Prelude

1) What is mathematics?

- For Euclid, mathematics consists of proofs and constructions.
- For Al-Khawarizmi, mathematics consists of calculations.
- For Leibniz, we can calculate whether a proof is correct. This will need a suitable language for writing proofs.
- Frege invented a universal characteristic. He called it Concept-script (Begriffsschrift)
- Gentzen's system of natural deduction allows us to write proofs in a way that is mathematically natural

2) Pronunciation guide.

$\perp$: absurdity

$\vdash$: turnstile

$\models$: models

$\forall$: for all

$\exists$: there is

t_A : the interpretation of t in A

$\models_A \phi$: A is a model of ϕ

$\approx$: has the same cardinality as

$\prec$: has smaller cardinality than

$\rightarrow$: arrow (...)

Chap 1

Informal natural deduction

In this course we shall study some ways of proving statements.
Of course not every statement can be proved; so we need to analyze the statements before we prove them.
Within propositional logic, we analyse statements down into shorter statements.
Later chapters will analyse statements into smaller expressions too, but the smaller expressions need not be statements

What is a statement ?

A string S of one or more words or symbols is a statement if it makes sense to put S in place of the "..." in the question

Is it true that ... ?

For exp, it makes sense to ask any of the questions
Is it true that π is rational ?
Is it true that differentiable functions are continuous?
Is it true that $f(x) > g(y)$?
So all of the following are statements :
π is rational.
Differentiable functions are continuous.
$f(x) > g(y)$.

The answers to the three questions are different :

- No.
- Yes.
- It depends on what f, g, x and y are.

On the other hand, none of the following questions make sense

Is it true that π ?
Is it true that Pythagoras' Theorem ?
Is it true that $3 + \cos\theta$?

So none of the expressions 'π', 'Pythagoras' Theorem' and '$3 + \cos\theta$' is a statement.

The above test assumes that we know what counts as a 'symbol'. In practice, we do know and a precise definition is hardly called for. But we will take for granted

(1) that a symbol can be written on a page — given enough paper, ink, time and patience ;
(2) that we know what counts as a finite string of symbols;
(3) that any set of symbols that we use can be listed, say as $s_0, s_1, s_2, \ldots$, indexed by natural numbers.

In some more advanced applications of logic it is necessary to call on a more abstract notion of symbol ; we will discuss this briefly in Sect 7.9.

1.1. Proofs and sequents

Definition 1.1.1
A mathematical proof is a proof of a statement; this statement is called the <u>conclusion</u> of the proof. The proof may use some assumptions that it takes for granted. These are called its <u>assumptions</u>. A proof is said to be a proof <u>of</u> its conclusion <u>from</u> its assumptions.

For example, here is a proof from a textbook of mathematics:

<u>Proposition</u> Let $z = r(\cos\theta + i\sin\theta)$, and let n be a positive integer. Then

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

<u>Proof</u> Applying Thm 6.1 with $z_1 = z_2 = z$ gives

$$z^2 = zz = rr(\cos(\theta+\theta) + i\sin(\theta+\theta)) = r^2(\cos 2\theta + i\sin 2\theta).$$

Repeating, we get

$$z^n = r\cdots r(\cos(\theta+\cdots+\theta) + i\sin(\theta+\cdots+\theta)) = r^n(\cos n\theta + i\sin n\theta). \square$$

The proof is the proof of the equation

$$(1.1) \qquad z^n = r^n(\cos n\theta + i\sin n\theta),$$

so this equation (1.1) is the conclusion of the proof.

- There are several assumptions:
- One assumption is stated at the beginning of the proposition:
  $z = r(\cos\theta + i\sin\theta)$, and n is a positive integer.
  (The word "Let" at the beginning of the proposition is a sign that what follows is an assumption.)
- Another assumption is Thm 6.1.
- Finally, there are a number of unstated assumptions about how to do arithmetic. For example, the proof assumes that if $a=b$ and $b=c$, then $a=c$. These assumptions are unstated because they can be taken for (*) granted between reader and writer.

(*) peut être considéré comme acquis

- When we use the tools of logic to analyse a proof, we usually need to write down statements that express the conclusion and all the assumptions, including unstated assumptions.

A proof P of a conclusion ψ *need not* show that ψ is true. All it shows is that ψ is true *if* the assumptions of P are true.

If we want to use P to show that ψ is true, we need to account for these assumptions. There are several ways of doing this.

- One is to show that an assumption says something that we can agree is true without needing argument. For exp, we need no argument to see that $0 = 0$.
- A second way of dealing with an assumption is to find another proof Q that shows the assumption must be true. In this case, the assumption is called a *lemma* for the proof P. The assumption no longer counts as an assumption of the longer proof consisting of P together with Q.
- Section 1.4 will introduce us to a third and *very important* way of dealing with assumptions, namely to *discharge* them; a discharged assumption is no longer needed for the conclusion. We will see that — just as with adding a proof of lemma — discharging an assumption of a proof will always involve putting the proof inside a larger proof. So mathematical proofs with assumptions are really pieces that are available to be fitted into longer proofs, like bricks in a construction kit.

Sequents

Definition 1.1.2

A <u>sequent</u> is an expression

$$(\Gamma \vdash \psi)$$

(or $\Gamma \vdash \psi$ when there is no ambiguity), where ψ is a statement (the <u>conclusion</u> of the sequent) and Γ is a set of statements (the <u>assumptions</u> of the sequent).

We read the sequent as

"Γ entails ψ".

The sequent $(\Gamma \longrightarrow \psi)$ means

(1.2) || There is a proof whose conclusion is ψ and whose undischarged assumptions are all in the set Γ.

When (1.2) is true, we say that the sequent is <u>correct</u>. The set Γ can be empty, in which case we write $(\vdash \psi)$ (read: turnstile ψ); this sequent is correct iff there is a proof of ψ with no undischarged assumptions.

- We can write down properties that sequents ought to have (*). For exp:

- SEQUENT RULE (<u>Axiom Rule</u>)

  If $\psi \in \Gamma$, then the sequent $(\Gamma \vdash \psi)$ is correct.

- SEQUENT RULE (<u>Transitive Rule</u>)

  If $(\Delta \vdash \psi)$ is correct and for every δ in Δ, $(\Gamma \vdash \delta)$ is correct, then $(\Gamma \vdash \psi)$ is correct.

Sequent rules like these will be at the heart of this course. But side ~~with~~ by side with them, we will introduce other rules called <u>natural deduction rules</u>.

The main difference will be that sequent rules are <u>about provability in general</u>, whereas natural deduction rules tell us <u>how we can build proofs</u> of a particular kind (called <u>derivations</u>) for the relevant sequents. These derivations, together with the rules for using them, form the <u>natural deduction calculus</u>. In later chapters we will redefine sequents so that they refer only to provability by natural deduction derivations within the natural deduction calculus.

This will have the result that the sequent rules will become provable consequences of the natural deduction rules.

(See Appendix A for a list of our natural deduction rules.)

(*) devraient avoir

Derivations are always written so that their conclusion is their bottom line. A derivation with conclusion ϕ is said to be a <u>derivation</u> <u>of ϕ</u>.
We can give one natural deduction rule straight away. It tells us how to write down derivations ~~to~~ justify the Axiom Rule for sequents.

NATURAL DEDUCTION RULE (Axiom Rule)

Let ϕ be a statement. Then

$$\phi$$

is a derivation. Its conclusion is ϕ, and it has one undischarged assumption, namely ϕ.

Sequent ruls and natural deduction rules were introduced in 1934 by Gerhard Gentzen as proof calculi. A <u>proof</u> <u>calculus</u> is a system of mathematical rules for proving theorems (See 2.9.)
(...)