1.2 Arguments introducing "and".

We shall study how the word "and" appears in arguments. We are mainly interested in this word where it appears between two statements, as, for example, in

(1.3) v is a vector and α is a scalar.

We shall write this sentence as

(1.4) (v is a vector $\wedge$ α is a scalar)

Note that the parentheses are an essential part of the notation.

Here are some typical examples:

(1.5) The function f is surjective and differentiable
(The function f is surjective $\wedge$ the function f is differentiable)

(1.6) $$2 < \sqrt{5} < 3$$
$$(2 < \sqrt{5} \wedge \sqrt{5} < 3)$$

(1.7) $$A = B$$
$$(A \subseteq B \wedge B \subseteq A)$$

Now consider how we prove a statement made by joining together two other statements with "and".

Example 1.2.1

We prove that $2 < 5 < \sqrt{3}$.

(1) We prove that $2 < \sqrt{5}$ as follows. We know that $4 < 5$. Taking positive square roots of these positive numbers, $2 = \sqrt{4} < \sqrt{5}$.

(2) We prove that $\sqrt{5} < 3$ as follows. We know that $5 < 9$. Taking positive square roots of these positive numbers, $\sqrt{5} < \sqrt{9} = 3$.

Then, if we put together a proof of ϕ and a proof of ψ, the result is a proof of $(\phi \wedge \psi)$. The assumptions of this proof of $(\phi \wedge \psi)$ consist of the assumptions of the proof of ϕ together with the assumptions of the proof of ψ.
We can write this fact down as a sequent rule:

SEQUENT RULE $(\wedge I)$
If $(\Gamma \vdash \phi)$ and $(\Delta \vdash \psi)$ are correct sequents, then $(\Gamma \cup \Delta \vdash (\phi \wedge \psi))$ is a correct sequent.

The name $(\wedge I)$ expresses that this is a rule about $\wedge$, and the symbol $\wedge$ is introduced (hence "I") in the last sequent of the rule. We refer to this rule as $\wedge$-introduction.

We also adopt a schematic notation for combining the proofs of ϕ and ψ:

$$\frac{\boxed{\begin{array}{c}\text{Proof of}\\ \phi\end{array}} \qquad \boxed{\begin{array}{c}\text{Proof of}\\ \psi\end{array}}}{(\phi \wedge \psi)}$$

This diagram represents a proof of $(\phi \wedge \psi)$, which appears at the bottom. This bottom expression is called the <u>conclusion</u> of the proof.

The box notation is a little heavy, so we adopt a lighter version. We write

$$\begin{array}{c} D \\ \phi \end{array} \tag{1.8}$$

to stand for a proof D whose conclusion is ϕ. Using this notation, we recast the picture above as a rule for forming proofs.

This new rule will be our second natural deduction. We give it the same label $(\wedge I)$ as the corresponding sequent rule above

NATURAL DEDUCTION RULE $(\wedge I)$ If

$$\begin{array}{ccc} D & \text{and} & D' \\ \phi & & \psi \end{array}$$

are derivations of ϕ and ψ respectively, then

$$\dfrac{\begin{array}{ccc} D & & D' \\ \phi & & \psi \end{array}}{(\phi \wedge \psi)}\;(\wedge I)$$

is a derivation of $(\phi \wedge \psi)$. Its undischarged assumptions are those of D together with those of D'.

Exp 1.2.2

Suppose

$$\begin{array}{c} \mathcal{D} \\ \phi \end{array}$$

is a derivation of ϕ. Then

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \phi\end{array} \qquad \begin{array}{c}\mathcal{D}\\ \phi\end{array}}{(\phi\wedge\phi)}\,(\wedge I)$$

is a derivation of $(\phi\wedge\phi)$. Its undischarged assumptions are those of $\mathcal{D}$.

Exp 1.2.3

Suppose

$$\begin{array}{c} \mathcal{D} \\ \phi \end{array}, \qquad \begin{array}{c} \mathcal{D}' \\ \psi \end{array} \quad \text{and} \quad \begin{array}{c} \mathcal{D}'' \\ \chi \end{array}$$

are respectively derivations of ϕ, ψ and χ. Then

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \phi\end{array} \qquad \begin{array}{c}\mathcal{D}'\\ \psi\end{array}}{(\phi\wedge\psi)}\,(\wedge I) \qquad \begin{array}{c}\mathcal{D}''\\ \chi\end{array}}{((\phi\wedge\psi)\wedge\chi)}\,(\wedge I)$$

is a derivation of $(\phi\wedge\psi)\wedge\chi$, got by applying $\wedge$-introduction twice; the second time we apply it with $\mathcal{D}''$ as the second derivation. The undischarged assumptions of this derivation are those of $\mathcal{D}$, those of $\mathcal{D}'$ and those of $\mathcal{D}''$.

Remark 1.2.4

The following points will apply (with obvious adjustments) to all future derivations too.

- The conclusion of a derivation is the statement written in the bottom line.
- If the conclusion of an application of $(\wedge I)$ is $(\phi \wedge \psi)$, then the derivation of ϕ must go on the left and the derivation of ψ on the right.
- In exp 1.2.2 we used the same derivation of ϕ twice. So the derivation must be written twice.
- As we go upwards in a derivation, it may branch. The derivation in exp 1.2.3 has at least three branches (...). The branches stay separate as we go up them; they never join up again. A derivation never branches downwards.
- The name of the rule used in the last step of a derivation is written at the right-hand side of the horizontal line above the conclusion of the derivation. In our formal definition of derivations (def 2.4.1) these rule labels will be essential parts of a derivation.

Now, by the Axiom Rule for natural deduction, ϕ by itself is a derivation of ϕ with undischarged assumption ϕ. So, in exp 1.2.3 the derivation D could be this derivation, and then there is nor need to write 'D'. Similarly, we can leave out 'D'' and 'D''', regarding ψ and χ as derivations with themselves as conclusions. The result is the derivation

$$(1.9) \qquad \cfrac{\cfrac{\phi \qquad \psi}{(\phi \wedge \psi)}\,(\wedge I) \qquad \chi}{((\phi \wedge \psi) \wedge \chi)}\,(\wedge I) \quad .$$

Now, the undischarged assumptions of this derivation are those of D, D' and D'' together; so they are ϕ, ψ and χ. Thus the derivation (1.9) shows that there is a proof of $((\phi \wedge \psi) \wedge \chi)$ with undischarged assumptions ϕ, ψ and χ. Therefore, the sequent

$$(1.10) \quad \{\phi, \psi, \chi\} \vdash ((\phi \wedge \psi) \wedge \chi)$$

is correct

likewise if we cut out the symbol 'D' from exp 1.2.2, what remains is a derivation of $(\phi \wedge \phi)$ from ϕ and ϕ, establishing the correctness of

(1.11) $\{\phi\} \vdash (\phi \wedge \phi)$.

Remark 1.2.5

The derivation (1.9) is a proof of its conclusion from certain assumptions. It is also a proof of the sequent (1.10), by showing that (1.10) is correct. In mathematics this is par for the course(*); the same argument can be used to establish many different things. But in logic, where we are comparing different proofs all the time, there is a danger of confusion. For mental hygiene we shall say that (1.9) is a derivation of its conclusion, but a proof of the sequent (1.10).

(*) normal (même si "embêtant").