60^\circ) \text{ A}$$

Notice that the current peak decreased from 10 A to 1 A with an increase of frequency from 5 to 50 rad/s.

Example :

A 0.1 H inductor has a current of $I = 15 \cos (20t + 10^\circ)$ A. What is the inductor voltage ?

Solution :

The reactance is $X_L = \omega L = 20 \times 0.1 = 2\ \Omega$. Consequently, $V_m = X_L I_m = 2 \times 15 = 30$ V. Then because v leads i by 90°, $v = 30 \cos (20t + 10^\circ + 90^\circ) = 30 \cos (20t + 100^\circ)$ V.

Now consider the power absorbed by an inductor having a voltage of $v = V_m \cos(\omega t + \theta)$ and a current of $i = I_m \sin(\omega t + \theta)$. The instantaneous power p is the product of instantaneous voltage and current :

$$P = vi = [V_m \cos(\omega t + \theta)][\, I_m \sin(\omega t + \theta)] = V_m I_m \cos(\omega t + \theta) \sin(\omega t + \theta)$$

From the trigonometric identity $\cos x \sin x = (\sin 2x) / 2$ and for $x = \omega t + \theta$, this result simplifies to :

$$p = \frac{V_m I_m}{2} \sin(2\omega t + 2\theta) = V_{eff} I_{eff} \sin(2\omega t + 2\theta)$$

This instantaneous power absorbed by inductor is sinusoidal at twice the frequency of either voltage or current. Being sinusoidal, its average is zero—$P_{av} = 0$ W—because a sinusoid has a zero average over a period. To repeat for emphasis, a sinusoidally excited inductor absorbs zero average power.

This instantaneous power has a peak or maximum value P_m or $V_m I_m/2$ that with the substitution of $V_m = \omega L I_m$ becomes :

$$P_m = \frac{\omega\, \mathrm{LI}_m^2}{2} = \mathrm{I}_{\mathrm{eff}}^2 \omega\, \mathrm{L} = \mathrm{I}_{\mathrm{eff}}^2\, \mathrm{X}_\mathrm{L}$$

The quantity $I_{eff}^2\, \mathrm{X}_\mathrm{L}$ is called reactive power because of its similarity to I^2R. We can get another expression for reactive power, which has the symbol Q, from the substitution $I_{eff} = V_{eff} / X_L$. The result is $Q = V^2_{eff} / X_L$. Although an inductor does not dissipate this reactive power, which is actually the peak power absorbed, and inductor requires current for this power, and it is this current that cases problems.

Now consider the energy absorbed by an inductor. Because a sinusoidally excited inductor absorbs zero average power, it does not dissipate energy. It does, however, alternating absorb and deliver energy as is evident from the instantaneous energy formula :

$$w = \frac{1}{2}\mathrm{Li}^2 = \frac{1}{2}\mathrm{LI}_m^2 \sin^2(\omega\, t + \theta) = \frac{1}{4}\mathrm{LI}_m^2\left[1 - \cos\left(2\,\omega\, t + 2\theta\right)\right]$$

The last step is from the trigonometric identity $\sin^2 x = (1 - \cos 2x)/2$ with $x = \omega t + \theta$. Because the cosine term cannot be grater than 1, the quantity in brackets is never negative, and so the energy absorbed is never negative. Also, each time that $\cos(2\,\omega t + 2\theta) = -1$, the energy stored has a peak of $LI^2_m/2$. This occurs twice each period of the

current, once for $I = I_m$ and the other for $I = - I_m$. Similarly, an inductor has zero energy twice each cycle. This occurs each time $i = 0$.

As should be apparent from $w = Li^2/2$, as the inductor current increases in magnitude, the inductor absorbs energy into its magnetic field. When the current decreases in magnitude, the inductor acts like a source of energy and delivers energy to the circuit from its magnetic field. But, of course, all the energy it delivers, it has previously received in electrical form.

Sinusoidal Current In a Capacitor :

If in the last section we interchange v and i and if we substitute C for L, the same material describes capacitor sinusoidal response. This follows from a comparison of the basic equations, $v = L\,di/dt$ and $i = C\,dv/dt$. For this reason, the following section on capacitor sinusoidal response strongly resembles the inductor material of the last section.

Consider the circuit of Fig. 6-15, in which a voltage source produces a sinusoidal voltage across a capacitor. What is the current ? This current is easy to find from the basic capacitor equation :

$$i = C\frac{dv}{dt} = C\frac{d}{dt}[V_m \sin(\omega t + \theta)] = \omega CV_m \cos(\omega t + \theta)$$

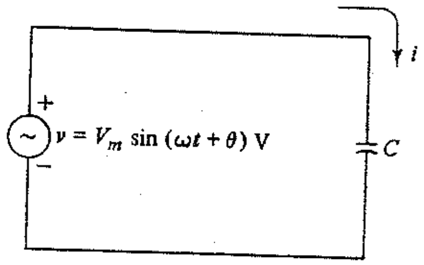


Figure 6-15

This capacitor current is sinusoidal and of the same frequency as the voltage. Further, viewing the capacitor current as the excitation and the capacitor voltage as the response, we find that a sinusoidal capacitor current produces a sinusoidal capacitor voltage of the same frequency. The sinusoidal is about the only practical excitation for which capacitor voltage and current have the same waveshape.

The multiplier ωCV_m is, the peak current :

$$I_m = \omega CV_m = \frac{V_m}{1/\omega C}$$

Comparing this with the resistor relation of $I = V_m/R$, we see that a capacitor has a current limiting action similar to that of a resistor with $1/\omega C$ corresponding to R. Because of this action, some electric circuits books have capacitive reactance defined as $1/\omega C$. But most electrical engineering circuits books include a negative sign and so have capacitive reactance defined as :

$$X_c = -\frac{1}{\omega C} .$$

for reasons having to do with phase shift, as we will study later. Although this is the definition we will adopt, we need not be concerned now about the negative sign. Of course, in this definition, X_c is the quantity symbol for capacetive reactance, and this reactance has the unit of ohm.

Dividing both sides of the equation relating V_m and I_m by $\sqrt{2}$ we get :

$$I = \omega CV$$

or $$V = I / \omega C$$

Where V and I are the effective values of voltage and current respectively.

Notice that $1/\omega C$ is inversely proportional to frequency and capacitance. Consequently, the greater the capacitance or the greater the frequency, the less the reactance, and so the greater current for the same voltage. At the extreme of frequency as the frequency approaches zero and becomes more and more like dc, $1/\omega C$ approaches infinity. This means that a capacitor acts more and more like an open circuit, in agreement with our dc results. On the other extreme, as the frequency gets very large, $1/\omega C$ approaches zero, which means that the capacitor approaches a short circuit.

From a comparison of $i = \omega C V_m \cos(\omega t + \theta)$ and $v = V_m \sin(\omega t + \theta)$ Clearly the capacitor current leads the capacitor voltage by 90°. Or, the capacitor voltage lags the capacitor current by 90°. This lead and lag are important to remember.

__Example__ **:**

A 0.01 μF capacitor has a voltage $v = 100 \sin(\omega t + 30^{\circ})$ V. Find the capacitor current for (a) $\omega = 1000$ rad/s and for (b) $\omega = 10^{7}$ rad/s.

Solution :

(a) For ω = 1000 rad/s , $\omega C = 1000 \times 10^{-8} = 10^{-5}$. This times the voltage peak produces a current peak of :

$$I_m = 10^{-5} \times 100 = 10^{-3} = 1 \text{ mA}$$

Then from the fact that capacitor current leads capacitor voltage by 90°, we get $i = \sin(1000t + 30° + 90°) = \cos(1000t + 30°)$mA.

(b) For $\omega = 10^7$ rad/s, $\omega C = 10^7 \times 10^{-8} = 0.1$, producing a current peak of $I_m = 0.1 \times 100 = 10$ A. Consequently, $i = 10 \sin(10^7 t + 30° + 90°)$ $= 10 \cos(10^7 t + 30°)$ A. Significantly, the current peak increased from 1 mA to 10 A with the increase of frequency from ω = 1000 to 10^7 rad/s.

Example :

A 1 μF capacitor carries a current of $i = 2\cos(1000t + 30°)$ A. What is the capacitor voltage ?

Solution :

For $C = 1\ \mu F$ and $\omega = 1000$ rad/s ,

$$\frac{1}{\omega C} = \frac{1}{1000\ x 10^{-6}} = 1000\ \Omega$$

So, the peak capacitor voltage is :

$$V_m = \frac{1}{\omega C}\ x\ I_m = 1000\ x\ 2 = 2000\ V$$

From this peak voltage and the fact that capacitor voltage lags capacitor current by 90^o , we get a capacitor voltage of :

$$V = 2000 \cos(1000t + 30^o - 90^o) = 2000 \cos(1000t - 60^o)\ V$$

Now consider the power absorbed by a capacitor with a voltage $v = V_m \sin(\omega t + \theta)$ and a current $i = I_m \cos(\omega t + \theta)$. The instantaneous power absorbed is :

$$P = vi = [V_m \sin(\omega t + \theta)][I_m \cos(\omega t + \theta)] = V_m I_m \sin(\omega t + \theta)\cos(\omega t + \theta)$$

Which simplifies to :

$$p = \frac{V_m I_m}{2} \sin(2\omega t + 2\theta) = V_{eff} I_{eff} \sin(2\omega t + 2\theta)$$

So, the instantaneous power absorbed by a capacitor is sinusoidal of twice the frequency of either voltage or current. Because the instantaneous power is sinusoidal, its average is zero : $P_{av} = 0$ W. To repeat, a sinusoidally excited capacitor absorbs zero average power.

This instantaneous power has a peak or maximum value P_m of $V_m I_m /2$, substitution of $V_m = I_m/\omega C$ into which produces :

$$P_m = \frac{I_m^2}{2\omega C} = \frac{I_{eff}^2}{\omega C} = \frac{-I_{eff}^2}{-\omega C} = -I_{eff}^2 X_C$$

The quantity $I_{eff}^2 X_C$ is the reactive power absorbed by a capacitor. Notice that this is the same formula as for an inductor except for X_C instead of X_L. Also, this reactive power is negative for a capacitor because X_C , the reactance, is negative. Capacitive reactive power has the same symbol Q as inductive reactive power : Q = $I_{eff}^2 X_C$. Also, Q = V_{eff}^2 / X_C is the same.

Now consider the energy absorbed by a capacitor. Since a sinusoidally excited capacitor absorbs zero average power, it does not dissipate energy. It does, however, alternately absorb and deliver energy. The instantaneous energy absorbed is :

$$w=\frac{1}{2}CV^2=\frac{1}{2}CV_m^2\sin^2(\omega t+\theta)=\frac{1}{4}CV_m^2\left[1-\cos\left(2\omega t+2\theta\right)\right]$$

Because the cosine term cannot be greater than 1, the quantity in brackets is never negative. Also, the peak energy stored is $CV_m^2/2$, which occurs each time the cosine term is – 1. These are the times at which the voltage is at its peaks, either positive or negative. The capacitor returns all this energy to the circuit each time the capacitor voltage is zero. As the capacitor voltage increases in magnitude, the capacitor absorbs increasing amounts of energy amounts of energy. And as the capacitor voltage decreases in magnitude, the capacitor returns energy to the circuit and in so doing acts like a source.

With this completion of our capacitor discussion, we now know how resistors, inductors, and capacitors individually respond to sinusoidal excitation. The next logical step is the analysis of sinusoidally excited circuits of mixed components.

Sinusoidal Current In RLC Series Combination :

Assume a current $I = I_m \sin(\omega t + \theta)$, passing through the circuit shown in Fig. 6-16. Then :

$$v = v_R + v_L + v_C$$

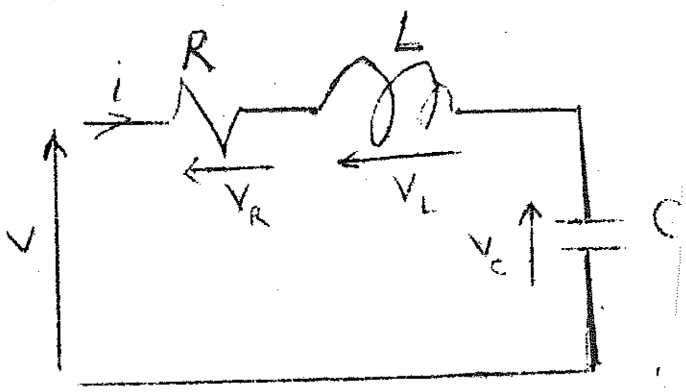


Figure 6-16

As previously mentioned :

$$v_R = R\, I_m \sin(\omega t + \theta)$$

$$v_L = \omega L\, I_m \sin(\omega t + \theta + \pi/2)$$

and $$v_C = -(I_m / \omega C) \sin(\omega t + \theta + \pi/2)$$

Substituting for v_R, v_L and v_C in the equation of voltage v, we get :

$$v = R\, I_m \sin(\omega t + \theta) + \omega L\, I_m \sin(\omega t + \theta + \frac{\Pi}{2}) - \frac{I_m}{\omega C} \sin(\omega t + \theta + \frac{\Pi}{2})$$

or $$v = (R \sin(\omega t + \theta) + (\omega L - \frac{1}{\omega C}) \cos(\omega t + \theta))\, I_m$$

or $$v = (R \sin(\omega t + \theta) + X \cos(\omega t + \theta))\, I_m$$

where :

$$x = x_L + x_C = \omega L - \frac{1}{\omega C}$$

X is known as the reactance of the circuit. The above equation of v can be put in the form :

$$v = I_m \sqrt{R^2 + X^2}\, \sin(\omega t + \theta + \phi)$$

or $$v = I_m \,.\, Z \sin(\omega t + \theta + \phi)$$

or $$v = V_m \sin(\omega t + \theta + \phi)$$

where

$$V_m = Z \,.\, I_m$$

And Z is the impedance of the circuit and is given by :

$$Z = \sqrt{R^2 + (\omega L - 1/\omega C)^2}$$

Also, ϕ is the angle by which the current lags the voltage and equals :

$$\phi = \tan^{-1} X/R$$

$$\therefore \quad R = Z \cos\phi \quad , \quad X = Z \sin\phi$$

Dividing the equation relating V_m , I_m by $\sqrt{2}$ we get :

$$V = Z\,I$$

In conclusion, we can say that the voltage and current, in general circuit containing R , L & C, are related by Ohm's law. Here Z replaces R (in d-c). There is a phase difference ϕ given by $\tan^{-1} X/R$.

SOLVED EXAMPLES

Example (1) :

a. Determine the angle at which the magnitude of the sinusoidal function v = 10 sin 377t is 4 V.

b. Determine the time at which the magnitude is attained.

Solutions :

a) $\alpha_1 = \sin^{-1} \dfrac{v}{E_m} \sin^{-1} \dfrac{4\,V}{10\,V} = \sin^{-1} 0.4 = 23.578°$

However Fig. 6-17 reveals that the magnitude of 4 V (positive) will be attained at two points between 0° and 180°. The second intersection is determined by :

$$\alpha_2 = 180^\circ - 23.578^\circ = 156.422^\circ$$

b) $\alpha = \omega t$ **and so** $t = \alpha/\omega$. **However,** α **, must be in radians.**

Thus,

$$\alpha\,(rad) = \frac{\Pi}{180^\circ}\,(23.578^\circ) = 0.411 \quad \text{rad}$$

$$t_1 = \frac{\alpha}{\omega} = \frac{0.411\,\text{rad}}{377\,\text{rad/s}} = 1.09\ \text{ms}$$

and

$$t_2 = \frac{\alpha}{\omega} = \frac{2.73\,\text{rad}}{377\,\text{rad/s}} = 7.24\ \text{ms}$$

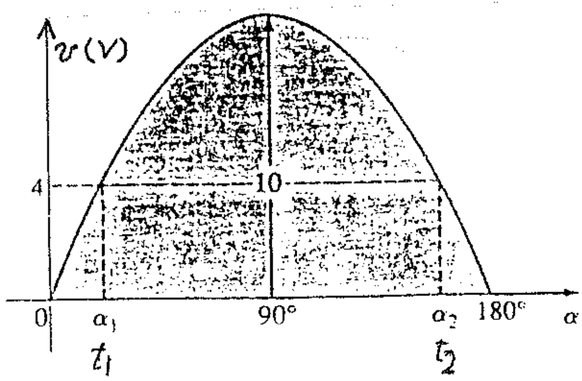


Figure 6-17

Example (2) :

What is the phase relationship between the sinusoidal waveforms of each of the following sets ?

a) $v = 10 \sin (\omega t + 30^{\circ})$

$i = 5 \sin (\omega t + 70^{\circ})$

b) $i = 15 \sin (\omega t + 60^{\circ})$

$v = 10 \sin (\omega t - 20^{\circ})$

c) $i = 2 \cos (\omega t + 10^{\circ})$

$v = 3 \sin (\omega t - 10^{\circ})$

d) $i = - \sin (\omega t + 30^{\circ})$

$v = 2 \sin (\omega t + 10^{\circ})$

e) $i = - 2 \cos (\omega t - 60^{\circ})$

$v = 3 \sin (\omega t - 150^{\circ})$

Solution :

a) i leads v by 40° , or v lags i by 40°.

b) i leads v by 80° , or v lags i by 80°.

c) $i = 2\cos(\omega t + 60^o) = 2\sin(\omega t + 10^o + 90^o)$

$$= 2\sin(\omega t + 100^o)$$

i leads v by 110^o , or v lags i by 110^o.

d) $-\sin(\omega t + 30^o) = \sin(\omega t + 30^o - 180^o)$

$$= \sin(\omega t - 150^o)$$

v leads i by 160^o , or i lags v by 160^o.

Or using :

Note

$$-\sin(\omega t + 30^o) = \sin(\omega t + 30^o + 180^o)$$

$$= \sin(\omega t + 210^o)$$

i leads v by 200^o , or v lags i by 200^o.

By choice

e) $i = -2\cos(\omega t - 60^o) = 2\cos(\omega t - 60^o - 180^o)$

$$= 2\cos(\omega t - 240^o)$$

However, $\cos\alpha = \sin(\alpha + 90^o)$

So that, $2\cos(\omega t - 240^o) = 2\sin(\omega t - 240^o + 90^o)$

$$= 2\sin(\omega t - 150^o)$$

v and i are in phase

Example (3) :

Determine the average value of the waveforms of Fig. 6-18.

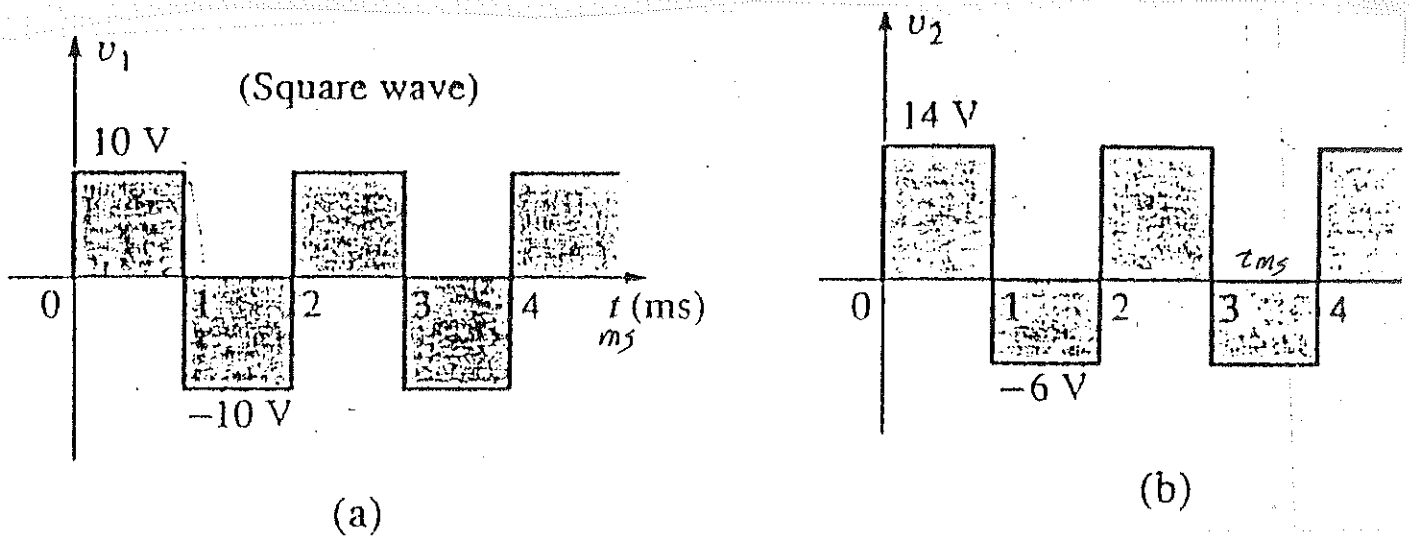


Figure 6-18

Solutions :

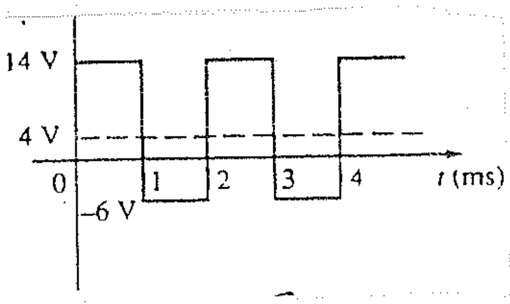
a) By inspection, the area above the axis equals the area below over one cycle, resulting in an average value of zero volts. The average value G is given by :

$$G = \frac{(10\text{ V})(1\text{ ms}) - (10\text{ V})(1\text{ ms})}{2\text{ ms}}$$

$$= \frac{0}{2\text{ ms}} = 0\text{ V}$$

b)

$$G = \frac{14\text{ V} - 6\text{ V}}{2} = \frac{8\text{ V}}{2} = 4\text{ V}$$



as shown in Fig. 6-18.

In reality, the waveform of Fig.6-18(b) is simply the square wave of Fig. 6-18(a) with a dc shift of 4 V. That is :

$$v_2 = v_1 + 4\ V$$

Example (4) :

Find the average values of the following waveforms over one full cycle :

a) Figure 6-19.

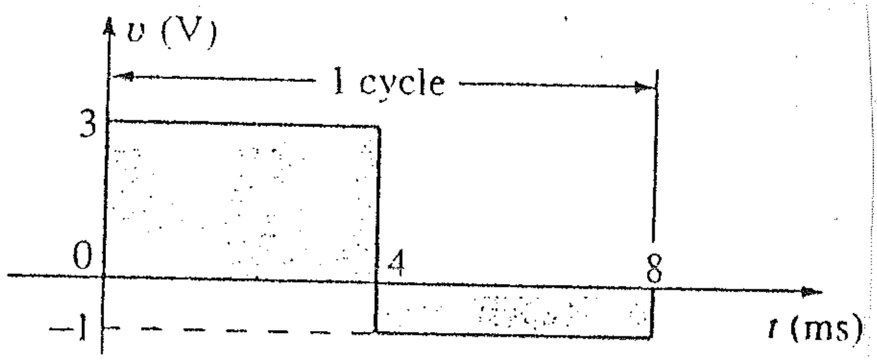


Figure 6-19

b) Figure 6-20.

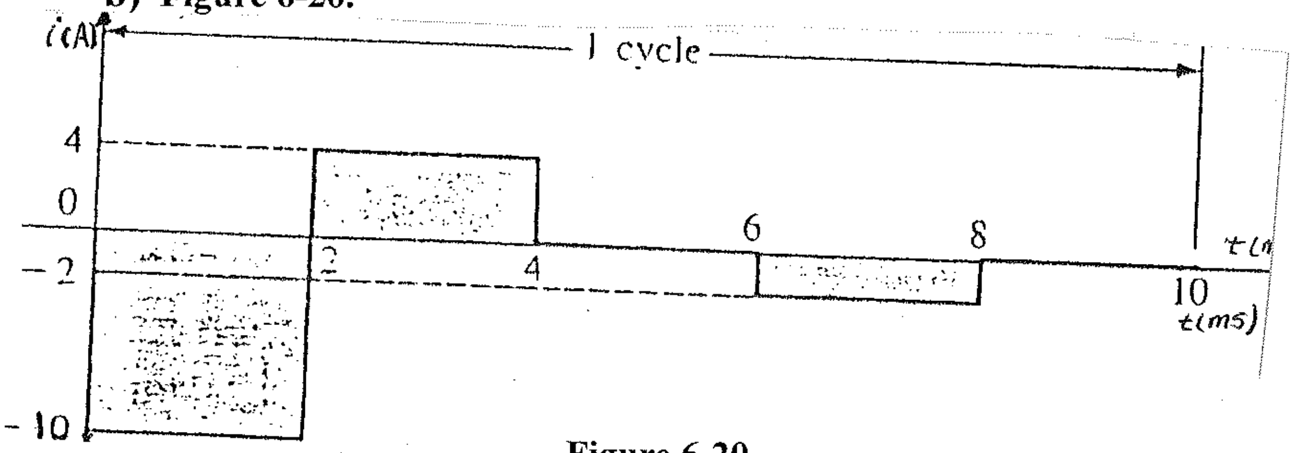


Figure 6-20

Solutions :

a) $$G = \frac{+(3\,\text{V})(4\,\text{ms}) - (1\,\text{V})(4\,\text{ms})}{8\,\text{ms}} = \frac{12\,\text{V} - 4\,\text{V}}{8} = 1\,\text{V}$$

b) $$G = \frac{-10 \times 2 + 4 \times 2 - 2 \times 2}{10} = -1.6\ \text{V}$$

Example (5) :

The 120 V dc source of Fig. 6-21(a) delivers 3.6 W to the load. Determine the peak value of the applied voltage (E_m) and the current (I_m) if the ac source (Fig. 6-21(b)) is to deliver the same power to the load.

Solution :

$$\mathbf{P_{dc} = V_{dc} I_{dc}}$$

and $$I_{dc} = \frac{P_{dc}}{V_{dc}} = \frac{3.6\,\text{W}}{120\,\text{V}} = 30\ \text{mA}$$

$$I_m = \sqrt{2}\,I_{dc} = (1.414)(30\,\text{mA}) = 42.42\ \text{mA}$$

$$E_m = \sqrt{2}\,E_{dc} = (1.414)(120\,\text{V}) = 169.68\ \text{V}$$

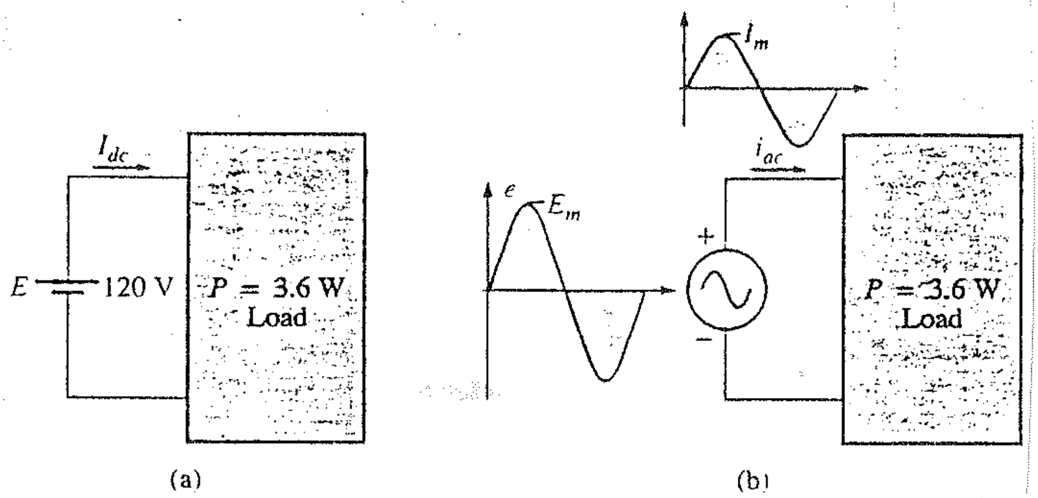


Figure 6-21

Example (6) :

Find the effective or rms value of the waveform of Fig. 6-23.

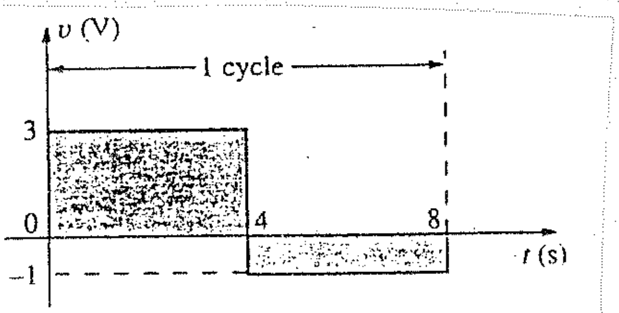


Figure 6-22

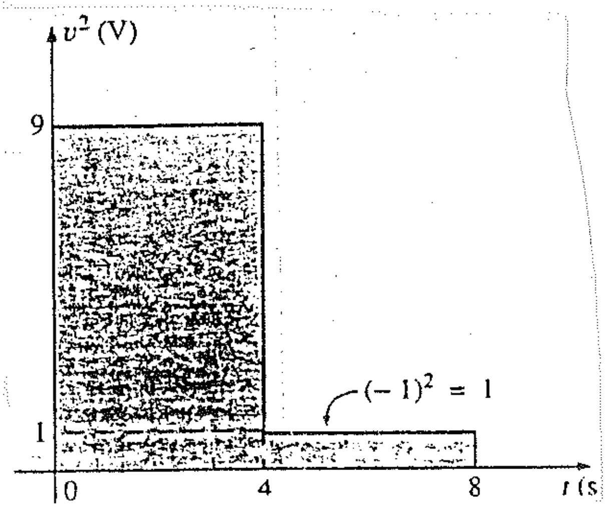


Figure 6-23

Solution :

v^2 (Figure 6-24)

$$V_{eff} = \sqrt{\frac{(9)(4)+(1)(4)}{8}} = \sqrt{\frac{40}{8}} = 2.236 \text{ V}$$

Example (7) :

Calculate the effective value of the voltage of Fig. 6-24.

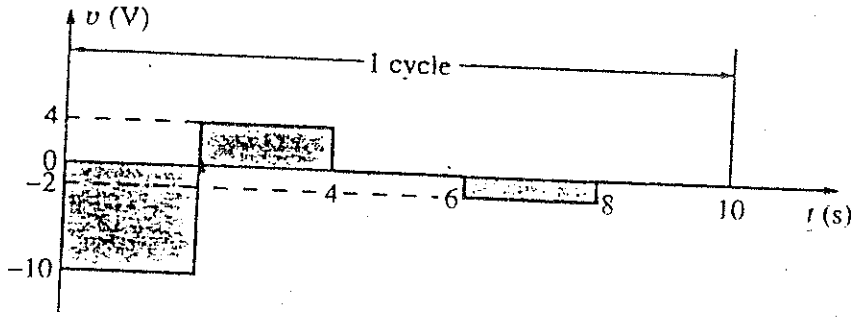


Figure 6-24

Solution :

v^2 (Figure 6-25)

$$V_{eff} = \sqrt{\frac{(1000)(2)+(16)(2)+(4)(2)}{10}} = \sqrt{\frac{240}{10}} = 4.899 \text{ V}$$

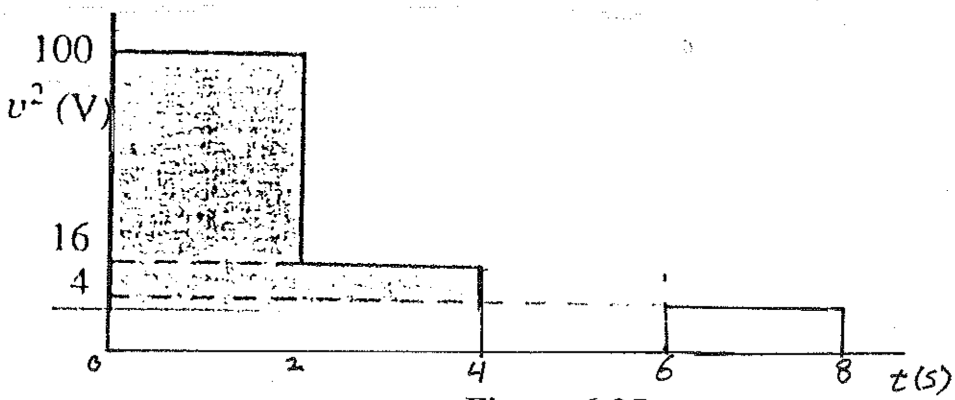


Figure 6-25

Example (8) :

Find the effective and average values of a half wave rectifier output.

Solution :

$$I_{eff}^2 = \frac{\int_o^{\Pi} i^2 \, d(\omega t)}{2\Pi}$$

Let $\quad \mathbf{i = I_m \sin \omega t}$

Then, $$I_{eff}^2 = \frac{\int_o^{\Pi} I_m^2 \sin^2 \omega t \, d(\omega t)}{2\Pi}$$

$$= \frac{I_m^2}{2\Pi} \int_o^{\Pi} \frac{(1 - \cos 2\omega t) \, d(\omega t)}{2} =$$

$$= \frac{I_m^2}{2 \times 2\Pi} \left[\omega t - \frac{\sin 2\omega t}{2} \right]_o^{\Pi} = \frac{I_m^2}{4}$$

$$= \frac{I_m^2}{2 \times 2\Pi} \left[\omega t - \frac{\sin 2\omega t}{2} \right]_o^{\Pi} = \frac{I_m^2}{4}$$

$$\mathbf{I_{eff} = R.\ M.\ S.\ value} = \frac{I_m}{2}$$

The average value is given by :

$$I_{av} = \int_{o}^{\Pi} i \, \frac{d(\omega t)}{2\Pi}$$

$$= \int_{o}^{\Pi} I_m \frac{\sin \omega t \, d(\omega t)}{2\Pi}$$

$$= I_m / \pi$$

Example (9) :

A lamp 100 V & 60 W is connected to AC source of 220 V. It is required to use a series resistor or inductor in order to limit the current of the lamp to its normal value at 50 C/S. Find R & L.

Solution :

The normal current in each case I is given by 60 W/100 V = 0.6 A

In case of a series resistor, its voltage is in phase with the lamp voltage as Fig. 6-27 (a, c) : So,

$$V_R + 100 = 220$$

$$V_R = 120 \text{ V}$$

Where V_R is the resistor voltage

$$\therefore\ R = \frac{120\ V}{0.6\ A} = 200$$

In case of inductor, its voltage V_L leads the lamp voltage by 90°. Their sum must equal the total voltage. This is shown in Fig. 6-27 (b,d), we can write :

$$V_L^2 + 100^2 = 220^2$$

$$V_L^2 = 48\ ,\ 400 - 10{,}000 = 38{,}400$$

$$\mathbf{V_L = 195.8\ V}$$

$$X_L = \frac{195.8\ V}{0.6\ A} = 326.3 = X_L$$

but $X_L = 2\ \pi\ x\ f\ x\ L$

$$\therefore\quad \mathbf{L} = \frac{326.3}{2\,\Pi\, x\, 50} = 1.038\ H$$

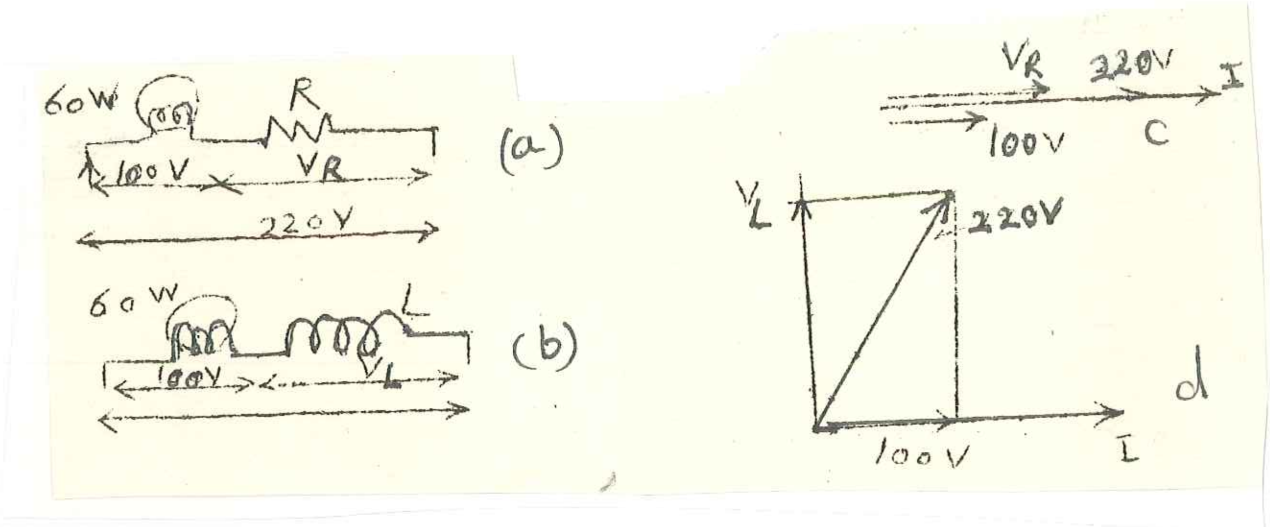


Figure 6-26

Example (10) :

An inductor draws a current of 5 A when it is connected to a 230 V, 50 c/s . If a resistor of 20 Ω resistance is connected in series with the inductor, the current drops to 4 A. Get the resistance and inductance of the inductor.

Solution :

The impedance of the inductor Z is given by :

$$Z = 230 \text{ V} / 5 \text{ A} = 46\ \Omega$$

If R & X are the resistance and reactance of the inductor respectively, then :

$$\sqrt{R^2 + X^2} = 46$$

$$R^2 + X^2 = 2116 \qquad \text{(A)}$$

In case of connecting $20\ \Omega$ in series with the inductor, the new impedance is given by :

$$\frac{230\ V}{4\ A} = 57.5$$

$$\sqrt{(R+20)^2 + X^2} = 57.5$$

$$(R+20)^2 + X^2 = 3306 \qquad \text{(B)}$$

Subtracting equation (A) from equation (B), then :

$$(R+20)^2 - R^2 = 1190$$

$$40\,R + 400 = 1190$$

$$\therefore \qquad R = 19.75 \ \Omega$$

From equation (B), we get :

$$19.75^2 + X^2 = 2116$$

$$X = 41.55 \ \Omega$$

From which we get :

$$L = \frac{X}{2\Pi f}$$

$$= \frac{41.55}{2\Pi \times 50}\ H$$

$$= 0.132\ H$$

Example (11) :

A resistor of 10 Ω is connected in series with capacitor of 400 μF and 60 V AC source. The current passing is 5 A. Get the source frequency f and the phase difference between the current and source voltage.

Solution :

The voltage of the resistor V_R is 5 x 15 = 50 V. It coinsides on the current, while the capacitor voltage V_C lags the current by 90°. This is shown in Fig. 6-28. We get :

$$V_C^2 + V_R^2 = 60^2$$

$$V_C^2 = 60^2 - 50^2 = 1100$$

$$V_C = 33.17 \ V$$

and $$= \frac{33.17\ V}{5\ A} = 6.634 \quad = X_C$$

Figure 6-27

But $$X_C = \frac{1}{2\Pi\ fC}$$

$$\mathbf{6.634} = \frac{1}{2\Pi \times 400 \times 10^{-6} \times f}$$

$$\mathbf{f} = \frac{10^{-6}}{2\Pi \times 400 \times 6.634}$$

= 60 c/s = supply frequency

If ϕ is the phase difference between V & I , Then :

$$\cos\phi = \frac{V_R}{60\ V}$$

$$= \frac{50}{60} = 0.8333$$

$\phi = 33.6^\circ$

PROBLEMS

1) Find the frequency and period of each of the following periodic functions.

[Hint : For parts (c) and (d) use trigonometric identities].

a) $4 \cos (15t + 35^{o})$

b) $6 + 8 \sin (377t - 20^{o})$

c) $4 \cos^{2} 7t$

d) $6 \sin 2t \cos 2t$

2) Find the period and frequency of a sinusoid having a radian frequency of :

a) 6π rad/s

b) 377 rad/s

c) 0.01 rad/s

d) 10^{6} rad/s

3) For the periodic waveform of Fig. 6-32

a) Find the period T.

b) How many cycles are shown ?

c) What is the frequency ?

d) Determine the positive amplitude and peak-to-peak value (think!).

4) Find the phase relations for the following pairs of sinusoids :

a) $v = 8 \sin(20t + 30^{\circ})$ V , $i = 6 \sin(20t - 25^{\circ})$ A

b) $v = 8 \sin(20\pi t + 30^{\circ})$ V , $i = 6 \cos(20\pi t - 35^{\circ})$ A

c) $v_1 = -11 \sin(377t - 45^{\circ})$ V , $v_2 = 23 \cos(377t + 37^{\circ})$ V

d) $i_1 = -3.6 \sin(754t + 15^{\circ})$ A , $i_2 = -7.8 \cos(754t - 35^{\circ})$ A

e) $v = -7.6 \sin(22t - 13^{\circ})$ V , $i = -4.3 \cos(11t + 22^{\circ})$ A

5) Write the analytical expression for the waveforms of Fig. 6-33 with the phase angle in degrees.

6) The sinusoidal voltage $v = 200 \sin(2\pi\, 1000t + 60^{\circ})$ is plotted in Fig. 6-34 Determine the time t_1 .

7) Find the average value of the periodic waveforms of Fig. 6-35 over one full cycle.

8) Find the average values of :

a) $6 - 8 \cos(377t + 10^{\circ})$ A

b) A sawtooth wave with peak of 10.

c) 3 V

d) $4 \cos^2 377t$ V [Hint: Use a trigonometric identity].

e) A periodic voltage that is 10 V for three-fourths of a period and is – 2 V for the remaining one-fourth of a period.

9) Find the effective values of the following sinusoidal waveforms :

a) $v = 20 \sin 754t$

b) $v = 7.07 \sin 377t$

c) $i = 0.006 \sin (400t + 20^{o})$

e) $i = 16 \times 10^{-3} \sin (377t - 10^{o})$

10) What are the average and effective values of the square wave of Fig. 6-36

11) Find the effective value of a periodic voltage that is 10 V for half a period and - 5 V for the second half-period. Then, repeat the calculation for the same first half-period value but for a second half- period value of 5 V instead of - 5 V.

12) For a 2 Ω resistor find the resistor currents corresponding to the following resistor voltages. Also, find the average powers dissipated.

a) $3 \cos 377t$ V

b) $4 \sin (20t - 10^{o})$ V

c) $-6 \cos (754t - 15^{o})$ μV

d) $20 \sin (100t + 45^{o})$ mV

13) For each pair of the following resistor voltages and currents, find the corresponding resistance and the average power dissipated.

a) $v = 20 \cos(377t + 10^{o})$ V and $i = 5 \cos(377t + 10^{o})$ A

b) $v = 3.6 \sin(754t + 10^{o})$ V and $i = 72 \sin(754t + 15^{o})$ A

14) For a 20 mH inductor find the inductor voltages corresponding to the following currents. Assume associated references.

a) $6 \sin 10t$ A

b) $7 \sin(50t + 10^{o})$ A

c) $0.8 \cos(100t + 45^{o})$ A

d) $-50 \sin(60t - 80^{o})$ A

15) For a 2 μF capacitor find the capacitor currents corresponding to the following voltages. Assume associated references.

a) $8 \sin 20t$ V

b) $10 \cos(1000t - 45^{o})$ V

b) $500 \cos(10^{6}t + 55^{o})$ V

d) $1250 \sin(377t - 25^{o})$ V

16. In Fig.6-37 the following readings are obtained

$V_1 = 60$ V, $f_1 = 50$ H_z, $I_1 = 10$ A

$V_2 = 60$ V, $f_2 = 100$ H_z, $I_2 = 6$ A

Where V, f and I are the voltage applied, frequency and measured current respectively. Get R & L.

17. Determine the period, frequency and the average of the saw tooth waveform shown in Fig. 6-38.

18-Find the average value of the periodic waveforms of Fig.6-39.

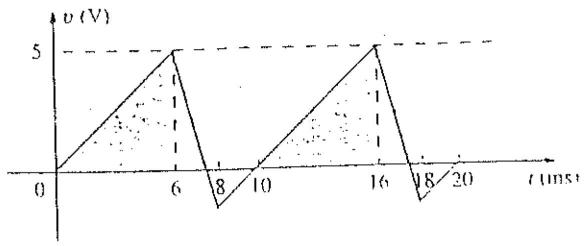


Figure 6-32

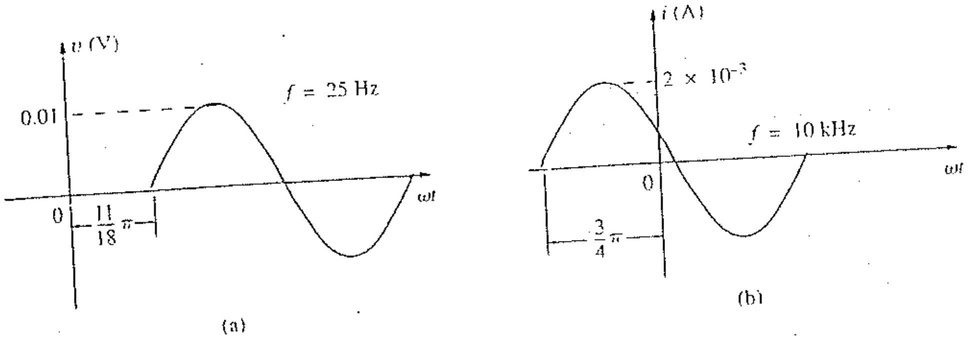


Figure 6-33

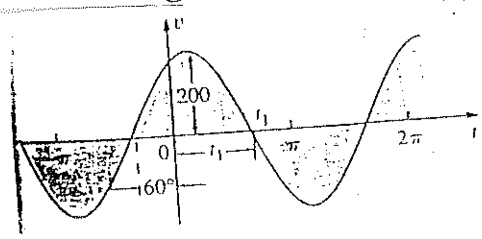


Figure 6-34

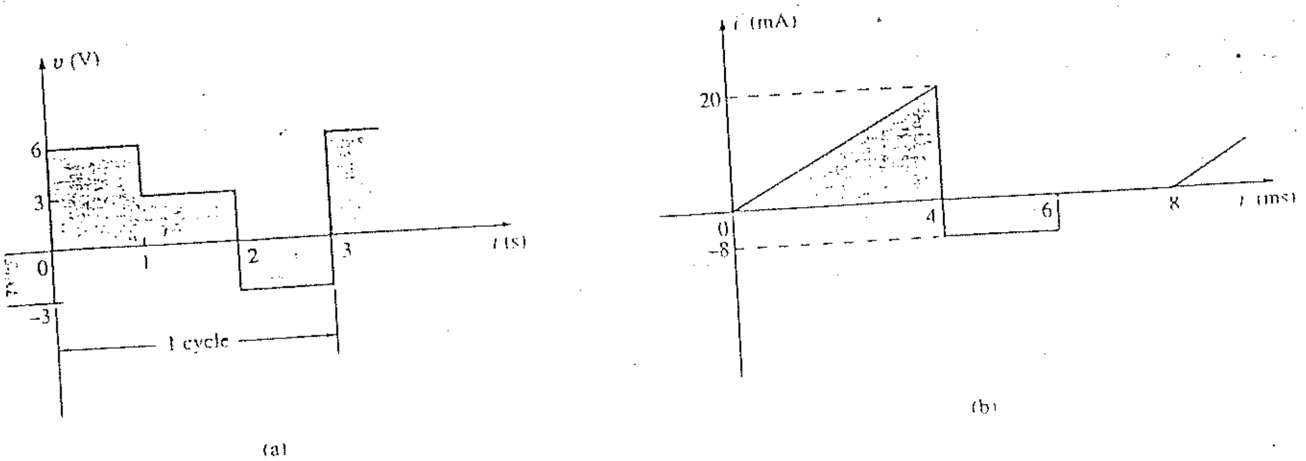


Figure 6-35

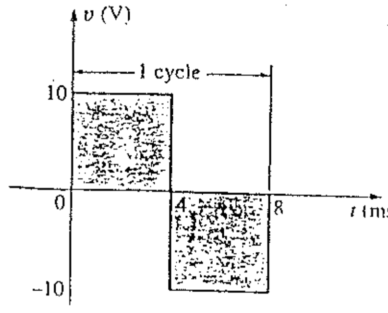


Figure 6-36

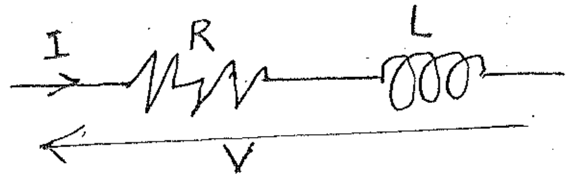


Figure 6-37

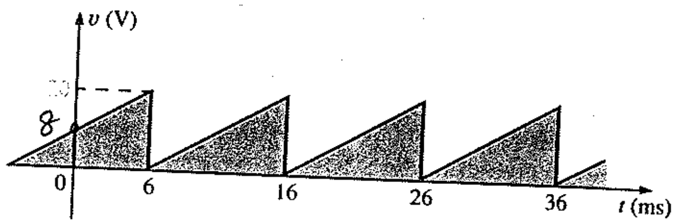


FIG. 6.38

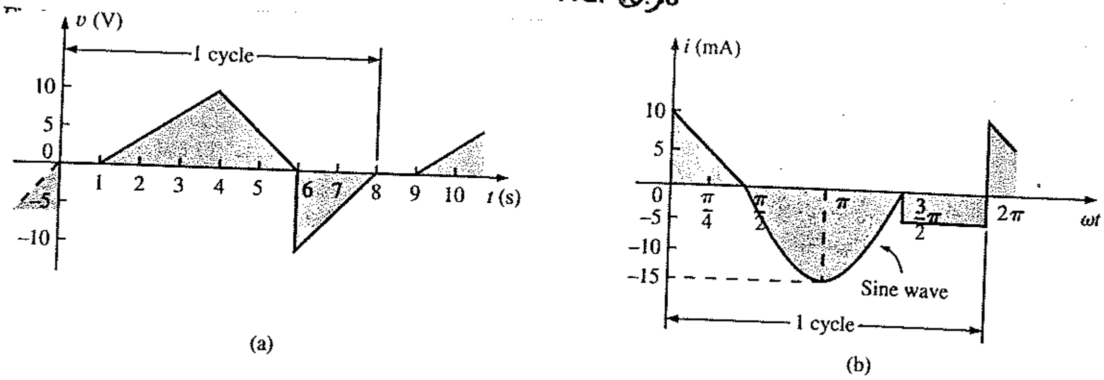


FIG. 6.39

Chapter Seven

COMPLEX ALGEBRA AND PHASORS

Introduction :

How do we analyze sinusoidally excited RLC circuits ? We could, of course, apply KVL and KCL. But they require differential equations and also summing of sinusoids of different phase angles. We can avoid both of these disadvantages by using complex algebra and phasors. As we will study, complex algebra is just a slight extension of the algebra of real numbers, the algebra we know so well. The extension is to complex numbers with their own special rules for operations. A knowledge of complex algebra is essential because we will be transforming sinusoids into complex numbers called phasors, and will be using complex algebra on these phasors.

Imaginary Numbers :

The numbers most familiar to us are real numbers. But these are not the only kind of numbers. There are also imaginary numbers. The name imaginary is misleading because it implies something

fictitious, and this is not true. Imaginary numbers are of just as much essence as real numbers, being different only in the rules they obey.

We need to distinguish imaginary numbers from real numbers because their rules for mathematical operations are different. Unfortunately, there is no one universally accepted way of representing imaginary numbers. In the electrical field, however, the use of the letter j is standard. For example, j3, j2.5, –j6, and –j0.01 are all imaginary numbers. Notice that the j goes to the left of the leftmost digit.

The rules for adding and subtracting imaginary numbers are the same as for real numbers except that the sums and differences are imaginary. To illustrate,

$$j6 + j4 = j10 \qquad j4 - j3 = j1 \qquad j2.5 - j10 = -j7.5$$

The multiplication rules, though, are different. The product of two imaginary numbers is a real number that is the negative of the product we would get if the numbers were real numbers instead. For instance,

$j4 \times j2 = -8 \qquad j6 \times (-j3) = 18 \qquad (-j4) \times (-j2.5) = -10$

This imaginary product rule has led some authors to say that $j = \sqrt{-1}$. Certainly it is true that this produces correct results. For example, $j3 \times j4 = (j)^2 \times 12 = -1 \times 12 = -12$.

One example of when it makes calculations easier is in the multiplication of more than two imaginary numbers. For these calculations we will sometimes use higher powers of j than $j^2 = -1$. So we should know them. They follow directly from $j = \sqrt{-1}$. Some of these are :

$j^3 = j^2 \times j = -j$, $j^4 = j^2 \times j^2 = -1 \times -1 = 1$ and $j^5 = j^4 \times j = 1 \times j = j$.

As illustrations of their use,

$$j2 \times j6 \times j4 = j^3\, 48 = -j48$$

and

$$j2 \times (-j3) \times j5 \times j4 = j^4(-120) = -120$$

It is possible to take the product of a real number and an imaginary number. The product is imaginary and otherwise the same as would be expected with two real numbers. So, 3 x j4 = j12 and 4.2 x (–j6) = –j25.2.

So much for multiplication. We will now consider division. In the division of two imaginary numbers, the quotient is real and the same as would be produced with real numbers. For example,

$$\frac{j6}{j2} = 3 \qquad \text{and} \qquad \frac{j10}{-j100} = -0.1$$

A convenient memory aid is to again treat j as a number and then to divide out the j's :

$$\frac{j20}{j5} = 4$$

The division of imaginary and real numbers is also defined. If the dividend is imaginary and the divisor real, the quotient is imaginary but otherwise the same as for real numbers. Again, a convenient memory aid is to consider j to be a number, as is done in the following examples :

$$\frac{j6}{3} = j2 \qquad \text{and} \qquad \frac{j2.5}{-0.5} = -j5$$

The only difference if the dividend is real and the divisor imaginary is that the result is negative of the above. For example,

$$\frac{6}{j3} = -j2 \qquad \frac{2.5}{-j0.5} = j5 \qquad \frac{16}{j4} = -j4$$

Another way of dividing when the denominator is imaginary is to multiply numerator and denominator by j 1 to make the denominator real, and proceed from there to divide. Making the denominator real is called rationalizing. Two examples follow :

$$\frac{4}{j2} = \frac{4 \times j1}{j2 \times j1} = \frac{j4}{-2} = -j2 \qquad \text{and} \qquad \frac{20}{-j2} = \frac{20 \times j1}{-j2 \times j1} = \frac{j20}{2} = j10$$

Complex Numbers And The Rectangular Form :

A complex number has a real part and an imaginary part. In one form, the rectangular form, it is written as the sum of the real part and imaginary part. By convention the real part is written first. For example, 3 + j4, –7 –j2, –0.5 + j100, and –2 + j0.6 are complex numbers in rectangular form.

A more general way of considering a complex number is as a point in the complex plane. As illustrated in Fig. 7-1, the complex plane has two perpendicular axes, one horizontal and one vertical. The horizontal axis is the real axis and the vertical axis is the imaginary axis (or j axis), as labeled. Both axes must have the same scale.

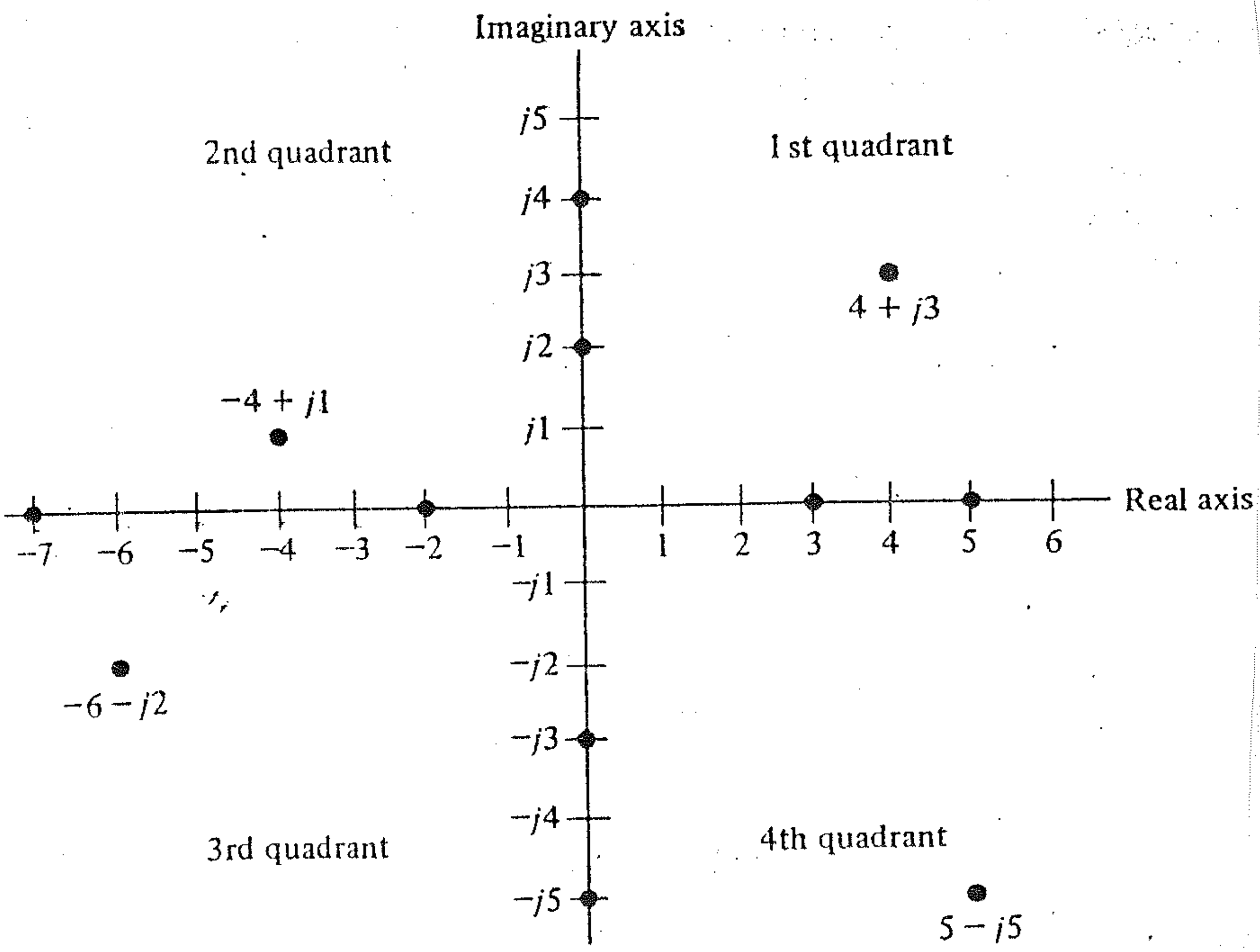


Figure 7-1

The axes divide the complex plane into four quadrants, as labeled. Real parts of complex numbers are positive to the right of the vertical axis in the first and fourth quadrants. They are negative to the left of the vertical axis in the second and third quadrants. Imaginary parts are positive above the horizontal axis in the first and second quadrants, and negative below the horizontal axis in the third and fourth quadrants.

Complex numbers with zero imaginary parts are purely real and so are points on the real axis. Figure 7-1 has four of these points : -7, -2, 3, and 5. Complex numbers with zero real parts, instead, are purely imaginary, and so are points on the imaginary or j axis. Figure 7-1 shows four of these points : $-j5$, $-j3$, $+j2$, and $j4$.

Other complex numbers have nonzero real and imaginary parts. Consequently, they correspond to points off the axes. The real part of each number gives the position to the right or to the left of the vertical axis and the imaginary part gives the position above or below the horizontal axis. Figure 7-1 shows four of these points, one in each quadrant.

The rectangular form is the only practical form for general addition and subtraction. As to be expected, addition and subtraction are applied separately to the real and imaginary parts.

Example :

Find the following :

(a) $(3 + j4) + (-6 + j8)$

(b) $(-6 + j10) + (4 - j7) + (2 - j4)$

(c) $4 + j8 + 5 - j2.4 - 0.6 - j0.8$

(d) $(4.2 - j3.8) - (-3.1 + j2.1)$

Solution :

As mentioned, the procedure is to separately add or subtract the real parts and imaginary parts. As a result,

(a) $(3 + j4) + (-6 + j8) = 3 - 6 + j(4 + 8) = -3 + j12$

(b) $(-6 + j10) + (4 - j7) + (2 - j4) = -6 + 4 + 2 + j(10 - 7 - 4)$

$= 0 - j1 = -j1$

(c) $4 + j8 + 5 - j2.4 - 0.6 - j0.8 = 4 + 5 - 0.6 + j8 - j2.4 - j0.8 = 8.4 + j4.8$ **. This part illustrates that parentheses are unnecessary in adding.**

(d) $(4.2 - j3.8) - (-3.1 + j2.1) = 4.2 - (-3.1) - j(3.8 + 2.1) = 7.3 - j5.9$

To multiply two complex numbers in rectangular form, we multiply the real part of one number times the entire second number, and then the imaginary part times the second number. Last, we add these two results.

Example :

Find the product of $3 + 2j$ **and** $-6 + j1$ **.**

Solution :

We take the real part, 3, of the first number times the second number and the imaginary part, j2, times the second number, and add :

$$(3 + 2j)(-6 + j1) = 3(-6 + j1) + 2j\,(-6 + j1) =$$

$$-18 + j3 - j12 - 2 = -20 - j9$$

As to be expected, the division of complex numbers in rectangular form is somewhat more difficult than multiplication. For division we first place the dividend and divisor in the usual ratio form with the dividend in the numerator and the divisor in the denominator. Then we multiply numerator and denominator by the conjugate of the denominator complex number. The conjugate of a complex number has the same real part but the negative of the imaginary part. As we will see, multiplying a complex number by its conjugate produces a real number equal to the sum of the squares of the real and imaginary parts. As a result of this multiplication, the denominator becomes real, making the division straightforward. This step of making the denominator real is the same rationalizing mentioned in the discussion of the division of imaginary numbers.

Example :

Calculate $(3 + j4) / (1 - j2)$.

Solution :

The denominator $1 - j2$ has a conjugate of $1 + j2$. Multiplying the numerator and denominator by this, we get :

$$\frac{(3 + j4)(1 + j2)}{(1 - j2)(1 + j2)} = \frac{3 + j6 + j4 - 8}{1 + j2 - j2 + 4} = \frac{-5 + j10}{1 + 4} = -1 + j2$$

Notice that multiplying the denominator by its conjugate produced a real number equal to the sum of the squares $(1)^2 + (2)^2$ of its real and imaginary parts, as always happens when multiplying by a conjugate.

When the dividend or divisor is a product of complex numbers, the procedure is to take the product of the numerator complex numbers to reduce the numerator to a single complex number and then do the same thing for the denominator. Then the division is the same as for the quotient of two complex numbers.

Example :

Find $(3 + j4)(-1 + j2) / (3 - j2)(-4 - j5)$.

Solution :

First we multiply :

$$\frac{(3+j4)(-1+j2)}{(3-j2)(-4-j5)} = \frac{-3+j6-j4-8}{-12-j15+j8-10} = \frac{-11+j2}{-22-j7}$$

and then divide in the usual way :

$$\frac{(-11+j2)(-22+j7)}{(-22-j7)(-22+j7)} = \frac{242-j77-j44-14}{(-22)^2+7^2} = \frac{228-j121}{533}$$

$$= \mathbf{0.428 - j0.227}$$

Exponential and Polar Forms :

We now consider the exponential form of complex numbers, and its shorthand version, the polar form. These forms are best for multiplication and division. But, except for one rare case, they are not useful for addition or subtraction.

The general exponential form is $Ae^{j\theta}$, in which A is the magnitude, θ the angle, and e = 2.718 ... , the base of the natural logarithm. Some examples of the exponential form are $4e^{j30^o}$, $-6e^{j60^o}$ and $8e^{j120^o}$.

The polar shorthand for $Ae^{j\theta}$ is $A\angle\theta$. It does not, for convenience, have e or the j. Some specific examples are $4e^{j30^o} = 4\angle 30^o$, $-6e^{j60^o}$ = $-6\angle 60^o$, and $8e^{j120^o} = -8\angle 120^o$. Both forms are equivalent, with the polar form being far more popular simply because it is easier to write.

That a number such as $4e^{j30^o}$ is a complex number becomes evident from Euler's identity : $e^{j\theta} = \cos\theta + j\sin\theta$. From Euler's identity, for example,

$$4e^{j30^o} = 4\angle 30^o = 4(\cos 30^o + j\sin 30^o) = 3.46 + j2$$

and

$$-6e^{j60^o} = -6\angle 60^o = -6(\cos 60^o + j\sin 60^o) = -3 - j5.2$$

Example :

Convert the following complex numbers to rectangular form :

(a) $3 \angle -20^\circ$

(b) $-6 \angle -120^\circ$

(c) $10 \angle -45^\circ$

(d) $120 \angle -180^\circ$

(e) $120 \angle 180^\circ$

(f) $80 \angle -90^\circ$

(g) $80 \angle 90^\circ$

Solution :

From Euler's identity ,

(a) $3 \angle -20^\circ = 3 \cos(-20^\circ) + 3j \sin(-20^\circ) = 2.82 - j1.03$

(b) $-6 \angle -120^\circ = -6 \cos(-120^\circ) + j(-6) \sin(-120^\circ) = 3 + j5.2$

(c) $10 \angle -45^\circ = 10 \cos(-45^\circ) + j10 \sin(-45^\circ) = 7.07 - j7.07$

(d) $120 \angle -180^\circ = 120 \cos(-180^\circ) + j120 \sin(-180^\circ) = -120$

(e) $120 \angle 180^\circ = 120 \cos 180^\circ + j120 \sin 180^\circ = -120$

(f) $80 \angle -90^\circ = 80 \cos(-90^\circ) + j80 \sin(-90^\circ) = -j80$

(g) $80 \angle 90^\circ = 80 \cos 90^\circ + j80 \sin 90^\circ = j80$

Particularly notice that the results of parts (d) and (e) show that a negative sign corresponds to an angle of either 180° or -180°. Put another way, $-1 = 1\angle 180^\circ = 1\angle -180^\circ$. Parts (f) and (g) indicate that an angle of -90° corresponds to multiplication by $-j1$ and that an angle of $+90^\circ$ corresponds to multiplication by $+j1$: $1\angle -90^\circ = -j$ and $1\angle 90^\circ = j1$.

We now know how to use Euler's identity to convert from the exponential and polar forms to the rectangular form, but how about going the other way ? How do we convert a complex number in rectangular form to exponential or polar form ? We will now derive the conversion formulas by considering the general complex number $x + jy$ in rectangular form. This is to be converted to the equivalent $Ae^{j\theta}$ in exponential form such that $x + jy = Ae^{j\theta}$. Presumably, x and y are known, and the problem is to find θ and A in terms of x and y. Using Euler's identity we can place the right side $Ae^{j\theta}$ in rectangular form :

$$x + jy = A\cos\theta + jA\sin\theta$$

Two complex numbers in rectangular form being equal means that the real parts are equal and that the imaginary parts are equal. Consequently, $x = A\cos\theta$ and $y = A\sin\theta$. By taking the ratio of these two equations, we can eliminate A :

$$\frac{A\ \sin\theta}{A\cos\theta} = \frac{y}{x}$$

But ,

$$\frac{\sin\theta}{\cos\theta} = \tan\theta$$

So ,

$$\tan\theta = \frac{y}{x} \qquad \text{and} \qquad \theta = \tan^{-1}\frac{y}{x}$$

which specifies that the angle of a complex number equals the arctangent of the quotient of the imaginary part divided by the real part.

With the angle known, we can easily find A from either $A\cos\theta = x$ or from $A\sin\theta = y$:

$$A = \frac{x}{\cos\theta} \qquad \text{or} \qquad \frac{y}{\sin\theta}$$

Another popular way to find A is to use a formula based on squaring both sides of $A\cos\theta = x$ and of $A\sin\theta = y$, and adding corresponding sides :

$$A^2(\cos^2\theta + \sin^2\theta) = A^2(1) = x^2 + y^2$$

By taking the square root of both sides we get the formula :

$$A = \sqrt{x^2 + y^2}$$

Example :

Convert the following complex numbers to polar form :

(a) 3 + j4

(b) 15 – j6

(c) 15 + j1

(d) 2 – j25

Solution :

(a) $\theta = \tan^{-1}\frac{4}{3} = 53.13^{\circ}$, $A = 3/\cos 53.13^{\circ} = 5$, so $3 + j4 = 5\angle 53.13^{\circ}$.

(b) $\theta = \tan^{-1} -\frac{6}{15} = -21.8^{\circ}$, $A = 15/\cos(-21.8^{\circ}) = 16.16$, so $15 - j6 =$

$16.16\angle -21.8^{\circ}$.

(c) when one part or a complex number in rectangular form is more than 10 times the other, often a reasonable approximation is to neglect the smaller part. Doing this here we get $15 + j1 \cong 15$. If we need more accuracy, we do as before : $\theta = \tan^{-1}\frac{1}{15} = 3.81^{\circ}$,

$A = 15/\cos 3.81^{\circ} = 15.03$, so $15 + j1 = 1.035\angle 3.81^{\circ}$.

(d) $2 - j25 \cong -j25 = 25\angle -90^{\circ}$ or, more accurately, $\theta = \tan^{-1}\frac{25}{2} = 3.81^{\circ}$,

$A = 25/\cos(-85.4^{\circ}) = 25.1$, so $2 - j25 = 25.1\angle -85.4^{\circ}$.

The exponential and polar forms may be easier to understand by considering a complex number to be more than just a point in the complex plane, and instead to be a line extending from the origin this point. Figure 7-2(a) illustrates this line for the general point x + jy. Usually, the line has, as shown, an arrowhead at the point even

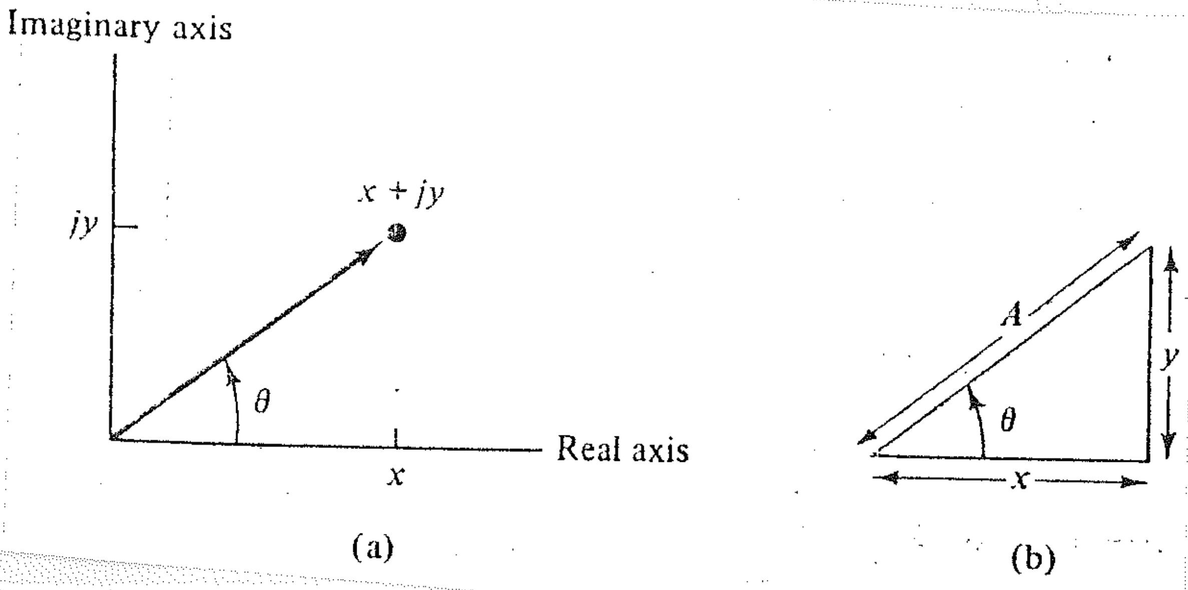


Figure 7-2

though the arrowhead has no significance. As shown in Fig. 7-2(b), the line forms a right triangle with its horizontal and vertical projections. This right triangle has a horizontal side x, a vertical side y, and a hypotenuse A. From elementary trigonometry, $x = \cos\theta$, $y = A \sin\theta$, and $A = \sqrt{x^2 + y^2}$, all in agreement with the results from Euler's identity. This line point of view is often helpful in finding the correct angles in the conversion from rectangular to polar form for complex numbers in the second and third quadrants.

Now we will consider mathematical operations with complex numbers in exponential and polar forms. Adding and subtracting in these forms is not practical except if the complex numbers have the

same angle or, almost equivalently, have angles that differ by integer multiples of 180^o. In this case the numbers are along the same line through the origin in the complex plane, with the result that adding and subtracting is similar to that with real numbers, with the adding and subtracting being performed basically on only the magnitudes, as the following example illustrates.

***Example* :**

Calculate $3\angle 45^o + 8\angle 225^o - 10\angle 45^o + 20\angle -135^o$.

***Solution* :**

All these complex numbers are along a line through the origin, which line is at an angle of 45^o in the first quadrant. Because they are along the same line, they can be added in polar form. For this addition it helps to have the same angle for each number. For some numbers this requires adding or subtracting 180^o and at the same time inserting a negative sign so as not to change the number. Here it is convenient to select the angle 45^o because two of the four numbers have this angle. Converting the other numbers to this angle, we have

$$8\angle 225^o = -8\angle 225^o - 180^o = -8\angle 45^o$$

and

$$20\angle -135^{\circ} = -20\angle -135^{\circ} + 180^{\circ} = -20\angle 45^{\circ}$$

Therefore,

$$3\angle 45^{\circ} + 8\angle 225^{\circ} - 10\angle 45^{\circ} + 20\angle -135^{\circ} = 3\angle 45^{\circ} - 8\angle 45^{\circ} - 10\angle 45^{\circ} - 20\angle 45^{\circ}$$

$$= (3 - 8 - 10 - 20)\angle 45^{\circ} = -35\angle 45^{\circ}$$

Because complex numbers that are to be added are seldom all along a line through the origin, we must usually use the rectangular form to add or subtract, and so convert to rectangular form any numbers that are in polar or exponential form. As mentioned, though, the polar and exponential forms are usually best for multiplication and division.

We will now consider the multiplication of two complex numbers $Ae^{j\theta}$ and $Be^{j\phi}$ in exponential form. By the law of exponents,

$$Ae^{j\theta} \times Be^{j\phi} = AB\, e^{j(\theta+\phi)}$$

Which product has a magnitude AB that is the product of the individual magnitudes and an angle θ and ϕ is the sum of the individual angles. In polar form this is :

$$A\angle\theta \ \times \ B\angle\phi \ = AB\angle\theta + \phi$$

***Example* :**

Find the products of :

(a) $(4\angle 30^\circ)(-5\angle 20^\circ)(6\angle -45^\circ)$

(b) $(3 + j4)(6 - j10)(-2 + j5)$

***Solution* :**

(a) Multiplying the magnitudes and adding the angles , we get :

$$(4)(-5)(6)\angle 30^\circ + 20^\circ - 45^\circ = -120^\circ\angle 5^\circ$$

(b) Converting these numbers from rectangular to polar form and then multiplying is easier than multiplying these numbers in rectangular form :

$$(3 + j4)(6 - j10)(-2 + j5) = (5\angle 53.1^\circ)(11.7\angle -59^\circ)(5.39\angle 112^\circ$$

$$= 5 \times 11.7 \times 5.39\angle 53.1^\circ - 59^\circ + 112^\circ = 315\angle 106^\circ$$

Division in exponential and polar form is about as easy as multiplication. To see this, consider the division of $Ae^{j\theta}$ and $Be^{j\phi}$:

$$\frac{Ae^{j\theta}}{Be^{j\phi}} = \frac{A}{B} e^{j(\theta - \phi)}$$

The quotient magnitude of A/B is quotient of the individual magnitudes, and by the law of exponents the quotient angle of $\theta - \phi$ is the difference of the individual angles. In polar form this is :

$$\frac{A/\theta}{B\angle\phi} = \frac{A}{B}\angle\theta - \phi$$

***Example* :**

Find the quotients in polar form of :

(a) $\frac{100/45^o}{20\angle 30^o}$

(b) $\frac{(0.6 - j2)(0.3 + j0.4)}{(0.8 - j0.7(0.6 + j0.9)}$

Solution :

(a) $\dfrac{100/45^o}{20\angle 30^o} = \dfrac{100}{20}\angle 45^o - 30^o = 5\angle 15^o$

(b) we will first convert the numbers to polar form and then combine the rules for multiplying and dividing in a rather obvious manner :

$$\frac{(0.6 - j2)(0.3 + j0.4)}{(0.8 - j0.7(0.6 + j0.9)} = \frac{(2.1\angle -73.3^o)(0.5\angle 35.1^o)}{(1.06\angle -41.2^o)(1.08\angle 56.3^o)}$$

$$= \frac{2.1 \text{ x } 0.5}{1.06 \text{ x } 1.08}\angle -73.3^\circ + 53.1^\circ - (-41.2^\circ) - 56.3^o$$

$$\mathbf{= 0.917\angle\text{-}35.3^o}$$

Phasors :

Having mastered complex algebra, we will now use it to find the sum and difference of sinusoids of the same frequency through use of complex numbers called phasors. By definition a phasor is a complex number associated with a sinusoidal voltage or current in a certain way that we will study.

For an understanding of phasors, consider the quantity $e^{j\omega t}$. It is a complex number, of course, as it is in exponential form with a magnitude of one. Also, it has an angle of ωt that increases linearly with time, thereby giving rotation to the line corresponding to $e^{j\omega t}$. Figure 7-3 shows this rotation in the complex plane. At $t = 0$ s the line is along the positive real axis. The angle increases with time, causing the line to rotate counterclockwise as illustrated. Because by Euler's identity $e^{j\omega t} = \cos \omega t + j \sin \omega t$, this line has sinusoidal projections on the two axes.

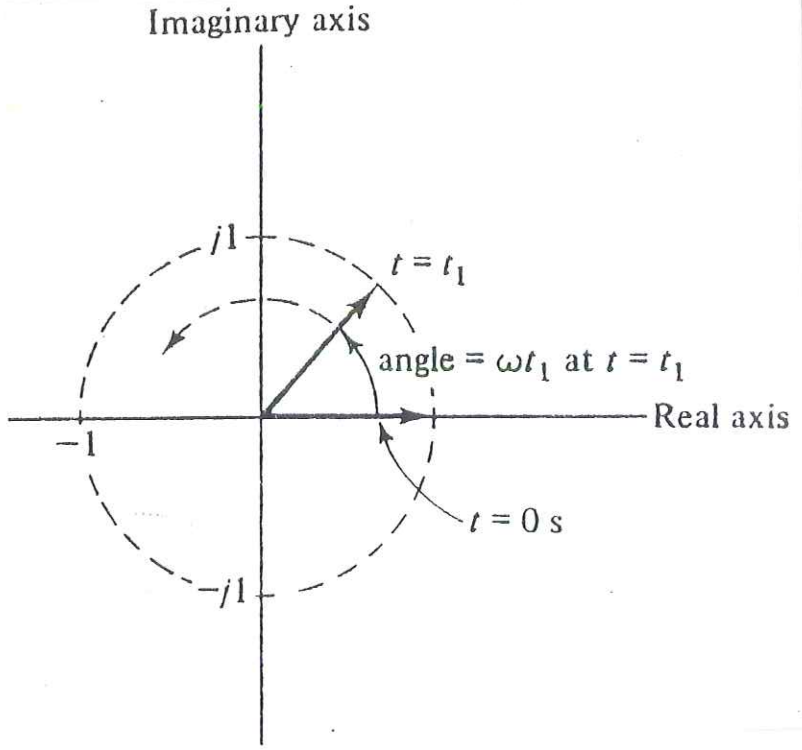


Figure 7-3

From this point of view, consider the sinusoid $i_1 = I_1 \sin(\omega t + \theta)$. It can be thought of as the projection of a rotating line $(I_1\angle\theta)e^{j\omega t}$ on the imaginary axis, as is evident from Euler's identity.

$$(I_1\angle\theta)e^{j\omega t} = I_1 e^{j\theta}\, e^{j\omega t} = I_1\, e^{j(\omega t+\theta)} = I_1 \cos(\omega t + \theta) + jI_1 \sin(\omega t + \theta)$$

The imaginary part is, of course the projection of the line on the vertical axis. This line has a length I_1 and at $t = 0$ s has an angle of θ with the positive real axis. Similarly, a current $i_2 = I_2 \sin(\omega t + \phi)$ is a projection on the vertical axis of a line of length I_2, which line has at $t = 0$ s an angle of ϕ with the positive real axis. Figure 7-4, illustrates both of these lines at $t = 0$ s. Remember that this is just for a moment of time. Actually, these lines rotate continuously in a counter clockwise direction. But they rotate at the same rate.

Now consider the rotating line that is the sum of these rotating lines : $(I_1\angle\theta)e^{j\omega t} + (I_2\angle\phi)e^{j\omega t}$. It is possible to sum these lines, even

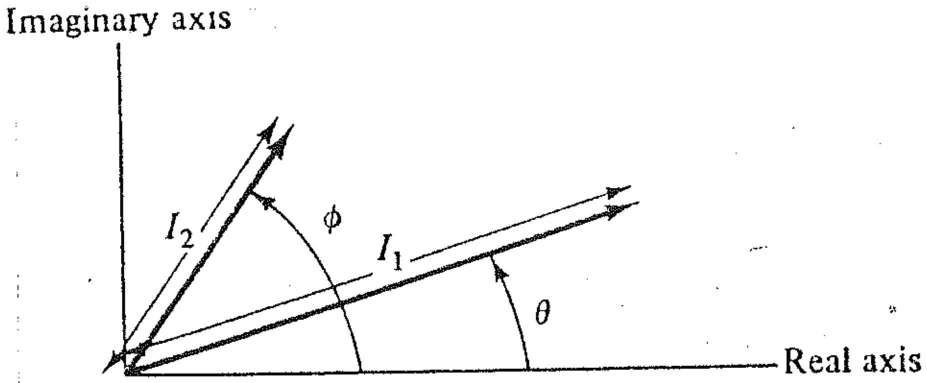


Figure 7-4

though they are rotating, because they rotate at the same speed and so always have the same angle $\phi - \theta$ between them. In other words, we can stop these rotating lines at any instant of time and add them. This stopping is analogous to taking a snapshot of them. Usually, for convenience, we stop them at $t = 0$ s.

We can find the sum of these two lines by placing one line at the end of the other as shown in Fig. 9-5, for $t = 0$ s. The result is a line $I\angle\psi$ in polar form, which number has a real part that is the sum of the real parts of $I_1\angle\theta$ and $I_2\angle\phi$ and an imaginary parts that is the sum of the of the imaginary parts of these numbers. Those familiar with vectors we know, that this is just vector addition.

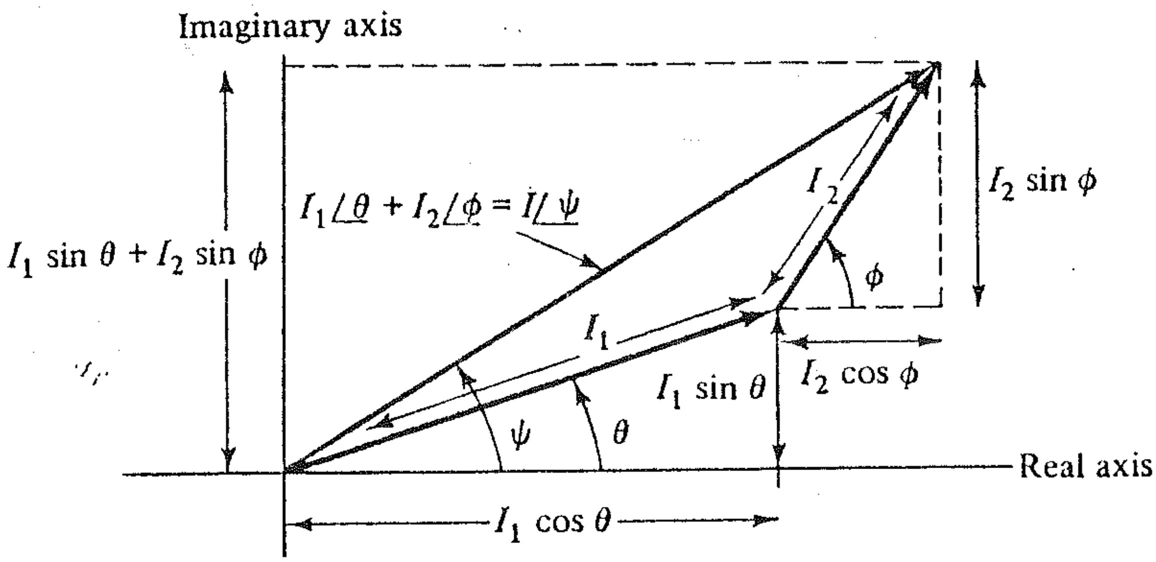


Figure 7-5

So, we have $I\angle\psi = I_1\angle\theta + I_2\angle\phi$. If we multiply both sides of this equation by $e^{j\omega t}$ and convert to exponential form, we get :

$$Ie^{j(\omega t+\psi)} = I_1e^{j(\omega t+\theta)} + I_2e^{j(\omega t+\phi)}$$

Then, by Euler's identity,

$$I\cos(\omega t+\psi) + jI\sin(\omega t+\psi) = I_1\cos(\omega t+\theta) + jI_1\sin(\omega t+\theta)$$
$$+ I_2\cos(\omega t+\phi) + jI_2\sin(\omega t+\phi)$$

By the definition of the equality of complex numbers, the imaginary part of the number on the left of the equal sign must equal the imaginary part of the number on the right :

$$I\sin(\omega t+\psi) = I_1\sin(\omega t+\theta) + I_2\sin(\omega t+\phi)$$

What does this result mean ? The significance is that we can get the peak value I and the phase angle ψ of the sum sinusoid by adding the complex numbers $I_1\angle\theta$ and $I_2\angle\phi$, which numbers are just from the peak values and phase angles of the individual sinusoids being added. This I and this ψ from the vector addition are really all we need to find the sum sinusoid since presumably we know ω. In other words, we can just add these complex numbers corresponding to the

individual sinusoids to get the peak value and phase angle of the sum sinusoid. Although in Fig. 7-5, we performed this addition graphically for purposes of explanation, this addition is almost always easier done analytically with the complex numbers in rectangular form. Even though we considered the addition of only two sine terms, the same method works regardless of the number of sine terms.

From this discussion we conclude that to add two or more sinusoids of the same frequency, we can :

(1) Convert all the sinusoids to sine terms.

(2) For each sine term form a complex number in polar form, the magnitude of which is the peak value and the angle of which is the phase angle.

(3) Convert these complex numbers to rectangular form and add them, and then convert this sum to polar form.

(4) Form a sine term from this polar number and the ω of the original sinusoids. The magnitude of the polar number is the sine-term peak and the angle is the sine-term phase angle. Of course, ω is its radian frequency. This resulting sine term is the sum of the original sinusoids.

Example :

What is the single sine-term equivalent of $3 \sin(2t + 30^o) + 4 \sin(2t + 60^o)$?

Solution :

We can go immediately to step 2 because the terms to be added are already in sine form. From step 2 the two complex numbers are $3\angle 30^o$ and $4\angle 60^o$. We convert these to rectangular form and then add them :

$$3\angle 30^o + 4\angle 60^o = (2.6 + j1.5) + (2 + j3.46) = 4.6 + j4.96$$

Next, we convert this sum to polar form : $4.6 + j4.96 = 6.76\angle 47.2^o$. This result tells us that the sum sine term has a peak value of 6.76 and a phase angle of 47.2^o. The only other information needed is the radian frequency, which is $\omega = 2$ rad/s, as in the original sinusoids. Consequently,

$$3 \sin(2t + 30^o) + 4 \sin(2t + 60^o) = 6.76 \sin(2t + 47.2^o)$$

Example :

What is the voltage v across the current source in the network of Fig. 7-6.

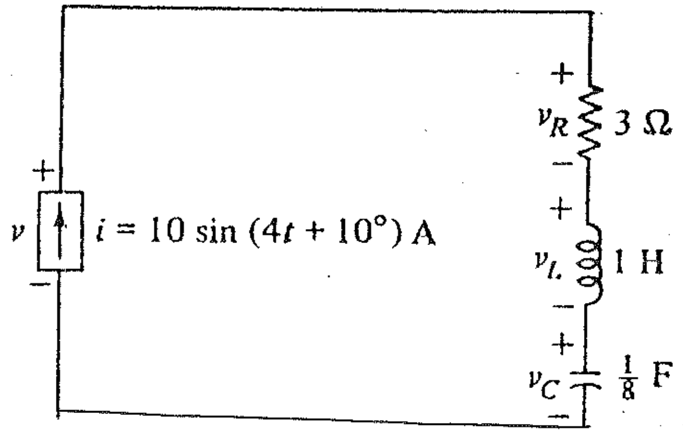


Figure 7-6.

Solution :

The voltage v equals the sum of the voltage drops, top to bottom, across the three components. From our studies we know how to find each of these voltages. The resistor voltage is in phase with the current and has a peak of $3 \times 10 = 30$ V : $v_R = 30 \sin (4t + 10^\circ)$ V. The inductor voltage has a peak of $\omega LI = 4 \times I = 4$ times the current peak and it leads the current by 90° : $v_L = (4)(10) \sin (4t + 10^\circ + 90^\circ) = 40 \sin (4t + 100^\circ)$ V. And the capacitor voltage has a peak of :

$$\frac{1}{\omega C} = \frac{1}{4 \times 1/8} = 2$$

times the current peak and lags the current by 90° : $v_C = (2)(10)\sin(4t + 10^\circ - 90^\circ) = 20\sin(4t - 80^\circ)$ V.

The voltage v across the current source is the sum of these voltages :

$$v = 30\sin(4t + 10^\circ) + 40\sin(4t + 100^\circ) + 20\sin(4t - 80^\circ)\ \text{V}$$

To sum these sine terms we find the corresponding complex numbers and add them :

$$30\angle 10^\circ + 40\angle 100^\circ + 20\angle{-80^\circ} = (29.5 + j5.21) + (-6.95 + j39.4) + (3.47 - j19.7)$$

$$= 26.1 + j24.9$$

Then we convert this sum to polar form :

$$26.1 + j24.9 = 36.1\angle 43.7^\circ$$

Finally, we get the corresponding sinusoid :

$$v = 36.1\sin(4t + 43.7^\circ)\ \text{V}$$

Solution :

(a) $V = \frac{30}{\sqrt{2}} \angle 10^\circ = 21.2 \angle 10^\circ$ V

(b) $V = -\ \frac{50}{\sqrt{2}} \angle -45^\circ + 90^\circ = -\ 35.4 \angle 45^\circ$ V

(c) $I = \frac{676}{\sqrt{2}} \angle 110^\circ + 90^\circ = 478 \angle 200^\circ = -478 \angle 20^\circ$ A

Example :

Find the sinusoids corresponding to the following phasors and frequencies :

(a) $V = 10\angle 20^\circ$ V and $f = 100$ Hz

(b) $I = 20\angle -45^\circ$ A and $f = 60$ Hz

(c) $V = -20\angle -30^\circ$ V and $\omega = 200$ rad/s

Solution :

For parts (a) and (b) we must convert from hertz to radians per second using $\omega = 2\pi f$. And for all three parts we must multiply the rms values by $\sqrt{2}$ to get the peak values. Then,

(a) $v = (10)(\sqrt{2}) \sin(2\pi 100t + 20^\circ) = 14.1 \sin(628t + 20^\circ)$ V

(b) $i = (20)(\sqrt{2}) \sin(2\pi 60t - 45^\circ) = 28.3 \sin(377t - 45^\circ)$ A

(c) $v = (-30)(\sqrt{2}) \sin(200t - 30^\circ) = -42.4 \sin(200t - 30^\circ)$ V

EXAMPLE Determine the current i_2 for the network of Fig. 7-7

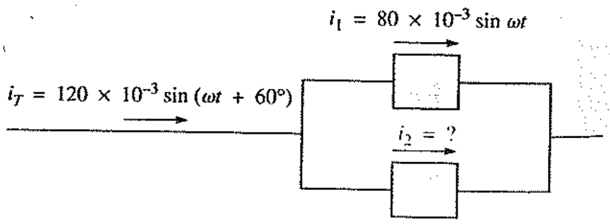


FIG. 7-7

Solution: Applying Kirchhoff's current law, we obtain

$$i_T = i_1 + i_2 \quad \text{or} \quad i_2 = i_T - i_1$$

Converting from the time to the phasor domain yields

$$i_T = 120 \times 10^{-3} \sin(\omega t + 60°) \Rightarrow 84.84 \text{ mA } \angle 60°$$
$$i_1 = 80 \times 10^{-3} \sin \omega t \Rightarrow 56.56 \text{ mA } \angle 0°$$

Converting from polar to rectangular form for subtraction yields

$$\mathbf{I}_T = 84.84 \text{ mA } \angle 60° = 42.42 \times 10^{-3} + j73.47 \times 10^{-3}$$
$$\mathbf{I}_1 = 56.56 \text{ mA } \angle 0° = 56.56 \times 10^{-3} + j0$$

Then

$$\mathbf{I}_2 = \mathbf{I}_T - \mathbf{I}_1$$
$$= (42.42 \times 10^{-3} + j73.47 \times 10^{-3}) - (56.56 \times 10^{-3} + j0)$$

and
$$\mathbf{I}_2 = -14.14 \times 10^{-3} + j73.47 \times 10^{-3}$$

Converting from rectangular to polar form, we have

$$\mathbf{I}_2 = 74.82 \text{ mA } \angle 100.89°$$

Converting from the phasor to the time domain, we have

$$\mathbf{I}_2 = 74.82 \text{ mA } \angle 100.89° \Rightarrow$$
$$i_2 = \sqrt{2}(74.82 \times 10^{-3}) \sin(\omega t + 100.89°)$$

and
$$i_2 = \mathbf{105.8 \times 10^{-3} \sin(\omega t + 100.89°)}$$

A plot of the three waveforms appears in Fig. 7-8. The waveforms clearly indicate that $i_T = i_1 + i_2$.

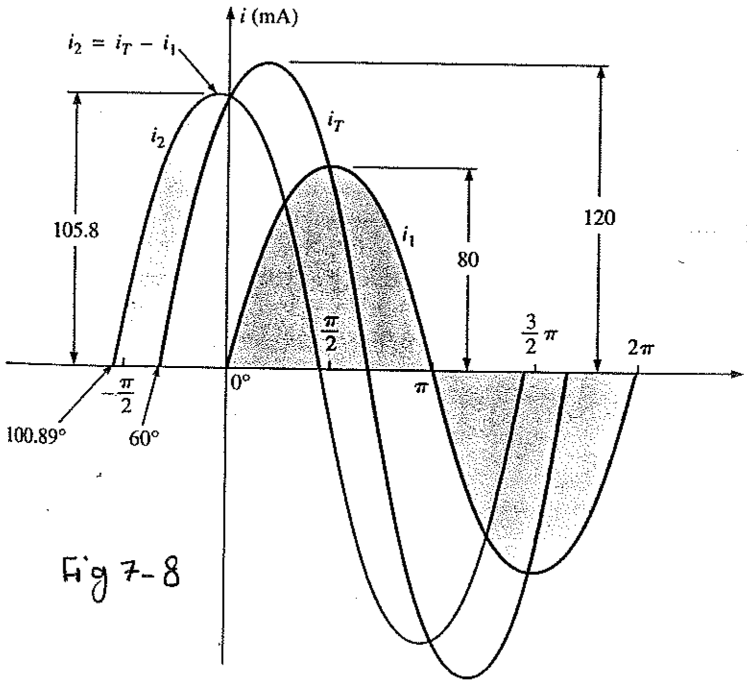


Fig 7-8

PROBLEMS

1) **Simplify each of the following to a complex number in rectangular form :**

(a) $(3 + j4) + (-6 + j2) - (10 - j3)$

(b) $4 + j6 - j10 - 22 + 4 - j3 - (6 + j8)$

(c) $(3 + j4)(3 - j4)$

2) **Express each of the following as a ratio of two complex numbers in rectangular form :**

(a) $\dfrac{3 + j2}{1 - j7} + \dfrac{4 - j5}{6 - j3}$ (b) $\dfrac{4 - j7}{-2 + j5} + \dfrac{3 - j2}{-5 - j4}$

3) **Convert the following to rectangular form :**

(a) $10\angle 60^\circ$ (b) $10\angle -60^\circ$

(c) $-100\angle 300^\circ$ (d) $3000\angle 420^\circ$

4) **Simplify each of the following to a complex number in rectangular form :**

(a) $4\angle 45^\circ + 6\angle 135^\circ$ (b) $4\angle -45^\circ + 8\angle 30^\circ - 10\angle 60^\circ + 20\angle 135^\circ$

5) **Convert the following from rectangular to polar form :**

a - $4 + j4$ b - $2 + j2$

c - $1000 + j400$ d - $0.001 + j0.0065$

These complex numbers corresponding to sinusoids are called phasors. Not all complex numbers, though, are phasors–just those corresponding to sinusoids.

We will use the sine basis and rms value for phasors because this is the most popular definition. So, to find a phasor corresponding to a sinusoid we will add just one step to what we have been doing, and that step is to divide the peak value by the square root of 2. And to go from a phasor to a sinusoid, we will have to remember to multiply the phasor magnitude by the square root of 2 to get the peak value.

Example :

Find phasors corresponding to the following sinusoids :

(a) $v = 30 \sin (100t + 10^\circ)$ V

(b) $v = -50 \cos (377t - 45^\circ)$ V

(c) $i = 676 \cos (2513t + 110^\circ)$ A

6) Convert the following from polar to rectangular form :

a - $6\angle 30^\circ$ b - $7.52\angle -125^\circ$

7) Division (express your answers in polar form) :

a - $(42\angle 10^\circ)(7\angle 60^\circ)$ b - $(8 + j8) / (2 + j2)$

8) Perform the following operations (express your answers in rectangular form) :

a - $\dfrac{(4 + j3) + (6 - j8)}{(3 + j3) - (2 + j3)}$ b - $\dfrac{8\angle 60^\circ}{(2\angle 0^\circ) + (100 + j100)}$

9) a - Determine a solution for x and y if :

$(x + j4) + (3x + jy) - j7 = 16\angle 0^\circ$

b- **Determine θ if :** $\dfrac{80\angle 0^\circ}{20\angle \theta} = 3.464 - j2$

10) Express the following in phasor form :

a - $\sqrt{2}\,(100)\sin(\omega t + 30^\circ)$ b - $3.6 \times 10^{-6}\ \cos\ (754t - 20^\circ)$

11) Express the following phasor currents and voltages as sine waves if the frequency is 60 Hz :

a - $I = 40\ A\ \angle 20^\circ$ b - $V = 120\ V\ \angle 0^\circ$

12) **For the system of Fig. 7-9, find the sinusoidal expression for the unknown voltage v_a if :**

$$e_{in} = 60 \sin(377t + 20^{\circ}) \quad , \quad v_b = 20 \sin 377t$$

13) **Find a single sine term equivalent of each of the following ::**

(a) $3 \sin \omega t + 4 \cos \omega t$

(b) $8 \sin 377t + 8 \cos 377t$

(c) $3 \sin(377t + 45^{\circ}) - 4 \cos(377t + 45^{\circ})$

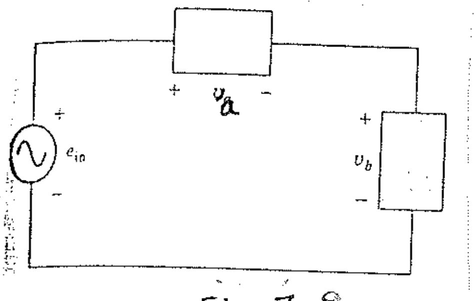


Fig 7-9

14) **Find phasors corresponding to the following :**

(a) $10\sqrt{2} \sin \omega t$ **(b)** $10\sqrt{2} \cos \omega t$

(c) $25\sqrt{2} \sin(20t + 30^{\circ})$

15) **Find the sinusoidal voltages corresponding to the following phasors. The frequency is 60 Hz :**

(a) $\frac{10}{\sqrt{2}} \angle 30^{\circ}$ V **(b)** $115 \angle 45^{\circ}$ V

16) For the system of Fig. 7-10 find the sinusoidal expression for the unknown current i_1 if

$$i_s = 20 \times 10^{-6} \sin(\omega t + 90^{\circ})$$
$$i_2 = 6 \times 10^{-6} \sin(\omega t - 60^{\circ})$$

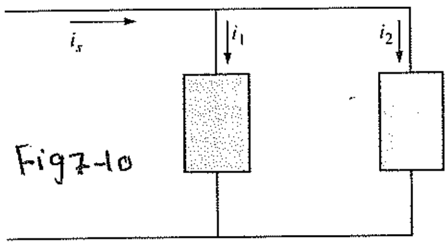


Fig7-10

17. Find the sinusoidal expression for the applied voltage e for the system of Fig. 7-11 if

$$v_a = 60 \sin(\omega t + 30^{\circ})$$
$$v_b = 30 \sin(\omega t - 30^{\circ})$$
$$v_c = 40 \sin(\omega t + 120^{\circ})$$

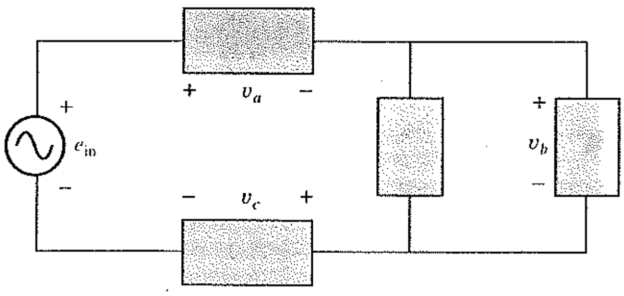


FIG. 7-11

Chapter Eight

SIMPLE AC CIRCUITS

Introduction :

In chap. 7, we learned how to relate complex numbers, called phasors, to sinusoids in a way such that by adding phasors we got a phasor corresponding to a sum of sinusoids of the same frequency. And from this sum phasor we easily got the sum sinusoid. Incidentally, mathematicians use the term transforming for a process like this. They say we transform the sinusoids to phasors, then add the phasors, and last we transform the sum phasor to a sinusoid.

In this chapter we extend the use of complex numbers to inductive and capacitive reactances, which, as discussed, have units of ohms. Further, we form new networks, called frequency-domain networks, that have phasors instead of voltages and currents, and reactances instead of inductors and capacitors. The resistors remain. Because in these networks the passive components all have the same ohm unit, they can be combined in much the same manner as resistors in dc circuits. The only difference is that the mathematical operations are with complex numbers instead of with real numbers. Except for this difference, everything we studied about dc networks-the theorems and all-apply as well to these frequency-domain networks.

It is important to remember that the complex numbers associated with passive components are not phasors, so we cannot relate them to sinusoids. Although we will use these complex numbers with phasors. We will never use them with sinusoids.

It is to be noticed that the solutions here are the voltages and currents occurring at steady-state i.e., the transient terms are negligible.

Reactance and Phasor Relations :

In this section we again consider the reactances of inductors and capacitors. But this time we relate them with voltage and current phasors instead of peak values.

We will consider the reactance of an inductor of L henrys first.

We have seen that for a current of $i = I_m \sin(\omega t + \theta)$, the inductor voltage $v = \omega L I_m \cos(\omega t + \theta)$, associated references assumed. These have phasors of :

$$\mathbf{I} = \frac{I_m}{\sqrt{2}}\angle\theta \text{ A} \qquad \textbf{and} \qquad \mathbf{V} = \frac{V_m}{\sqrt{2}}\angle\theta + 90^\circ \text{ V}$$

Now if we take the ratio of this voltage phasor and this current phasor, the $\sqrt{2}$ in each divides out, leaving

$$\frac{V}{I} = \frac{\omega L I_m \angle\theta + 90^\circ}{I_m\angle\theta} = \omega L\angle 90^\circ\ \Omega$$

The result of $\omega L\angle 90^\circ$ in polar form is $j\omega L = jX_L$ in rectangular form. As defined $X_L = \omega L$ is inductive reactance. The j multiplier corresponds to $1\angle 90^\circ$ and so gives the 90° angle difference between the voltage and current phasors.

If we treat the inductor voltage phasor as a voltage and the inductor current phasor as a current, the $j\omega L$ relating them is similar to a resistance in that it is the ratio of a voltage to a current

and has the unit of ohm. Extending this concept, we will change our original circuit, called a time-domain circuit, by replacing each inductor voltage v by a phasor V, each inductor current i by a phasor I, and each inductor L by jωL. In doing this we form a frequency-domain network.

Figure 8-1 (a), illustrates the time – domain inductor circuit representation for which $v = V_m \cos(\omega t + \theta)$ and $i = I_m \cos(\omega t + \theta)$.

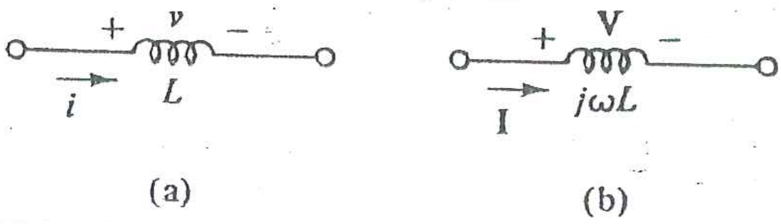


Figure 8-1

Figure 8-1(b), is the corresponding frequency-domain circuit representation in which $V = (V_m/\sqrt{2})\angle\theta + 90^\circ$ and $I = (I_m/\sqrt{2})\angle\theta$.

Figure 8-2, shows the phase relationship between the inductor current and voltage phasors in what is called a phasor diagram. The only important fact of this particular phasor diagram is that it shows that the voltage phasor leads the current phasor by 90°. In the time domain this corresponds to the voltage leading the current by 90°.

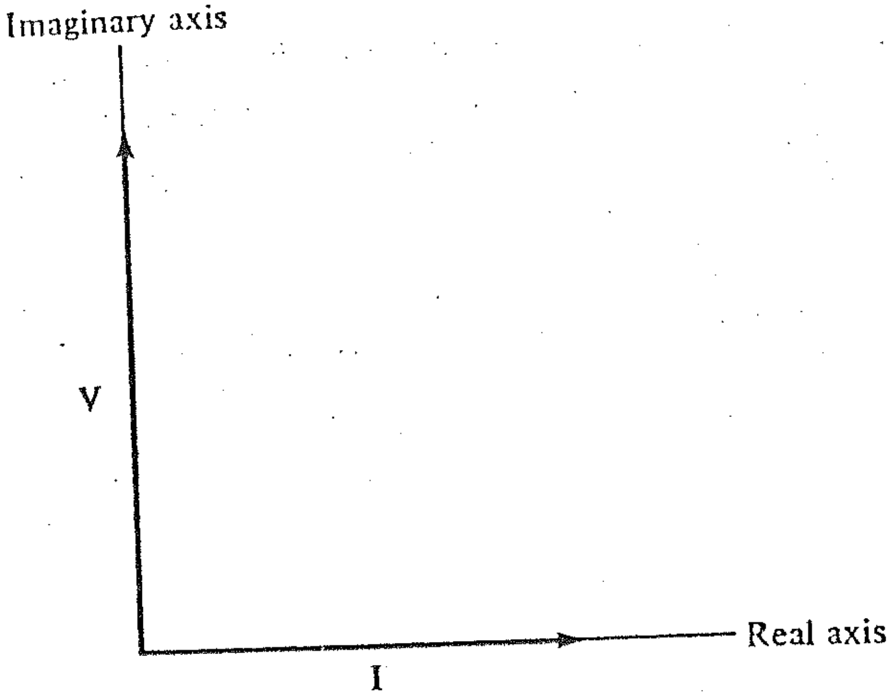


Figure 8-2

As in all phasors diagrams, the relative positions of the phasors are what is important and not the actual positions. Here the current phasors is rather arbitrarily positioned along the positive real axis, which is a reference axis because we measure angles from it. The reason the actual phasor positions are not important is because they correspond to counterclockwise rotating lines, all rotating at the same rate, as we have mentioned. In using phasors we are merely stopping these lines at a somewhat arbitrary instant of time. But, as we will see, it is convenient to select this time such that for a series circuit the current phasor is along the positive real axis and for a parallel circuit the voltage phasor is along this axis. For other

circuits we sometimes position a source quantity along the positive real axis. Often it does not matter if any phasor happens to be along this axis. To summarize, we can place any phasor, regardless of its angle, along the positive real axis. But all phasors must have correct relative angles. This will become clearer with examples of phasors diagrams.

Now consider capacitors and capacitive reactance. The explanation that follows is similar to that for inductors. If a capacitor of C farads has a voltage of $v = V_m \sin(\omega t + \theta)$, the capacitor current is $i = \omega C V_m \cos(\omega t + \theta)$. The corresponding phasors are

$$\mathbf{V} = \frac{V_m}{\sqrt{2}} \angle \theta \ \ \mathrm{V} \qquad \text{and} \qquad \mathbf{I} = \frac{\omega C V_m}{\sqrt{2}} \angle \theta + 90^\circ \ \ \mathrm{A}$$

In the ratio of the voltage to current phasors each $\sqrt{2}$ divides out, leaving

$$\frac{V}{I} = \frac{V_m \angle \theta}{\omega C V_m \angle \theta + 90^\circ} = \frac{1}{\omega C \angle 90^\circ} = \frac{1}{\mathrm{j}\omega C} \ \ \Omega$$

Multiplying numerator and denominator by j1, we get :

$$\frac{j1 \times 1}{j1 \times j\omega C} = -\frac{j1}{\omega C} = jX_c$$

As defined in chapter 8, the quantity $X_c = -1 / \omega C$ is the capacitive reactance. Be reminded, however, that some authors define this reactance as $1 / \omega C$, without the negative sign. In the numerator of $-j1 / \omega C$, the $-j1$ equals $1 \angle -90^{\circ}$ and so gives the angle difference between the voltage and current phasors.

Figure 8-3(a), illustrates the time-domain capacitor network representation and Fig. 8-3(b), the corresponding frequency-domain network representation. Figure 8-3(c), shows the phasor diagram.

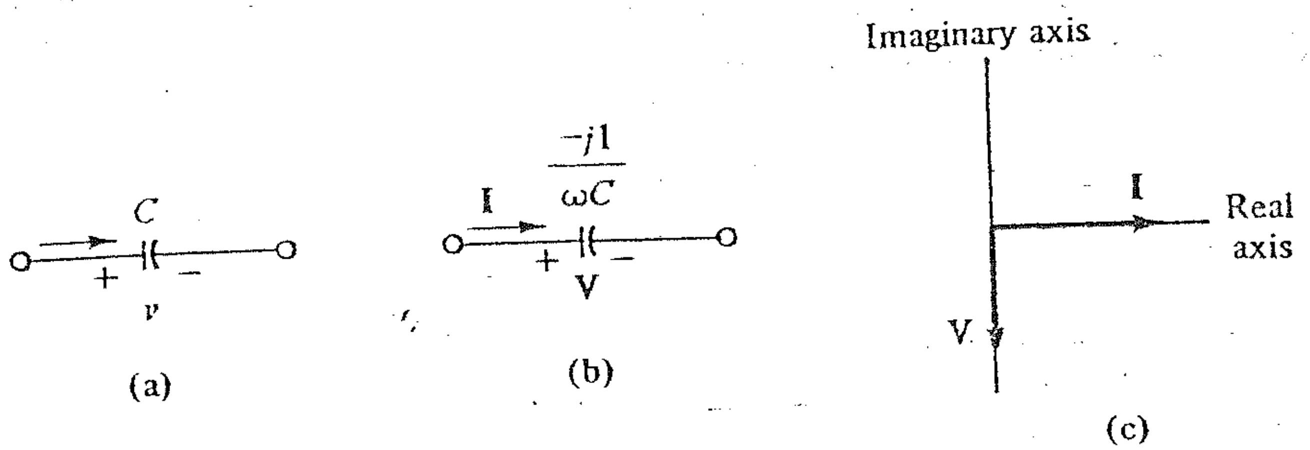


Figure 8-3

Series Circuits :

Suppose we want to find the steady-state current i, the sinusoidal current, in the circuit of Fig. 8-4(a). The first step is to make a corresponding frequency-domain circuit as in Fig. 8-3(b), in which the current and voltages are replaced by corresponding phasors, the inductor is replaced by $j\omega L$, but the resistor remains the same. Here, $\omega = 2$ rad/s from the source sine-term argument and $L = 2$ H, and so $j\omega L = j2 \times 2 = j4\ \Omega$.

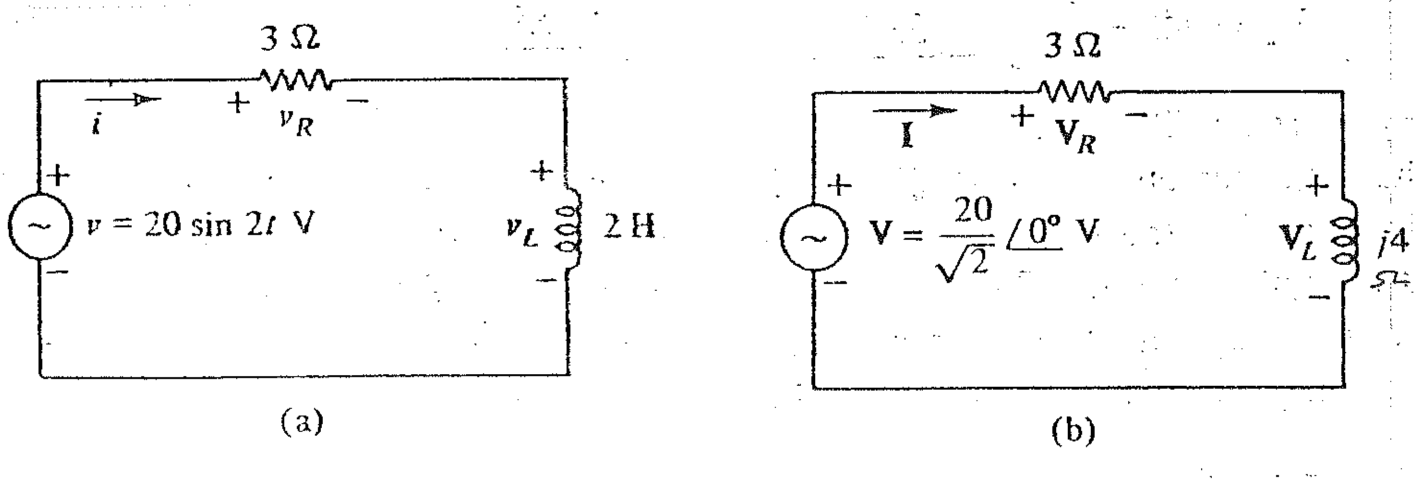


Figure 8-4

The next step in this analysis is to apply KVL to this frequency-domain circuit. We need some justification for this step because KVL applies only to voltages, and phasors of voltages are not really voltages themselves. Our justification is that we could use KVL on the circuit of Fig. 8-4(a), to sum sinusoids. Then we could replace these sinusoids by their corresponding phasors. The result would be a sum of phasors, the same as we would get by skipping the step of summing sinusoids and applying KVL directly to the voltage phasors instead. This is why KVL is valid for frequency-domain circuits. (For similar reasons, KCL is also valid). So, we can use KVL on the frequency-domain circuit of Fig. 8-4(b). Doing this, we get

$$V = V_R + V_L$$

The third step is to substitute in for the phasors. Clearly, $V = (20/\sqrt{2})\angle 0^\circ$, $V_R = 3\,I$, and $V_L = j4I$, with the drops as indicated. Usually, we do not draw in the polarity markings for the drops because the polarities are so obvious. With these substitutions the KVL equation becomes

$$\frac{20}{\sqrt{2}}\angle 0^\circ = 3I + j4I = (3 + j4)I$$

Next, we solve for I :

$$\frac{(20/\sqrt{2})\angle 0^{o}}{3+j4}=\frac{(20/\sqrt{2})\angle 0^{o}}{5\angle 53.1^{o}}=\frac{4}{\sqrt{2}}\angle -53.1^{o}$$

And finally we transform back to a sinusoid : $I = 4 \sin(2t - 53.1^{o})$ A.

If the resistor and inductor voltages are also of interest, they are easy to find from the current. The resistor voltage is just $v_R = 3I = 12 \sin (2t - 53.1^{o})$ V. The inductor voltage is a little more difficult to find because it is more than just a constant times the current. The inductor voltage peak is $\omega L = 4$ times the current peak of 4, and the inductor voltage leads the current by 90°. So,

$$\mathbf{V_L = (4 \times 4) \sin (2t - 53.1^{o} + 90^{o}) = 16 \sin (2t + 36.9^{o})\ V}$$

Another way of finding this voltage is from the inductor voltage phasors V_L :

$$V_L = j4I = \frac{16}{\sqrt{2}}\angle -53.1^{o} + 90^{o} = \frac{16}{\sqrt{2}}\angle 36.9^{o}$$

from which $v_L = 16 \sin (2t + 36.9^{o})$ V.

Notice that the current $i = 4 \sin(2t - 53.1^\circ)$ lags the applied voltage $v = 20 \sin 2t$ by an angle less than 90°. A circuit is said to be an inductive circuit if the current lags the applied voltage. The more the inductive effect, the greater the lag. Of course, the circuit appears to be purely inductive for a current lag of 90°.

The phasor diagram of Fig. 8-5 readily illustrates this lag as well as the angle and magnitude relations for the three voltage phasors of Fig. 8-4(b). Because the current is common to all components in this series circuit, it is convenient to reference all phasors with respect to the current phasor and so place it along the positive real axis. Recall our discussion that we can do this despite the fact that the current phase angle is not zero degrees. Also remember that the current phasor length has no relation to the voltage phasor lengths because there are two different scale-one for amperes and one for volts.

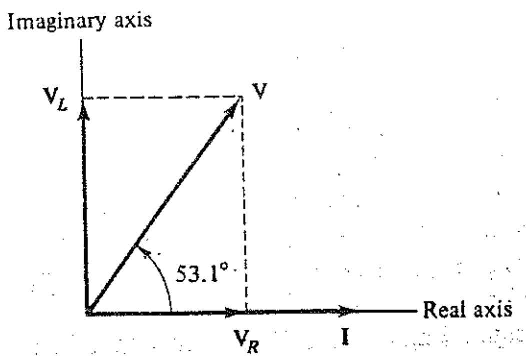


Figure 8-5

This phasor diagram illustrates the fact that V_R has the same angle as I and that V_L has an angle 90° greater than that of I. This diagram also gives information about magnitudes and illustrates that the vector sum of V_R and V_L equals the applied voltage phasor V, which, in turn, has an angle 53.1° greater than that of I. Notice also that the three voltage phasors form a right triangle. Thus, if we know any two of them, we can use the Pythagorean theorem to find the magnitude of the third.

Another point of interest regarding the solutions is that if only rms values are wanted, we can get these by deleting the angle parts of the phasors. Here then, $I = 4/\sqrt{2} = 2.83$ A, $V_R = 12/\sqrt{2} = 8.49$ V, and $V_L = 16/\sqrt{2} = 11.3$ V.

<u>Example</u> :

A sinusoidally excited series RL circuit has a source rms voltage of 20 V and a resistor rms voltage of 16 V. What is the inductor rms voltage ?

Solution :

From the fact that the voltage phasors have a right triangle relation (Fig. 8-5), we can find the inductor rms voltage from Phythagoras's theorem :

$$V_L = \sqrt{20^2 - 16^2} = \sqrt{144} = 12\text{ V}$$

Turning now to series RC circuits, we will find the current and also the resistor and capacitor voltages in the circuit of Fig. 8-6(a). The first step in the analysis is drawing the frequency-domain circuit diagram shown in Fig. 8-6(b). The only feature perhaps not readily apparent in it is the replacement of the capacitor by $-j1/\omega C$. Here, $\omega = 3$ rad/s from the sine term argument and $C = \frac{1}{18}$ F from the circuit diagram. So,

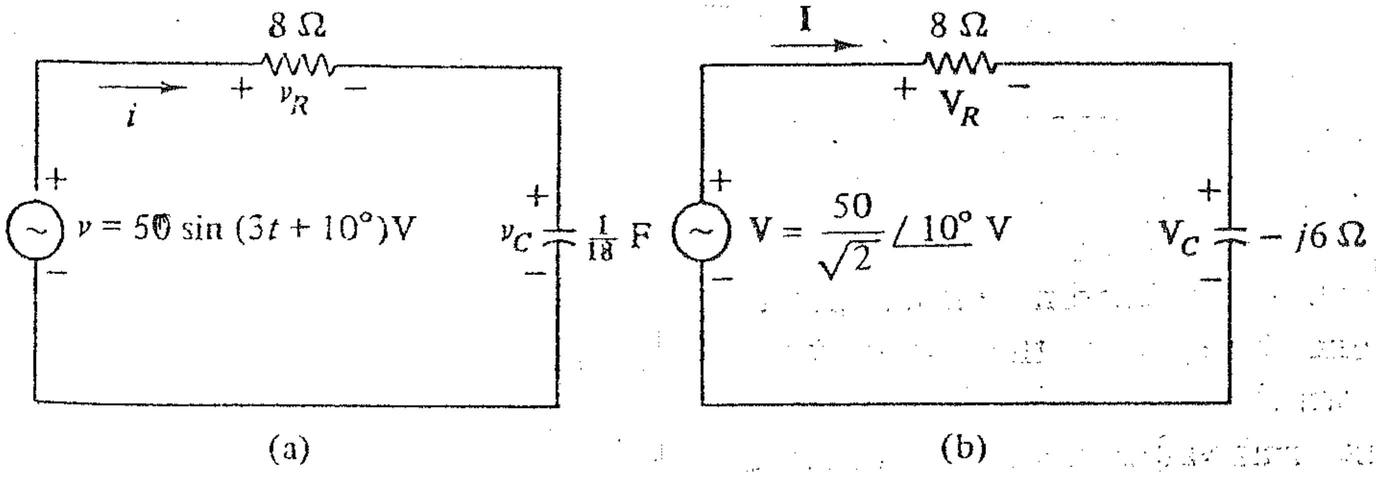


Figure 8-6

$$\frac{-j1}{\omega C} = \frac{-j1}{3 \times \frac{1}{18}} = -j6\ \Omega$$

To find I we first apply KVL to the circuit of Fig. 8-6 (b) :

$V = V_R + V_C$. Then we substitute in for the phasors,

$$\frac{50}{\sqrt{2}}\angle 10^\circ = 8I + (-j6)I$$

and solve for I :

$$\mathbf{I} = \frac{(50/\sqrt{2})\angle 10^\circ}{8 - j6} = \frac{50\angle 10^\circ}{\sqrt{2}\angle -36.9^\circ} = \frac{5}{\sqrt{2}}\angle 46.9^\circ \text{ A}$$

Finally, we transform back to a sinusoid : $i = 5 \sin(3t + 46.9^\circ)$ A.

The resistor voltage is

$$v_R = 8i = 40 \sin(3t + 46.9^\circ) \text{ V}$$

And the capacitor voltage phasor is

$$V_C = -j6I = (-j6)\left(\frac{5}{\sqrt{2}}\angle 46.9^\circ\right) = \frac{30}{\sqrt{2}}\angle 46.9^\circ - 90^\circ = \frac{30}{\sqrt{2}}\angle -43.1^\circ \text{ V}$$

from which $v_C = 30 \sin(3t - 43.1^\circ)$ V.

In this circuit the current leads the applied voltage by an angle less than 90°. A circuit with a leading current is called a capacitive circuit. The more the capacitive effect, the more the lead with the circuit appearing purely capacitive for a current lead of 90°.

Figure 8-7, shows the phasor diagram. The resistor voltage phasor has the same angle as the current phasor, but the capacitor voltage phasor has an angle 90° behind these. Also, the vector sum of V_R and V_C equals the applied voltage phasor, which lags I by 36.9°.

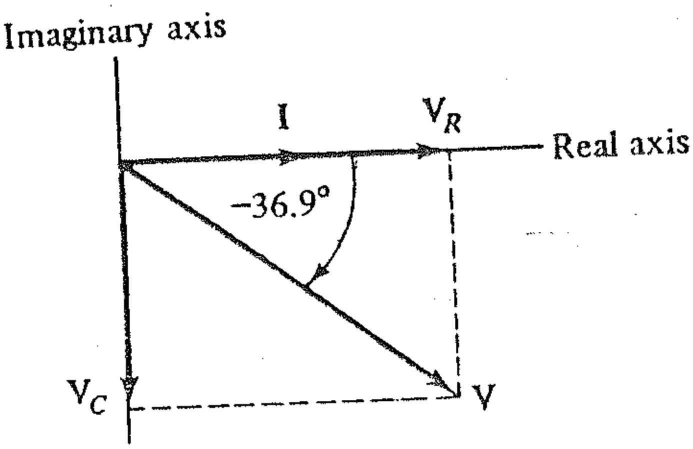


Figure 8-7

Again, rms values instead of instantaneous values may be only of interest. In this case, finding the sinusoids is more work than necessary.

Example :

A sinusoidally excited series RC circuit has a resistance of 6 Ω and a capacitive reactance of – 8 Ω. If the applied rms voltage is 100 V, what are the rms resistor and capacitor voltages and the rms current ?

Solution :

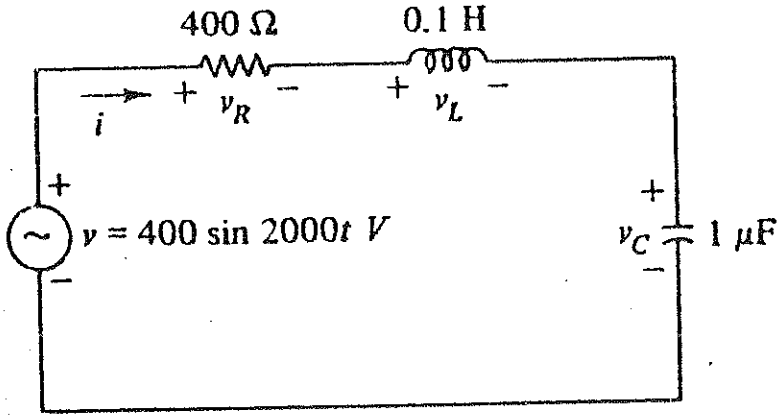
The rms current times the resistance equals the rms resistor voltage : $V_R = 6I$. This current times the magnitude of capacitive reactance equals the rms capacitor voltage : $V_C = 8I$. Then from the right-triangle relation of voltage phasors as illustrated in Fig. 8-7,

$$100 = \sqrt{(6I)^2 + (8I)^2} = \sqrt{100I^2} = 10I$$

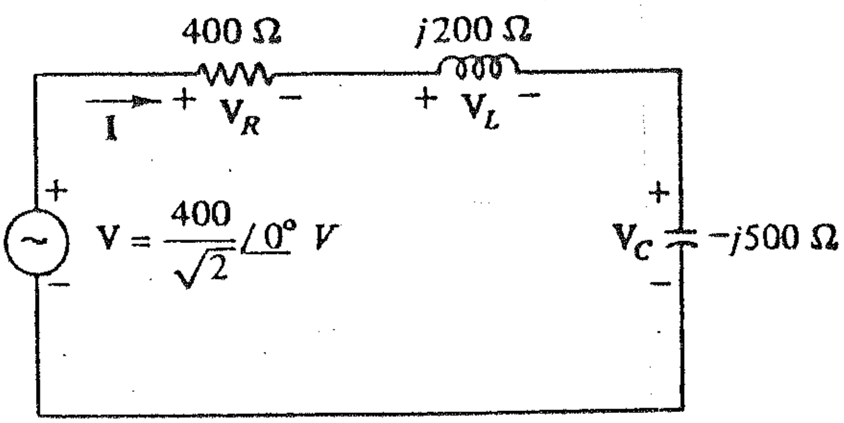
from which the rms current is $I = 10$ A. So, the rms resistor voltage is $6I = 60$ V and the rms capacitor voltage is $8I = 80$ V.

We have studied RL and RC series circuits. The remaining series circuit to be studied is RLC. For this we will find the current and the unknown voltages in the circuit of Fig. 8-8(a). Figure 8-8(b), shows the corresponding frequency-domain circuit with

$$j\omega L = j(2000)(0.1) = j200 \qquad \text{and} \qquad \frac{-j1}{\omega C} = \frac{-j1}{(2000)(10^{-6})} = -j500$$



(a)



(b)

Figure 8-8

By KVL applied to the circuit of Fig. 10-8(b), $V = V_R + V_L + V_C$, or

$$\frac{400}{\sqrt{2}}\angle 0^\circ = 400I + j200I - j500I = (400 - j300)I$$

and

$$I = \frac{400\angle 0^o}{\sqrt{2}\,(400 - j300)} = \frac{400\,\angle 0^o}{\sqrt{2}\,(500\angle -36.9^\circ)} = \frac{0.8}{\sqrt{2}}\angle 36.9^\circ \ \text{A}$$

The corresponding sinusoid is $i = 0.8 \sin (2000t + 36.9^o)$ A. With this current leading the applied voltage, the circuit is capacitive. This is also evident from the fact that the magnitude of the capacitive reactance is greater than that of the inductive reactance.

The resistor voltage is

$$v_R = 400I = 320 \sin (2000t + 36.9^o) \text{ V}$$

and the inductor voltage phasor is

$$V_L = (j200)\left(\frac{0.8}{\sqrt{2}}\angle 36.9^\circ\right) = \frac{160}{\sqrt{2}}\angle 126.9^\circ \ \text{V}$$

from which

$$v_L = 160 \sin (2000t + 126.9^\circ) = 160 \cos (2000t + 36.9^\circ) \text{ V}$$

The capacitor voltage phasor is

$$\mathbf{V_C} = (-j500)\left(\frac{0.8}{\sqrt{2}}\angle 36.9^\circ\right) = \frac{400}{\sqrt{2}}\angle -53.1^\circ \text{ V}$$

So, $v_C = 400 \sin (2000t - 53.1^\circ)$ V.

Figure 8-9, illustrates the phasor diagram with the current along the positive real axis, as is most convenient for phasor diagrams of series circuits. This phasor diagram shows that the resistor voltage is in phase with the current, the inductor voltage leads the current by 90°, and the capacitor voltage lags it by 90°. Notice especially that the inductor and capacitor voltage phasors differ in angle by 180°, which means that the inductor voltage is 180° out of phase with the capacitor voltage. In other words, the two voltages oppose, as they always do in a series circuit that is sinusoidally excited. From this partial cancellation it should be apparent that if the capacitive and inductive reactances have the same magnitude, the inductor and capacitor voltages completely cancel to produce zero volts across the

two components even though each component may have a large voltage across it. The circuit is then in resonance, a subject we will study later.

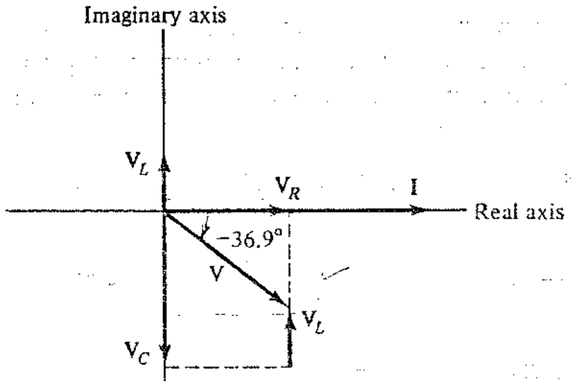


Figure 8-9

Impedance :

The KVL analysis method of the last section requires more work than necessary. A better way is to use the fact that resistances have the same unit of ohm and so can be combined in much the same way that resistances can be combined in dc circuits. The result of this combining is called impedance. It has the quantity symbol Z and the unit of ohm. For a network with a single source, impedance relates the input voltage and current phasors according to

$$V = IZ$$

In general, and not just for series circuits,

$$Z = R + jX$$

in which R, the real part, is called resistance and X, the imaginary part, is called reactance. For the simple series circuit of Fig. 8-4(b), R = 3 Ω, the resistance of the resistor, and X = 4 Ω, the reactance of the inductor. But for more complex circuits, the resistance R, the real part of the impedance, does not equal any individual resistance or combination of individual resistances. Nor does the reactance X, the imaginary part of the impedance, equal any individual reactance or combination of individual reactances.

We will study the use of impedance, which use closely parallels the use of resistance in dc circuits.

For the circuit of Fig. 8-4(b), we could have started with

$$\mathbf{Z} = 3 + j4 = 5\angle 53.1^\circ\ \Omega$$

and then use **Z** to find **I**,

$$\mathbf{I} = \frac{V}{Z} = \frac{\left(20/\sqrt{2}\right)\angle 0^\circ}{5\angle 53.1^\circ} = \frac{4}{\sqrt{2}}\angle -53.1^\circ\ \text{A}$$

and proceed as before.

Notice that Z divides into the input voltage phasor but not into the input voltage itself. Except for the case when the impedance is purely real-has zero reactance-it is always wrong to either divide or multiply a sinusoid by an impedance. Instead, in general we must limit our use of impedances to phasors. Also, notice that the impedance angle of 53.1° is the angle by which the input voltage leads the input current. In general, the impedance angle is the angle by which the voltage leads the current, as should be evident from Z = V/I.

Similarly, in the circuit of Fig. 8-8(b), we could have gotten the impedance

$$\mathbf{Z} = 400 + j200 - j500 = 400 - j300 = 500\angle -36.9^\circ\ \Omega$$

and found **I** from it

$$\mathbf{I} = \frac{V}{Z} = \frac{\left(400/\sqrt{2}\right)\angle 0^\circ}{500\angle -36.9^\circ} = \frac{0.8}{\sqrt{2}}\angle 36.9^\circ\ \mathrm{A}$$

and continue as before.

Because the proof for combining resistances of dc circuits is basically applicable to the proof that impedances combine in the same manner, we will omit this proof and will just accept the fact that the use of impedances in ac frequency-domain circuits is the same as the use of resistances in dc circuits. Consequently, the total impedance Z_T of series branches equals the sum of the individual branch impedances :

$$Z_T = Z_1 + Z_2 + Z_3 + \ldots + Z_N$$

and for two parallel branches,

$$Z_T = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

Of course, not everything is the same with impedances as with resistances. Impedances are usually complex while resistances are always real. Also, with impedances we find voltage and current phasors instead of actual voltages and currents. With resistances, however, we can find actual voltages and currents.

Now returning to similarities, there is an input impedance corresponding to the input resistance we studied. This input impedance, also called driving-point impedance or just impedance, is valid for two-terminal networks that do not have any independent sources. But these networks can have dependent sources.

Example :

Find the input impedance Z of the network of Fig. 8-10.

Solution :

Proceeding as with resistances,

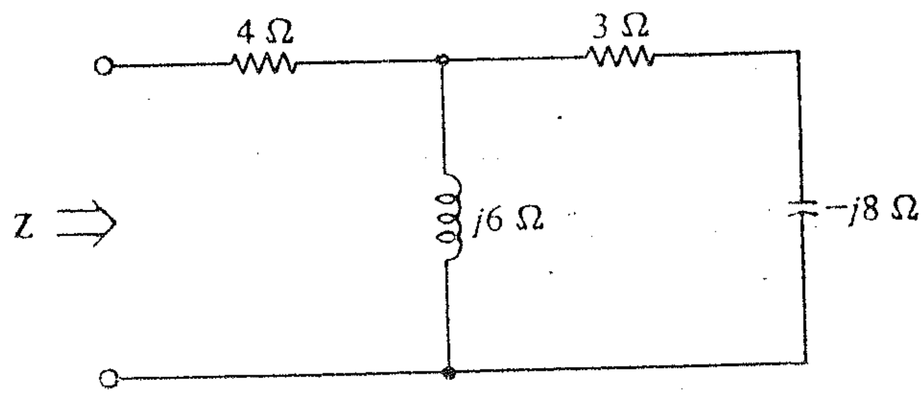


Figure 8-10

$$\mathbf{Z} = 4 + j6\|(3 - j8) = 4 + \frac{j6(3 - j8)}{j6 + 3 - j8} = 4 + \frac{48 + j18}{3 - j2}$$

Rationalizing :

$$\mathbf{Z} = 4 + \frac{(48 + j18)(3 + j2)}{(3 - j2)(3 + j2)} = 4 + \frac{108 + j150}{9 + 4} = 12.3 + j11.5\ \Omega$$

The resistance is 12.3 Ω and the reactance is 11.5 Ω. Because the reactance is positive, the network is inductive.

In this example, observe that the resistance of 12.3 Ω is not just a combination of the 4 Ω and 3 Ω individual resistances, and the 11.5 Ω reactance is not just a combination of the 6 Ω and the – 8 Ω reactances. Instead, the input resistance and reactance are both combinations of the individual resistances and reactances.

Interestingly, the simple circuit of Fig. 8-11 has the same impedance as that of Fig. 8-10. This simpler circuit could replace the more complex one and perform the same function for any network connected to the input terminals, provided that the frequency remains the same. A change in frequency would change the individual reactances in the circuit of Fig. 8-10, and this change, in turn, would change the input resistance and reactance.

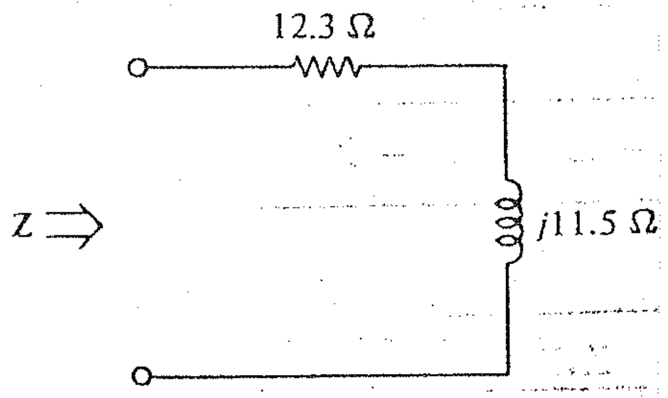


Figure 8-11

An impedance diagram helps in understanding impedance. This diagram is placed on an impedance plane, which, as illustrated in Fig. 8-12, has a horizontal axis called the resistance axis, indicated by R, and a vertical axis called the reactance axis, indicated by jX. Shown is a diagram of Z = 12.3 + j11.5 = 16.8∠43° Ω. Not suprisingly, an impedance diagram is similar to a diagram of a complex number in the complex plane. About the only difference is that an impedance diagram has axes scaled in ohms.

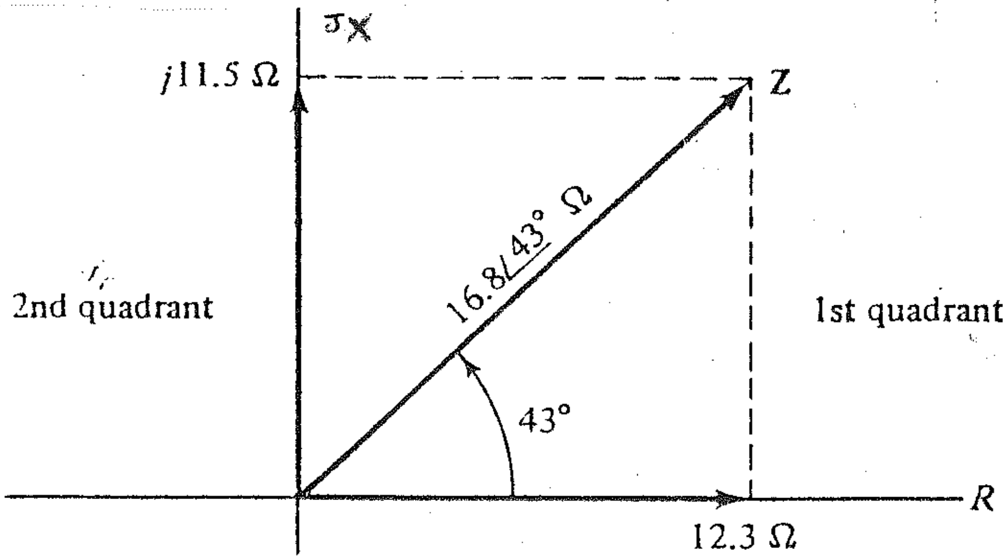


Figure 8-12

It should be apparent that an inductive circuit has a diagram in the first quadrant and a capacitive network a diagram in the fourth quadrant. For a diagram to be in either the second or third quadrant, a circuit must have negative resistance. This resistance requires at least one dependent source, as shown by the following examples.

Example :

The circuit of Fig. 8-13, has a dependent source that produces a voltage that is 2.5 times the applied voltage. Find the input impedance and draw its impedance diagram.

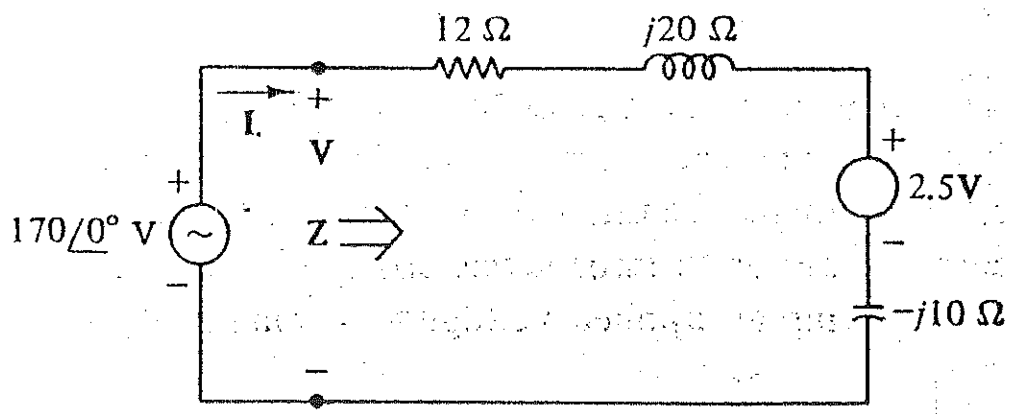


Figure 8-13

<u>***Solution*** </u>**:**

By KVL,

$$V = 12I + j20I + 2.5V + (-j10)I$$

Simplifying,

$$V - 2.5V = (12 + j20 - j10)I$$

From which

$$Z = \frac{V}{I} = \frac{12 + j10}{-1.5} = -8 - j6.67 = 10.4\angle{-140.2^\circ}\ \Omega$$

Figure 8-14, shows the impedance diagram. This diagram is in the third quadrant because the resistance and reactance are both negative.

An impedance triangle is often a more convenient form of diagram than the impedance diagram. The triangle contains only the lines, really vectors, corresponding to R, jX, and Z, with the vector

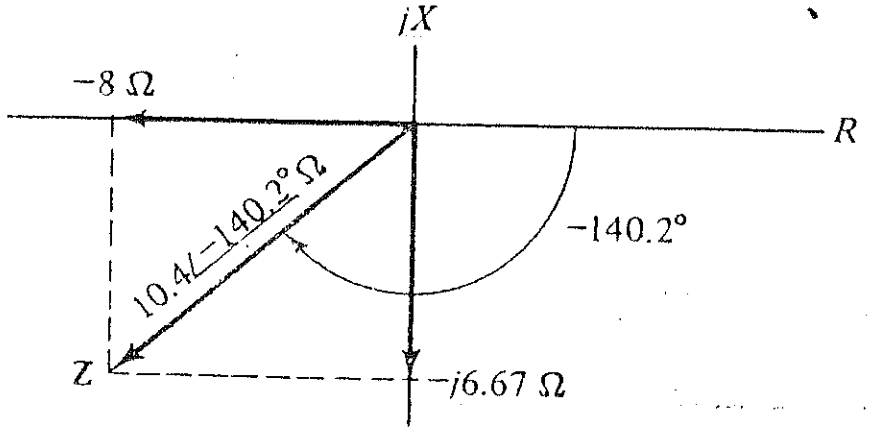


Figure 8-14

for jX drawn at the end of the R vector in conventional vector additional manner. Figure 8-15(a), illustrates an impedance triangle for $Z = 3 + j4\ \Omega$ and Fig. 10-15(b), one for $Z = 3 - j4\ \Omega$.

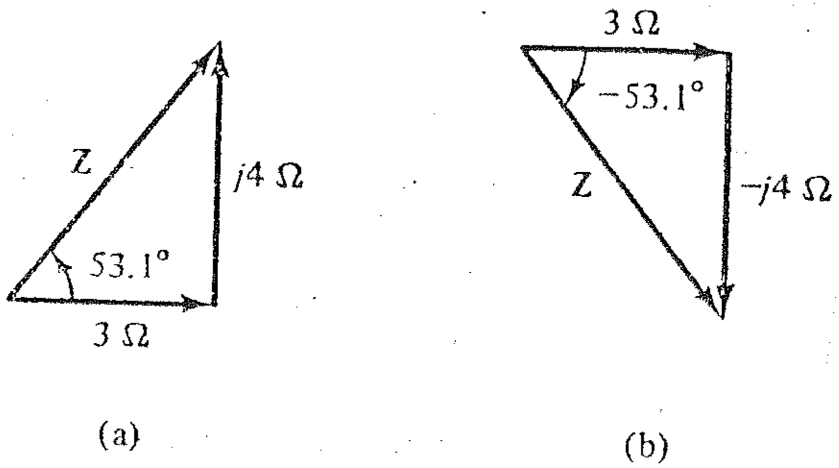


Figure 8-15

Voltage Division :

The voltage division rule for AC circuits should be apparent from our study of this rule for dc circuits and the fact that everything we learned from DC circuits applies to frequency-domain circuits with voltage and current phasors instead of voltages and currents, and with impedances instead of resistances. Consequently, for a series circuit having an applied voltage with phasor V_S, the voltage phasor V_X across one of the series impedances Z_X is

$$V_X = \frac{Z_X}{Z_T} V_S$$

in which Z_T is the sum of the series impedances. Also, V_X and V_S must have opposing polarities around a loop : one is a rise and the other a drop. If the polarities are not opposing, the right side of this equation has a factor of –1.

Example :

What is the voltage v_L across the inductor in the circuit of Figure 8-16(a) ?

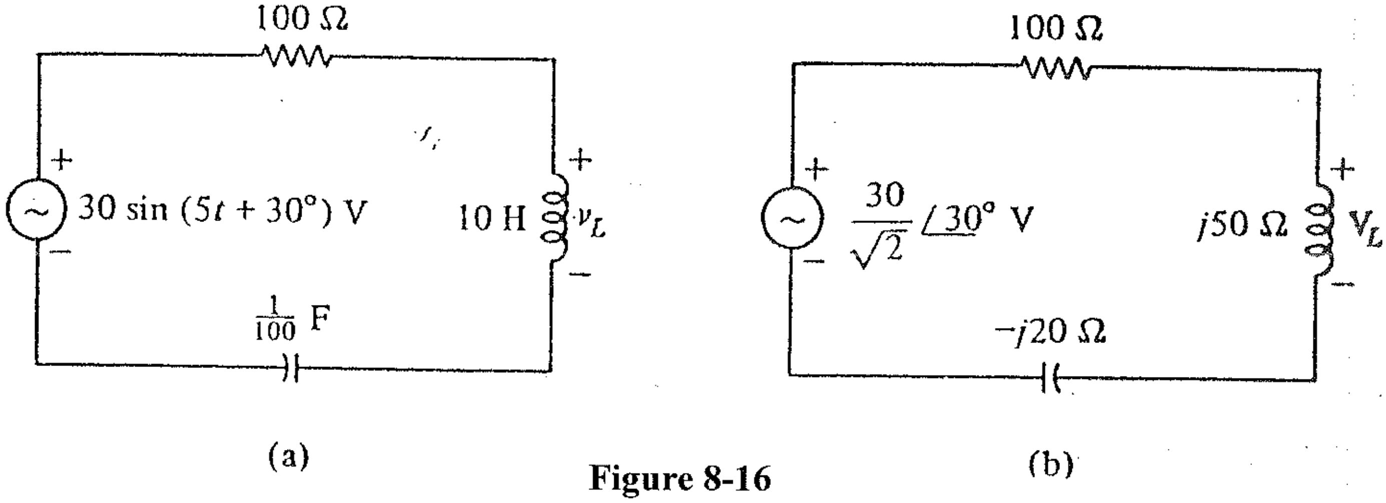


Figure 8-16

Solution :

The corresponding frequency-domain circuit is in Fig. 8-16(b). By the voltage division rule,

$$V_L = \frac{j50}{100 + j50 - j20} \times \frac{30}{\sqrt{2}}\angle 30° = \frac{1500\angle 120°}{\sqrt{2}\,(100 + j30)} = \frac{1500\angle 120°}{\sqrt{2}\,(104.4\angle 16.7°)}$$

$$= \frac{14.4}{\sqrt{2}}\angle 103.3° \text{ V}$$

which results in $v_L = 14.4 \sin (5t + 103.3°) = 14.4 \cos (5t + 13.3°)$ V.

Parallel Circuits :

Suppose that the voltage v is of interest in the parallel circuit of Fig. 8-17(a). The first step in finding v is constructing the corresponding frequency-domain circuit of Fig. 8-17(b).

This being a parallel circuit, we will use KCL, which is just as applicable to current phasors as KVL is to voltage phasors. Here, $I = I_R + I_L$. Then substituting in for the phasors,

$$\frac{20}{\sqrt{2}}\angle 0^\circ = \frac{V}{1/3} + \frac{V}{j1/4} = (3 - j4)\,V$$

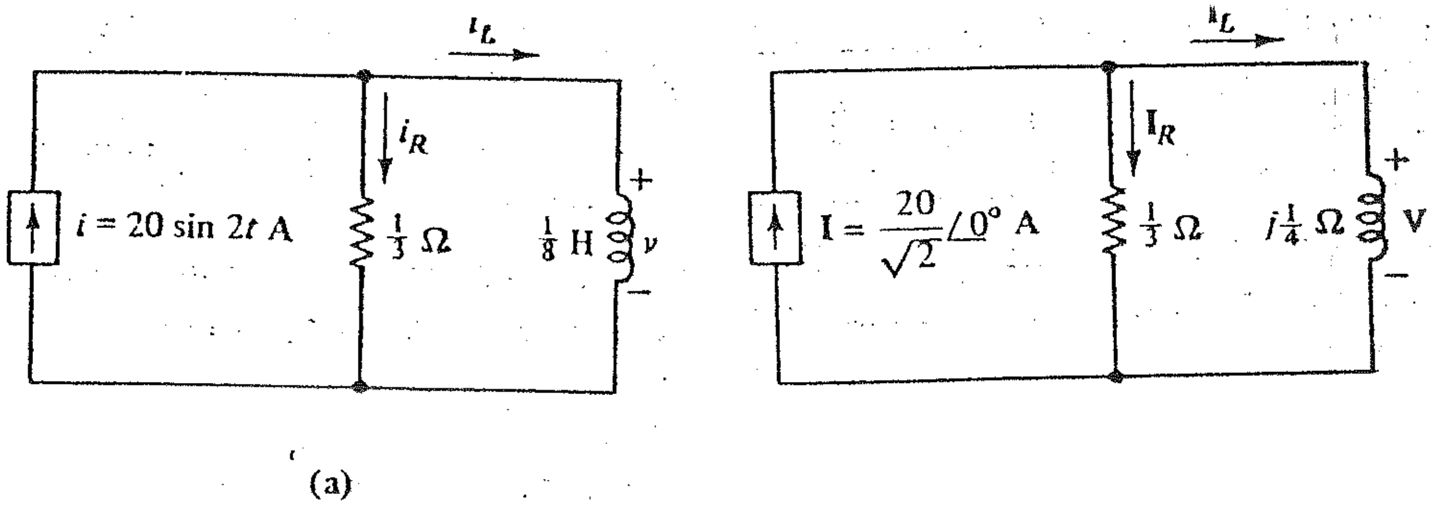


Figure 8-17

Figure 8-18, shows the phasor diagram based on the voltage phasor being the reference phasor, as is convenient for parallel

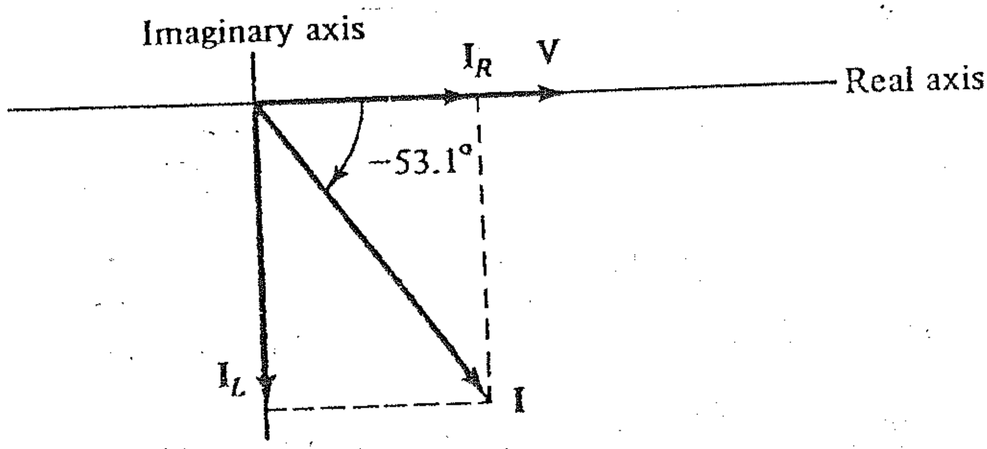


Figure 8-18

networks. The resistor current, being in phase with the voltage, has a phasor at the same angle as the voltage phasor- both are along the positive real axis. The inductor current lags the voltage by 90^0 and so has a phasor extending along the negative imaginary axis. Vector addition of the resistor current and inductor current phasors produces the input current phasor, which lags the voltage phasor by 53.1^o.

Turning now to the parallel RL circuit of Fig. 8-19(a), we will find the voltage and unknown currents. The corresponding frequency-domain circuit is in Fig. 8-19(b).

Solving for V,

$$\mathbf{V} = \frac{20\angle 0^\circ}{\sqrt{2}\,(3 - j4)} = \frac{20\angle 0^\circ}{\sqrt{2}\left(5\angle -53.1^\circ\right)} = \frac{4}{\sqrt{2}}\angle 53.1^\circ \text{ V}$$

from which $v = 4 \sin(2t + 53.1^\circ)$ V. Predictably, this voltage leads the input current in this inductive circuit.

With V known, the branch currents are easy to find. But, of course, for the resistor current we may prefer to use v.

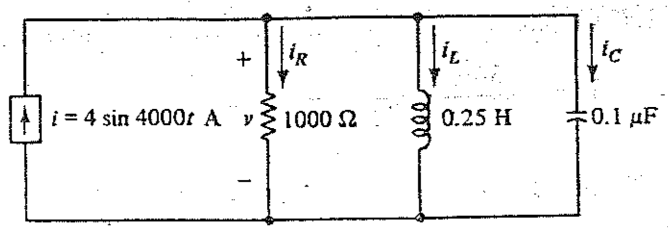
$$i_R = \frac{v}{R} = \frac{4 \sin(2t + 53.1^\circ)}{\frac{1}{3}} = 12 \sin(2t + 53.1^\circ) \text{ A}$$

Also,

$$\mathbf{I}_L = \frac{\mathbf{V}}{j\omega L} = \frac{4\angle 53.1^\circ}{\sqrt{2}\,(j1/4)} = \frac{4\angle 53.1^\circ}{\sqrt{2}\left(0.25\angle 90^\circ\right)} = \frac{16}{\sqrt{2}}\angle -36.9^\circ \text{ A}$$

So, $i_L = 16 \sin(2t - 36.9^\circ)$ A.

In Fig (8-19) a, Find i_R, i_L, i_C.



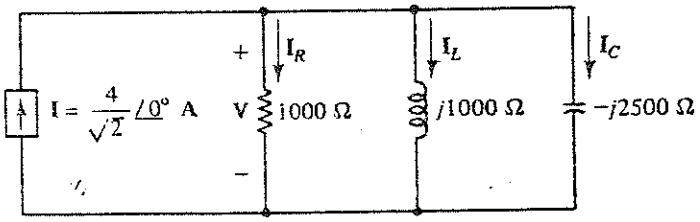


Figure 8-19

By KCL, $\mathbf{I} = \mathbf{I}_R + \mathbf{I}_L + \mathbf{I}_C$, or

$$\frac{4}{\sqrt{2}}\angle 0° = \frac{V}{1000} + \frac{V}{j1000} + \frac{V}{-j2500} = (0.001 - j0.001 + j0.0004)\ V$$

$$= \mathbf{(0.001 - j0.0006)\ V}$$

and

$$V = \frac{4\angle 0°}{\sqrt{2}\,(0.001 - j0.0006)} = \frac{4\angle 0°}{\sqrt{2}\left(0.001\ \ 17\angle -31°\right)} = \frac{3430}{\sqrt{2}}\angle 31°\ V$$

The corresponding sinusoid is $v = 3430 \sin(4000t + 31°)$ V.

Because this voltage leads the input current, the circuit is inductive. The cause for this is that the magnitude of the inductive reactance is smaller than that of the capacitive reactance. This, in turn, means that greater current flows through the inductor, giving the input current a greater inductive current component than capacitive current component.

The resistor current is

$$i_R = \frac{3430 \sin (4000t + 31^\circ)}{1000} = 3.43 \sin (4000t + 31^\circ) \text{ A}$$

The inductor current phasor is

$$\mathbf{I_L} = \frac{3430 \angle 31^\circ}{\sqrt{2}\,(j1000)} = \frac{3.43}{\sqrt{2}} \angle -59^\circ \text{ A}$$

from which i_L = 3.43 sin (4000t – 59°) A. And capacitor current phasor is

$$\mathbf{I_C} = \frac{3430 \angle 31^\circ}{\sqrt{2}\,(-j2500)} = \frac{1.37}{\sqrt{2}} \angle 121^\circ \text{ A}$$

So,

$$\mathbf{I_C = 1.37 \sin (4000t + 121^\circ) = 1.37 \cos (4000t + 31^\circ) \ A}$$

Figure 8-20 illustrates the phasor diagram with the voltage phasor as the reference, as is most convenient for parallel circuits. This phasor diagram shows that the inductor current lags the voltage by 90^o, and that the capacitor current leads the voltage 90^o. The resistor current is, of course, in phase with the voltage, and so their phasors coincide. Notice especially that the capacitor and inductor current phasors differ in angle by 180^o. Thus their corresponding currents tend to cancel in the input current. If the inductive and capacitive reactances have the same magnitude, the inductor and capacitor current peaks are the same, and the inductor and capacitor currents completely cancel : $i_L + i_C = 0$, even though each is not zero. Then the circuit is in resonance.

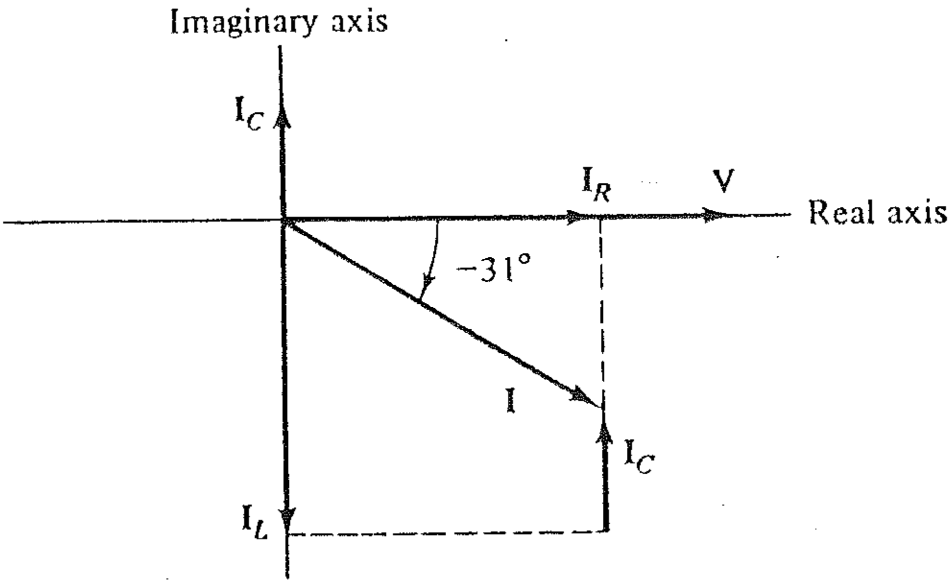


Figure 8-20

Admittance :

For the analysis of the parallel circuits of the last section, we could have used admittance of KCL. Admittance, which has the quantity symbol Y, is the inverse of impedance. So, it has the unit of siemens.

$$Y = \frac{1}{Z}$$

It follows that because $V = IZ$ and $Y = 1/Z$, that

$$I = YV$$

It should be fairly apparent that admittance corresponds to conductance of dc relative networks in the same way that impedance does to resistance. Consequently, admittances of parallel components add.

$$Y_r = Y_1 + Y_2 + Y_3 + \ldots + Y_N$$

As an illustration, for the circuit of Fig. 8-17(b), we can invert the resistance to get 3 S and invert the reactance with the j to get – j4 S. Then we can add to get $Y = 3 - j4$ S.

In general, and not just for parallel circuits, Y = G + jB, in which G, the real part of admittance, is called conductance, and B, the imaginary part, is called susceptance. For the simple parallel circuit of Fig. 8-17(b), G = 3 S, which is the conductance of the resistor, and B = - 4 S, the susceptance of the inductor. But for more complex circuits we cannot associate G with any individual resistors or B with any individual inductors of capacitors.

Another important fact is that G # 1/R, except for a purely real impedance. Further, B # 1/X for any network. These inequalities are easy to demonstrate starting with

$$Y = \frac{1}{Z} = \frac{1}{R + jX}$$

Rationalizing,

$$Y = G + jB = \frac{1}{R + jX} x \frac{R - jX}{R - jX} = \frac{R}{R^2 + X^2} + \frac{-jX}{R^2 + X^2}$$

Equating real imaginary parts :

$$G = \frac{R}{R^2 + X^2} \qquad \text{and} \qquad B = \frac{-X}{R^2 + X^2}$$

The equations show that G # 1/R except for networks for which X = 0 Ω, and B # 1/X for any network. Also note that for the same network, B and X have opposite signs. This also follows from the fact that Y being the inverse of Z has an angle that is the negative of the angle of Z.

If we had used admittance in the analysis of the circuit of Figure 8-17(b), we would have first found the admittance,

$$Y = \frac{1}{R} + \frac{1}{j\omega L} = 3 - j4 = 5\angle -53.1^\circ \text{ S}$$

then would have used it and the current phasor to find the voltage phasor,

$$\mathbf{V} = \frac{I}{Y} = \frac{20\angle 0^\circ}{\sqrt{2}\left(5\angle -53.1^\circ\right)} = \frac{4}{\sqrt{2}}\angle 53.1^\circ \text{ V}$$

and continued as before.

For the circuit of Fig. 8-19(b), we could have used admittance.

Here,

$$\mathbf{Y} = \frac{1}{R} + \frac{1}{j\omega L} + j\omega C = 0.001 - j0.001 + j0.0004$$

$$= \mathbf{0.001 - j0.0006 = 0.00117\angle\text{-}31^\circ\ S}$$

and

$$\mathbf{V} = \frac{I}{Y} = \frac{4\angle 0^\circ}{\sqrt{2}\left(0.001\,17\angle 31^\circ\right)} = \frac{3430}{\sqrt{2}}\angle 31^\circ\ \mathrm{V}$$

as we got before.

Notice that in finding the admittances of parallel circuits, we use $-\,j1/\omega L$ and $j\omega C$ instead of the $j\omega L$ and $-\,j1/\omega C$ that we use in finding the impedances of series circuits.

In general, using admittances in frequency-domain networks is basically the same as using conductances in DC networks. Naturally, one difference is that admittances are usually complex while conductances are real. Aslo,with admittances we use and find voltage

and current phasors instead of actual voltages and currents as we do with conductances in dc networks. Otherwise, though, the uses are the same. For this reason, a two-terminal network has an input or driving-point admittance that is the inverse of the input or driving-point impedance. Finally, in finding either the input impedance or admittance of a ladder-type network by combining components, we will often use both impedances and admittances–impedances for combining series components and admittances for combining parallel components.

Example :

Find the admittance Y of the network of Fig. 8-21.

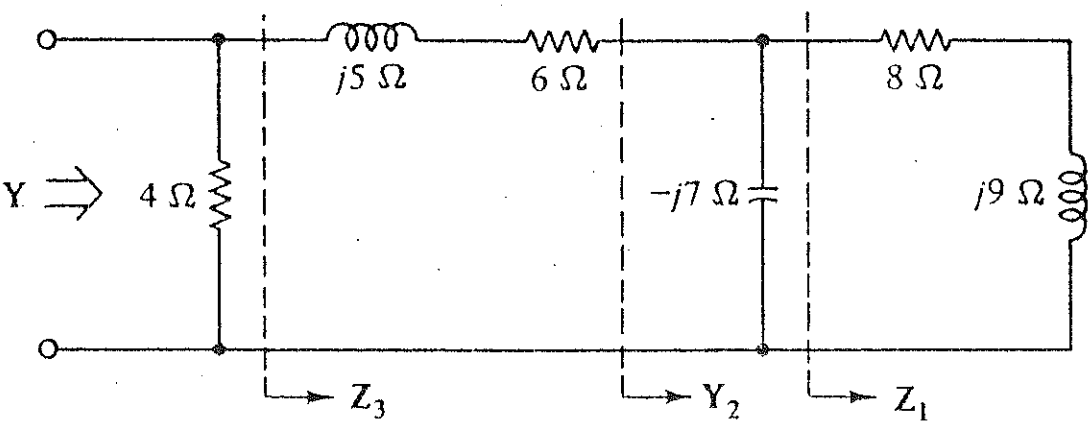


Figure 8-21

Solution :

Figure 8-21, has some dashed lines along with designations for impedances or admittances to the right of these lines. Although we will not make it a practice to always use such lines, they are helpful the first time through. At the right end, the impedance Z_1 of the two series branches is $Z_1 = 8 + j9\ \Omega$. We can then invert Z_1 and $-j7$ of the capacitor to combine admittances of parallel branches :

$$Y_2 = \frac{1}{-j7} + \frac{1}{8 + j9} = \frac{8 + j9 - j7}{-j7\,(8 + j9)} = \frac{8 + j2}{63 - j56}$$

We then invert Y_2 to get an impedance, and add this to $j5 + 6$:

$$Z_3 = j5 + 6 + \frac{63 - j56}{8 + j2} = \frac{(j5 + 6)(8 + j2) + 63 - j56}{8 + j2} = \frac{101 - j4}{8 + j2}$$

Finally, we invert Z_3 to have an admittance for combining with the ¼ S conductance of the left parallel branch :

$$Y = \frac{1}{4} + \frac{8 + j2}{101 - j4} = \frac{101 - j4 + 4(8 + j2)}{4(101 - j4)} = \frac{133 + j4}{404 - j16}$$

In polar form this is :

$$\mathbf{Y} = \frac{133.1\angle 1.72^\circ}{404.3\angle -2.27^\circ} = 0.329\angle 3.99^\circ \text{ S}$$

Notice in the preceding example that we used the rectangular form of the complex numbers even though there were some multiplications. It was easier to keep the numbers in rectangular form for the additions than it was to convert the numbers to polar form for the multiplications and then convert the products back to rectangular form for the additions.

As to be expected, there is an admittance diagram on an admittance plane that has conductance for its horizontal axis and j times susceptance for its vertical axis, and there is a corresponding admittance triangle. Figure 8-22(a), shows an admittance diagram for $Y = 3 - j4 = 5\angle{-53.1^\circ}$ S and Fig. 8-22(b), the corresponding admittance triangle.

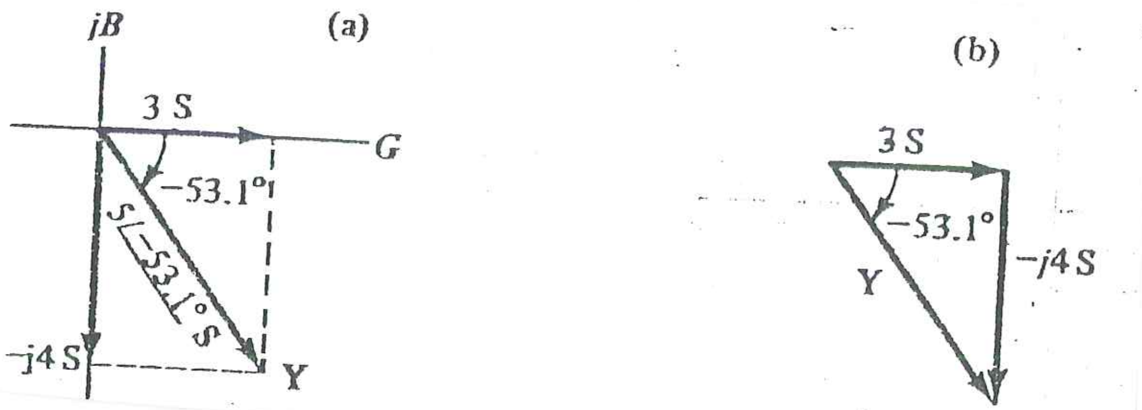


Figure 8-22

Current Division :

The current division rule applies to AC frequency-domain circuits in the same manner as to DC circuits. Consequently, if a parallel circuit has a current I_S flowing into it, the current I_X is :

$$I_X = \frac{Y_X}{Y_T} I_S$$

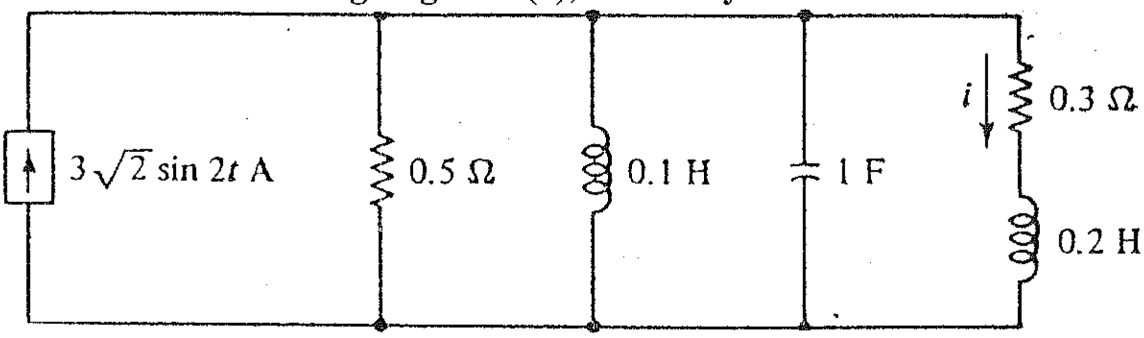
In which Y_T is the sum of the parallel admittances. For the special case of two parallel branches with impedances Z_1 and Z_2 ,

$$I_1 = \frac{Z_2}{Z_1 + Z_2} I_S$$

***Example* :**

In the circuit of Fig. Fig. 8-23(a), find **I** by current division.

(a)



(b)

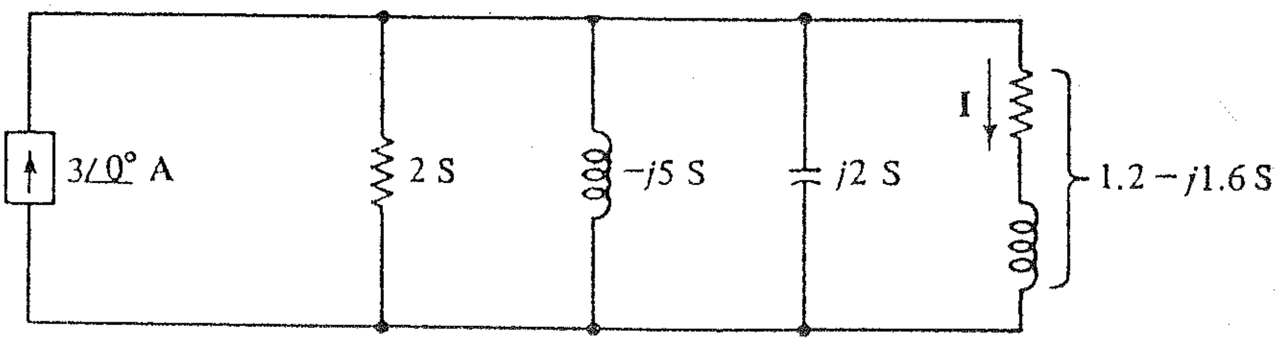


Fig. 8-23

Solution :

Figure 8-23(b), shows the corresponding frequency-domain circuit with the right branch of 0.3 + j0.4 Ω converted to admittance :

$$\frac{1}{0.3 + j0.4} x \frac{0.3 - j0.4}{0.3 - j0.4} = \frac{0.3 - j0.4}{0.09 + 0.16} = 1.2 - j1.6 \text{ S}$$

By current division,

$$\mathbf{I} = \frac{(1.2 - j1.6)(3\angle 0^\circ)}{2 - j5 + j2 + 1.2 - j1.6} = \frac{3.6 - j4.8}{3.2 - j4.6} = \frac{6\angle -53.1^\circ}{5.6\angle -55.2^\circ} = 1.07\angle 2.1^\circ \text{ A}$$

The corresponding sinusoid is

$$\mathbf{i = 1.07\sqrt{2}\ \sin(2t + 2.1^\circ) = 1.52\ \sin(2t + 2.1^\circ)\ A}$$

Example :

What is the current phasor I in the circuit of Fig. 8-24 ?

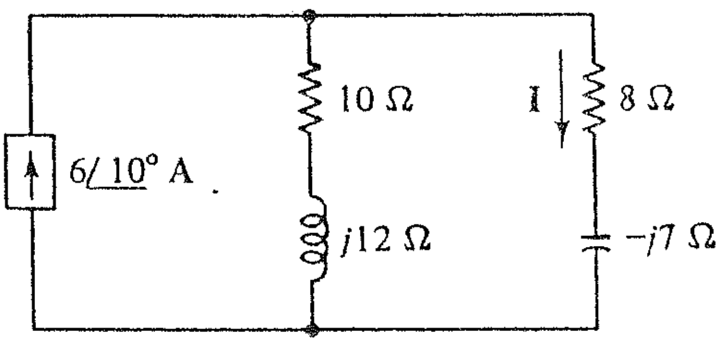


Figure 8-24

***Solution* :**

From the two-branch impedance approach for current division,

$$\mathbf{I} = \frac{(10 + j12)(6\angle 10°)}{10 + j12 + 8 - j7} = \frac{(15.6\angle 50.2°)(6\angle 10°)}{18.7\angle 15.5°} = 5\angle 44.7°\ \text{A}$$

AC Circuit Power Absorption :

Consider a two-terminal circuit with an applied voltage $v = V_m \cos(\omega t + \theta)$ and an input current $i = I_m \cos \omega t$ with associated references. This current is phase-displaced from the applied voltage by the angle θ, which may be either positive or negative. If the circuit has no independent sources, this angle is the impedance angle of the circuit input impedance. Then if this angle is between 0^{o} and 90^{o}, the voltage leads the current and the circuit is inductive overall. That is, although the circuit may contain inductors and capacitors, as well as resistors and other circuit components, it is the inductors that dominate over the capacitors. If, however, this angle is between 0^{o} and -90^{o}, the voltage lags the current and so the circuit is basically capacitive.

As we shall see, circuits having values of θ greater than 90° or less than -90° do not absorb average power, but deliver average power instead. Such circuits, called active circuits, must contain sources, either independent or dependent, or both. Circuits without sources are passive circuits, and they have impedance angle, values of θ, no greater than 90° and no less than -90°. They cannot deliver average power. Of course, inductive and capacitive circuits are passive circuits.

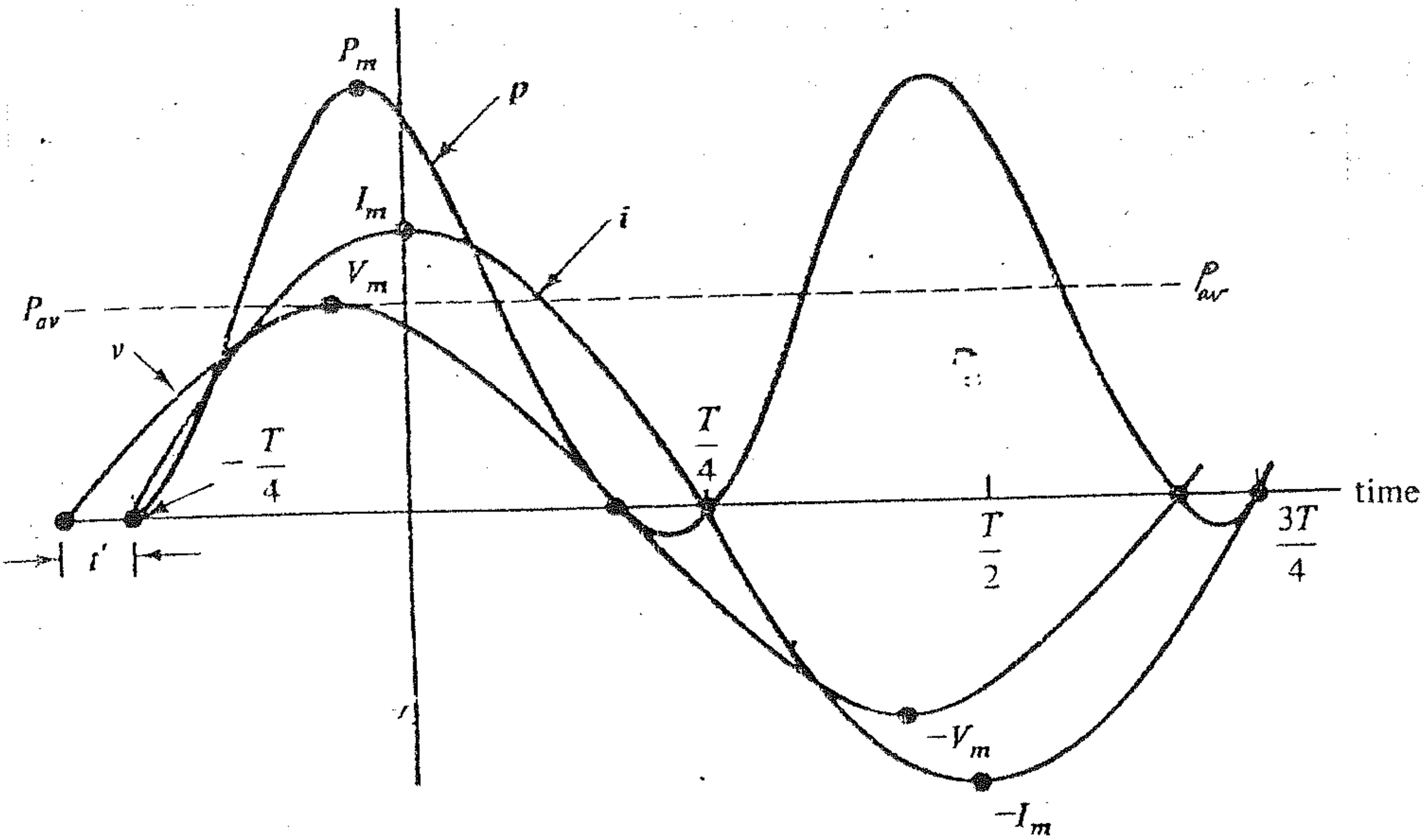


Figure 8-25

Figure 8-25, is a plot of an inductive circuit voltage and current as well as of the instantaneous power absorbed. The shown time difference t' between the times that the voltage and current are zero corresponds to the phase shift angle θ but is in seconds rather than in radians or degrees. Specifically, $t' = \theta / \omega$.

As easy and accurate way of getting p is by using a trigonometric identity on the product voltage-current expression of

$$p = vi = V_m \cos(\omega t + \theta) \times I_m \cos \omega t = V_m I_m \cos(\omega t + \theta) \cos \omega t$$

The desired trigonometric identity is

$$\text{Cos } A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

Here, either $A = \omega t + \theta$ and $B = \omega t$ or $A = \omega t$ and $B = \omega t + \theta$. The result is the same with either assignment. With the first assignment,

$$p = V_m I_m \cos(\omega t + \theta) \cos \omega t = \frac{V_m I_m}{2}[\cos(2\omega t + \theta) + \cos \theta]$$

But,

$$\frac{V_m I_m}{2} = \frac{V_m}{\sqrt{2}} \times \frac{I_m}{\sqrt{2}} = V_{eff} I_{eff}$$

So,

$$p = V_{eff} I_{eff} \cos(2\omega t + \theta) + V_{eff} I_{eff} \cos\theta$$

The first term varies sinusoidally between $V_{eff} I_{eff}$ and $-V_{eff} I_{eff}$ at twice the frequency of either voltage or current. Being sinusoidal, this first term has a zero average value. The second term, though, is a constant and so must be the average value of the power absorbed, or the true power.

$$P_{av} = V_{eff} I_{eff} \cos\theta$$

If the circuit is passive, $\cos\theta$ is positive and the P_{av} is positive and above the horizontal axis, as shown in Fig. 8-2 5 But if the circuit, is active, $\cos\theta$ is negative and average value is below the horizontal axis.

The term $\cos\theta$ is important enough to have its own name : power factor. For a passive circuit the power factor is the cosine of the impedance angle. Of course, this angle can be either positive or negative, depending on whether a circuit is inductive or capacitive. But because $\cos(-\theta) = \cos\theta$, the sign of the impedance angle has no effect on the power factor. And yet, as we will see, it is important to be able to distinguish power factors of inductive loads from those for capacitive loads. The power factor of an inductive load is called a lagging power factor because the current lags the voltage. And the power factor of a capacitive load is called a leading power factor because the current leads the voltage. Whether a power factor is leading or lagging is not important for the following examples on power absorption but will be important for our study of reactive power.

Example :

Find the average power absorbed by a circuit having an applied voltage of $v = 300\sqrt{2}\cos(377t + 10^\circ)$ V and an input current of $i = 50\sqrt{2}\cos(377t - 45^\circ)$ A. The references are associated.

Solution :

Here, V_{eff} = 300 V and I_{eff} = 50 A, and the impedance angle is the voltage phase angle minus the current phase angle : $\theta = 10^{\circ} - (-45^{\circ}) = 55^{\circ}$, making the power factor $\cos 55^{\circ} = 0.574$.

Consequently, the average power absorbed is

$$P_{av} = V_{eff} I_{eff} \cos\theta = (300)(50)(0.574) = 8604 \text{ W}$$

Example :

Find the average power delivered by the source in circuit of Figure 8-26(a).

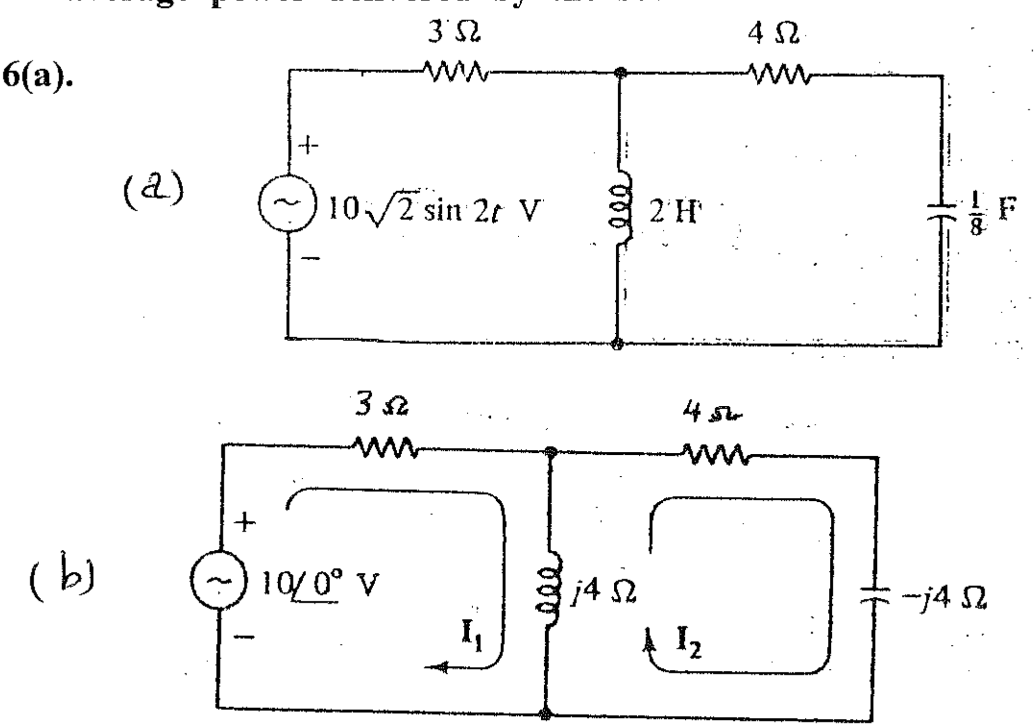


Figure 8-26

Solution :

There are several ways of finding the power. One way is to use the formula $P_{av} = V_{eff} I_{eff} \cos\theta$. Another way is to add the powers absorbed individually by the 3 Ω and 4 Ω resistors. Regardless of the approach, we need the corresponding frequency-domain circuit in Figure 8-26(b).

The first approach requires only the input impedance. This impedance has the angle needed for the power factor, and its magnitude divided into the applied rms voltage gives the rms input current : $I_1 = 10/Z$. Here,

$$\mathbf{Z = 3 + j4 \| (4 - j4) = 3 + 4 + j4 = 7 + j4 = 8.06\angle 29.7^{\circ}\ \Omega}$$

making

$$I_1 = \frac{10}{8.06} = 1.24 \text{ A}$$

Consequently,

$$\mathbf{P_{av} = (10)(1.24) \cos 29.7^{\circ} = 10.8 \text{ W}}$$

The second approach requires the rms currents in the two resistors. We have already found the rms current in the 3 Ω resistor: $I_1 = 1.24$. By current division on I_1 we can get the rms current I_2 in the 4 Ω resistor. In using current division on rms currents, however, we must have only magnitudes of impedances–no angles–because magnitudes are necessary to make the rms currents purely real and possitive, as all rms quantities are. So here, with magnitudes designated by vertical lines,

$$I_2 = \frac{|j4|}{|4 + j4 - j4|} I_1 = |j1| I_1 = I_1 = 1.24 \text{ A}$$

Both currents have the same effective or rms values, which is unusual and has no special significance. From these values,

$$P_{av} = (1.24)^2(3) + (1.24)^2(4) = 10.8 \text{ W}$$

which checks with the other calculation.

For the circuit of Fig. 8-26, of this last example, and in general, there are two other ways of finding the power. The equations needed are slight variations of $P_{av} = V_{eff} I_{eff} \cos\theta$. For one equation we substitute in $V_{eff} = I_{eff}Z$, in which Z is the magnitude of impedance, and get :

$$P_{av} = (I_{eff}Z)I_{eff} \cos\theta = I^2_{eff}Z \cos \theta$$

But $Z \cos\theta = R$, as may be recalled from the definition of resistance being the real part of the input impedance. As a result,

$$P_{av} = I^2_{eff}R$$

So, another way of finding P_{av} absorbed is to find the input resistance and the rms value of the input current and then to multiply this resistance by the square of this current. As an illustration, for the last example the input current is 1.24 A and the input resistance is $8.06 \times \cos 29.7^\circ = 7\ \Omega$. The result is $P_{av} = I^2{}_1R = (1.24)^2(7) = 10.8$ W, the same as previously found.

Still another way of finding P_{av} is substituting $I_{eff} = YV_{eff}$ into $P_{av} = V_{eff}\, I_{eff} \cos\theta$, this Y being the magnitude of the input admittance. With this substitution,

$$P_{av} = V_{eff}\,(V_{eff}Y)\ \cos\theta = V^2_{eff}Y\cos\theta$$

But $Y\cos\theta$ is G, the input conductance. Consequently, the average power absorbed is the square of the rms applied voltage times the input conductance :

$$P_{av} = G\,V^2_{eff}$$

In the last example,

$$Y = \frac{1}{Z} = \frac{1}{8.06\angle 29.7^\circ} = 0.124\angle -29.7^\circ = 0.108 - j0.06\ S$$

From which G = 0.108 S. It follows that $P_{av} = (10)^2(0.108) = 10.8$ W

Reactive Power :

The average power is often called the real power or true power. These names distinguish it from the reactive power that we considered briefly in Chapter 6. There we considered it for individual inductors and capacitors. But here we will consider it for two-terminal circuits in general. Incidentally, the motivation for studying reactive power is that this concept is useful for power and current calculations and for understanding power systems in general. Reactive power has the quantity symbol Q. This comes from the first letter of the word quadrature, which means at right angles to. In the next section we will learn what this refers to. Q has the unit of volt-amperes reactive, the symbol for which is var.

Q is defined as :

$$Q = V_{eff} I_{eff} \sin\theta$$

For a two-terminal circuit having an applied effective or rms voltage V_{eff} and an input effective or rms current I_{eff}, The angle θ is the angle by which the applied voltage leads the input current, associated references assumed. Of course, for passive two-terminal circuits, this angle θ is the impedance angle.

The equation differs from the similarly appearing P_{av} equation only in that the Q equation has the sine of the impedance angle instead of the cosine of this angle. The quantity sin θ is called the reactive factor. For inductive circuits the angle θ is positive, making sin θ and thus Q positive. On the other hand, for capacitive circuits this angle is negative, making sin θ and thus Q also negative. Rather arbitrarily, a positive Q is associated with var consumption and a negative Q with var generation. Although this selection is somewhat arbitrary, it does correspond to the signs for real power : a load with positive watts absorbs real power and a load with negative watts generates real power.

For a purely resistive circuit, $\theta = 0^\circ$, making $\sin\theta = 0$ and $Q = 0$. This result suggests that Q depends on the effects of inductors and capacitors. Moreover, the Q-reactance relations are identical to the P-resistance relations with the substitution of X for R :

$$Q = I^2_{eff}X \qquad \text{and} \qquad Q = \frac{V_{eff}^2}{Z^2} X$$

In general, the total vars absorbed in a circuit is the algebraic sum of the individual inductor and capacitor vars, just as the total watts absorbed is the sum of the watts absorbed by the individual resistors. Summing up the individual vars is sometimes done. Most often, however, we can find the total vars easier by inserting the input quantities into $Q = V_{eff} I_{eff} \sin\theta$.

Example :

What are the vars consumed by the circuit to right of terminals a-b in the circuit of Figure 8-27 ?

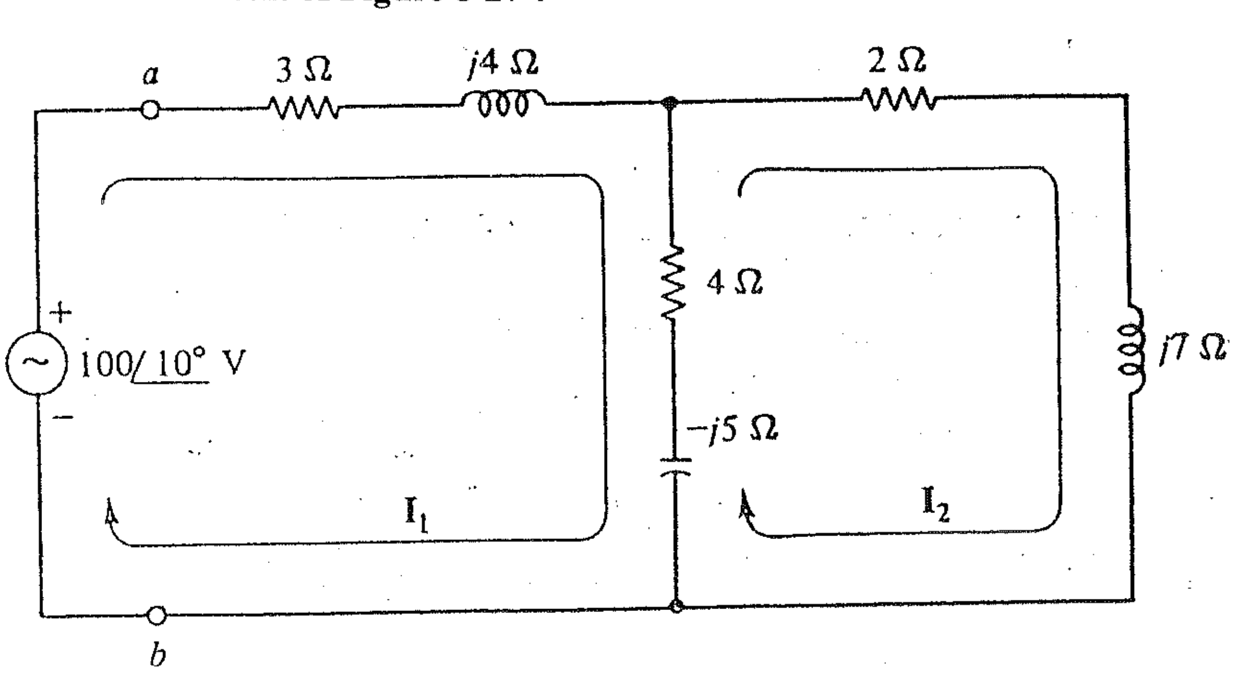


Figure 8-27

Solution :

We will find the vars two ways: first using $Q = V_{eff} I_{eff} \sin\theta$, and then by summing the various $I^2_{eff}X$'s. For the first approach we will find the input impedance and then use it to get the reactive factor and the input effective current. This input impedance is :

$$Z = 3 + j4 + (4 - j5)\|(2 + j7) = 10.35 + j4.55 = 11.3\angle 23.7^\circ\ \Omega$$

The reactive factor is the sine of the impedance angle : $\sin 23.7^\circ =$ 0.402, and the effective input current is the effective applied voltage divided by the impedance magnitude : $I_{1_{eff}}$ = 100/11.3 = 8.84 A. So,

$$Q = V_{eff} I_{eff} \sin\theta = (100)(8.84)(0.402) = 355 \text{ var}$$

For the second approach we need the effective currents in the three reactances. The effective current in the j4 Ω reactance is the 8.84 A input current that we have found.

That in the - j5 Ω reactance is, by current division,

$$\frac{|2 + j7|}{|4 - j5 + 2 + j7|} \times 8.84 = \left|\frac{2+j7}{6+j2}\right| \times 8.84 = \frac{\sqrt{53}}{\sqrt{40}} \times 8.84 = 10.2\ A$$

and the effective current through the j7 Ω reactance is

$$\frac{|4 - j5|}{|4 - j5 + 2 + j7|} \text{ x } 8.84 = \left|\frac{4-j5}{6+j2}\right| \text{ x } 8.84 = \frac{\sqrt{41}}{\sqrt{40}} \text{ x } 8.84 = 8.95 \ A$$

Summing the vars :

$$Q = (8.84)^2(4) + (10.2)^2(-5) + (8.95)^2(7) = 313 - 520 + 561 = 354 \text{ var}$$

Which is within roundoff error of the first result.

As with power, there are two other equations for getting vars from the inputs. These equations are slight variations of $Q = V_{eff} I_{eff} \sin\theta$ in which, remember, V_{eff} and I_{eff} are the rms input voltage and current and probably not just the voltage across and the current through a single component. To get one equation we substitute in $V_{eff} = I_{eff}Z$:

$$Q = (I_{eff}Z)\, I_{eff} \sin\theta = I^2_{eff} Z \sin\theta$$

$$\text{But} \quad Z\sin\theta = X, \text{ so,}$$

$$Q = I^2_{eff} X$$

in which X is the input reactance of the circuit and not necessarily just the reactance of a single inductor or capacitor.

For the other equation we substitute in $I_{eff} = V_{eff}Y$:

$$Q = V_{eff}(V_{eff}Y)\sin\theta = V^2_{eff}Y\sin\theta$$

We can simplify this by using the fact that the admittance angle is the negative of the impedance angle, and that the susceptance is the magnitude of the admittance times the sine of the admittance angle : $B = Y\sin(-\theta) = -Y\sin\theta$. With this substitution, we get :

$$Q = -V^2_{eff}B$$

Now we will verify these equations using quantities from the last example. For the circuit of Fig. 10-27, we found that $X = 4.55\ \Omega$ and $I = 8.84$ A. So, from $Q = I^2_{eff}X$ we have $Q = (8.84)^2(4.55) = 356$ var, which checks. For the same example,

$$Y = \frac{1}{Z} = \frac{1}{10.35 + j4.55} = 0.081 - j0.0356\ \text{S}$$

The susceptance thus is $B = -0.0356$ S, from which

$$Q = -V^2_{eff}B = -(100)^2(-0.0356) = 356\ \text{var}$$

Complex Power and Apparent Power :

We will now study the relations among real power, reactive power, and another power called complex power. For this, consider the impedance triangle of Fig. 10-28(a), for a load $Z = R + jX$. If we multiply each side by the square of the effective load current, we get the triangle of Fig. 8-28(b). Note that this multiplication does not change the angle θ, the impedance angle. The horizontal side $I^2_{eff} R$ is clearly $P_{av} = P$, the real power absorbed by the load. The vertical side is $I^2_{eff} X$, which is the reactive power Q absorbed by the load. Because this reactive power side is at a right angle to the real power side, reactive power is also called quadrature power. The hypotenuse $I^2_{eff} Z$ is called complex power. It has the quantity symbol S and the unit of volt-ampere, with symbol V.A. The three power quantities are shown in Fig. 8-28(c), which is the same as Fig. 8-28 (b), except for the labeling of the sides. From Fig. 8-28(c), $S = P + jQ$. For obvious reasons the triangle of Fig. 8-28(c) is called the power triangle.

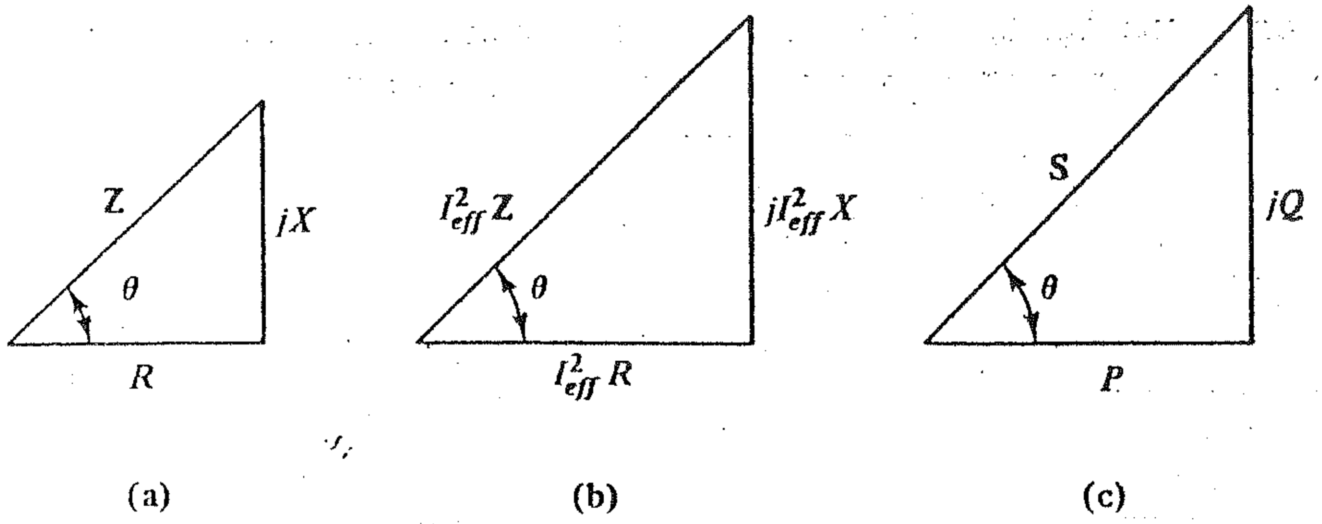


Figure 8-28

The magnitude of S, S, is called apparent power. This name comes from this magnitude equaling the product of the rms applied voltage and rms input current :

$$S = |I^2Z| = |IZ| \times I = VI$$

And from the fact that in DC circuits this product is the power absorbed. That is, by comparison with the DC case, the complex power magnitude appears to be the power absorbed in an AC circuit– hence the name apparent power. Actually, of course, the real power absorbed in an AC circuit is this apparent power multiplied by the power factor.

Another way of finding complex power is by taking the product of the applied voltage phasor and the conjugate of the input current phasor. We need to justify this fact both as to magnitude and angle. Recall that the conjugate of a complex number in polar form differs only in the sign of the angle, and in rectangular form differs only in the sign of the imaginary part. For example, $5\angle -63^\circ$ is the conjugate of $5\angle 63^\circ$ and $5 + j8$ is the conjugate of $5 - j8$. Because the input current phasor and its conjugate have the same magnitude, the product of this conjugate and the applied voltage phasor has a magnitude equal to the rms applied voltage times the rms input current. This magnitude is VI, the apparent power and also magnitude of the complex power S. Now as regards angle, the angle of the product of the applied voltage phasor and the conjugate of the input current phasor is the angle of the voltage phasor minus the angle of the current phasor. The minus sign occurs because of using the conjugate of the current phasor. This difference angle is, course the impedance angle θ. The whole reason for using the conjugate of the current is to get this difference angle. In summary, we have shown that $S = VI^*$

The asterisk (*) in I* designates the conjugate of I.

Example :

A circuit with 100 $\sqrt{2}$ sin (377t + 30°) V applied has an input current of 50 $\sqrt{2}$ sin (377t + 60°) A. Assuming associated references, what is the complex power absorbed by the circuit ?

Solution :

The corresponding voltage phasor is $100\angle 30^{\circ}$ and the corresponding current phasor is $50\angle 60^{\circ}$. Changing the sign of the current angle to get the conjugate of the current phasor, we have for complex power,

$$S = VI^* = (100\angle 30^{\circ})(50\angle -60^{\circ}) = 5000\angle -30^{\circ}) = 4330 - j2500 \ \text{V.A}$$

from which P = 4330 W and Q = - 2500 var

Suppose that we do not know the input voltage and current phasors. How then can we find the complex power absorbed ? One way is to take the vector sum of the individual component complex powers, if they are known. This approach is valid because the total real power is the sum of the individual component real powers, and the total reactive power is the sum of the individual reactive powers.

Example :

The circuit of Fig. 8-29, has two components with indicated loads. Find the total complex power absorbed and the input rms current I.

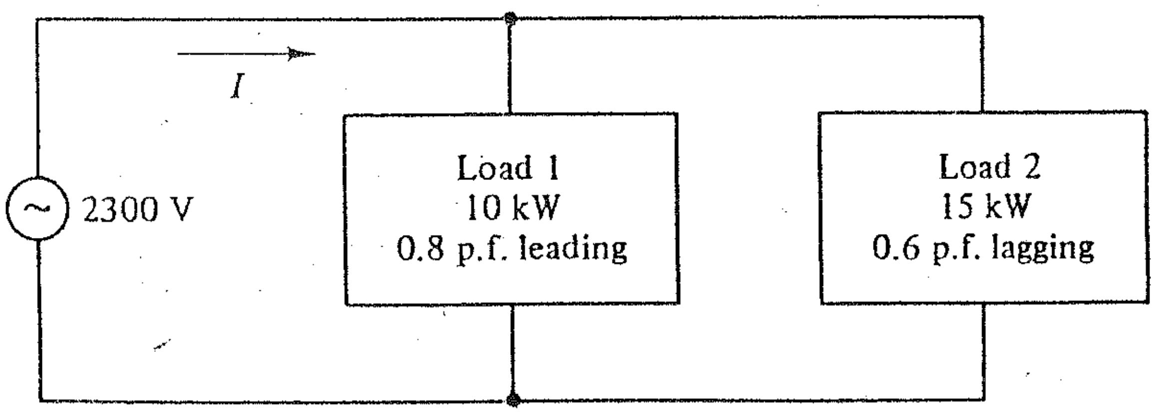


Figure 8-29

Solution :

The best approach here is to find the complex power of both loads, add the complex powers to get the total complex power, and then divide the apparent power by the applied rms voltage to get the rms input current. To get the complex power of each load, we will

use the complex power triangle knowledge that we learned from Fig. 8-28. From Fig. 8-28(c), clearly the apparent power for each load is the real power divided by cos θ, the power factor, which is the cosine of the impedance angle. The corresponding complex power is this apparent power at an angle that is the arc cosine of the power factor for a lagging power factor and the negative of this for a leading power factor. So here, with S_1 and S_2 the designators for the complex powers of loads 1 and 2, respectively.

$$S_1 = \frac{10\ 000}{0.8} \angle -\cos^{-1} 0.8 = 12\ 500\angle -36.9^\circ = 10\ 000 - j7500 \text{ V.A}$$

and

$$S_2 = \frac{15\ 000}{0.6} \angle \cos^{-1} 0.6 = 25\ 000\angle 53.1^\circ = 15\ 000 + j20\ 000 \text{ V.A}$$

giving a total complex power of

$$\mathbf{S = S_1 + S_2 = 10\ 000 - j7500 + 15\ 000 + j20\ 000}$$

$$\mathbf{= 25\ 000 + j12\ 500 = 27\ 951\angle 26.6^\circ\ \text{V. A}}$$

The magnitude, 27 951, of this complex power is the apparent power. This, divided by the applied rms voltage, is the input rms current :

$$\mathbf{I} = \frac{27\ 951}{2300} = 12.2 \text{ A}$$

PROBLEMS

1) A series RLC circuit has 100 V rms applied. If the resistor and inductor rms voltages are 60 V and 100 V, respectively, what is the capacitor rms voltage ? Further, find all voltage phasors given that the current phasor has a 30° angle.

2) Find the impedances in polar form of the series combination of 1 Ω resistor, a 1 mH inductor, and a 10 μF capacitor at frequencies of 100 Hz, 5 kHz & 1 MHz.

3) Repeat Prob. 2 with the components in parallel instead of in series.

4) A coil energized by 120 V at 60 Hz draws a 5 A current that lags the applied voltage by 60°. What are the coil resistance and inductance ?

5) Two elements in series draw a current of 12 sin (200t + 20°) A in response to an applied voltage of 48 sin (200t + 60°) V. Find the two elements.

6) A series RC circuit has a 10 Ω resistor and a 0.1 μF capacitor. At what frequency does the current lead the applied voltage by 60° ?

7) Three components in series produce a $60\angle -30^\circ$ Ω impedance at 1 krad/s. If one component is a 9 mH inductor, what are the other two ?

8) Find the total impedance in polar form of three series-connected impedances : $8\angle 60^\circ$ Ω , $10\angle -40^\circ$ Ω , and $12\angle -75^\circ$ Ω.

9) A capacitor is in series with a 2 H coil having a 10 Ω winding resistance. What capacitance makes the combination purely resistive at 1 kHz ?

10) A 0.1 H coil with a 20 Ω winding resistance is shunted by a 20 Ω resistor. What series RL circuit has the same impedance at 60 Hz ?

11) For the circuit of Fig. 8-30 find the input impedance at 10 krad/s. Note that the dependent source controlling voltage is across the resistor and inductor.

12) A voltage source of 100 sin $(1000t + 10^\circ)$ V, a 10 Ω resistor, and a 10 mH inductor are in series. Find the current out of the positive source terminal and also the voltage drop across the resistor and inductor.

13) A parallel RLC circuit has resistor, inductor, and capacitor rms currents of 30 A, 40 A, and 50 A, respectively. What is the input rms current ? Also, find all current phasor given that the capacitor current phasor has a -60° angle.

14) A capacitor and resistor in parallel have an admittance of $120\angle 60^\circ$ S at 60 Hz. Find the capacitance and resistance.

15) **Find the frequency at which a 2 mH inductor and a parallel 1 μF capacitor appear to be an open circuit.**

16) **For the circuit of Fig. 8-31 find the input admittance at 20 krad/s.**

17) **Find the type and impedance in ohms of series circuit elements that must be in the closed container of Fig. 8-32 in order for the indicated voltages and currents to exist at the input terminals. (Find the simplest series circuit that will satisfy the indicated conditions).**

18) **For the circuit of Fig. 8-33**

(a) Find the total impedance Z_T in polar form.

(b) Draw the impedance diagram.

(c) Find the value of C in microfarads and L in henries.

(d) Find the current I and the voltages V_R, V_L and V_C in phasor form.

(e) Draw the phasor diagram of the voltages E, V_R, V_L and V_C and the current I.

(f) Verify Kirchoff's voltage law around the closed loop.

(g) Find the average power delivered to the circuit.

(h) Find the power factor of the circuit and indicate whether it is leading or lagging.

(i) Find the sinusoidal expressions for the voltages and currents.

(j) Plot the waveforms for the voltages and current on the same set of axes.

19) Find the series element or elements that must be in the enclosed container of Fig. 8-34 satisfy the following conditions :

(a) Average power to circuit = 300 W.

(b) Circuit has a lagging power factor.

20) For the network of Fig. 8-35 :

(a) Calculate E, I_R and I_L in phasor form.

(b) Calculate the total power factor and indicate whether it is leading or lagging.

(c) Calculate the average power delivered to the circuit.

(d) Draw the admittance diagram.

(e) Draw the phasor diagram of the currents I_S, I_R and I_L and the voltage E.

(f) Find the current I_C for each capacitor using only Kirchhoff's current law.

(g) Find the series circuit of one resistive and reactive element that will have the same impedance as the original circuit.

22) For the element or elements that must be in the closed container of Fig. 8-36, to satisfy the following conditions. (Find the simplest parallel circuit that will satisfy the indicated conditions).

(a)Average power to the circuit = 3000 W.

(b) Circuit has a lagging power factor.

23) Find the network of Fig. 8-37

(a) Find the current I_1.

(b) Calculate the voltage V_C using the voltage divider rule.

(c) Find the voltage V_{ab}.

24. Find the average power delivered to R_4 in fig 8.38.

25. For the circuit of Fig. 8.39
   a. Find the total admittance Y_T in polar form.
   b. Find the value of C in microfarads and L in henries.
   c. Find the voltage E and currents I_R, I_L and I_C in phasor from.
   d. Verify Kirchhoff's current law at one node.
   e. Find the average power delivered to the circuit.
   f. Find the power factor of the circuit and indicate whether it is leading or lagging.
   g. Find the sinusoidal expressions for the currents and voltage.

26. For the network of Fig. 8.40.
   a. Find the total impedance Z_1 and the admittance Y_T .
   b. Find the source current I, in phasor form.
   c. Find the current I_1, and I_2, in phasor form.
   d. Find the voltage V_1, and V_{ab} in phasor form.
   e. Find the average power delivered to the network.
   f. Find the power factor of the network and indicate whether it is leading or lagging.

27. Find the current I_s for the network of Fig. 8.41, find the effect of one reactive element on the resulting calculations.

28. Find the average power delivered to R_4 in Fig. 8.42, check active and reactive power balance.

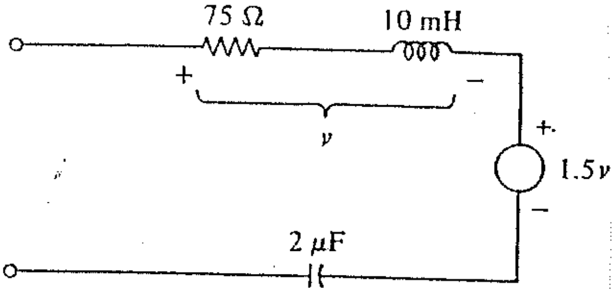


Fig. 8-30

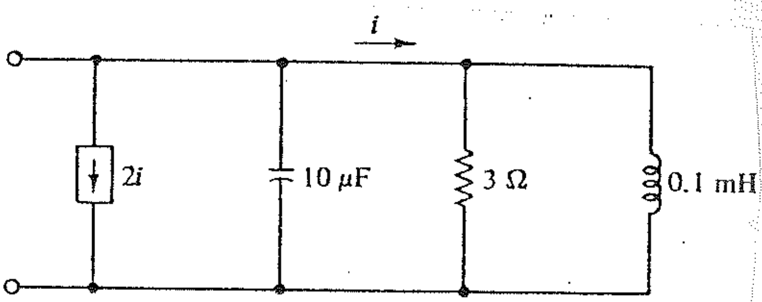


Fig. 8-31

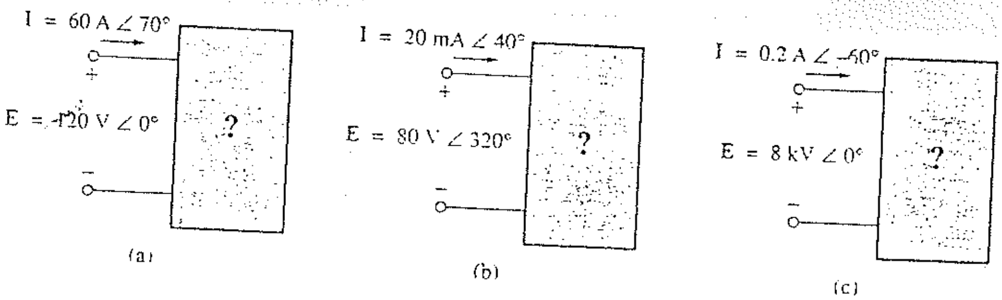


Fig. 8-32

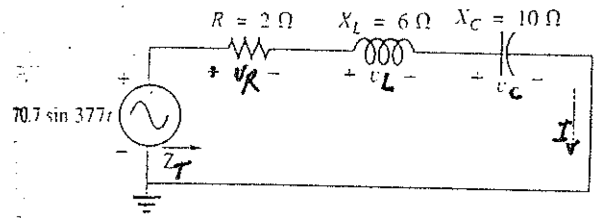


Fig. 8-33

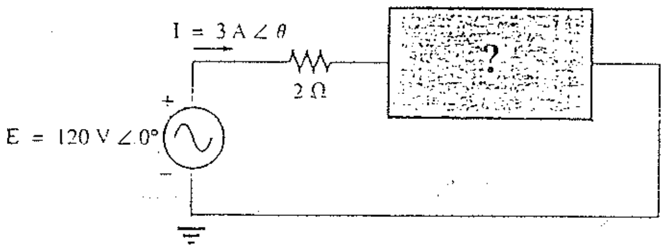


Fig. 8-34

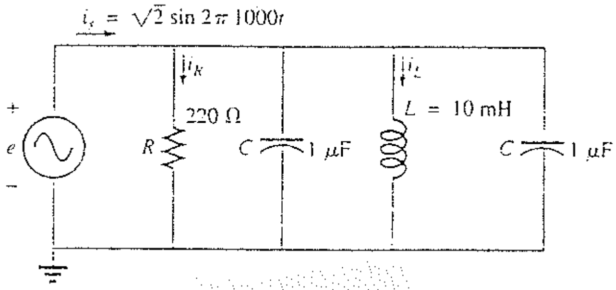


Fig. 8-35

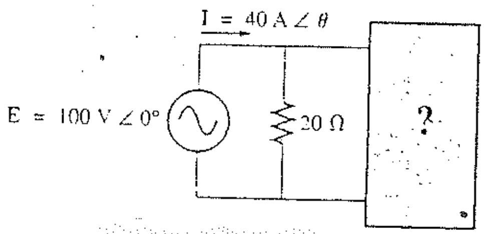


Fig. 8-36

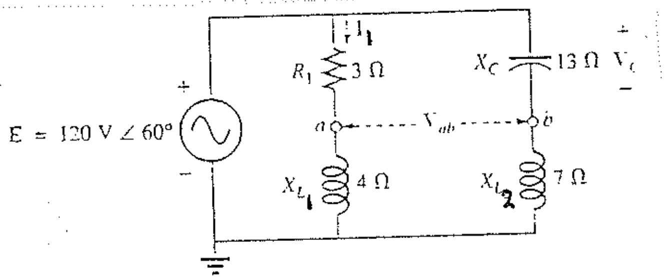


Fig. 8-37

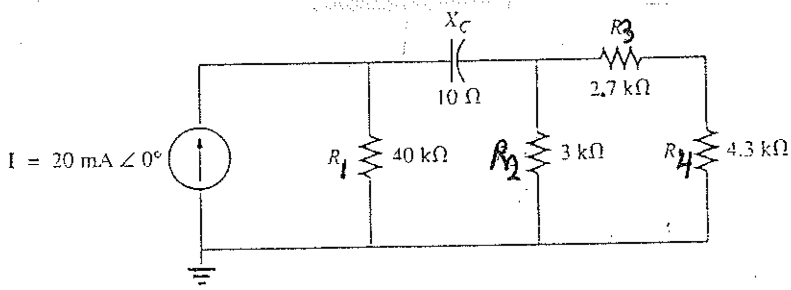


Fig. 8-38

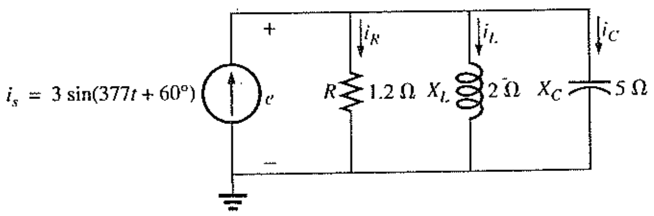


FIG. 8.39

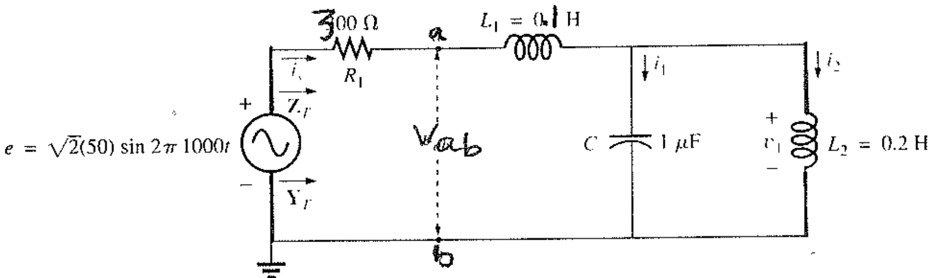


FIG. 8.40

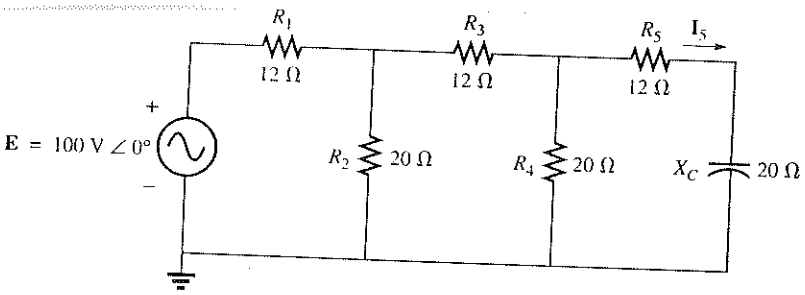


FIG. 8.41

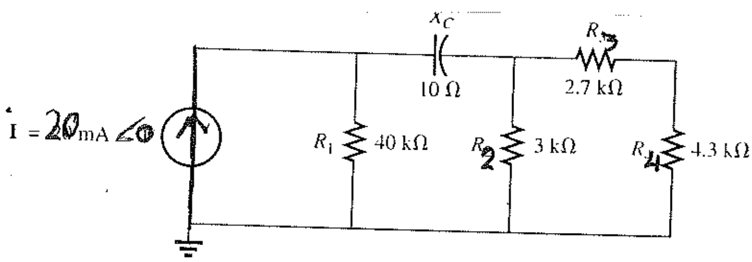


FIG. 8.42

Chapter Nine

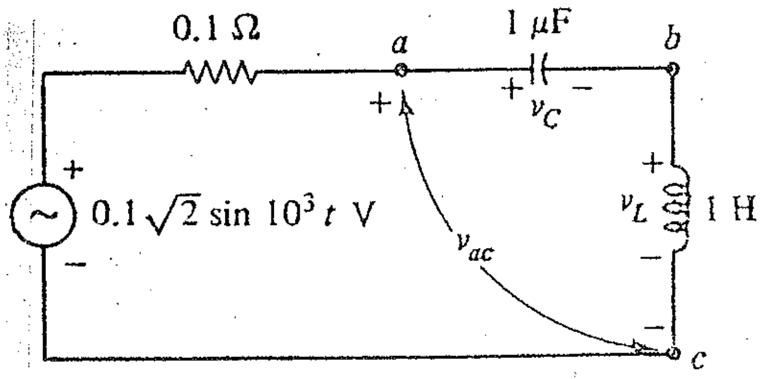
RESONANCE

Introduction :

In an ac circuit the energy stored in inductors and capacitors can interact at certain frequencies to produce responses that may be suppressing when first encountered.

Example :

For the circuit of Fig. 9-1(a), find the capacitor voltage v_C , the inductor voltage v_L , and the sum v_{ac} of these two voltages.



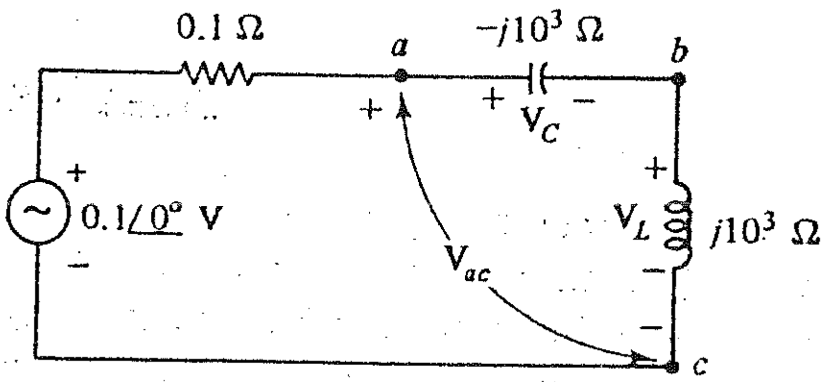
(a)

Figure 9-1

Solution :

As usual, the first step in the analysis is making a corresponding frequency-domain circuit, as illustrated in Fig. 9-1(b). Then by voltage division,

$$V_C = \frac{-j10^3}{0.1 - j10^3 + j10^3} \times 0.1 = -j10^3 = 10^3\angle -90^o \ \text{V}$$



(b)

Figure 9-1 (cont.)

also,

$$V_L = \frac{j10^3}{0.1 - j10^3 + j10^3} \times 0.1 = j10^3 = 10^3\angle 90^o \ \text{V}$$

and

$$\mathbf{V_{ac} = V_C + V_L = -j10^3 + j10^3 = 0 \ V}$$

The corresponding functions of time are

$$v_C = 1000\sqrt{2}\ \sin(10^3t - 90^\circ) = -\,1000\sqrt{2}\ \cos 10^3t\ \text{ V}$$

$$v_L = 1000\sqrt{2}\ \sin(10^3t + 90^\circ) = 1000\sqrt{2}\ \cos 10^3t\ \text{ V}$$

and $\quad v_{ac} = 0\ \text{ V}$

With such a small applied voltage, the circuit of Fig. 9-1(a), might seem almost obviously safe to handle. But not so! Look at the large voltages across the capacitor and inductor. Each of these voltages has an rms value that is 10000 times that of the source voltage. It may be unsafe to place our hands across nodes a and b or across nodes b and c, there being 1000 V across these nodes before touching them. Naturally, placing our hands across these nodes places the resistance of our body in the circuit, and that changes the circuit so that the voltages probably drop–perhaps to a safe level. Regardless, although not a good idea, it is safe to place our hands across nodes a and c because there is 0 V across these nodes. This 0 V results from the capacitor and inductor voltages being equal in magnitude but 180° out of phase, which means that they cancel when added. In addition to this voltage cancellation, the circuit has reactance cancellation with the result that the input impedance is purely resistance (0.1 Ω).

This is an example of a resonant circuit. In general, an ac circuit with inductors and capacitors is in resonance if the input impedance is purely resistive. The frequency of operation at which this occurs is the resonant frequency, which is designated by f_r in hertz and by ω_r in radians per second.

The circuit of Fig. 9-1(a), is a series resonant circuit. We will study it in the next section. Following that is a presentation of two types of parallel resonant circuits.

Series Resonant Circuit :

Although the series RLC circuit of Fig. 9-2, is often called a series resonant circuit, it is in resonance only when energized by a sinusoidal source operating at the resonant frequency f_r–the frequency at which the inductive and capacitive reactances add to zero. We can use this zero reactance relationship to solve for the resonant frequency in terms of L and C. Summing the reactances we get $\omega_r L - 1/\omega_r C = 0$, or $\omega_r L = 1/\omega_r C$, which results in $\omega^2{}_r = 1/LC$. Taking the square root of this, we get the two important equations for the series resonant frequencies in terms of L and C :

$$\omega_r = \frac{1}{\sqrt{LC}} \qquad \textbf{and} \qquad f_r = \frac{1}{2\Pi\sqrt{LC}}$$

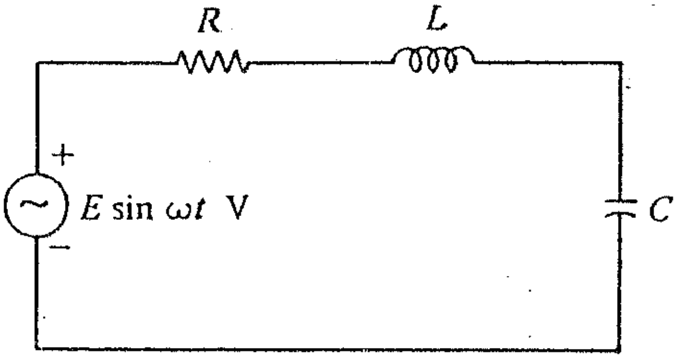


Figure 9-2

<u>Example</u> :

A series RLC circuit is resonant at 455 kHz. If the inductance is 2 μH, what is the capacitance ?

<u>Solution</u> :

From the resonant frequency formula squared,

$$\mathbf{C} = \frac{1}{(2\Pi f_r)^2 L} = \frac{1}{\left(2\Pi \text{ x } 455 \text{ x } 10^3\right)^2\left(2 \text{ x } 10^{-6}\right)} = 0.0612\ \mu\text{F}$$

For a better understanding of series resonance and the operation of a series RLC circuit at different frequencies, consider Fig. 9-3(a), which has three graphs, one of inductive reactance, one of capacitive

reactance, and one of their sum, all plotted against frequency. The inductive reactance of $X_L = \omega L = 2\pi fL$, being directly proportional to frequency, is a straight line through the origin, increasing with frequency at a slope of $2\pi L$. In contrast, the capacitive reactance, $X_C = -1/\omega C = -1/2\pi fC$, is inversely proportional to frequency, having large negative values at frequencies near zero (dc), and approaching zero for large frequencies.

Being inversely proportional to frequency, the capacitive reactance dominates the sum at low frequencies. In fact, this reactance approaches negative infinity as the frequency approaches zero or dc. But for increasing frequencies, the capacitive reactance decreases in magnitude while the inductive reactance increases. At f_r the resonant frequency, the inductive reactance has sufficiently increased and the magnitude of the capacitive reactance has sufficiently decreased such that their magnitudes are equal, and so their sum is zero. At higher frequencies the inductive reactance dominates. It follows, then, as shown in Fig. 9-3(b), that the angle θ of $j(X_L + X_C)$ is -90° at low frequencies, frequencies lower than the resonant frequency, because it is at these frequencies that the capacitive reactance dominates.

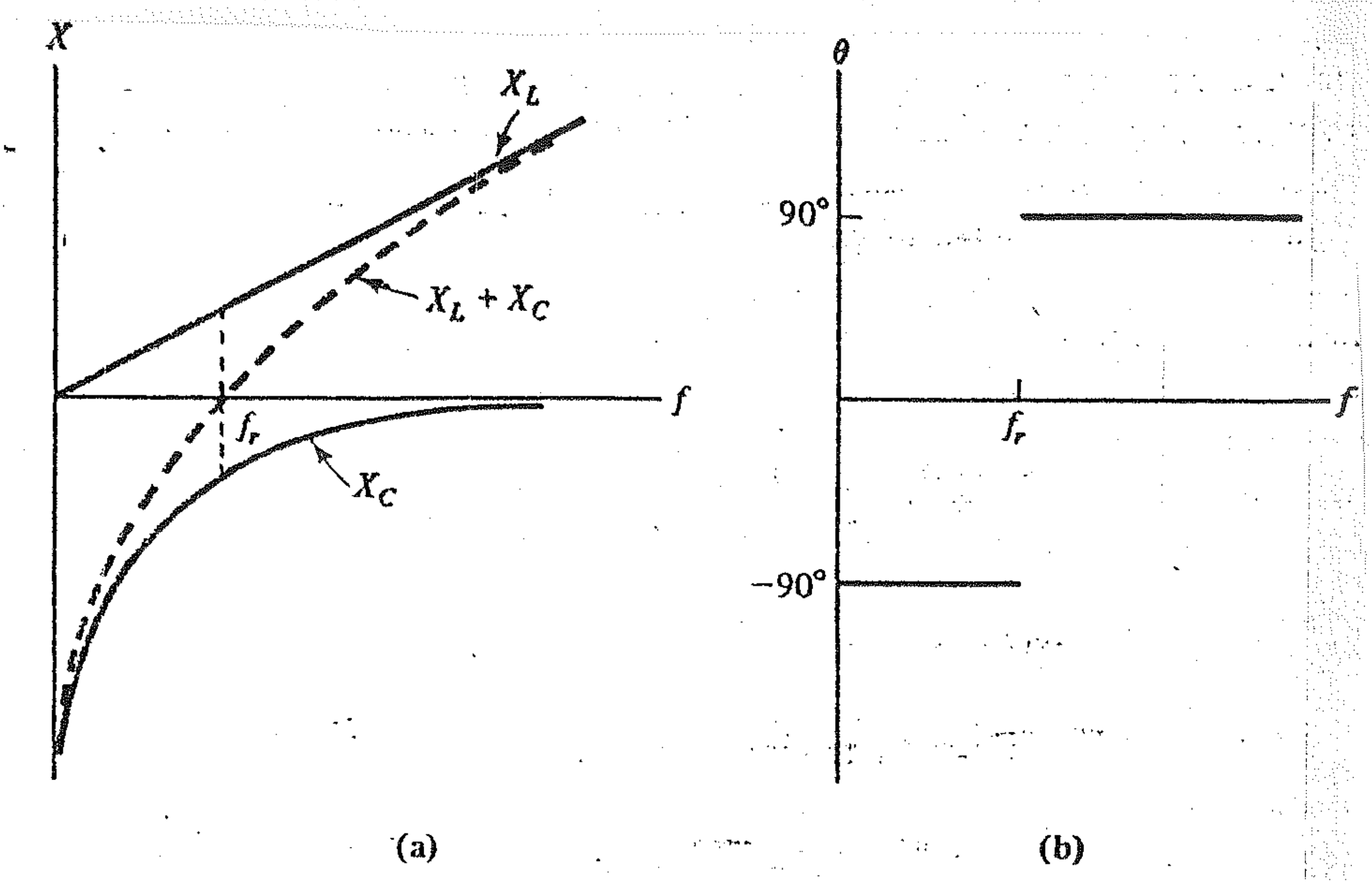


Figure 9-3

As frequencies greater than the resonant frequency, the angle of the reactance sum is 90°, corresponding to the dominate inductive reactance. Incidentally, if the angle was that of impedance instead of just the reactance, the resistance would have a smoothing effect on the plot of Fig. 9-3(b), gradually bringing the angle to 0° at the resonant frequency, and causing a gradual increase in angle magnitude both to the left and to the right of the resonant frequency.

From what has been discussed it should be clear that a series RLC circuit is capacitive at frequencies lower than the resonant frequency, is inductive at frequencies higher than the resonant frequency, and is purely resistive at the resonant frequency. Also, at the resonant frequency the input impedance is equal to the series resistance.

Quality Factor :

In Fig. 9-4, the variation of circuit current I with variation of ω, for different values of R, is plotted for R.L.C circuit in series.

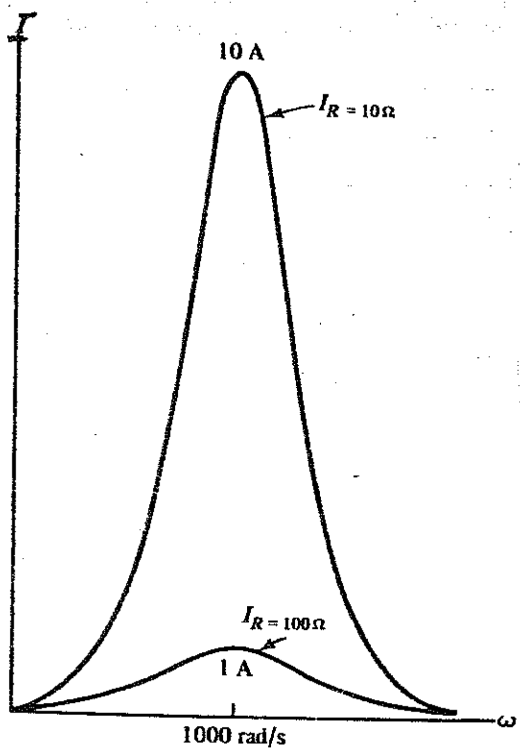


Figure 9-4

The narrowness as well as the heights of the resonance curves indicate the circuit selectivity or discriminating action, This selectivity is, in fact, the entire reason for using a resonant circuit. An indicator of selectivity is the quality factor Q of a circuit. The general definition for the Q of any circuit, and not just a series RLC circuit, is :

$$\mathbf{Q} = \frac{\textit{resonant} \text{ frequency}}{\textit{bandwidth}} = \frac{\omega_r}{\omega_2 - \omega_1} = \frac{f_r}{f_2 - f_1}$$

in which the bandwidth is the frequency range between half-power frequencies ω_2 and ω_1 in radians per second and between f_2 and f_1 in hertz. By definition, the half-power frequencies are the frequencies at which the average power absorbed is one-half the average power absorbed at the resonant frequency if the input has the same peak at these frequencies.

We will consider these half-power frequencies more. In terms of current for a series RLC circuit, and as Fig. 9-5 illustrates, these half-power frequencies are the frequencies at which the rms current is down to $1/\sqrt{2}$ of its resonant frequency maximum value I_r. The reason for this result is fairly apparent. At the resonant frequency the average power dissipated in R is $I^2_r R$. For the current $I_r\sqrt{2}$ at

a half-power frequency, the average power dissipated is $P = (I_r\sqrt{2})^2R$ $= I^2_rR/2$, one-half the average power dissipated at the resonant frequency. We shall see later that for the current to be decreased by a factor of $1/\sqrt{2}$ in a series RLC circuit, the magnitude of the reactance equals the resistance.

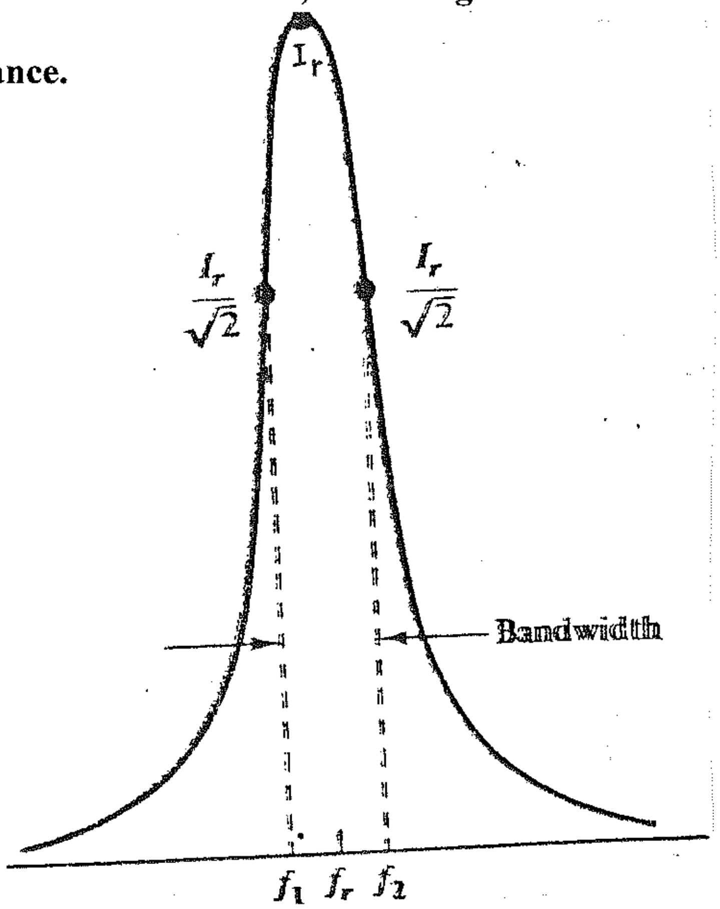


Figure 9-5

As is apparent from the definition of Q, the smaller the bandwidth for a given resonant frequency, the greater the Q. This fact verifies that Q is a measure of selectivity. The more the value of Q is, the narrower is the curve produced by a given bandwidth since the bandwidth becomes a smaller fraction of the frequency range being considered.

Now having some understanding of bandwidth and half-power frequencies, we will derive equations for these half-power frequencies in terms of R, L, and C for a series RLC circuit. Recall that at both half-power frequencies, the current is less than its resonant value by a factor of $\sqrt{2}$, assuming as always, sinusoidal input voltages of the same peak value at the different frequencies. This decrease means that at these frequencies the impedance magnitude is greater than R, the impedance at resonance, by a factor of $\sqrt{2}$. Specifically, at the half-power frequencies,

$$\left|R + j(X_L + X_C)\right| = \sqrt{2} \times R \qquad \text{or} \qquad \sqrt{R^2 + (X_L + X_C)^2} = \sqrt{2} \times R$$

Squaring,

$$R^2 + (X_L + X_C)^2 = 2R^2$$

and simplifying,

$$(X_L + X_C)^2 = R^2$$

Taking the square root :

$$X_L + X_C = \pm R$$

Taking the square root results in both a positive and a negative sign for R. The negative sign applies at the lower half-power frequency, ω_1 at which the magnitude of X_C exceeds that of X_L : $\omega_1 L - 1/\omega_1 C = -R$. The positive sign applies at the upper half-power frequency ω_2 at which X_L is greater than the magnitude of X_C : $\omega_2 L - 1/\omega_2 C = R$. We need to work with each equation separately to solve for the half-power frequencies.

We will first work with the lower half-power frequency equation, $\omega_1 L - 1/\omega_1 C = -R$. If we divide through by L and multiply through by ω_1, and then rearrange the equation, the result is :

$$\omega_1^2 + \frac{R}{L}\omega_1 - \frac{1}{LC} = 0$$

Then from the quadratic formula,

$$\omega_1 = \frac{-(R/L) \pm \sqrt{(R/L)^2 + 4/LC}}{2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

In this result only the positive sign is valid because it makes ω_1 positive, while the negative sign makes ω_1 negative, negative frequencies have no physical significance. Consequently,

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

and

$$f_1 = \frac{1}{2\Pi}\left[-\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}\right]$$

Now working with the upper half-power frequency equation, $\omega_2 L - 1/\omega_2 C = R$, in a similar fashion, we get :

$$\omega^2_2 - \frac{R}{L}\omega_2 - \frac{1}{LC} = 0$$

which solves to

$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

and

$$f_2 = \frac{1}{2\Pi}\left[\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}\right]$$

We can use these results to find the bandwidth (BW) in terms of R, L, and C :

$$BW = \omega_2 - \omega_1 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} - \left[-\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \right]$$

Which simplifies to :

$$BW = \frac{R}{L} \ rad/s \qquad \text{and} \qquad f_2 - f_1 = \frac{R}{2\Pi L} \ \text{Hz}$$

Example :

A series RLC circuit with a 10 Ω resistor has a resonant radian frequency of 1000 rad/s and a bandwidth of 10 rad/s. Find the capacitance and inductance.

Solution :

From the bandwidth equation BW = R/L, L = R/BW = 10/10 = 1 H. And from the resonant frequency equation $\omega_r = 1/\sqrt{LC}$,

$$C = \frac{1}{L\omega_r^2} = \frac{1}{(1)(1000)^2} = 1 \ \mu F$$

Intuitively, one might think that the resonant frequency bisects the passband and so is halfway between the half-power frequencies. But not so, although for a high-Q circuit (Q > 10), it is close to being halfway. We can see this by finding the half-power frequencies in terms of the bandwidth and the resonant frequency. For this we will substitute BW = R/L and $\omega_r = 1/\sqrt{LC}$ into the half-power frequency equations. The result is :

$$\omega_1 = -\frac{BW}{2} + \sqrt{\left(\frac{BW}{2}\right)^2 + \omega_r^2} \quad \text{and} \quad \omega_2 = \frac{BW}{2} + \sqrt{\left(\frac{BW}{2}\right)^2 + \omega_r^2}$$

which show that the half-power frequencies are symmetrical about $\sqrt{(BW/2)^2 + \omega_r^2}$ and not ω_r. Observe that if it were not for the $(BW/2)^2$ term under the square-root sign ω_r would be midway between ω_1 and ω_2, for without this term, $\omega_1 = \omega_r - BW/2$ and $\omega_2 = \omega_r + BW/2$. Also note that this is approximately so if $\omega_r > (BW/2)$, as it is for high-Q circuits.

Example :

A series RLC circuit has a resonant frequency of 10^4 rad/s and a bandwidth of 200 rad/s. What are the half-power or cutoff frequencies ?

Solution :

The lower half-power frequency is

$$\omega_1 = -\frac{BW}{2} + \sqrt{\left(\frac{BW}{2}\right)^2 + \omega_r^2} = -100 + \sqrt{(100)^2 + (10^4)^2} = 9900.5 \text{ rad/s}$$

and the upper half-power frequency is

$$\omega_2 = \frac{BW}{2} + \sqrt{\left(\frac{BW}{2}\right)^2 + \omega_r^2} = 100 + \sqrt{(100)^2 + (10^4)^2} = 100.5 \text{ rad/s}$$

Turning now to Q, we know that in general it equals the resonant frequency divided by the bandwidth. But specifically, what is it in terms of component value for a series RLC circuit ? There are three popular equations, one of which we can get by substituting R/L for the bandwidth :

$$\mathbf{Q} = \frac{\omega_r}{BW} = \frac{\omega_r}{R/L} \qquad \text{and} \qquad Q = \frac{\omega_r L}{R}$$

That is, the Q of a series RLC circuit equals the inductive reactance at resonance divided by the circuit resistance. We can get another important equation from this equation by substituting $1/\omega_r C$ for $\omega_r L$:

$$\mathbf{Q} = \frac{1}{\omega_r CR}$$

Finally, with the substitution of $\omega_r = 1/\sqrt{LC}$ into either of these equations, we get the third important equation :

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

These equations are also useful for finding circuit component values that give a desired Q.

<u>Example</u> :

A series RLC circuit has a Q of 100, a resonant frequency of 1 MHz, and an inductance of 1 mH. Find the capacitance and resistance.

<u>Solution</u> :

First finding C from the basic f_r formula,

$$C = \frac{1}{(2\Pi f_r)^2 L} = \frac{1}{(2\Pi 10^6)^2(10^{-3})} = 25.3 \text{ pF}$$

Then finding R from a Q formula,

$$R = \frac{\omega_r L}{Q} = \frac{(2\Pi 10^6)^2(10^{-3})}{100} = 62.8\ \Omega$$

We will consider just one other equation for Q. And this one is completely general, applying not only to all electrical circuits but to mechanical systems and other systems as well. Incidentally, for electrical circuits it is related to the vars-power definition. This equation for Q is :

$$Q = 2\pi \frac{\textit{total energy stored}}{\text{energy dissipated per cycle}}$$

With resonance conditions assumed.

It is easy to prove that this equation is valid for the series RLC circuit. As regards the numerator, the total energy stored is equal to that stored in the inductor when the current is at its peak value, because then, as will be shown, no energy is stored in the capacitor. If the peak current is I_m , this stored energy is $(½)LI^2_m$. In terms of the rms current I_r , this is LI^2_r since $I_m = \sqrt{2}\, I_r$. Now, as regards the denominator, the energy dissipated in the resistor per cycle is the average power times the period : I^2_rR x T, in which T is the period. But the period equals the inverse of the frequency in hertz : $T = 1/f_r = 2\pi/\omega_r$. Consequently, the energy dissipated per cycle is $I^2_rR \times 2\pi/\omega_r$, and

$$2\pi\ \frac{\textit{total energy stored}}{\text{energy dissipated per cycle}} = 2\Pi\ \frac{\mathrm{L}\mathrm{I}_r^2}{\left(\mathrm{I}_r^2 R\right)\left(2\Pi/\omega_r\right)} = \frac{\omega_r L}{R} = \mathrm{Q}$$

The proof is complete.

Three-Branch Parallel Resonant Circuit :

Figure 9-6, shows a three-branch parallel RLC circuit energized by a variable frequency but constant peak sinusoidal current source. Included also is an ammeter for measuring the current flow to the inductor and capacitor. This circuit is the dual of the series RLC circuit of Fig. 9-2, in that all of the series circuit discussion applies

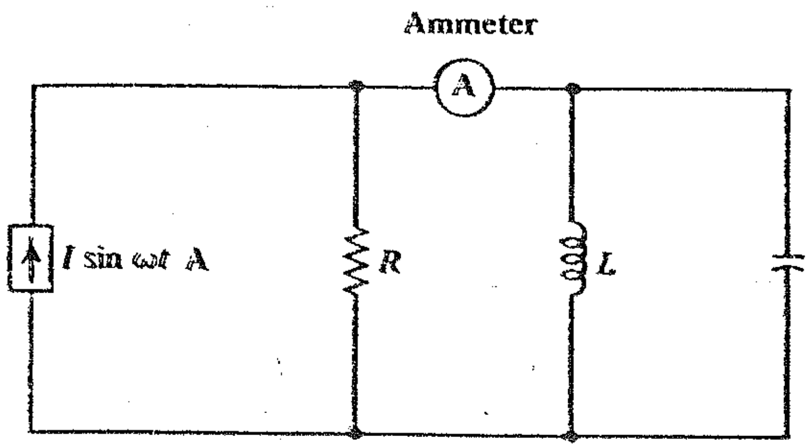


Figure 9-6

as well to this circuit with the interchange of some words : voltage and current, conductance and resistance, and inductance and capacitance. For example, this parallel circuit has a resonant current rise instead of voltage rise. Also, its $Q = \omega_r C/G$ instead of $\omega_r L/R$, and so on.

By definition, the resonant frequency of this circuit is the frequency at which the circuit appears purely resistive. In turn this means that the capacitive and inductive susceptances add to zero : $\omega_r C - 1/\omega_r L = 0$, from which

$$\omega_r = \frac{1}{\sqrt{LC}}$$

the same as for the series RLC circuit.

At the resonant frequency the total impedance is just R because the parallel capacitor and inductor together produce an open circuit as a result of their suscaptance adding to zero. This open circuit causes a zero ammeter reading. At other frequencies, however, the impedance is less because of the shunting by L and C. In fact, at the frequency extreme of 0 rad/s (dc), the impedance magnitude is zero because of shorting by the inductor. At the other frequency extreme of infinity, the impedance is also zero because of shorting by the capacitor. In between these extremes the impedance has a peak equal to R at the resonant frequency ω_r as shown in Fig. 9-7. Also illustrated are the half-power frequencies ω_1 and ω_2 which we will

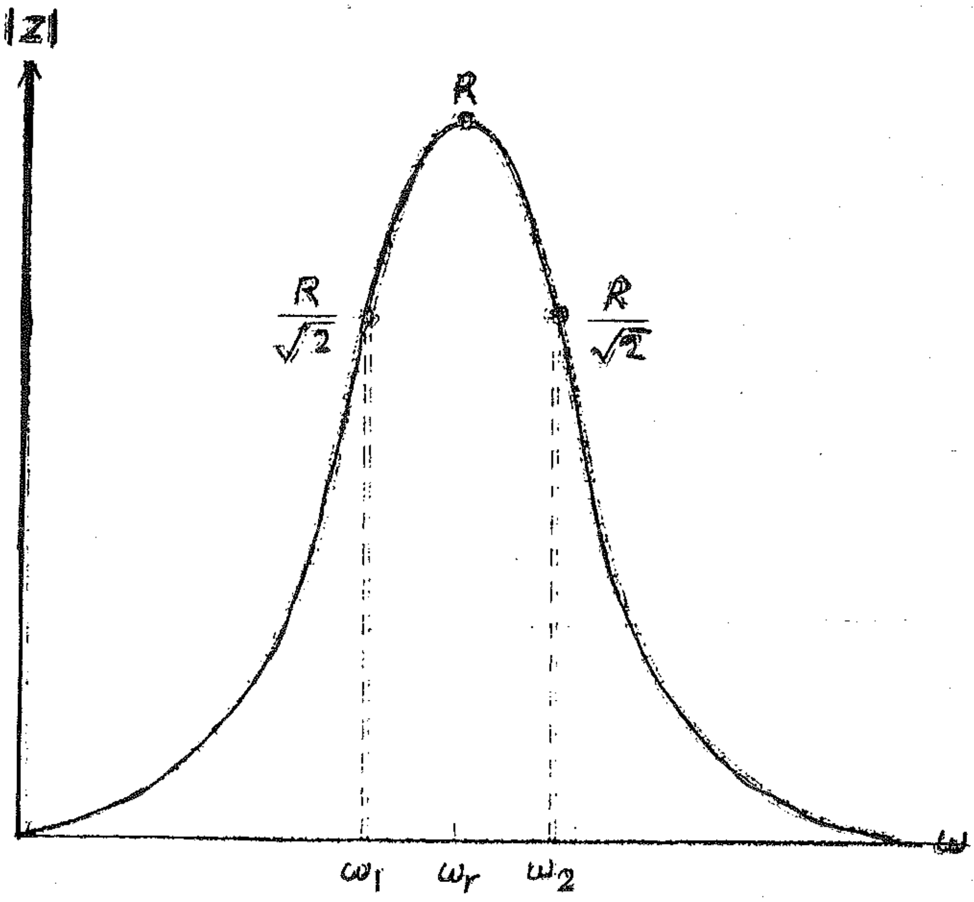


Figure 9-7

now show occur at those frequencies at which the impedance magnitude is down to $R/\sqrt{2}\ \Omega$.

At the half-power frequencies the power dissipation in R must, of course, be one – half that at the resonant frequency. In turn, this

means that the voltage magnitude (there is only one voltage across this parallel combination) must be $V_r/\sqrt{2}$ for half-power, in which V_r is the rms voltage at resonance. Checking, we see that this voltage produces an average power of :

$$P = \frac{\left(V_r/\sqrt{2}\right)^2}{R} = \frac{1}{2}\frac{V_r^2}{R}$$

which is one-half the average power absorbed at the resonant frequency. The voltage drop by a factor of $1/\sqrt{2}$ clearly requires an impedance magnitude drop by a factor of $1/\sqrt{2}$, assuming the same rms input current at ω_1 and ω_2 as at ω_r . So, in Fig. 9-7, ω_1 and ω_2 are the frequencies corresponding to impedance magnitude points of $R/\sqrt{2}$, and the proof is complete.

We can use the inverse of $R/\sqrt{2}$ to get two equations for the Q of the circuit of Fig. 9-6. This inverse is the magnitude of the admittance at the half-power frequencies. So,

$$\left|\frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)\right| = \frac{\sqrt{2}}{R}$$

in which ω is either half-power radian frequency. We could solve this equation for the half-power frequencies in terms of R, L, and C

in a similar fashion as we did for the series RLC circuit. Then by subtracting the equation for the upper half-power frequency from the equation for the lower half-power frequency, we would have the bandwidth as a function of R, L, and C. Then we could substitute the equation for bandwidth into the general definition of Q: $Q = \omega_r/BW$. If we did all this, the result would be :

$$Q = \frac{R}{\omega_r L} = \omega_r CR$$

which are just the inverses of the corresponding series RLC equations for Q. Resistance being in the numerators of these equations should not be too surprising because the input impedance is just R at the resonant frequency. This, in turn, means that the larger the R, the greater impedance peak in Fig. 9-7, as to be expected for a higher-Q circuit.

Now we will consider the resonant current rise in the circuit of Fig. 9-6. At the resonant frequency all the source current flows through the resistor to produce a voltage of IR, which voltage also appears across the inductor and capacitor. This voltage times the capacitance susceptance is the rms capacitor current: $I_C = (IR)(\omega_r C) = (\omega_r CR)I$. But $Q = \omega_r CR$. So,

$$I_C = QI$$

Which specifies that the rms capacitor current at resonance is Q times the source current, this despite the fact that the ammeter in Fig. 9-6, reads zero. The inductor rms current is the same :

$$\mathbf{I}_L = \frac{RI}{\omega_r L} = QI$$

With these inductor and capacitor currents possibly being quite large, how is it that no source current flows to be inductor and capacitor ? The answer to this is that although the inductor and capacitor currents are equal in rms values, they are 180° out of phase –the inductor current lags the voltage by 90° and the capacitor current leads it by 90°. Being 180° out of phase and having the same rms values, the inductor and capacitor currents add to zero, which is the reason for the zero ammeter reading.

The parallel inductor and capacitor together form a tank circuit that at the resonant frequency stores a constant amount of energy, as must be the case with no current flow from the source. As the voltage magnitude decreases, energy flows from the capacitor to the inductor. When the capacitor voltage gets to zero, all the energy that was stored in the capacitor becomes stored in the inductor magnetic

field. On the other hand, as the voltage magnitude increases, the inductor gives up energy to the capacitor, with all the energy going to the capacitor when the voltage is at a peak, either positive or negative. At such times the inductor current is zero because this current lags the voltage by 90°. This energy shuttles back and forth between inductor and capacitor, with possibly a large circulating current in the tank circuit but absolutely no current flow from the source to the parallel inductor-capacitor combination.

Example :

If the circuit of Fig. 9-6 has an rms source current of 10 A and component values of R = 10 Ω, L = 0.25 mH, and C = 1000 μF, and if the circuit is in resonance, what are the rms component currents and the rms voltage ?

Solution :

Of course, at resonance all the 10 A source current flows through the resistor. Consequently, this current produces an rms voltage of 10 x 10 = 100 V. The inductor and capacitor rms currents are each Q times the rms source current. This Q is $\omega_r CR = (1/\sqrt{LC})(CR) = 20$. So, these currents are 20 x 10 = 200 A each.

One last general comment about the three-branch parallel RLC circuits. Below the resonant frequency, the inductive susceptance magnitude is greater than the capacitive susceptance magnitude, making the circuit inductive. Above the resonant frequency the capacitive susceptance magnitude is greater, making the circuit capacitive. Put another way, at frequencies below resonance more current flows through the inductor than through the capacitor because the inductor reactance is less than that of the capacitor. With a greater current through the inductor, the circuit is inductive. At frequencies above the resonant frequency, more current flows through the capacitor than through the inductor because the capacitor reactance is less than that of the inductor. With a greater current through the capacitor, the circuit is capacitive. As to be expected, this inductive and capacitive characteristic is just the opposite of that for the series RLC circuit.

The three-branch parallel RLC circuit is not completely practical because the inductor is ideal in the sense of zero resistance. The resistance of an actual inductor is seldom negligible, the principal exception being for a parallel RLC circuit in which the coil power loss is negligible in comparison with the shunt resistor power loss. Omitting resistance in the capacitor branch is, in contrast, a sound approximation because an actual capacitor is nearly ideal. The two-branch parallel RLC circuit of the next section is approximately practical because it includes the inductor series resistance.

Two-Branch Parallel Resonant Circuit :

Figure 9-8, illustrates the two-branch parallel resonant circuit that we will, for brevity, often refer to as a tank circuit. In it the resistance is that of the coil. Almost always the coil has a high Q; that is, $Q = \omega L/R > 10$ for the frequencies of interest.

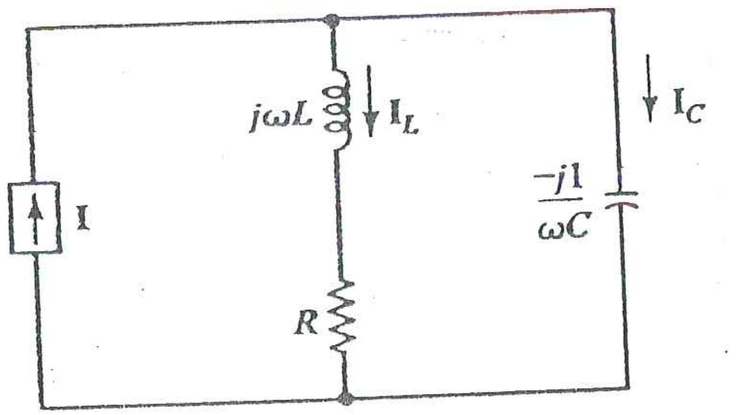


Figure 9-8

As is true for the other resonant circuits, the resonant frequency here is that frequency at which the tank circuit appears purely resistive to the source. This means that the admittance is purely real or, in other words, the susceptance is zero. We can use this fact to solve for the resonant frequency in terms of the circuit components. The admittance is :

$$\mathbf{Y} = \frac{1}{R + j\omega_r L} + j\omega_r C$$

Rationalizing :

$$\mathbf{Y} = \frac{R - j\omega_r L}{R^2 + \omega^2{}_r L^2} + j\omega_r C = \frac{\mathrm{R}}{R^2 + \omega^2{}_r L^2} - \frac{j\omega_r L}{R^2 + \omega^2{}_r L^2} + j\omega_r C$$

Setting the imaginary part equal to zero :

$$\frac{\omega_r L}{R^2 + \omega^2{}_r L^2} + \omega_r C = 0 \quad \text{or} \quad R^2 + \omega^2{}_r L^2 = \frac{\mathrm{L}}{C}$$

Solving for ω_r results in

$$\omega_r = \sqrt{\frac{1}{LC} - \left(\frac{\mathrm{R}}{\mathrm{L}}\right)^2}$$

But if, as is usually the case, the coil is high Q so that $\omega_r L >> R$,

then in our deriving equation we could have used the approximation

$$R^2 + \omega_r^2 L \cong \omega_r^2 L^2$$

and gotten

$$\omega_r^2 L^2 = \frac{L}{C}$$

which simplifies to

$$\omega_r = \frac{1}{\sqrt{LC}}$$

the same as for series RLC circuit and the three-branch parallel RLC circuit. Note that this frequency is only slightly greater than the exact resonant frequency.

Example :

A tank circuit has component values of $L = 4$ H, $C = 1\mu$F, and $R = 100\ \Omega$. What is the resonant frequency ?

Solution :

The exact resonant frequency is

$$\omega_r = \sqrt{\frac{1}{LC} - \left(\frac{R}{L}\right)^2} = \sqrt{\frac{1}{4 \times 10^{-6}} - \left(\frac{100}{4}\right)^2} = 499.37 \text{ rad/s}$$

By the approximate equation the result is

$$\omega_r = \frac{1}{\sqrt{4 \times 10^{-6}}} = 500 \text{ rad/s}$$

which shows that the error in resonant frequency is very small in using the approximate formula even when the coil Q is as low as 20, its value here.

We can use the same admittance equation to find the impedance at the resonant frequency. With the imaginary part zero, as it is at the resonant frequency, what is left is

$$\mathbf{Y} = \frac{R}{R^2 + \omega_r^2 L^2}$$

The denominator $R^2 + \omega^2{}_r L^2$ equals L/C, as appears in one step in the resonant frequency derivation. Substituting this in produces

$$\mathbf{Y} = \frac{R}{L/C} = \frac{RC}{L}$$

Finally, the impedance Z, being the inverse of Y, is

$$\mathbf{Z} = \frac{L}{RC}$$

Now, what about the Q of this circuit ? This Q is easy to find the energy stored and dissipated if we assume, as is approximately true for a high-Q coil in the circuit, that the stored energy is a constant, shuttling back and forth between the capacitor and inductor. Then it follows that the energy stored is that in the inductor current when is at its peak value. This energy stored is $\frac{1}{2} L(\sqrt{2}\ I_L)^2 = LI^2_L$, with the $\sqrt{2}$ occurring because the peak current is $\sqrt{2}$ times I_L , the rms current. Of course, the average power dissipated in the resistor is $I^2_L R$, and so the energy dissipated per cycle is $I^2_L R \times T = (I^2_L R) \times (2\pi/\omega_r)$.

Then,

$$Q = 2\pi \frac{\text{energy stored}}{\text{energy dissipated per cycle}} = 2\Pi \frac{LI_L^2}{(LI_L^2 R)(2\Pi/\omega_r)}$$

from which we can get :

$$Q = \frac{\omega_r L}{R}$$

This is the same result as for a series RLC circuit. Incidentally, an exact derivation, not assuming a high-Q coil, gives the same result. Note also that the Q of this circuit is the same as the Q of the coil.

Example :

A tank circuit has a 1 pF capacitor, if the circuit is to be resonant at 1 MHz and to have a bandwidth of 50 kHz, what are L and R ?

Solution :

We can calculate L from the approximate resonant frequency formula $\omega_r = 1/\sqrt{LC}$ if the circuit has a high Q. Checking this, we have :

$$\mathbf{Q} = \frac{f_r}{BW} = \frac{10^6}{50 \times 10^3} = 20$$

Since Q > 10, we can use the approximate resonant frequency formula $\omega_r = 1/\sqrt{LC}$ and get a fairly accurate value for L. Squaring this formula and rearranging, we get :

$$\mathbf{L} = \frac{1}{\omega^2{}_r C} = \frac{1}{(2\Pi \times 10^6)^2(10^{12})} = 25.3 \text{ mH}$$

Finally, from Q and this L and $\omega_r L/R = Q$.

$$R = \frac{\omega_r L}{Q} = \frac{(2\Pi \times 10^6)(25.3 \times 10^{-3})}{20} = 7.95\ \text{k}\Omega$$

The impedance at resonance, $Z = L/RC$, is easier to remember if expressed in terms of Q. This we can do by multiplying numerator and denominator by $\omega_r R$:

$$Z = \frac{L}{RC} = \frac{\omega_r LR}{R\omega_r RC} = \left(\frac{\omega_r L}{R}\right)\left(\frac{1}{\omega_r RC}\right)R$$

But $Q = \omega_r L/R = 1/\omega_r RC$. Consequently,

$$Z = Q^2 R$$

Remember this important equation–that for a practical tank circuit, the impedance at resonance is Q^2 times the coil resistance. For a high-Q circuit this impedance can be very large compared to the impedances at frequencies away from the resonant frequency. There is impedance peaking at the resonant frequency.

<u>Example</u> :

Find the branch currents I_L and I_C in the circuit of Fig. 9-6 for $I = 10\angle 0^\circ$ A, $j\omega L = j40\ \Omega$, $-j1/\omega C = -j40\ \Omega$, and $R = 1\ \Omega$.

Solution :

By current division ,

$$\mathbf{I_L} = \frac{-j40}{1 + j40 - j40} \times 10\angle 0^\circ = 400\angle -90^\circ \text{ A}$$

and

$$\mathbf{I_C} = \frac{1 + j40}{1 + j40 - j40} \times 10\angle 0^\circ = 400.1\angle 88.6^\circ \text{ A}$$

Notice that each of these branch has an rms value of approximately Q = 40 times the input current. Further, the current phasors have angles that are approximately 180° apart.

PROBLEMS

1) What is the resonant frequency of a series circuit of a 2.7 Ω resistor, a 10 pF capacitor, and a 1 mH inductor ?

2) What capacitance tunes a 10 μH coil to series resonance at 10^5 kHz.

3) In a home AM radio the station dial is mounted on the shaft of a variable capacitor having a maximum capacitance of 365 pF. What is the associated inductance required to produce resonance at the lowest AM radio frequency of 540 kHz ?

4) A 100 Ω resistor, a 0.01 H inductor, and a 0.01 μF capacitor are in series with a 10 V source operating at the resonant frequency. Find the rms inductor voltage. Then reduce the resistance by a factor of 10 and find this inductor voltage again. Finally, in the original circuit, increase the inductance by a factor of 10, to 0.1 H, decrease the capacitance by the same factor, to 0.001 μF, to maintain the same resonant frequency, and find the inductor voltage.

5) In the circuit shown in Fig. 11-9, the frequency is 2000 Hz. Get C necessary to get maximum power of the resistor 10 Ω. Also give the value of maximum power.

6) A series RLC circuit has $L = 25$ mH and $C = 75$ μF. The phase angle is 25° lag at $\omega = 2000$ rad/s. Get the frequency at which the phase angle equals lead. What is the resonant frequency ?

7) Find the inductance L in the circuit shown in Fig. 11-10, to get a resonance at $\omega = 5000$ rad/s.

8) Get the quality factor Q for the RLC circuit by three different methods if $R = 20\ \Omega$, $L = 0.05$ H and $C = 1\ \mu F$.

9) What is the bandwidth of a series resonant circuit having a Q of 100 and a resonant frequency of 1 MHz ?

10) Find the Q of a series resonant circuit having a resonant frequency of 700 kHz and bandwidth of 10 kHz. Then find the cutoff frequencies by using an approximation.

11) Find the exact cutoff frequencies for the circuit of Problem 10.

12) Find the resonant frequency and the exact half-power frequencies for a series RLC circuit in which $R = 1\ \Omega$, $L = 1$ mH, and $C = 0.1\ \mu F$.

13) A series RLC circuit with a resonant radian frequency of 10^4 rad/s and a bandwidth of 100 rad/s has a 1 μF capacitor. Find the inductance and resistance.

14) A series RLC circuit with a resonant frequency of 1 MHz and a Q of 50 has a 1 mH inductor. What are the resistance and capacitance ?

15) A series RLC circuit is energized at its resonant frequency by a source of voltage of 4 sin 9000t V. If the resistance is 10 Ω and the capacitance is 0.2 μF, what is the capacitor voltage as a function time ?

16) A certain series RLC circuit in resonance draws 4 A with 12 V applied. If the capacitive reactance is −100 Ω, find the circuit Q.

17) A certain series RLC circuit carries a 5 A current with 15 V applied at the resonant frequency of 377 rad/s. If the inductor and capacitor store a total of 50 J of electrical energy, what is the circuit Q and what are the inductance and capacitance ?

18) Design a series RLC circuit that carries 5 A in response to 20 V applied at the resonant frequency of 10^4 Hz. The circuit is to have a Q of 100.

19) A current source supplies 0.3 A at the resonant frequency to the parallel combination of a 0.4 μF capacitor, a 1 k Ω resistor, and a 1 mH coil having negligible resistance. Find the voltage and the unknown currents.

20) A certain tank circuit consists of a 1 μF capacitor in parallel with a 400 μH coil having 0.1 Ω of resistance. Find the resonant frequency, the Q, and the impedance of the tank circuit at the resonant frequency.

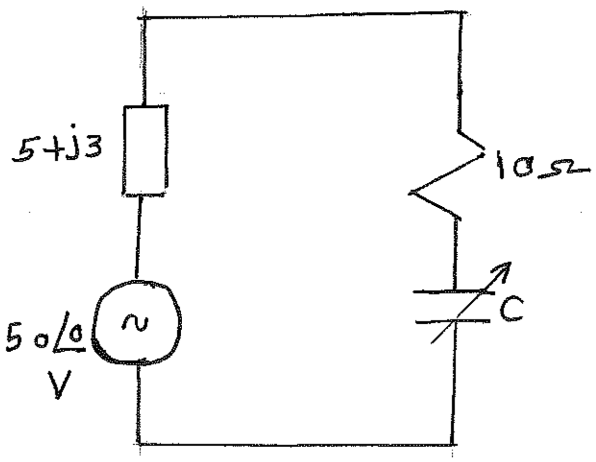


Figure 9-9

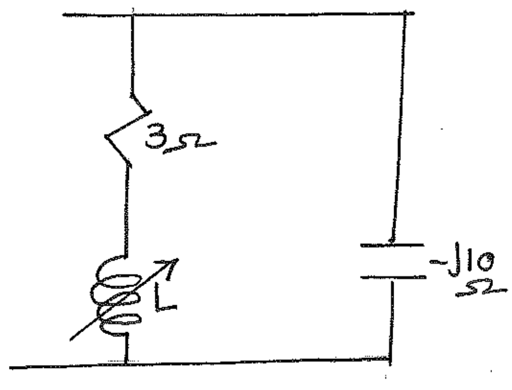


Figure 9-10

Chapter Ten

MAGNETIC CIRCUITS

Introduction :

Magnetism plays an integral part in almost every electrical device used today in industry, research, or the home. Generators, motors, transformers, circuit breaker, televisions, computers, tape recorders, and telephones all employ magnetic effects to perform a variety of important tasks.

There is a great deal of similarity between the analyses of electric circuits and magnetic circuits. This will be demonstrated later in this chapter when we compare the basic equations and methods used to solve magnetic circuits with those used for electric circuits.

Magnetic Fields :

In the region surrounding a permanent magnet there exists a magnetic field, which can be represented by magnetic flux lines similar to electric flux lines. Magnetic flux lines, however, do not have origins or terminating points like electric flux lines but exist in continuous loops, as shown in Fig. 10-1. The symbol for magnetic flux is the Greek letter Φ (phi).

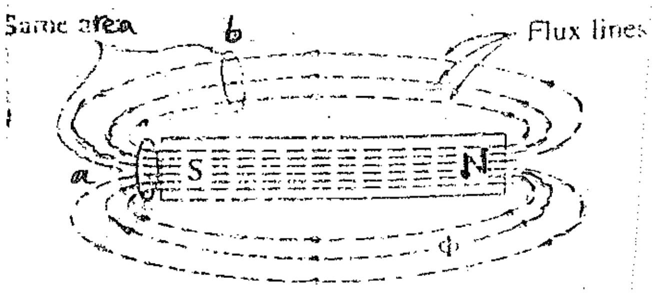


Figure 10-1

The magnetic flux lines radiate from the north pole to the south pole, returning to the north pole through the metallic bar. Note the equal spacing between the flux lines within the core and the symmetric distribution outside the magnetic material. These are additional properties of magnetic flux lines in homogeneous (that is materials having uniform structure or composition throughout). It is also important to realize that the continuous magnetic flux line will strive to occupy as small an area as possible. This will result in magnetic flux lines of minimum length between the like poles, as shown in Fig. 10-3. The strength of a magnetic field in a particular region is directly related to the density of flux lines in that region. In Fig. 10-1, for example, the magnetic field strength at a is twice at b since there are twice as many magnetic flux lines associated with the perpendicular plane at a than at b. Recall from childhood experiments how the strength of permanent magnets was always stronger near the poles.

If unlike poles of two permanent magnets are brought together, the magnets will attract, and the flux distribution will be as shown in Fig. 10-2. If like poles are brought together, the magnets will repel, and the flux distribution will be as shown in Fig. 10-3.

If a nonmagnetic material, such as glass or copper, is placed in the flux paths surrounding a permanent magnet, there will be an almost unnoticeable change in the flux distribution (Fig. 10-4). However, if

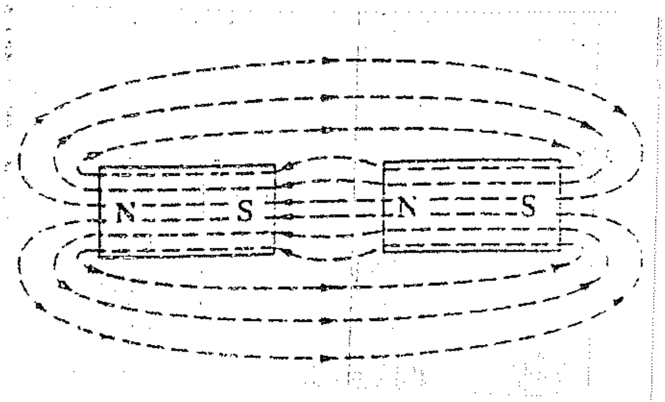


Figure 10-2

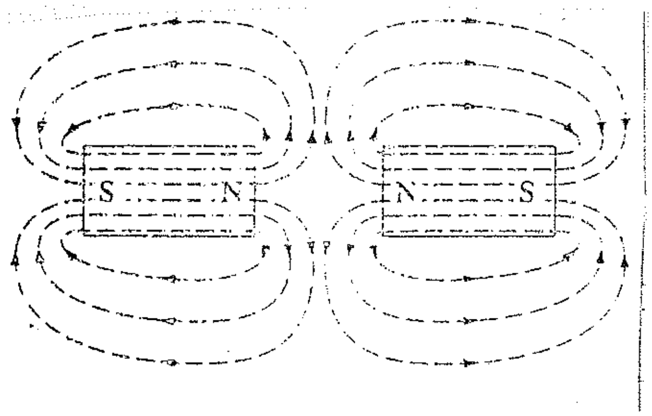


Figure 10-3

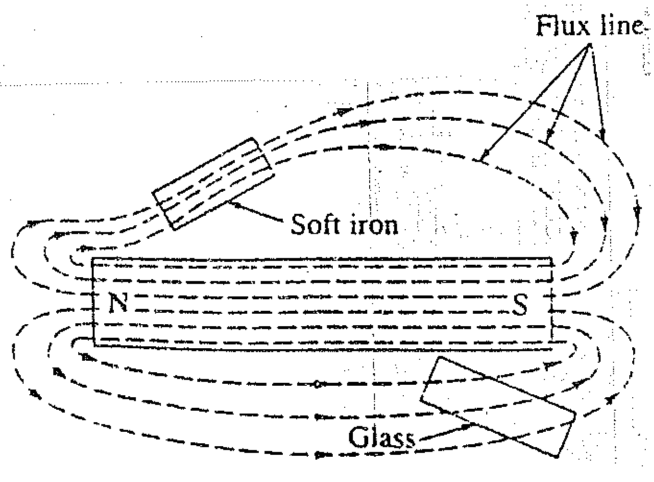


Figure 10-4

a magnetic material, such as soft iron, is placed in the flux path, the flux lines will pass through the soft iron rather the surrounding air because flux lines pass with greater ease through magnetic materials than through air. The principle is put to use in the shielding of sensitive electrical elements and instruments that can be affected by stray magnetic fields (Fig. 10-5).

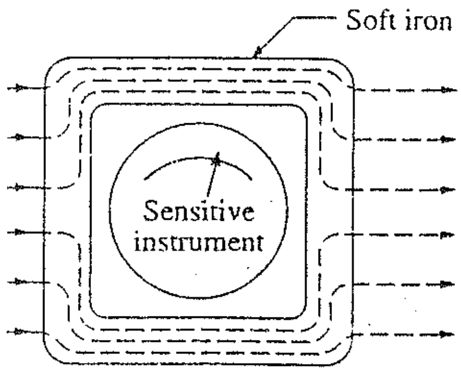


Figure 10-5

As indicated in the introduction, a magnetic field (represented by concentric magnetic flux lines, as in Fig. 10-6) is present around every wire that carries an electric current. The direction of the magnetic flux lines can be found simply by placing the thumb of right hand in the direction of conventional current flow and noting the direction of the fingers. (This method is commonly called the right-

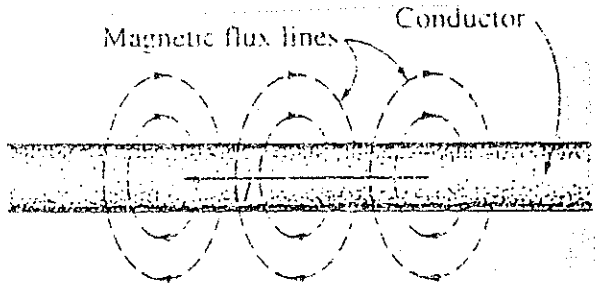


Figure 10-6

hand rule). If the conductor is wound in a single-turn coil (Fig. 10-7), the resulting flux will flow in a common direction through the center of the coil.

A coil of more than one turn would produce a magnetic field that would exist in a continuous path through and around the coil (Fig. 10-8).

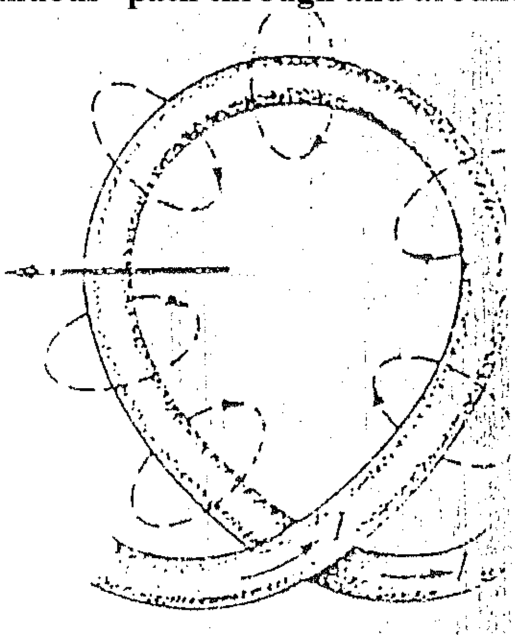


Figure 10-7

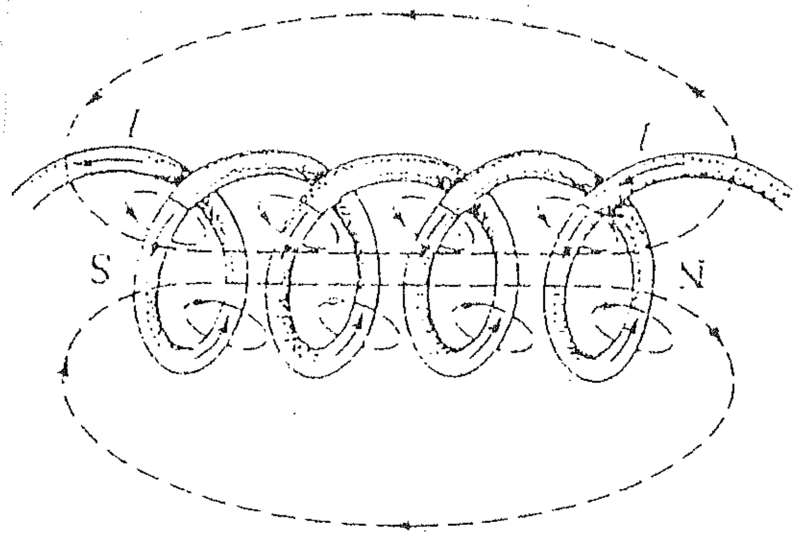


Figure 10-8

The flux distribution of the coil is quite similar to that of the permanent magnet. The flux lines leaving the coil from the left and entering to the right simulate a north and south pole, respectively. The field strength of the iron, steel, or cobalt, within the coil to increase the flux density within the coil. By increasing the field strength with the addition of the core, we have devised an electromagnet (Fig. 10-9) which, in addition to having all the properties of a permanent magnet, also has a field strength that can be varied by changing one of the component values (current, turns, and so on). Of course, current must pass through the coil of the electromagnet in order for magnetic flux to be developed, whereas there is no need for the coil or current in the permanent magnet. The

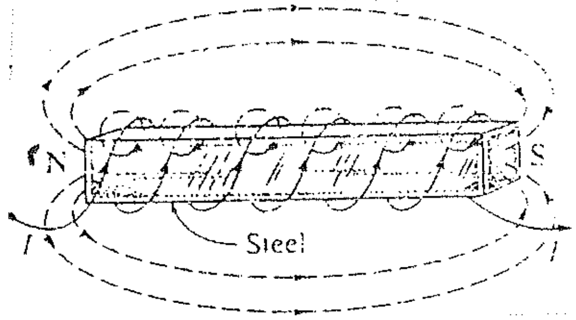


Figure 10-9

direction of flux lines can be determined for the electromagnet (or in any core with a wrapping of turns) by placing the fingers of the right hand in the direction of current flow around the core. The thumb will then point in the direction of the north pole of the induced magnetic flux. This is demonstrated in Fig. 10-10.

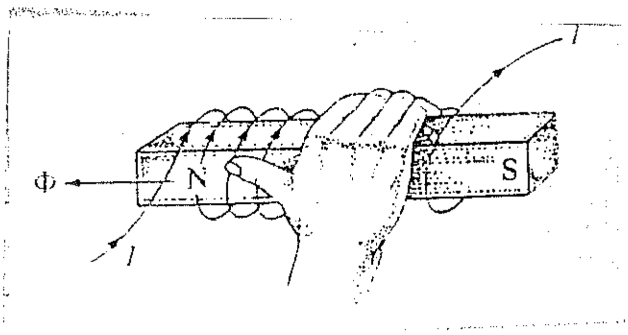


Fig. 10-10

Flux Density :

In the SI system of units, magnetic flux is measured in webers (Wb) and has the symbol Φ. The number of flux lines per unit area is called the flux density and is denoted by the capital letter B. Its magnitude is determined by the following equation :

$$B = \frac{\Phi}{A} \tag{10.1}$$

B = teslas (T)

Φ = webers (Wb)

A = square meters (m^2)

where Φ is the number of flux lines passing through the area A (Fig. 10-11). The flux density at position a in Fig. 10-1, is twice that at b because twice as many flux lines are passing through the same area.

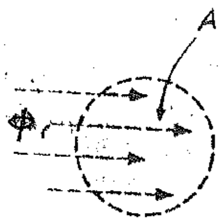


Figure 10-11

As noted in Eqn. (10.1), magnetic flux density in the SI system of units is measured in teslas, for which the symbol is T. By definition,

$$1 \text{ tesla (T)} = 1 \text{ Wb/m}^2$$

Example :

Determine the flux density B in a core having uniform cross section of $1.2 \times 10^{-3}\ m^2$ and the flux through it is 6×10^{-5} Wb.

Solution :

$$\mathbf{B} = \frac{\Phi}{A} = \frac{6 \times 10^{-5}\ \text{Wb}}{1.2 \times 10^{-3}\ \text{m}^2} = 5 \times 10^{-2}\ \text{T}$$

Permeability :

If cores of different materials with the same physical dimensions are used in the electromagnet previously described, the strength of the magnet will vary in accordance with the core used. This variation in strength is due to the greater or lesser numbers of flux lines passing through the core. Materials in which flux lines can readily be set up are said to be magnetic and to have high permeability. The permeability (μ) of a material, therefore, is a measure of the ease with which magnetic flux lines can be established in the material. It is similar in many respects to conductivity in electric circuits. The permeability of free space μ_o (vacuum) is :

$$\mu_o = 4\pi \times 10^{-7} \text{ weber/ampere-meter}$$

As indicted above, μ has the units of Wb/A. m. Practically speaking, the permeability of all nonmagnetic materials, such as copper, aluminum, wood, glass, and air, is the same as that for free space. Materials that have permeabilities slightly less than that of free space are said to be diamagnetic, and those with permeabilities slightly greater than that of free space are said to be paramagnetic. Magnetic materials, such as iron, nickel, steel, cobalt, and alloys of these metals, have permeabilities hundreds and even thousands of times that of free space. Materials with these very high permeabilities are referred to as ferromagnetic.

The ratio of the permeability of a material to that of free space is called its relative permeability : that is ,

$$\mu_r = \frac{\mu}{\mu_o} \qquad (10.2)$$

In general, for ferromagnetic materials, $\mu_r \geq 100$, and for nonmagnetic materials $\mu_r = 1$.

Reluctance :

The resistance of material to the flow of charge (current) is determined for electric circuits by the equation :

$$R = \rho \frac{\ell}{A} \qquad (\text{ohms}, \Omega)$$

The reluctance of a material to the setting up of magnetic flux lines in the material is determined by the following equation :

$$\Re = \frac{\ell}{\mu A} \qquad (\text{rels. or At/Wb}) \qquad (10.3)$$

where $\Re$ is the reluctance, ℓ is the length of the magnetic path, and A is its cross-sectional area. The t in the units At/Wb is the number of turns of the applied winding. More is said about ampere-turns (At) in the next section. Note that the resistance and reluctance are inversely proportional to the area, indicating that an increase in area will result in a reduction in each and an increase in the desired result : current and flux. For an increase in length the opposite is true, and the desired effect is reduced. The reluctance, however, is inversely proportional to the permeability, while the resistance is directly proportional to the resistivity. The larger the μ

or the smaller the ρ, the smaller the reluctance and resistance, respectively. Obviously, therefore, materials with high permeability, such as the ferromagnetics, have very small reluctances and will result in an increased measure of flux through the core. There is no widely accepted unit for reluctance, although the rel and the At/Wb are usually applied.

Ohm's Law For Magnetic Circuits :

The equation between the cause and opposition is given by :

$$\text{Effect} = \frac{cause}{opposition}$$

For magnetic circuits, the effect desired is the flux Φ. The cause is the magnetomotive force (mmf) F, which is the external force (or "Pressure") required to set up the magnetic flux lines within the magnetic material. The opposition to the setting up of the flux Φ is the reluctance $\Re$.

Substituting, we have :

$$\Phi = \frac{\Im}{\Re} \qquad (10.4)$$

The magnetomotive force $\Im$ is proportional to the product of the number of turns around the core (in which the flux is to be established) and the current through the turns of wire (Fig. 10-12). In equation form,

$$\Im = NI \quad \text{(ampere-turns. At)} \qquad (10.5)$$

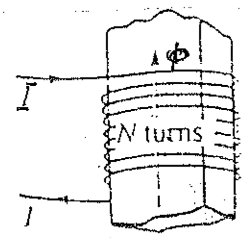


Figure 10-12

The equation clearly indicates that an increase in the number of turns or the current through the wire will result in an increased "pressure" on the system to establish flux lines through the core.

Magnetizing Force :

The magnetomotive force per unit length is called the magnetizing force (H). In equation form :

$$H = \frac{\Im}{\ell} \quad \text{(At/m)} \qquad (10.6)$$

Substituting for the magnetomotive force will result in :

$$\mathbf{H} = \frac{NI}{\ell} \quad \text{(At/m)} \tag{10.7}$$

For the magnetic circuit of Fig. 10-13, if NI = 40 At and I = 0.2 m, then ,

$$\mathbf{H} = \frac{NI}{\ell} = \frac{40\ \text{At}}{0.2\ \text{m}} = 200 \quad \text{At/m}$$

In words, the result indicates that there are 200 At of "pressure" per meter to establish flux in the core.

Note in Fig. 10-13 that the direction of the flux Φ can be determined by placing the fingers of the right hand in the direction of current around the core and noting the direction of the thumb. It is interesting to realize that the magnetizing force is independent of the type of core material–it is determined solely by the number of turns, the current, and the length of the core,

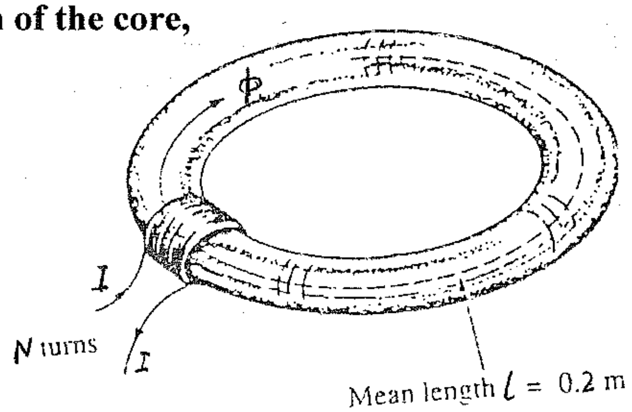


Figure 10-13

The flux density and the magnetizing force are related by the following equation :

$$B = \mu H \qquad (10.8)$$

Hysteresis :

A curve of the flux density B versus the magnetizing force H of a material is of particular importance to the engineer. A typical B-H curve for a ferromagnetic material such as steel can be derived using the setup of Fig. 10-14.

The core is initially unmagnetized and the current $I = 0$. If the current I is increased to some value above zero, the magnetizing force H will increase to a value determined by :

$$H\uparrow = \frac{NI\uparrow}{l}$$

The flux Φ and the flux density $B (B = \Phi/A)$ will also increase with the current I (or H). If the material has no residual magnetism and the magnetizing force H is increased from zero to some value H_a, the B-H curve will follow the path shown in Fig. 12-15 between o and a. If the magnetizing force H is increased until saturation (H_s) occurs, the curve will continue as shown in the figure to point b. When saturation occurs, the flux density has, for all practical purposes,

reached its maximum value. Any further increase in current through the coil increasing H = NI/I will result in a very small increase in flux density B.

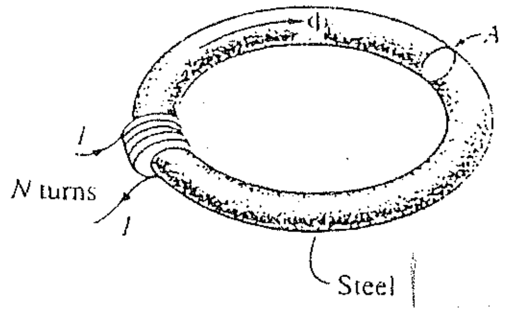


Figure 10-14

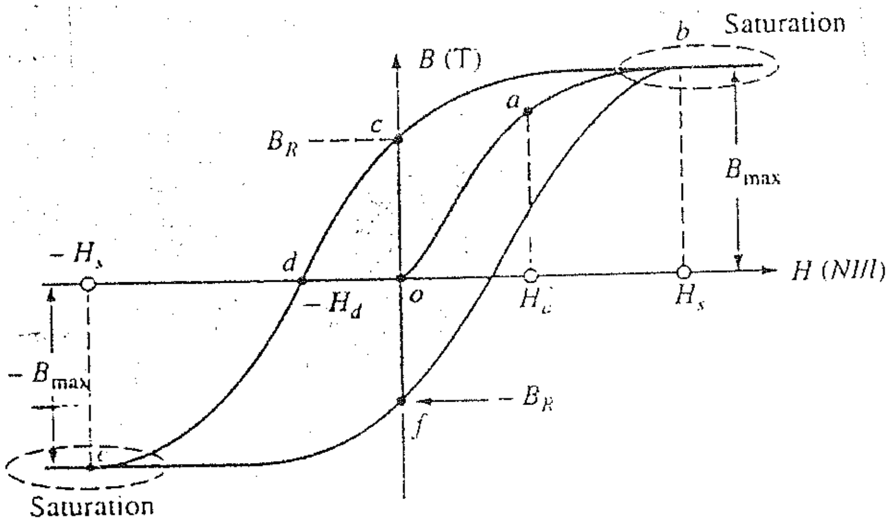


Figure 10-15

If the magnetizing force is reduced to zero by letting I decrease to zero, the curve will follow the path of the curve between b and c. The flux density B_R which remains when the magnetizing force is zero, is called the residual flux density. It is this residual flux density that makes it possible to create permanent magnets. If the coil is now

removed from the core of Fig. 10-14, the core will still have the magnetic properties determined by the residual flux density, a measure of its “retentivity”. If the current I is reversed, developing a magnetizing force, $-H$, the flux density B will decrease with increase in I. Eventually, the flux density will be zero when $-H_d$ (the portion of curve from c to d) is reached. The magnetizing force $-H_d$ required to “coerce” the flux density to reduce its level to zero is called the coercive force, a measure of the coercivity of the magnetic sample. As the force $-H$ is increased until saturation again occurs and is then reversed and brought back to zero, the path def will result. If the magnetizing force is increased in the positive direction (+H), the curve will trace the path shown from f to b. The entire curve represented by bcdefb is called the hysteresis curve for the ferromagnetic material, from the Greek hysterein, meaning “to lag behind”. The flux density B lagged behind the magnetizing force H during the entire plotting of the curve. When H was zero at c, B was not zero but had only begun to decline. Long after H had passed through zero and had become equal to $-H_d$ did the flux density B finally become equal to zero.

If the entire cycle is repeated, the curve obtained for the same core will be determined by the maximum H applied. Three hysteresis loops for the same material for maximum values of H less than the saturation value are shown in Fig. 10-16. In addition, the saturation curve is repeated for comparison purposes.

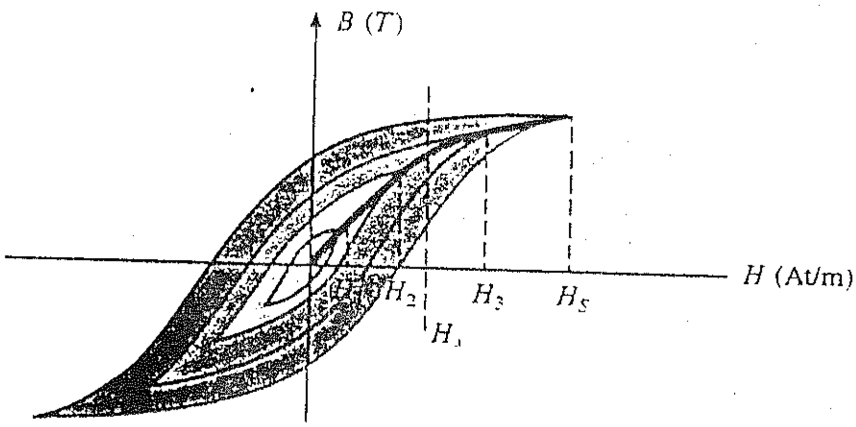


Figure 10-16

Note from the various curves that for a particular value of H, say, H_x , the value of B can vary widely, as determined by the history of the core. In an effort to assign a particular value of B to each value of H, we compromise by connecting the tips of the hysteresis loops. The resulting curve, shown by the heavy, solid line in Fig. 10-16, and for various materials in Fig. 10-17, is called the normal magnetization curve.

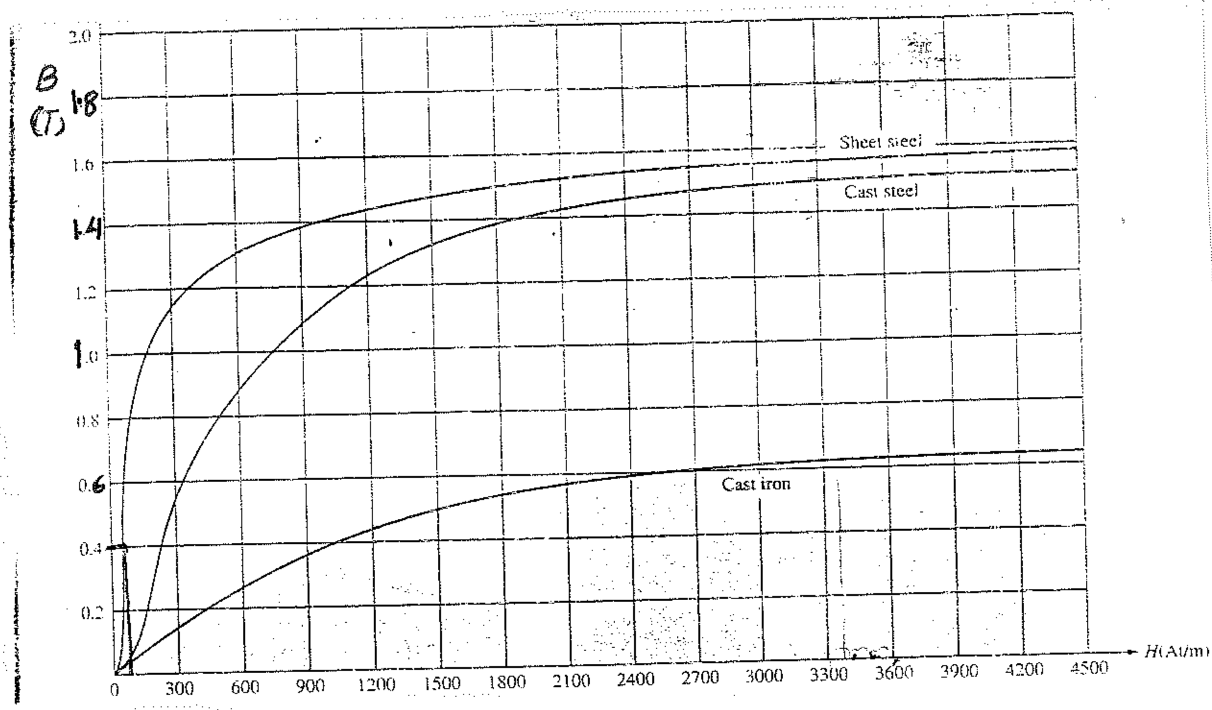


Figure 10-17

Ampere's Circuital Law :

It was mentioned in the introduction to this chapter that there is a board similarity between the analysis of electric and magnetic circuits. This has already been demonstrated to some extent for the quantities in Table 10.1.

Table 10.1

	Electric Circuits	Magnetic Circuits
Cause	E	$\mathfrak{I}$
Effect	I	Φ
Opposition	R	$\mathfrak{R}$

If we apply the " cause" analogy to Kirchhoff's voltage law ($\Sigma_C V = 0$), we obtain the following :

$$\Sigma_C \mathfrak{I} = 0 \qquad \text{(for magnetic circuits)} \qquad (10.9)$$

which, in words, states that the algebraic sum of the rises and drops of the mmf around a closed loop of a magnetic circuit is equal to zero, that is, the sum of the mmf rises equals the sum of the mmf drops around a closed loop.

Equation (10.9), is referred to as Ampere's circuital law. When it is applied to magnetic circuits, sources of mmf are expressed by the equation :

$$\mathfrak{I} = NI \qquad \text{(At)} \qquad (10.10)$$

The equation for the mmf drop across a portion of a magnetic circuit can be found by applying the relationships listed in Table 10.1. That is, for electric circuits.

$$V = IR$$

Resulting in the following for magnetic circuits :

$$\mathfrak{I} = \Phi\mathfrak{R} \qquad \text{(At)} \qquad (10.11)$$

where Φ is the flux passing through a section of the magnetic circuit and $\mathfrak{R}$ is the reluctance of that section. The reluctance, however, is seldom calculated in the analysis of magnetic circuits. A more practical equation for the mmf drop is :

$$\Im = H\,\ell \qquad \text{(At)} \qquad (10.12)$$

as derived from Eqn. (10.6), where H, is the magnetizing force on a section of a magnetic circuit and ℓ is the length of the section. As an example of Eqn. (10.9), consider the magnetic circuit appearing in Fig. 10.18, constructed of three different ferromagnetic materials.

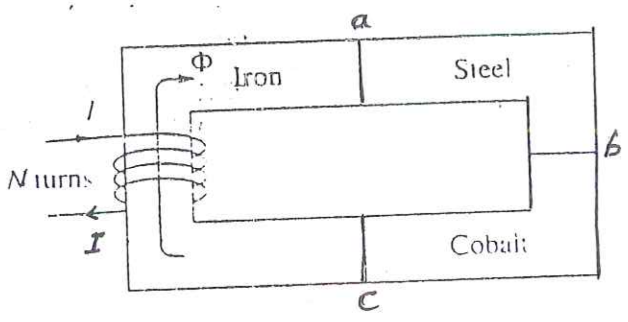


Figure 10-18

Applying Ampere's law, we have :

$$\Sigma \,.\, \Im = 0$$

$$\underbrace{+NI}_{Rise} - \underbrace{H_{ab}\, I_{ab}}_{\text{Drop}} + \underbrace{H_{bc}\, \ell_{bc}}_{Drop} + \underbrace{H_{cd}\, \ell_{cd}}_{\text{Drop}} = 0$$

or

$$\underbrace{NI}_{\text{Im pressed } mml} = \underbrace{H_{ab}\, I_{ab} + H_{bc}\, \ell_{bc} + H_{cd}\, \ell_{cd}}_{mml \text{ drops}}$$

All the terms of the equation are known except the magnetizing force for each portion of the magnetic circuit, which can be found by using the B-H curve if the flux density B is known.

The Flux Φ :

If we continue to apply the relationships described in the previous section to Kirchhoff's current law, we will find that the sum of the fluxes entering a junction is equal to the sum of the fluxes leaving a junction: that is, for the circuit of Fig. 10-19.

$$\Phi_a = \Phi_b + \Phi_c \qquad \text{(at junction a)}$$

or

$$\Phi_b = \Phi_c + \Phi_a \qquad \text{(at junction b)}$$

both of which are equivalent.

Series Magnetic Circuits :

Determining NI :

We are now in a position to solve a few magnetic circuit problems, which are basically of two types. In one type, Φ is given, and the impressed mmf NI must be computed. This is the type of

problem encountered in the design of motors, generators, and transformers. In the other type, NI is given, and the flux Φ of the magnetic circuit must be found. This type of problem is encountered primarily in the design of magnetic amplifiers and is more difficult since the approach is " hit or miss".

As indicated in earlier discussions, the value of μ will vary from point to point along the magnetization curve. This eliminates the possibility of finding the reluctance of each "branch" or the "total reluctance" of a network as was done for electric circuits where ρ had a fixed value for any applied current or voltage. If the total reluctance could be determined, Φ could then be determined using the Ohm's law analogy for magnetic circuits.

For magnetic circuits, the level of B or H is determined from the other using the B-H curve, and μ is seldom calculated unless asked for.

An approach frequency employed in the analysis of magnetic circuits is the table method. Before a problem is analyzed in detail, a table is prepared listing in the extreme left-hand column the various

sections of the magnetic circuit. The columns on the right are reserved for the quantities to be found for each section. In this way, the individual doing the problem can keep track of what is required to complete the problem and also of what the next step should be.

This section will consider only series magnetic circuits in which the flux Φ is the same throughout. In each example, the magnitude of the magnetomotive force is to be determined.

Example **:**

For the series magnetic circuit of Fig. 10-20 :

a) Find the value of I required to develop a magnetic flux of $\Phi = 4 \times 10^{-4}$ Wb.

b) Determine μ and μ_r for the material under these conditions.

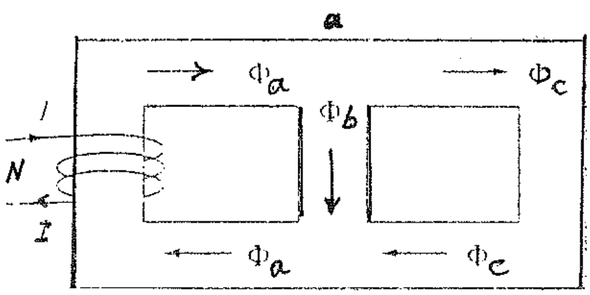


Figure 10-19

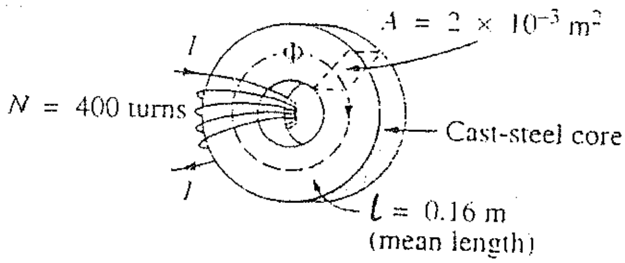


Figure 10-20

Solutions :

The magnetic circuit can be represented by the system shown in Fig. 10-21(a). The electric circuit analogy is shown in Fig. 10-21(b). Analogies of this type can be very helpful in the solution of magnetic circuits. Table 10.2 is for part (a) of this problem. The table is fairly trivial for this example but it does define the quantities to be found.

Table 10.2

Section	Φ (Wb)	A (m^2)	B (T)	H (At/m)	ℓ (m)	$H\ell$ (At)
One continuous section	4×10^{-4}	2×10^{-3}			0.16	

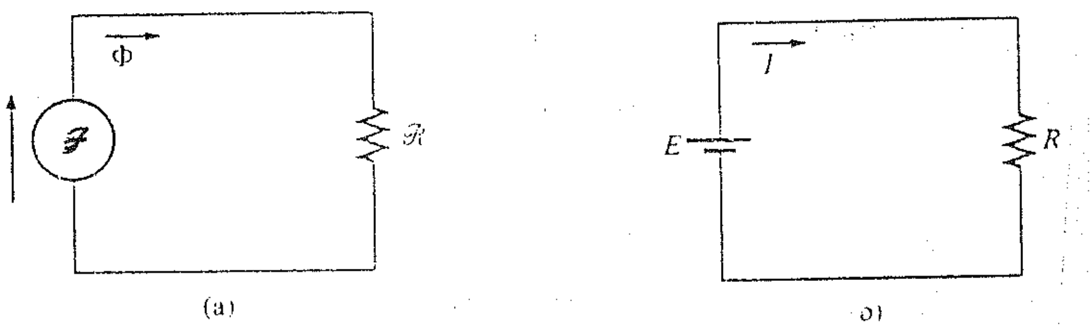


Figure 10-21

a- The flux density B is :

$$\mathbf{B} = \frac{\Phi}{A} = \frac{4 \times 10^{-4}\ \text{Wb}}{2 \times 10^{-3}\ \text{m}^2} = 2 \times 10^{-1}\ \text{T} = 0.2\ \text{T}$$

Using the B-H curves of Fig. 10-17, we can determine the magnetizing force H :

$$\textbf{H (cast steel)} = \textbf{170 At/m}$$

Applying Ampere's circuital law yields :

$$\textbf{NI} = \textbf{H}\,\ell$$

And

$$\textbf{I} = \frac{H\ell}{N} = \frac{(170\ \text{At/m})(0.16\,\text{m})}{400\ \text{t}} = 68\ \text{mA}$$

(Recall that t represents turns)

The permeability of the material can be found using Eqn. (10.8) :

$$\mu = \frac{B}{H} = \frac{0.2\,\text{T}}{170\ \text{At/m}} = 1.176 \times 10^{-3}\ \ \text{Wb/A.m}$$

and the relative permeability is

$$\mu_r = \frac{\mu}{\mu_o} = \frac{1.176 \times 10^{-3}}{4\Pi \times 10^{-3}} = 935.83$$

Example :

Determine the secondary current I_2 for the transformer of Fig. 10-22, if the resultant clockwise flux in the core is 1.5×10^{-5} Wb.

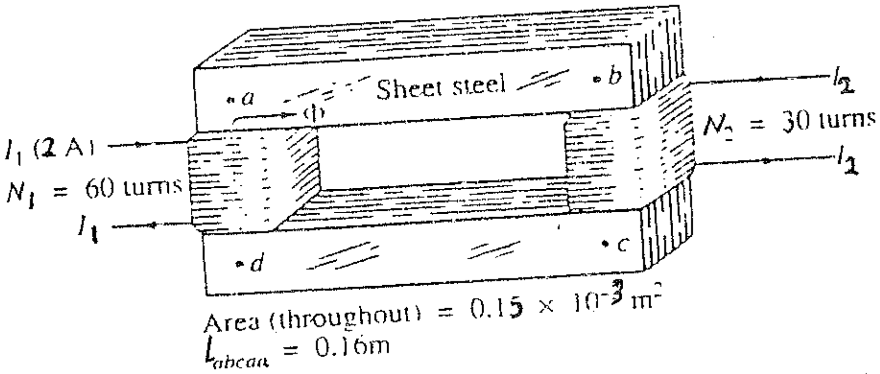


Figure 10-22

Solution :

This is the first example with two magnetizing force to consider. In the analogies of Fig. 10-23, you will note that the resulting flux of each is opposing, just as the two sources of voltage are opposing in the electric circuit analogy.

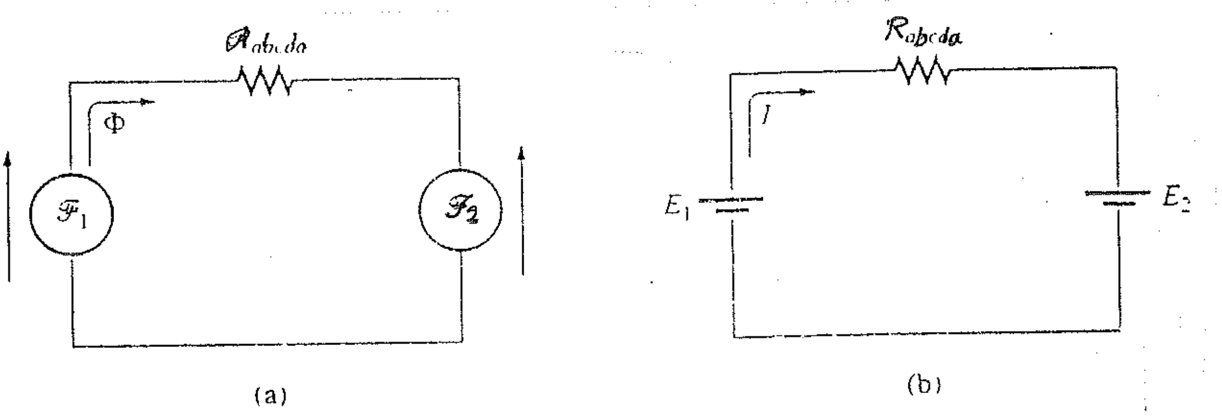


Figure 10-23

The structural data appear in Table 10.3

Table 10.3

Section	Φ (Wb)	A (m^2)	B (T)	H (At/m)	ℓ (m)	$H\ell$ (At)
Abcda	1.5×10^{-5}	0.15×10^{-3}			0.16	

The flux density throughout is :

$$B = \frac{\Phi}{A} = \frac{1.5 \text{ x } 10^{-5} \text{ Wb}}{0.15 \text{ x } 10^{-3} \text{ m}^2} = 10 \text{ x } 10^{-2} \text{ T} = 0.10 \text{ T}$$

and

$$\text{H (from Fig. 12-17)} \cong \frac{1}{5} \text{(100 At/m)} = 20 \text{ At/m}$$

Applying Ampere's circuit law

$$N_1 I_1 - N_2 I_2 = H_{abcda} \, \ell_{abcda}$$

$$(60 \text{ t})(2\text{A}) - (30 \text{ t})(I_2) = (20 \text{ At/m})(0.16 \text{ m})$$

$$120 \text{ At} = (30 \text{ t})I_2 = 3.2 \text{ At}$$

and

$$(30 \text{ t})I_2 = 120 \text{ At} - 3.2 \text{ At}$$

or

$$I_2 = \frac{116.8 \text{ At}}{30 \text{ t}} = 3.89 \text{ A}$$

Because of the nonlinearity of the B-H curve, it is not possible to apply superposition to magnetic circuits that is, in the previous example, we cannot consider the effects of each source independently and then find the total effects by using superposition.

Air Gaps :

Before continuing with the illustrative examples, let us consider the effects an air gap has on a magnetic circuit. Note the presence of air gaps in the magnetic circuits of the motor and meter. The spreading of the flux lines outside the common area of the core for the air gap in Fig. 10-24(a), is known as fringing. For our purposes, we shall neglect this effect and assume the flux distribution to be as in Fig. 10-24(b).

The flux density of the air gap in Fig. 10-24(b), is given by :

$$B_g = \frac{\Phi_g}{A_g} \qquad (10.13)$$

Where, for our purposes,

$$\Phi_R = \Phi_{core}$$

and

$$A_g = A_{core}$$

For most practical applications, the permeability of air is taken to be equal to that of free space. The magnetizing force of the air gap is then determined by :

$$H_g = \frac{B_g}{\mu_o} \qquad (10.14)$$

And the mmf drop across the air gap is equal to $H_g l_g$. An equation for H_g is as follows :

$$H_g = \frac{B_g}{\mu_o} = \frac{B_g}{4\Pi \times 10^{-7}}$$

and

$$H_g = (7.96 \times 10^5)B_g \quad \text{(At/m)} \qquad (10.15)$$

Example :

Find the value of I required to establish a magnetic flux of $\Phi = 0.75 \times 10^{-4}$ wb in the series magnetic circuit of Fig. 10-25.

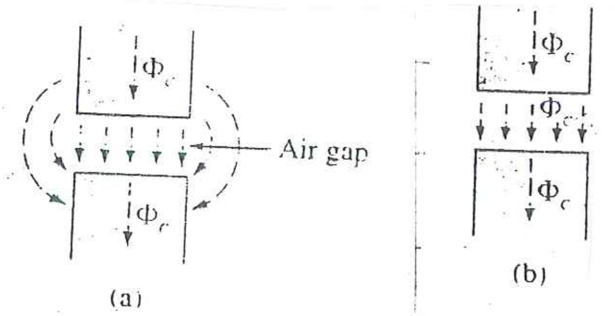


Figure 10-24

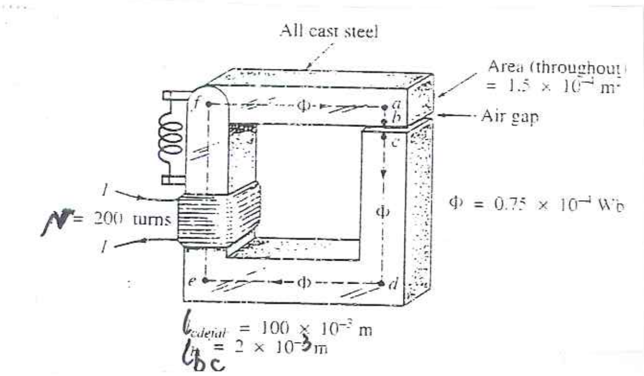


Figure 10-25

Solution :

An equivalent magnetic circuit and its electric circuit analogy are shown in Fig. 10-26.

The flux ensity for each section is

$$B = \frac{\Phi}{A} = \frac{0.75 \text{ x } 10^{-4} \text{ Wb}}{1.5 \text{ x } 10^{-4} \text{ m}^2} = 0.5 \text{ T}$$

From the B-H curves of Fig. 10-17 :

$$\mathbf{H \text{ (cast steel)} \cong 280 \text{ At/m}}$$

Applying Eqn. (10-15),

$$\mathbf{H_g = (7.96 \text{ x } 10^5)B_g = (7.96 \text{ x } 10^5)(0.5 \text{ T}) = 3.98 \text{ x } 10^5 \quad At/m}$$

The mmf drops are :

$$\mathbf{H_{core} l_{core} = (280 \text{ At/m})(100 \text{ x } 10^{-3} \text{ m}) = 28 \text{ At}}$$

$$\mathbf{H_g l_g = (3.98 \text{ x } 10^5 \text{ At/m})(2 \text{ x } 10^{-3} \text{ m}) = 796 \text{ At}}$$

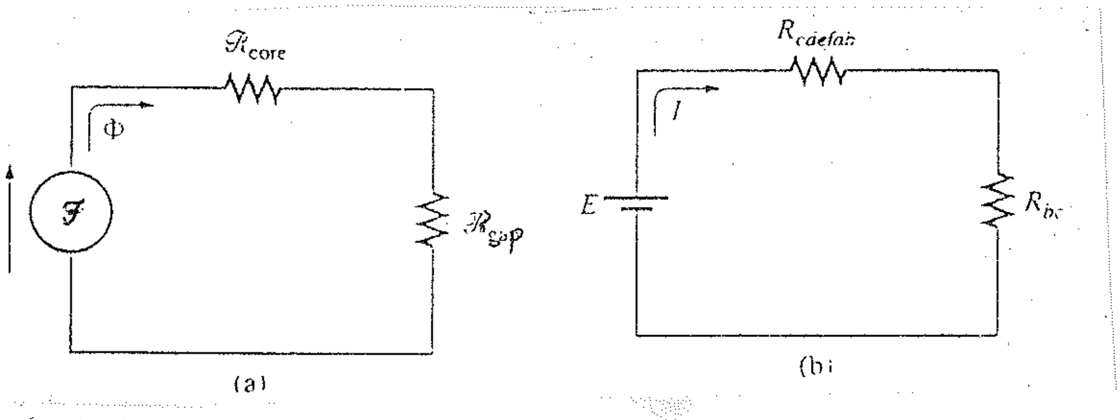


Figure 10-26

Applying Ampere's circuit law

$$NI = H_{core}\, l_{core} + H_g\, l_g$$

$$= 28\ At + 796\ At$$

$$(200\ t)I = 824\ At$$

$$I = 4.12\ A$$

Note from the above that the air gap requires the biggest share (by far) of the impressed NI due to the fact that air is nonmagnetic.

Series-Parallel Magnetic Circuits :

As one might expect, the close analogies between electric and magnetic circuits will eventually lead to series-parallel magnetic circuits. In fact, the electric circuit analogy will prove helpful in defining the procedure to follow toward a solution.

Example :

Determine the current I required to establish a flux of 1.5 x 10^{-4} Wb in the section of the core indicated in Fig. 10-27.

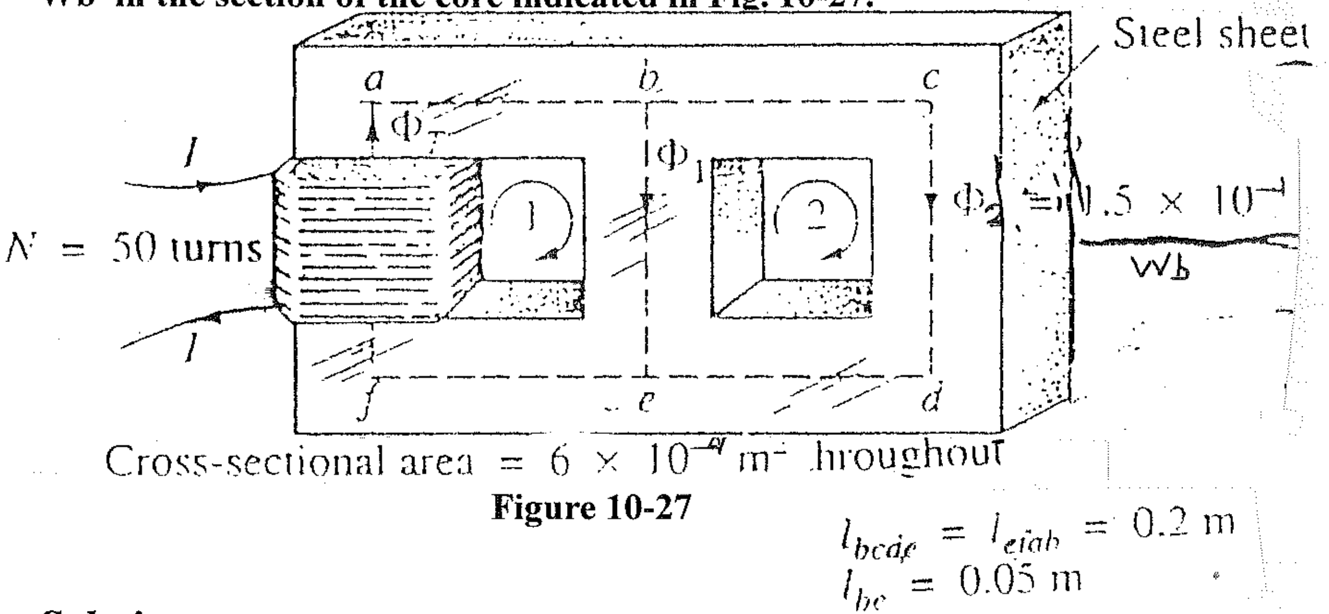


Figure 10-27

Solution :

The equivalent magnetic circuit and the electric circuit analogy appear in Fig. 10-28, we have :

$$B_2 = \frac{\Phi_2}{A} = \frac{1.5 \times 10^{-4}\ \text{Wb}}{6 \times 10^{-4}\ \text{m}^2} = 0.25\ \text{T}$$

From Fig. 10-17,

$$H_{bcde} \cong 40\ \text{At/m}$$

Applying Ampere's circuital law around loop 2 of Figs. 10-27, and 10-28.

$$\Sigma \, . \, \mathfrak{I} = 0$$

$$\mathbf{H_{be}l_{be}} \; - \; \mathbf{H_{bcde}l_{bcde}} = 0$$

$$\mathbf{H_{be}(0.05\ m)} \; - \; \mathbf{(40\ At/m)(0.2\ m)} = 0$$

$$\mathbf{H_{bc}} = \frac{8\ \text{At}}{0.05\ \text{m}} = 160\ \text{At/m}$$

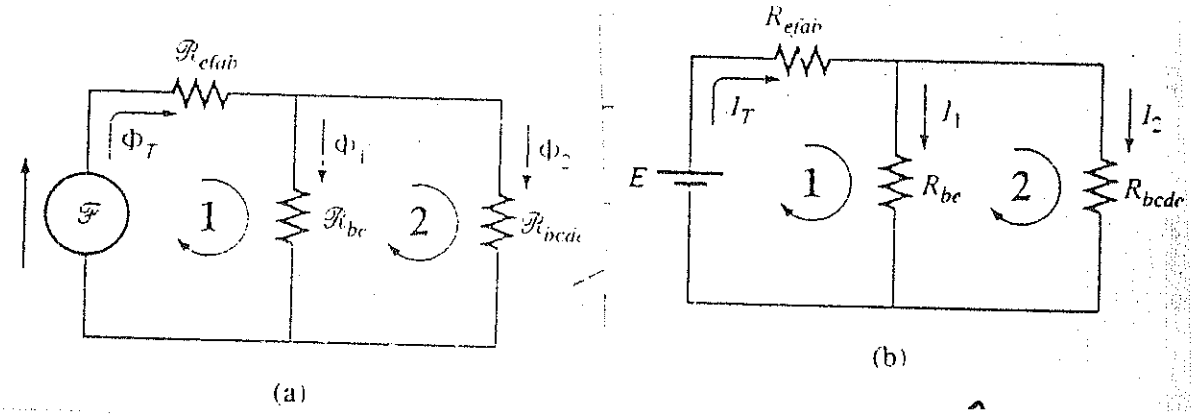


Figure 10-28

From Fig. 12-17,

$$\mathbf{B_1 \cong 0.97\ T}$$

and

$$\mathbf{\Phi_1 = B_1A = (0.97\ T)(6 \times 10^{-4}\ m^2) = 5.82 \times 10^{-4}\ Wb}$$

The results are then entered in Table 10.4.

Table 10.4

Section	Φ (Wb)	A (m^2)	B (T)	H (At/m)	ℓ (m)	$H\ell$ (At)
Bcde	1.5×10^{-4}	6×10^{-4}	0.25	40	0.2	8
Be	5.82×10^{-4}	6×10^{-4}	0.97 T	160	0.05	8
Efab		6×10^{-4}			0.2	

The table reveals that we must now turn attention to section efab :

$$\Phi_T = \Phi_1 + \Phi_2 = 5.82 \times 10^{-4}\ \text{Wb} + 1.5 \times\ 0^{-4}\ \text{Wb}$$

$$= 7.32 \times 10^{-4}\ \text{Wb}$$

$$B_T = \frac{\Phi_T}{A} = \frac{7.32 \times 10^{-4}\ \text{Wb}}{6 \times 10^{-4}\ \text{m}^2}$$

$$= 1.22\ \text{T}$$

From Fig. 10-17,

$$H_{efab} \cong 400\ \text{At}$$

Applying Ampere's circuital law,

$$+ NI - H_{efab}I_{efab} - H_{be}I_{be} = 0$$

$$NI = (400\ \text{At/m})(0.2\ \text{m}) + (160\ \text{At/m})(0.05\ \text{m})$$

$$(50\ t)I = 80\ \text{At} + 8\ \text{At}$$

$$I = \frac{88\ At}{50\ t} = 1.76\ \text{A}$$

Determining Φ :

The examples of this section are of the second type, where NI is given and the flux Φ must be found. This is a relatively straightforward problem if only one magnetic section is involved. Then,

$$H = \frac{NI}{\ell} \qquad H \to B \text{ (B-H curve)}$$

and

$$\Phi = BA$$

Example :

Calculate the magnetic flux Φ for the magnetic circuit of Figure 10-29.

Solution :

By Ampere's circuital law,

$$NI = H_{abcda}\ell_{abcda}$$

$$H_{abcda} = \frac{NI}{\ell_{abcda}} = \frac{(60\text{ t})(5\text{ A})}{0.3\,m}$$

$$= \frac{300\text{ At}}{0.3\,m} = 1000\text{ At/m}$$

$$H_{abcda} \text{ (from Fig. 10.17)} \cong 0.39\text{ T}$$

Since $B = \Phi/A$, we have :

$$\Phi = BA = (0.39\ T)(2 \times 10^{-4}\ m^2) = 0.78 \times 10^{-4}\ Wb$$

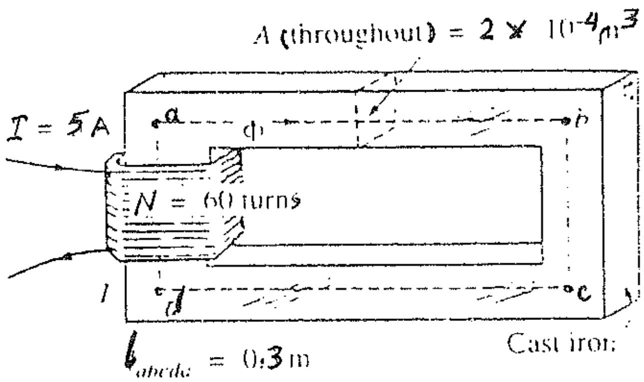


Figure 10-29

PROBLEMS

1) For the electromagnet of Fig. 10-30 :

a- Find the flux density in the core.

b-Sketch the magnetic flux lines and indicate their direction.

c- Indicate the north and south poles of the magnet.

2) Find the reluctance of a magnetic circuit if a magnetic flux Φ = 4.2×10^{-4} Wb is established by an impressed mmf of 400 At.

3) For the series magnetic circuit of Fig. 10-31, determine the current I necessary to establish the indicated flux.

4) a- Find the number of turns N_1 required to establish a flux $\Phi = 4.2 \times 10^{-4}$ Wb in the magnetic circuit of Fig. 10-32.

b- Fine the permeability μ of the material.

5) a- Find the current I required to establish a flux $\Phi = 4.2 \times 10^{-4}$ Wb in the magnetic circuit of Fig. 10-33.

b- Compare the mmf drop across the air gap to that across the rest of the magnetic circuit. Discuss your results using the value of μ for each material.

6) Determine the current I_1 required to establish a flux of $\Phi = 2 \times 10^{-4}$ Wb in the magnetic circuit of Fig. 10-34.

7) For the series-parallel magnetic circuit of Fig. 10-35, find the value of I required to establish a flux in the gap $\Phi_g = 2 \times 10^{-4}$ Wb.

8) **Find the magnetic flux Φ established in the series magnetic circuit of Fig. 10-36.**

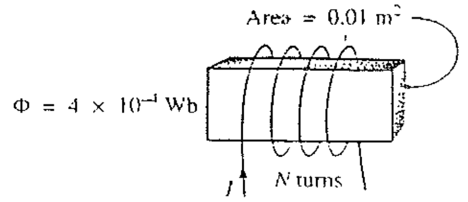


Figure 10-30

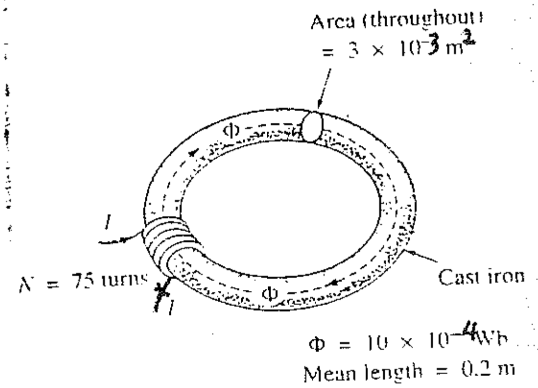


Figure 10-31

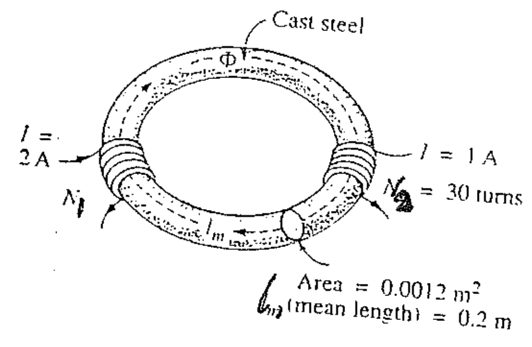


Figure 10-32

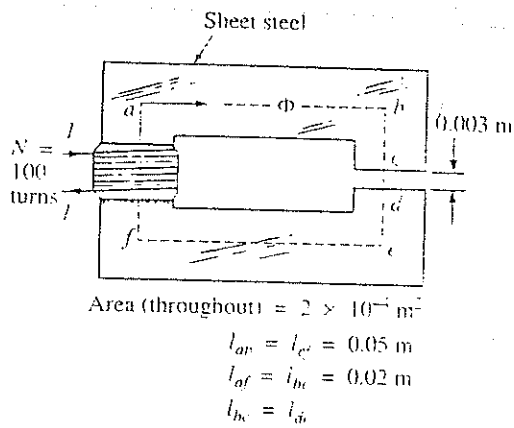


Figure 10-33

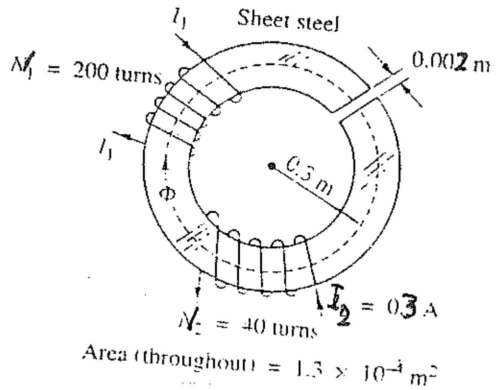


Figure 10-34

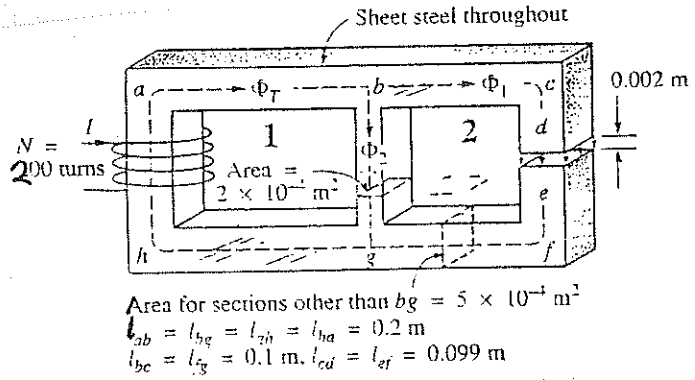


Figure 10-35

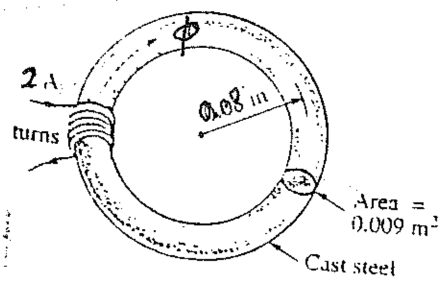


Figure 10-36

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