Work sheet N 2
Integrals depending on one parameter

Exercise 1. Consider the following functions:

$$F(x) = \int_1^2 \frac{e^{-x^2t^3}}{1+t^2}dt \quad \text{and} \quad G(x) = \int_0^1 \frac{e^{x(t+1)}}{t+1}dt.$$

1) Study the continuity and differentiability of F and G over $\mathbb{R}$.
2) Deduce an explicit expression for G'.

Exercise 2. Let the function $F(x) = \int_x^{x^2} \frac{e^{-t}}{\sqrt{t}}dt$; $x > 0$.
1) Study the differentiability of F.
2) Calculate F'.

Exercise 3. Let the function $F(x) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-x\sin t}dt$
1) Show that F is continuous on $\mathbb{R}$.
2) Show that F is differentiable on $\mathbb{R}$ and find the expression of F'.
3) Show that F is of class $\mathcal{C}^2$ and verifies

$$xF''(x) + F'(x) - xF(x) = 0, \quad \forall x \in \mathbb{R}.$$

Exercise 4. Study the continuity over $\mathbb{R}$ of the following functions:

$$F(x) = \int_1^{+\infty} \frac{\sin x + \sin t}{1+x^2+t^2}dt, \quad x \in \mathbb{R} \quad \text{and} \quad G(x) = \int_1^{+\infty} \frac{\sin t}{e^{xt}-1}dt, \quad x > 0.$$

Exercise 5. Consider the following function:

$$F(x) = \int_1^{+\infty} \frac{\sin t}{x^2+t^2}dt, \quad x \geq 0.$$

1) Study the continuity of F.
2) Study the differentiability of F.

Exercise 6. Let

$$F(x) = \int_1^{+\infty} \frac{e^{-xt}}{(1+t)\sqrt{t}}dt.$$

1) Show that $D_f = [0, +\infty[$.
2) Show that F is continuous on $[0, +\infty[$.
3) Show that F is differentiable on $]0, +\infty[$.

Exercise 7. We define $f(t,x) = e^{-t^2}\cos(xt)$ and $F(x) = \displaystyle\int_0^{+\infty} e^{-t^2}\cos(xt)dt$.

1) Show that F is well-defined on $\mathbb{R}$.
2) Show that F is of class C^1 on $\mathbb{R}$.
3) Using an integration by parts, show that $F'(x) = \frac{1}{2}xF(x)$.
4) Deduce the expression of F given that $\displaystyle\int_0^{+\infty} e^{-t^2}dt = \frac{\sqrt{\pi}}{2}$.