Theory of Computing — TD 1

Introduction

2ND YEAR - ENSIA

PRE-TUTORIAL EXERCISE

Before the tutorial session, you need to work on the following questions:

- Prove by Induction : If $C(n) = 1^3 + 2^3 + \cdots + n^3$, Then : $C(n) = \frac{1}{4}n^2(n+1)^2$.
- If C is a set with c elements, how many elements are in the power set of C? Prove by Induction.

EXERCISES

Exercise C1 (Logic and Proofs) :

Prove the following statements:

1. By Contrapositive(Contraposition) : If n is an integer for which n^2 is odd, then n is odd.
2. By Contradiction: If n is an integer for which n^2 is odd, then n is odd.
3. By Contradiction : The Square Root of 2 is Irrational. Hints, an Irrational number is the one that we cannot write as a ratio of two integers.

Exercise C2 (Sets and Functions) :

1. Write formal descriptions of the following sets.
   a. The set containing all natural numbers that are less than 5
   b. The set containing the string aba
   c. The set containing the empty string

2. If A has a elements and B has b elements, how many elements are in $A \times B$? Explain your answer.

Exercise C3 (Languages) :

1. Given the following formal definition of the language L over the alphabet {0,1}, such that $L = \{w \mid w = w^R$, w^R is the reversed string of $w \}$
   a. Is this language finite
   b. List examples of words from this Language.

2. Enumerate a few words from this language.
   Given the following language $L = \{x \mid$ there is w such that xw=algeria}. Enumerate all possible strings belonging to the language L.

3. Show using mathematical induction that for every $x \in \{a, b\}^*$ such that x begins with a and ends with b, x contains the substring ab.

4. Consider the language L of all strings of a's and b's that do not end with b and do not contain the substring *bb*.
   a. Is the language L finite ?
   b. Find a finite language **S** such that $L = S^*$.

5. Give an example of two languages L_1 and L_2 such that $L_1^* \cup L_2^* \neq (L_1 \cup L_2)^*$

Exercise P1 (Optional) :

Prove the following :

1. By Contrapositive: For every three positive integers i, j, and n, if $i*j=n$, then $i \le \sqrt{n}$ or $j \le \sqrt{n}$
2. By Direct Proof : If n is an odd integer, then n^2 is an odd integer
3. By Induction that for every integer $n \ge 4$, $n! > 2^n$.
4. By Contradiction : There exists no integers *a* and *b* for which $21a + 30b = 1$
5. $n \in \mathbb{N}$. If $2^n - 1$ is prime, then n is prime
6. Without using Induction, find the formula for $1^3 + 2^3 + 3^3 + 4^3 + \ldots + n^3 = ?$

Exercise P2 (Optional):

1. Let L_1 and L_2 be subsets of $\{a, b\}^*$.
   a. Show that if $L_1 \subseteq L_2$, then $L_1^* \subseteq L_2^*$.
   b. Show that $L_1^* \cup L_2^* \subseteq (L_1 \cup L_2)^*$.

2. Let L_1 , L_2 , and L_3 be languages over some alphabet . In each case below, two languages are given. Say what the relationship is between them. (Are they always equal? If not, is one always a subset of the other?) Give reasons for your answers, including counterexamples if appropriate.
   a. $L_1 (L_2 \cap L_3)$ **vs** $L_1 L_2 \cap L_1 L_3$
   b. $L_1^* \cap L_2^*$ **vs** $(L_1 \cap L_2)^*$
   c. $L_1^* L_2^*$ **vs** $(L_1 L_2)^*$

Exercise P3 (Optional):

Pages of a book are numbered sequentially starting with 1. If the total number of decimal digits used is equal to 1578, how many pages are there in the book?

Exercise P4 (Optional) :

Find the error in the following proof that **2 = 1**. : *Consider the equation* $a = b$. *Multiply both sides by* a *to obtain* $a^2 = ab$. *Subtract* b^2 *from both sides to get :* $a^2 - b^2 = ab - b^2$. *Now factor each side,* $(a + b)(a - b) = b(a - b)$, *and divide each side by* $(a - b)$ *to get* $a + b = b$. *Finally, let* a *and* b *equal 1, which shows that* $2 = 1$.

Exercise P5 (Optional) :

Without the help of a computer or calculator, find the total sum of the digits in all integers from 1 to a million, inclusive.

Exercise P6 (Optional) :

Suppose A is a set having n elements.

1. How many relations are there on A?
2. How many reflexive relations are there on A?
3. How many symmetric relations are there on A?
4. How many relations are there on A that are both reflexive and symmetric?

Exercise P7 (Optional) :

There are five items of different weights and a two-pan balance scale with no weights. Order the items in increasing order of their weights, making no more than seven weighings.