3. Arguments eliminating 'and'

Often in arguments we rely a statement of the form (ϕ and ψ) to justify the next step in the argument. The simplest examples are where the next step is to deduce ϕ, or to deduce ψ.

Exp 1.3.1

We prove that every prime greater than 2 is odd.

Let p be a prime greater than 2. Since p is prime, p is not divisible by any integer n with $1 < n < p$. Since p is greater than 2, $1 < 2 < p$. So p is not divisible by 2, in other words, p is odd.

In this argument we assume

(1.12) (p is prime $\wedge$ p is greater than 2)

From (1.12), we deduce

(1.13) p is prime.

From this example, we extract another natural deduction rule :

NATURAL DEDUCTION RULE $(\wedge E)$

If

$$\begin{array}{c} D \\ (\phi \wedge \psi) \end{array}$$

is a derivation of $(\phi \wedge \psi)$, then

$$\dfrac{\begin{array}{c} D \\ (\phi \wedge \psi) \end{array}}{\phi}(\wedge E) \quad \text{and} \quad \dfrac{\begin{array}{c} D \\ (\phi \wedge \psi) \end{array}}{\psi}(\wedge E)$$

are derivations of ϕ and ψ, respectively. Their undischarged assumptions are those of D.
In the label $(\wedge E)$ the E stands for Elimination, and this rule is known as $\wedge$- elimination.

In sequent terms, this natural deduction rule tells us :

SEQUENT RULE ($\wedge$E)

If the sequent $(\Gamma \vdash (\phi \wedge \psi))$ is correct, then so are both the sequents $(\Gamma \vdash \phi)$ and $(\Gamma \vdash \psi)$.

We can use both of the rules $(\wedge I)$ and $(\wedge E)$ in a single derivation:

Exp 1.3.2

$$
\dfrac{\dfrac{(\phi \wedge \psi)}{\psi}\,(\wedge E) \qquad \dfrac{(\phi \wedge \psi)}{\phi}\,(\wedge E)}{(\psi \wedge \phi)}\,(\wedge I)
$$

This derivation proves the sequent

$$\{(\phi \wedge \psi)\} \vdash (\psi \wedge \phi).$$

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Exp 1.3.3

$$
\dfrac{
\dfrac{\dfrac{(\phi \wedge (\psi \wedge \chi))}{\phi}\,(\wedge E) \qquad \dfrac{\dfrac{(\phi \wedge (\psi \wedge \chi))}{(\psi \wedge \chi)}\,(\wedge E)}{\psi}\,(\wedge E)}{(\phi \wedge \psi)}\,(\wedge I)
\qquad
\dfrac{\dfrac{(\phi \wedge (\psi \wedge \chi))}{(\psi \wedge \chi)}\,(\wedge E)}{\chi}\,(\wedge E)
}{((\phi \wedge \psi) \wedge \chi)}\,(\wedge I)
$$

This derivation proves the sequent

$$\{ (\phi \wedge (\psi \wedge \chi)) \} \vdash ((\phi \wedge \psi) \wedge \chi).$$