

Mathematical analysis 3

Chapter 2 : Integrals Depending On a Parameter



2024/2025

Course outline

1 Introduction

2 Proper Integrals Depending on a Parameter

- Continuity of Proper Integrals Depending on a parameter
- Integration of Integrals Depending on a parameter
- Differentiation of Integrals Depending on a Parameter

3 Improper Integrals Depending on a Parameter

- Continuity of Improper Integrals depending on one parameter
- Differentiation of Improper Integrals depending on one parameter

Introduction and motivation

- In this chapter, we focus on the study of the **continuity** and **differentiability** of functions defined by

$$x \mapsto F(x) = \int_a^b f(t, x) dt; \quad x \mapsto F(x) = \int_{u(x)}^{v(x)} f(t, x) dt,$$

(where f is a real function of two variables), In both cases:

- The integral is proper (Riemann)
- The integral is improper
- The integrals depending on a parameter are used in various fields

Differential equations, partial differential equations, probability and statistics, physics and engineering, economics and finance, computer science, machine learning, artificial intelligence, robotics, biology,...

- The determination of **the domain of definition** of the function F is not always evident, as generally one cannot provide an expression for F without using the symbol \int .
- Suppose $x \rightarrow F(x)$ is well-defined on a certain interval $I \subset \mathbb{R}$. Many natural questions arise:

- If f is **continuous**, will F be **continuous**? And for every $x_0 \in I$ and $[\alpha, \beta] \subset I$, is it true that

$$\lim_{x \rightarrow x_0} \int_a^b f(t, x) dt \stackrel{?}{=} \int_a^b \lim_{x \rightarrow x_0} f(t, x) dt,$$

$$\int_\alpha^\beta \int_a^b f(t, x) dt dx \stackrel{?}{=} \int_a^b \int_\alpha^\beta f(t, x) dx dt.$$

- If f is **differentiable**, will F be **differentiable**? And is it true that

$$\frac{\partial}{\partial x} \int_a^b f(t, x) dt \stackrel{?}{=} \int_a^b \frac{\partial}{\partial x} f(t, x) dt.$$

Example. Consider the function

$$F(x) = \int_0^{+\infty} x \sin(x) e^{-tx^2} dt.$$

We have

- For $x = 0$, $F(0) = 0$,
- If $x \neq 0$, We have

$$F(x) = x \sin(x) \int_0^{+\infty} e^{-tx^2} dt = x \sin(x) \left[\frac{-e^{-tx^2}}{x^2} \right]_0^{+\infty} = \frac{\sin(x)}{x}.$$

Therefore, $D_F = \mathbb{R}$.

The function $(t, x) \mapsto f(t, x) = x \sin(x) e^{-tx^2}$ is continuous on \mathbb{R}^2 , but the function F is not continuous on \mathbb{R} . Indeed, the issue arises at $x = 0$:

$$\lim_{x \rightarrow 0} F(x) = \lim_{x \rightarrow 0} \int_0^{+\infty} f(t, x) dt = \lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1 \neq 0 = \int_0^{+\infty} \lim_{x \rightarrow 0} f(t, x) dt = F(0).$$

Course outline

1 Introduction

2 Proper Integrals Depending on a Parameter

- Continuity of Proper Integrals Depending on a parameter
- Integration of Integrals Depending on a parameter
- Differentiation of Integrals Depending on a Parameter

3 Improper Integrals Depending on a Parameter

- Continuity of Improper Integrals depending on one parameter
- Differentiation of Improper Integrals depending on one parameter

Introduction

Proper integrals depending on a parameter

Throughout this section, we adopt the following notations:

- I is an interval of \mathbb{R} ,
- $a, b \in \mathbb{R}$, and $\Delta = [a, b] \times I$,
- f is a function of two variables such that

$$f : [a, b] \times I \rightarrow \mathbb{R} \quad (t, x) \mapsto f(t, x)$$

- $F(x) = \int_a^b f(t, x) dt.$

Course outline

1 Introduction

2 Proper Integrals Depending on a Parameter

- Continuity of Proper Integrals Depending on a parameter
- Integration of Integrals Depending on a parameter
- Differentiation of Integrals Depending on a Parameter

3 Improper Integrals Depending on a Parameter

- Continuity of Improper Integrals depending on one parameter
- Differentiation of Improper Integrals depending on one parameter

Continuity of Proper Integrals Depending on a parameter

Theorem (Continuity Preservation under Integration)

If f is a continuous function on $\Delta = [a, b] \times I$, then the function F defined for every $x \in I$ by

$$F(x) = \int_a^b f(t, x) dt$$

is continuous on I , and

$$\forall x_0 \in I, \lim_{x \rightarrow x_0} \int_a^b f(t, x) dt = \int_a^b \lim_{x \rightarrow x_0} f(t, x) dt = \int_a^b f(t, x_0) dt.$$

Continuity of Proper Integrals Depending on a parameter

Example.

Let

$$F(x) = \int_0^{\pi} \sin(x+t)e^{xt^2} dt.$$

- 1) Show that F is continuous on \mathbb{R} .
- 2) Calculate $\lim_{x \rightarrow 0} F(x)$.

Solution.

1) The integral is a proper Riemann integral. Set $f(t, x) = \sin(x+t)e^{xt^2}$. Since $(t, x) \mapsto f(t, x)$ is continuous on $\Delta = [0, \pi] \times \mathbb{R}$ (composition of continuous functions), then according to the previous theorem, the function $x \mapsto F(x)$ is continuous on \mathbb{R} .

Continuity of Proper Integrals Depending on a parameter

2) Consequently, we deduce the continuity of F at 0 . Therefore,

$$\lim_{x \rightarrow 0} F(x) = F(0) = \int_0^{\pi} \sin(t) dt = 2.$$

In this example, we were able to calculate the limit of F at 0 without having an explicit formula.

Generalization of the continuity theorem

The following result generalizes the theorem of continuity preservation under the integral sign \int :

Theorem

If f is a continuous function on $\Delta = [a, b] \times I$, $x \mapsto u(x)$ and $x \mapsto v(x)$ are two continuous functions on I within $[a, b]$, then the function F defined for every $x \in I$ by

$$F(x) = \int_{u(x)}^{v(x)} f(t, x) dt$$

is continuous on I .

Course outline

1 Introduction

2 Proper Integrals Depending on a Parameter

- Continuity of Proper Integrals Depending on a parameter
- **Integration of Integrals Depending on a parameter**
- Differentiation of Integrals Depending on a Parameter

3 Improper Integrals Depending on a Parameter

- Continuity of Improper Integrals depending on one parameter
- Differentiation of Improper Integrals depending on one parameter

Integration of Integrals Depending on a parameter

Is $F(x) = \int_a^b f(t, x) dt$ integrable over $[\alpha, \beta] \subset I$? And

$$\int_{\beta}^{\alpha} F(x) dx = ???$$

Theorem (Integration under the integral sign)

If f is a continuous function on $\Delta = [a, b] \times I$, then the function F defined for every $x \in I$ by

$$F(x) = \int_a^b f(t, x) dt$$

is integrable over any closed and bounded interval $[\alpha, \beta] \subset I$, and

$$\int_{\beta}^{\alpha} F(x) dx = \int_{\beta}^{\alpha} \left(\int_a^b f(t, x) dt \right) dx = \int_a^b \left(\int_{\beta}^{\alpha} f(t, x) dx \right) dt$$

Course outline

1 Introduction

2 Proper Integrals Depending on a Parameter

- Continuity of Proper Integrals Depending on a parameter
- Integration of Integrals Depending on a parameter
- Differentiation of Integrals Depending on a Parameter

3 Improper Integrals Depending on a Parameter

- Continuity of Improper Integrals depending on one parameter
- Differentiation of Improper Integrals depending on one parameter

Differentiation of Integrals Depending on a Parameter

Theorem

If f is a continuous function on $\Delta = [a, b] \times I$, and $(t, x) \mapsto \frac{\partial f}{\partial x}(t, x)$ exists and is continuous on Δ , then the function F defined for every $x \in I$ by

$$F(x) = \int_a^b f(t, x) dt$$

is of class C^1 on I , and

$$\forall x \in I, F'(x) = \left(\int_a^b f(t, x) dt \right)' = \int_a^b \frac{\partial f}{\partial x}(t, x) dt.$$

Corollary

If f is of class C^n (with $n \in \mathbb{N} \cup \{+\infty\}$) on Δ , then F is of class C^n on I , and for all $x \in I$, we have

$$F^{(n)}(x) = \left(\int_a^b f(t, x) dt \right)^{(n)} = \int_a^b \frac{\partial^n f}{\partial x^n}(t, x) dt.$$

Differentiation of Integrals Depending on a Parameter

Example.

Let's consider the function

$$f(t, x) = \frac{1}{t^2 + x^2}$$

and let

$$F(x) = \int_0^1 \frac{1}{t^2 + x^2} dt.$$

- 1) Show that F is of class C^1 on $]0, +\infty[$ and calculate $F'(x)$.
- 2) Calculate F directly and deduce the value of

$$\int_0^1 \frac{1}{(t^2 + x^2)^2} dt.$$

Differentiation of Integrals Depending on a Parameter

Solution. The integral is a proper Riemann integral.

1) We have:

- f is a continuous function on $[0, 1] \times]0, +\infty[$,
- $(t, x) \mapsto \frac{\partial f}{\partial x}(t, x) = \frac{-2x}{(t^2 + x^2)^2}$ exist and is continuous on $[0, 1] \times]0, +\infty[$,

Then the function $x \mapsto F(x)$ is of class C^1 on $]0, +\infty[$ and

$$F'(x) = \int_0^1 \frac{-2x}{(t^2 + x^2)^2} dt.$$

Differentiation of Integrals Depending on a Parameter

2) We can easily verify from the definition of F that $F(x) = \frac{1}{x} \arctan \frac{1}{x}$,
thus

$$F'(x) = -\frac{1}{x^2} \arctan \frac{1}{x} - \frac{1}{x(1+x^2)}.$$

Hence, we have:

$$\int_0^1 \frac{1}{(t^2 + x^2)^2} dt = \frac{1}{2x^3} \arctan \frac{1}{x} + \frac{1}{2x^2} (1 + x^2).$$

Differentiation of Integrals Depending on a Parameter

The following result generalizes the theorem of derivative preservation under integration to parameter-dependent integrals with parameterized bounds.

Theorem

If f and $\frac{\partial f}{\partial x}$ are continuous on $\Delta = [a, b] \times I$, $x \mapsto u(x)$ and $x \mapsto v(x)$ are two C^1 functions on I with values in $[a, b]$, then the function F defined on I by

$$F(x) = \int_{u(x)}^{v(x)} f(t, x) dt$$

is of class C^1 on I , and

$$F'(x) = \int_{u(x)}^{v(x)} \frac{\partial f}{\partial x}(t, x) dt + v'(x)f(v(x), x) - u'(x)f(u(x), x).$$

Differentiation of Integrals Depending on a Parameter

Example.

Let's consider the following function:

$$F(x) = \int_0^x \frac{1}{t^2 + x^2 + 1} dt.$$

Show that F is differentiable on $[0, +\infty[$ and calculate $F'(x)$.

Answer: Let $\beta > 0$. Apply the previous theorem for

$$f(t, x) = \frac{1}{t^2 + x^2 + 1}, \quad [a, b] = [0, \beta], \quad I = [0, \beta].$$

We have: f and $\frac{\partial f}{\partial x}$ are continuous on $\Delta = [0, \beta] \times [0, \beta]$, $x \mapsto u(x) = 0$ and $x \mapsto v(x) = x$ are two C^1 functions on $[0, \beta]$ with values in $[0, \beta]$.

Differentiation of Integrals Depending on a Parameter

Then the function F is of class C^1 on every $[0, \beta] \subset [0, +\infty[$ and

$$\forall x \in [0, \beta], F'(x) = \int_0^x \frac{-2x}{(t^2 + x^2 + 1)^2} dt + \frac{1}{2x^2 + 1}.$$

By extension, we deduce that F is of class C^1 on $[0, +\infty[$ and

$$\forall x \in [0, +\infty[, F'(x) = \int_0^x \frac{-2x}{(t^2 + x^2 + 1)^2} dt + \frac{1}{2x^2 + 1}.$$

Differentiation of Integrals Depending on a Parameter

Example.

Let's consider the function

$$F(x) = \int_0^1 \frac{1}{(t^2 + x^2)(t^2 + 1)} dt.$$

1) Show that F is continuous on \mathbb{R}^* .

2) Deduce the value of $\int_0^1 \frac{1}{(t^2 + 1)^2} dt$.

Differentiation of Integrals Depending on a Parameter

We set

$$f(t, x) = \frac{1}{(t^2 + x^2)(t^2 + 1)}.$$

The function

$$F(x) = \int_0^1 \frac{1}{(t^2 + x^2)(t^2 + 1)} dt$$

is a proper parameterized integral.

1) First, let's study the continuity of F on \mathbb{R}_+^* . Since f is continuous as a quotient of continuous functions on $[0, 1] \times \mathbb{R}_+^*$ with a non-zero denominator, by using the theorem of continuity preservation under the integral sign for proper parameterized integrals, we deduce that F is continuous on \mathbb{R}_+^* . Furthermore, since the function F is even, we conclude that F is continuous on all of \mathbb{R}^* .

Differentiation of Integrals Depending on a Parameter

2) From the previous question, we have in particular

$$\lim_{x \rightarrow 1} F(x) = F(1) = \int_0^1 \frac{1}{(t^2 + 1)^2} dt.$$

Furthermore,

$$f(t, x) = \frac{1}{(t^2 + x^2)(t^2 + 1)} = \frac{1}{x^2 - 1} \left(\frac{1}{t^2 + 1} - \frac{1}{t^2 + x^2} \right)$$

for $x \neq 1$ and $x \neq -1$.

Differentiation of Integrals Depending on a Parameter

Then

$$\begin{aligned}\lim_{x \rightarrow 1} F(x) &= \lim_{x \rightarrow 1} \frac{1}{x^2 - 1} \left(\int_0^1 \frac{1}{t^2 + 1} dt - \int_0^1 \frac{1}{t^2 + x^2} dt \right) \\&= \lim_{x \rightarrow 1} \frac{1}{x^2 - 1} \left[\arctan(t) \Big|_0^1 - \frac{1}{x} \arctan\left(\frac{t}{x}\right) \Big|_0^1 \right] \\&= \lim_{x \rightarrow 1} \frac{1}{x^2 - 1} \left(\frac{\pi}{4} - \frac{1}{x} \arctan\left(\frac{1}{x}\right) \right) \\&= \lim_{x \rightarrow 1} \frac{\frac{\pi}{4} - \frac{1}{x} \arctan\left(\frac{1}{x}\right)}{x^2 - 1}\end{aligned}$$

Differentiation of Integrals Depending on a Parameter

Using L'Hopital's rule, it follows that

$$\lim_{x \rightarrow 1} F(x) = \lim_{x \rightarrow 1} \frac{-\left(\frac{-1}{x^2} \arctan\left(\frac{1}{x}\right) + \frac{1}{x} \frac{\frac{-1}{x^2}}{\left(\frac{1}{x}\right)^2 + 1}\right)}{2x} = \frac{\pi}{8} + \frac{1}{4}.$$

Conclusion: From (1) and (2), we deduce that

$$\int_0^1 \frac{1}{(t^2 + 1)^2} dt = \frac{\pi}{8} + \frac{1}{4}.$$

Course outline

1 Introduction

2 Proper Integrals Depending on a Parameter

- Continuity of Proper Integrals Depending on a parameter
- Integration of Integrals Depending on a parameter
- Differentiation of Integrals Depending on a Parameter

3 Improper Integrals Depending on a Parameter

- Continuity of Improper Integrals depending on one parameter
- Differentiation of Improper Integrals depending on one parameter

Improper integrals depending on a parameter

Throughout this section, we adopt the following notations:

- I is any interval in \mathbb{R} ,
- $a \in \mathbb{R}$, $b \in \overline{\mathbb{R}}$, and $\Delta = [a, b[\times I$, where $[a, b[$ is open on the side of b ,
- f is a function of two variables such that

$$\begin{aligned} f: [a, b[\times I &\rightarrow \mathbb{R} \\ (t, x) &\mapsto f(t, x). \end{aligned}$$

- $F(x) = \int_a^b f(t, x) dt$

Improper Integrals Depending on a Parameter

- for all $x \in I$, $t \mapsto f(t, x)$ is assumed to be locally Riemann integrable on $[a, b[$ according to t ($f \in R_{loc}([a, b[)$ with respect to t)
- if $F(x) = \int_a^b f(t, x) dt$, then its domain of definition is given by

$$D_F = \{x \in I \mid \int_a^b f(t, x) dt \text{ converges}\}.$$

Improper Integrals Depending on a Parameter

In this paragraph, we focus on improper integrals dependent on a parameter. In this case, for example, the continuity of f cannot systematically imply that F is well-defined on I , nor that it is continuous. To see this, let's consider the example of the function

$$F(x) = \int_0^{+\infty} x \sin(x) \exp(-tx^2) dt.$$

By calculation, we have

$$F(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

Note: Although $f(t, x) = x \exp(-tx^2) \sin(x)$ is C^∞ on $[0, +\infty[\times \mathbb{R}$, the function F is not continuous at 0 .

To preserve the "analytic" properties under the integral sign, we will need to add certain hypotheses called domination hypotheses.

Dominated convergence

Definition

If there exists a function $\varphi : t \mapsto \varphi(t)$ (independent of x) piecewise continuous on $[a, b[$ such that

- $\forall (t, x) \in [a, b[\times A, |f(t, x)| \leq \varphi(t),$
- $\int_a^b \varphi(t) dt$ is convergent.

then we say that the improper parametrized integral $\int_a^b f(t, x) dt$ satisfies the criterion of dominated convergence on I .

Proposition

Dominated convergence \Rightarrow Simple convergence

Dominated convergence

Example. Let's consider the integral $\int_1^{+\infty} \frac{\sin(xt)}{t^2} dt$.

We set

$$f(t, x) = \frac{\sin(xt)}{t^2}$$

then

- $\forall (t, x) \in [1, +\infty[\times \mathbb{R} : |f(t, x)| \leq \frac{1}{t^2},$

- $\int_1^{+\infty} \frac{1}{t^2} dt$ converges

Therefore,

$$\int_1^{+\infty} \frac{\sin(xt)}{t^2} dt.$$

satisfies the criterion of dominated convergence on \mathbb{R} .

Course outline

1 Introduction

2 Proper Integrals Depending on a Parameter

- Continuity of Proper Integrals Depending on a parameter
- Integration of Integrals Depending on a parameter
- Differentiation of Integrals Depending on a Parameter

3 Improper Integrals Depending on a Parameter

- Continuity of Improper Integrals depending on one parameter
- Differentiation of Improper Integrals depending on one parameter

Continuity of an improper integral depending on one parameter

Theorem

If

- f is a continuous function on $\Delta = [a, b] \times I$,
- $F(x) = \int_a^b f(t, x) dt$ satisfies the criterion of dominated convergence on I ,

Then F is well-defined and continuous on I , and integrable over any $[\alpha, \beta] \subseteq I$.

Continuity of an improper integral depending on one parameter

Example.

Let's Consider the function $F(x) = \int_1^{+\infty} \sin(3xt)e^{-t^2} dt$.

- 1) Determine the domain of definition D of F .
- 2) Study the continuity of F on D .

Answer:

1) The parameterized integral is improper and poses a problem at $+\infty$.

Let

$$f(t, x) = \sin(3xt)e^{-t^2}.$$

We have

$$\forall (t, x) \in \Delta = [1, +\infty[\times \mathbb{R}, |f(t, x)| \leq e^{-t^2} \leq e^{-t},$$

and

$$\int_1^{+\infty} e^{-t} dt \text{ converges}$$

Continuity of an improper integral depending on one parameter

Therefore, $\int_1^{\infty} \sin(3xt)e^{-t^2} dt$ satisfies the criterion of dominated convergence, which implies its simple convergence. Thus, $D = \mathbb{R}$.

2) We have

- $(t, x) \mapsto f(t, x)$ is continuous on Δ ,
- $F(x) = \int_1^{\infty} f(t, x) dt$ satisfies the criterion of dominated convergence on \mathbb{R} .

Verifying the hypotheses of the theorem of the conservation of continuity under the integral sign, the function F is well-defined and continuous on \mathbb{R} .

Continuity of an improper integral depending on one parameter

Remark

As continuity is a local property, using covering, the theorem remains valid if we replace the hypothesis of dominated convergence over I by domination over any $[\alpha, \beta]$ of I .

Continuity of an improper integral depending on one parameter

Example.

Study the continuity of the function $F(x) = \int_0^{+\infty} \frac{e^{-xt}}{1+t} dt$ on $]0, +\infty[$.

Let's set $f(t, x) = \frac{e^{-xt}}{1+t}$ and take $0 < \alpha < \beta$. We have:

- f is continuous on $[0, +\infty[\times]0, +\infty[$,
- For all $t \in [0, +\infty[$, for all $x \in [\alpha, \beta] \subset]0, +\infty[$,

$$|f(t, x)| \leq e^{-xt} \leq e^{-\alpha t},$$

- The integral $\int_0^{\infty} e^{-\alpha t} dt$ converges.

Verifying the hypotheses of the continuity preservation theorem, the function F is well-defined and continuous on any $[\alpha, \beta] \subset]0, +\infty[$.

Thus, by covering, we deduce the continuity of F on $]0, +\infty[$.

Course outline

1 Introduction

2 Proper Integrals Depending on a Parameter

- Continuity of Proper Integrals Depending on a parameter
- Integration of Integrals Depending on a parameter
- Differentiation of Integrals Depending on a Parameter

3 Improper Integrals Depending on a Parameter

- Continuity of Improper Integrals depending on one parameter
- Differentiation of Improper Integrals depending on one parameter

Differentiation of an improper integral depending on one parameter

Theorem

If

- $\exists x_0 \in I$ such that $\int_a^b f(t, x_0) dt$ is *convergent*,
- f is *continuous* on $\Delta = [a, b[\times I$,
- The partial derivative $(t, x) \mapsto \frac{\partial f}{\partial x}(t, x)$ exists and is *continuous* on Δ ,
- $\int_a^b \frac{\partial f}{\partial x}(t, x) dt$ satisfies the *dominated convergence criterion* on I ,

then

- $\int_a^b f(t, x) dt$ *converges*,
- The function $F(x) = \int_a^b f(t, x) dt$ is *well-defined* and of *class C^1* on I ,
and

$$F'(x) = \int_a^b \frac{\partial f}{\partial x}(t, x) dt.$$

Differentiation of an improper integral depending on one parameter

Example.

Study the differentiability of $F(x) = \int_1^{+\infty} \frac{\sin(xt)}{t^3} dt$ on \mathbb{R} .

Differentiation of an improper integral depending on one parameter

Answer. The integral is improper and poses a problem at $+\infty$.

We have

$$f(t, x) = \frac{\sin(xt)}{t^3}, \quad \frac{\partial f}{\partial x}(t, x) = \frac{\cos(xt)}{t^2}, \quad \forall (t, x) \in \Delta = [1, +\infty[\times \mathbb{R}$$

- There exists $x_0 = 0 \in \mathbb{R}$ such that $\int_1^\infty f(t, 0) dt = 1$ is convergent,
- f and $\frac{\partial f}{\partial x}$ are continuous over Δ .
- $\forall (t, x) \in \Delta: \left| \frac{\partial f}{\partial x}(t, x) \right| \leq \frac{1}{t^2}$, and $\int_1^\infty \frac{1}{t^2} dt$ converges. Therefore, $\int_1^\infty \frac{\partial f}{\partial x}(t, x) dt$ satisfies the dominated convergence criterion over \mathbb{R} .

Then the function F is well-defined and of class C^1 over \mathbb{R} and

$$F'(x) = \int_1^\infty \frac{\cos(xt)}{t^2} dt.$$

Differentiation of an improper integral depending on one parameter

Example.

Let the function F be given by:

$$F(x) = \int_1^{\infty} \frac{\cos(xt)}{t(t^2 + 1)} dt;$$

- 1) Find the domain of definition D of the function F .
- 2) Study the continuity and then the differentiability of F on D :

Answer

- Let $f(t, x) = \frac{\cos(xt)}{t(t^2 + 1)}$ on $\Delta = [1, +\infty[\times \mathbb{R}$,
- $f \in R_{loc}$ with respect to the variable t .
- F is an improper parametrized integral.

$$D_F = \left\{ x \in \mathbb{R} : \int_1^{+\infty} \frac{\cos(xt)}{t(t^2 + 1)} dt \text{ converges} \right\}.$$

Differentiation of an improper integral depending on one parameter

- We have:

$$\frac{\cos(xt)}{t(t^2 + 1)} \leq \frac{1}{t(t^2 + 1)} \leq \frac{1}{t^3} = \varphi(t) \forall x \in \mathbb{R}, \forall t \in [1, +\infty[,$$

moreover,

$$\int_1^\infty \frac{1}{t^3} dt \text{ converges}$$

Hence F verifies the criterion for dominated convergence over \mathbb{R} . We obtain simple convergence over \mathbb{R} . So $D = \mathbb{R}$.

Differentiation of an improper integral depending on one parameter

2) Let's apply the theorem of continuity preservation for an improper parameterized integral:

- f is continuous on $U = \mathbb{R}^* \times \mathbb{R}$, $\Delta \subseteq U$, as composed and ratio of continuous functions.
- $\int_1^\infty f(t;x) dt$ verifies the criterion for dominated convergence over \mathbb{R} (from question 1).

Then F is continuous on \mathbb{R} .

Differentiation of an improper integral depending on one parameter

We have

$$\frac{\partial f}{\partial x}(t; x) = \frac{-t \sin(xt)}{t(t^2 + 1)} = \frac{-\sin(xt)}{t^2 + 1} \quad \text{for } [1; +\infty[\times \mathbb{R};$$

and

$$\left| \frac{-\sin(xt)}{t^2 + 1} \right| \leq \frac{1}{t^2 + 1} \leq \frac{1}{t^2} = \psi(t) \quad \text{for } t \in \mathbb{R} \text{ and } t \in [1; +\infty[;$$

Moreover, we have

$$\int_1^{+\infty} \frac{1}{t^2} dt \text{ converges.}$$

This implies the dominated convergence of

$$\int_1^{+\infty} \frac{\partial f}{\partial x}(x; t) dt \text{ over } \mathbb{R}.$$

Differentiation of an improper integral depending on one parameter

We apply the theorem of conservation of differentiability:

- f is C^1 on U where f and $\frac{\partial f}{\partial x}$ are continuous, as they are composed, multiplied, or ratios of C^1 functions.
- The integral

$$\int_1^{+\infty} \frac{\partial f}{\partial x}(t; x) dt$$

satisfies the criterion of dominated convergence on R .

- F is defined on \mathbb{R} so $\forall x_0 \in \mathbb{R} \int_1^{+\infty} f(t; x_0) dt$ is convergent.

Then F is differentiable on \mathbb{R} and

$$F'(x) = \int_1^{+\infty} \frac{\partial f}{\partial x}(t; x) dt = \int_1^{+\infty} \frac{\sin(xt)}{t^2 + 1} dt.$$