## 1.6 Arguments using 'not'.

If $\phi$ is a statement, we write $(\neg\phi)$ for the statement expressing that $\phi$ is not true. The symbol $\neg$ is pronounced 'not' or 'negation', and $(\neg\phi)$ is called the negation of $\phi$. (...)

(1.20) If that's justice then I'm a banana.

He meant 'That's not justice'. He was using the following device.

We write $\perp$ (pronounced 'absurdity') for a statement which is definitely false, for example, '$0=1$' or 'I'm a banana'. In derivations we shall treat $(\neg\phi)$ exactly as if it was written $(\phi \to \perp)$.

NATURAL DEDUCTION RULE ($\neg E$) If

$$\begin{array}{ccc} \mathcal{D} & & \mathcal{D}' \\ \phi & \text{and} & (\neg\phi) \end{array}$$

are derivations of $\phi$ and $(\neg\phi)$ respectively, then

$$\dfrac{\begin{array}{ccc} \mathcal{D} & & \mathcal{D}' \\ \phi & & (\neg\phi) \end{array}}{\bot}\ (\neg E)$$

is a derivation of $\bot$. Its undischarged assumptions are those of $\mathcal{D}$ together with those of $\mathcal{D}'$.

For the second rule, suppose we want to prove $(\neg \phi)$. Then we proceed as if we were using $(\to I)$ to prove $(\phi \to \bot)$. In other words, we assume $\phi$ and deduce $\bot$. The assumption $\phi$ is discharged after it has been used.

NATURAL DEDUCTION RULE $(\neg I)$ Suppose

$$\begin{array}{c} D \\ \bot \end{array}$$

is a derivation of $\bot$, and $\phi$ is a statement. Then the following is a derivation of $(\neg \phi)$:

$$\dfrac{\begin{array}{c} \cancel{\phi} \\ D \\ \bot \end{array}}{(\neg \phi)} \; (\neg I)$$

Its undischarged assumptions are those of D, except possibly $\phi$.

## Example 1.6.1

The following derivation proves $\vdash (\phi \to (\neg(\neg\phi)))$.

$$\cfrac{\cfrac{\cfrac{\cancel{\phi}^{\,2} \qquad\qquad \cancel{(\neg\phi)}^{\,1}}{\bot}\,(\neg E)}{(\neg(\neg\phi))}\,(\neg I)\ 1}{(\phi \to (\neg(\neg\phi)))}\,(\to I)\ 2$$

At the application of $(\neg I)$ we discharge the assumption $(\neg\phi)$ to get the conclusion $(\neg(\neg\phi))$.

## Example 1.6.2

**Theorem** There are infinitely many prime numbers

**Proof** Assume not. Then there are only finitely many prime numbers

$$p_1, \dots, p_n.$$

Consider the integer

$$q = (p_1 \times \dots \times p_n) + 1.$$

The integer $q$ must have at least one prime factor $r$. But then $r$ is one of the $p_i$, so it cannot be a factor of $q$. Hence $r$ both is and is not a factor of $q$; absurd! So our assumption is false, and the theorem is true. $\square$

A close inspection of this argument shows that we prove the theorem $\phi$ by assuming $(\neg\phi)$ and deducing an absurdity. The assumption $(\neg\phi)$ is then discharged. This is known as reductio ad absurdum, RAA for short. In natural deduction form it comes out as follows.

NATURAL DEDUCTION RULE (RAA)

Suppose we have a derivation

$$\begin{array}{c} D \\ \bot \end{array}$$

whose conclusion is $\bot$. Then there is a derivation

$$\dfrac{\begin{array}{c} \cancel{(\neg\phi)} \\ D \\ \bot \end{array}}{\phi}\,(RAA)$$

Its undischarged assumptions are those of $D$, except possibly $(\neg\phi)$.

## Example 1.6.3

We prove $\vdash ((\neg(\neg\phi))\to\phi)$.

$$2\ \dfrac{1\ \dfrac{\dfrac{\cancel{(\neg\phi)}^{1} \qquad\qquad \cancel{(\neg(\neg\phi))}^{2}}{\bot}\,(\neg E)}{\phi}\,(RAA)}{((\neg(\neg\phi))\to\phi)}\,(\to I)$$

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Here are the sequent rules corresponding to $(\neg E)$, $(\neg I)$ and (RAA):

SEQUENT RULE $(\neg E)$ If $(\Gamma \vdash \phi)$ and $(\Delta \vdash (\neg \phi))$ are both correct sequents, then the sequent $(\Gamma \cup \Delta \vdash \bot)$ is correct.

SEQUENT RULE $(\neg I)$ If the sequent $(\Gamma \cup \{\phi\} \vdash \bot)$ is correct, then so is the sequent $(\Gamma \vdash (\neg \phi))$.

SEQUENT RULE (RAA) If the sequent $(\Gamma \cup \{(\neg \phi)\} \vdash \bot)$ is correct, then so is the sequent $(\Gamma \vdash \phi)$.