The National School of Artificial Intelligence | **2023-2024** | **Probability and Statistics Semestre 3**

Name:........................... .... **Group:**.......................

Final Exam
(2h00)

Problem 1 *A disillusioned examiner distributes the marks randomly. He assigns each candidate a mark chosen at random from the integers* $0, 1, 2, \cdots, 19$*. He considers that, since perfection is not of this world, mark* 20 *should not be awarded. He grades the candidates independently of each other. Candidates with a mark of* 10 *or above, and only those with a mark of* 10 *or above, are admitted. Candidates who obtain a mark of 0, and only they, will lose the right to take the exam again.*

(Un examinateur désabusé distribue les notes aléatoirement. Il attribue à chaque candidat une note choisie au hasard parmi les entiers $0, 1, 2, \cdots, 19$*. Il considère que, la perfection n'étant pas de ce monde, la note* 20 *ne doit pas être attribuée. Il note les candidats indépendamment les uns des autres. Seront déclarés admis les candidats obtenant un note supérieure ou égale à* 10*, et eux seulement. Les candidats obtenant la note* 0*, et eux seulement, perdront le droit de se présenter une nouvelle fois à cet examen.)*

1. *Arslane is one of the candidates. Calculate the probability of each of the following events:*

   A : *Arslane will be admitted.* B : *Arslane will lose the right to reapply.* C : *Arslane will obtain* 9 *or* 10*.*

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2. *There are exactly* 100 *candidates.* X *is the random variable equal to the number of candidates who will be admitted.* Y *is the random variable equal to the number of candidates who will lose the right to reapply.*

   a. *Give the probability distribution of* X*, its mathematical expectation and variance.*

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   b. *Propose a suitable approximation for* X *and use this approximation to estimate the probability that at least* 55 *candidates will be admitted.*

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c. *Give the probability distribution of Y, its mathematical expectation and variance.*

d. *Propose a suitable approximation for Y and use this approximation to estimate the probability that there are at most 2 candidates losing the right to run again.*

3. There are exactly 100 candidates, numbered from 1 to 100. X_i is the random variable equal to the mark that will be awarded to candidate number i with $(1 \leq i \leq 100)$. $\overline{X}_{100}$ is the random variable equal to the average of the 100 marks that will be given.

a. *Give the expectation and the variance of $X_i, (1 \leq i \leq 100)$.*

We recall that: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ *and* $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

b. *Give the expectation and the variance of* $\overline{X}_{100}$.

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Problem 2 *A real random variable* X *is said to follow an exponential distribution with parameter* $\lambda\,(\lambda > 0)$, *noted* $\mathcal{E}(\lambda)$, *if the density of the distribution of* X *is defined by*

$$f_X(x) = \lambda e^{-\lambda x} \mathbb{I}_{]0,\infty[}(x).$$

A. 1. *Determine* F_X *the cumulative function of* X. *Find* x_0 *such that* $F_X(x_0) = \frac{1}{2}$.

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2. *For* $k \in \mathbb{N}^*$, *calculate* $\mathbb{E}\left[X^k\right]$. *Deduce* $Var(X)$.

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3. *Determine* φ_x *the characteristic function of* X.

4. *For* $x > 0$ *and* $y > 0$, *calculate* $\mathbb{P}\left(X > x + y \,|X > x\right)$.

5. *Determine the distribution of* $[X]$, *where* $[X]$ *is the integer part of* X.

6. *Determine the distribution of* $Z = X - [X]$.

B. *Let* X_1 *and* X_2 *be two independent real random variables with respective distributions:* $\mathcal{E}(\lambda)$ *and* $\mathcal{E}(\mu)$. *We set:*

$$S = X_1 + X_2, U = \frac{X_1}{S}, V = \frac{X_2}{S}, T = \sup\{U, V\} \text{ and } W = \inf\{U, V\}.$$

1. *Determine the distribution of* (S, U).

2. *Determine* f_U *the marginal distribution of* U*. Are* S *and* U *independent?*

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3. *Express f_T the density of the distribution of T as a function of f_U.*

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4. *Express f_W the density of the distribution of W as a function of f_U.*

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C. *Let $Y_1, \cdots, Y_n$ be independent real random variables with the same distribution $\mathcal{E}(\lambda)$.*

1. *Determine φ_{S_n} the characteristic function of $S_n = Y_1 + \cdots + Y_n$. Deduce $\mathbb{E}[S_n]$ and $Var(S_n)$.*

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2. *Determine the distibution of* $Y_{(1)} = \inf\{Y_1, Y_2, \cdots, Y_n\}$.

D. *Let* X *be a random variable following the* $\mathcal{E}(\lambda)$ *distribution and* Y *a random variable following, conditionally to* $\{X = x\}$, *a uniform distribution on* $[x, 1 + x]$

1. *Calculate* $\mathbb{E}[Y]$ *and* $Var(Y)$.

2. *Determine the density of the distribution of* Y.

3. *Let* $Y_1, \cdots, Y_n$ *be independent random variables with the same distribution as* Y.

   a. *Show that* $T_n = \frac{1}{n}\sum_{i=1}^{n}\left(Y_i - \frac{1}{2}\right)$ *converges in probability to a random variable to be determined.*

   b. *Show that* $W_n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(Y_i - \frac{1}{2} - \frac{1}{\lambda}\right)$ *converges in law to a random variable with normal distribution* $\mathcal{N}(\mu, \sigma)$, *where* μ *and* σ *are constants to be determined.*