1.7 Arguments using 'or'

We write $(\phi \vee \psi)$ for 'Either ϕ or ψ or both'. The symbol '$\vee$' is read as 'or'. For example,

$$(x = 0 \vee x > 0) \text{ says}$$

Either $x = 0$ or x is greater than 0 or both.

In this case (as often), the 'or both' doesn't arise and can be ignored. The whole statement is equivalent to '$x \geqslant 0$'.

SEQUENT RULE ($\vee$I)

If at least one of $(\Gamma \vdash \phi)$ and $(\Gamma \vdash \psi)$ is a correct sequent, then the sequent $(\Gamma \vdash (\phi \vee \psi))$ is correct.

NATURAL DEDUCTION RULE ($\vee I$) If

$$\begin{array}{c} \mathcal{D} \\ \phi \end{array}$$

is a derivation with conclusion ϕ, then

$$\dfrac{\begin{array}{c} \mathcal{D} \\ \phi \end{array}}{(\phi \vee \psi)}$$

is a derivation of $(\phi \vee \psi)$. Its undischarged assumptions are those of $\mathcal{D}$. Similarly, if

$$\begin{array}{c} \mathcal{D} \\ \psi \end{array}$$

is a derivation with conclusion ψ, then

$$\dfrac{\begin{array}{c} \mathcal{D} \\ \psi \end{array}}{(\phi \vee \psi)}$$

is a derivation with conclusion $(\phi \vee \psi)$. Its undischarged assumptions are those of $\mathcal{D}$.

Example 1.7.1

We prove the sequent $\vdash (\neg(\neg(\phi \vee (\neg\phi))))$.

$$
\dfrac{
  \dfrac{
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        \dfrac{
          \dfrac{\cancel{\phi}^{\,①}}{(\phi \vee (\neg\phi))}\,(\vee I)
          \qquad
          \cancel{(\neg(\phi \vee (\neg\phi)))}^{\,②}
        }{\bot}\,(\neg E)
      }{①\ (\neg\phi)}\,(\neg I)
    }{(\phi \vee (\neg\phi))}\,(\vee I)
    \qquad
    \cancel{(\neg(\phi \vee (\neg\phi)))}^{\,②}
  }{\bot}\,(\neg E)
}{②\ (\neg(\neg(\phi \vee (\neg\phi))))}\,(\neg I)
\tag{1.21}
$$

Example 1.7.1 - bis

In this example, we prove a contradiction from the negation of the conclusion, and then finish with (RAA).

$$3\ \dfrac{\dfrac{1\ \dfrac{\dfrac{\dfrac{\cancel{(\neg\phi)}^{1}}{(\phi\vee(\neg\phi))}\,(\vee I)\qquad \cancel{(\neg(\phi\vee(\neg\phi)))}^{3}}{\bot}\,(\neg E)}{\phi}\,(RAA)\qquad 2\ \dfrac{\dfrac{\dfrac{\cancel{\phi}^{2}}{(\phi\vee(\neg\phi))}\,(\vee I)\qquad \cancel{(\neg(\phi\vee(\neg\phi)))}^{3}}{\bot}\,(\neg E)}{(\neg\phi)}\,(\neg I)}{\bot}\,(\neg E)}{(\phi\vee(\neg\phi))}\,(RAA) \tag{1.22}$$

NATURAL DEDUCTION RULE ($\lor E$)

Given derivations

$$\begin{array}{c} \mathcal{D} \\ (\phi \lor \psi) \end{array}, \quad \begin{array}{c} \mathcal{D}' \\ \chi \end{array} \quad \text{and} \quad \begin{array}{c} \mathcal{D}'' \\ \chi \end{array}$$

we have a derivation

$$\dfrac{\begin{array}{c} \\ \mathcal{D} \\ (\phi \lor \psi) \end{array} \qquad \begin{array}{c} \cancel{\phi} \\ \mathcal{D}' \\ \chi \end{array} \qquad \begin{array}{c} \cancel{\psi} \\ \mathcal{D}'' \\ \chi \end{array}}{\chi} \; (\lor E)$$

Its undischarged assumptions are those of $\mathcal{D}$, those of $\mathcal{D}'$ except possibly ϕ, and those of $\mathcal{D}''$ except possibly ψ.

SEQUENT RULE ($\vee$E)

If $(\Gamma \cup \{\phi\} \vdash \chi)$ and $(\Delta \cup \{\psi\} \vdash \chi)$ are correct sequents, then the sequent

$$(\Gamma \cup \Delta \cup \{(\phi \vee \psi)\} \vdash \chi)$$

is correct.