The National School of Artificial Intelligence | **2023-2024** | **Probability Semestre 3**

Resit Exam
(2h00)

Exercise 1 (3 points) *Let f be defined by:* $f(x) = \begin{cases} 0 \text{ if } x < 0 \\ xe^{-\frac{x^2}{2}} \text{ if } x \geq 0 \end{cases}$.

1. *Show that f is a probability distribution.*
2. *Let X be a real random variable having f as a distribution. Determine the distribution of $Y = X^2$.*
3. *Determine $\mathbb{E}[Y]$.*

Exercise 2 (4 points) *Let X and Y be two independent random variables with the same distribution, given by*

$P(X=1) = P(X=-1) = \frac{1}{4}$ *and* $P(X=0) = \frac{1}{2}$.

1. *Determine the distribution of the couple $(X+Y, XY)$, its marginale distributions, $\mathbb{E}[X+Y]$ and $\mathbb{E}[XY]$.*
2. *Calculate $Cov(X, X+Y)$. Are the random variables X and $X+Y$ indépendent?*

Exercise 3 (13 points) *Let f be a probability distribution on $\mathbb{R}$ with cumulative distribution function F, verifying*

$$\mu = \int_{-\infty}^{+\infty} x f(x)\, dx < \infty \text{ and } \sigma^2 = \int_{-\infty}^{+\infty} (x-\mu)^2 f(x)\, dx < \infty.$$

For $\theta > 0$, we define by μ_θ the uniform distribution on $]-\theta, \theta[$, defined by: $u_\theta(x) = \frac{1}{2\theta}\mathbb{I}_{]-\theta,\theta[}(x)$.
We define the function h_θ by

$$\forall x \in \mathbb{R}, h_\theta(x) = f * u_\theta(x) = \int_{-\infty}^{+\infty} f(t)\, u_\theta(x-t)\, dt.$$

1. *Show that h_θ is a probability distribution on $\mathbb{R}$.*

   Express h_θ in function of F. Let T be a random variable with distribution h_θ, calculate $\mathbb{E}[T]$ and $Var(T)$.

2. *Let $X_1, X_2, \cdots, X_n$ be independent random variables. We suppose that for all fixed $k \in \{1, \cdots, n-1\}$, $X_1, \cdots, X_k$ follow the same distribution f and $X_{k+1}, \cdots, X_n$ follow the same distribution h_θ.*

   We put

$$\begin{aligned} S_k &= \sum_{i=1}^{k} X_i, S_k^\star = \sum_{i=k+1}^{n} X_i, S_n = \sum_{i=1}^{n} X_i \\ \overline{X}_k &= \frac{S_k}{k}, \overline{X}_k^\star = \frac{S_k^\star}{n-k}, \overline{X}_n = \frac{S_n}{n}. \end{aligned}$$

   We suppose that when n goes to infinity, $\frac{k}{n}$ converges to $\tau \in]0,1[$.

a. *Determine* $\mathbb{E}[S_n]$ *and* $Var(S_n)$.

b. *Study the quadratic mean convergence of* $\overline{X}_n$.

c. *Study the convegence in law of* $U_n = \frac{1}{\sqrt{n}}(S_n - n\mu)$.

d. *For* $\eta > 0$, *calculate* $\lim\limits_{n\longrightarrow\infty} \mathbb{P}(|U_n| \geq \eta)$.

e. *We note by* ρ_n *correlation coefficient between* $\overline{X}_k$ *and* $\overline{X}_n$, $\rho_n^\star$ *correlation coefficient between* $\overline{X}_k^\star$ *and* $\overline{X}_n$. *Calculate* $\lim\limits_{n\longrightarrow\infty} \rho_n$ *et* $\lim\limits_{n\longrightarrow\infty} \rho_n^\star$.