1.5 Arguments using 'if and only if'

We write

(1.18) $$(\phi \leftrightarrow \psi)$$

for ~~'ϕ if and only if ψ'~~.

~~We saw already that 'ϕ if and only if ψ' expresses the same as~~

(1.19) $$(\text{if } \phi \text{ then } \psi \wedge \text{if } \psi \text{ then } \phi).$$

So, we can use the introduction and elimination rules for $\wedge$ to devise introduction and elimination rules for $\leftrightarrow$, as follows.

NATURAL DEDUCTION RULE $(\leftrightarrow I)$ If

$$\begin{array}{ccc} \mathcal{D} & \text{and} & \mathcal{D}' \\ (\phi \to \psi) & & (\psi \to \phi) \end{array}$$

are derivations of $(\phi \to \psi)$ and $(\psi \to \phi)$, respectively, then

$$\frac{\begin{array}{cc} \mathcal{D} & \mathcal{D}' \\ (\phi \to \psi) & (\psi \to \phi) \end{array}}{(\phi \leftrightarrow \psi)} \; (\leftrightarrow I)$$

is a derivation of $(\phi \leftrightarrow \psi)$. Its undischarged assumptions are those of $\mathcal{D}$ together with those of $\mathcal{D}'$.

NATURAL DEDUCTION RULE ($\leftrightarrow$E) If

$$\begin{array}{c} \mathcal{D} \\ (\phi \leftrightarrow \psi) \end{array}$$

is a derivation of $(\phi \leftrightarrow \psi)$, then

$$\dfrac{\begin{array}{c} \mathcal{D} \\ (\phi \leftrightarrow \psi) \end{array}}{(\phi \to \psi)}(\leftrightarrow E) \quad \text{and} \quad \dfrac{\begin{array}{c} \mathcal{D} \\ (\phi \leftrightarrow \psi) \end{array}}{(\psi \to \phi)}(\leftrightarrow E)$$

are derivations of $(\phi \to \psi)$ and $(\psi \to \phi)$, respectively. Their undischarged assumptions are those of $\mathcal{D}$.

Example 1.5.1

A proof of the sequent $\{(\phi \leftrightarrow \psi)\} \vdash (\psi \leftrightarrow \phi)$.

$$\dfrac{\dfrac{(\phi \leftrightarrow \psi)}{(\psi \rightarrow \phi)}\,(\leftrightarrow E) \qquad\qquad \dfrac{(\phi \leftrightarrow \psi)}{(\phi \rightarrow \psi)}\,(\leftrightarrow E)}{(\psi \leftrightarrow \phi)}\,(\leftrightarrow I)$$