Worksheet 1 : Informal Natural Deduction

Exercise 1 : What are the conclusion and the assumptions of the following argument ?

Theorem. Let r be a positive real number. Then r has a square root.

Proof. Write $f(x) = x^2 - r$ for any real x. Then f is a continuous function on $\mathbb{R}$. If $x = 0$, then $f(x) = 0 - r < 0$ since r is positive. If x is very large, then $f(x) = x^2 - r > 0$. So, by the Intermediate Value Theorem, there must be x such that $f(x) = 0$. For this value x, we have

$$r = r + 0 = r + f(x) = r + (x^2 - r) = x^2.$$

Exercise 2 : A first-year calculus textbook contains the following paragraph :
Given that

$$1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{4} \quad \text{for all} \quad x \neq 0,$$

we calculate $\lim_{x \to 0} u(x)$. Since

$$\lim_{x \to 0}(1 - \frac{x^2}{4}) = 1 \quad \text{and} \quad \lim_{x \to 0}(1 + \frac{x^2}{4}) = 1,$$

the Sandwich Theorem implies that $\lim_{x \to 0} u(x) = 1$.
What is the conclusion of this argument, and what are the assumptions ?

Exercise 3 : From your understanding of mathematical arguments, which (if any) of the following possible sequent rules seem to be true ? Give reasons.

Possible sequent rule A : If the sequent $(\Gamma \vdash \psi)$ is a correct sequent, and every statement in Γ is also in Δ, then the sequent $(\Delta \vdash \psi)$ is also correct.

Possible sequent rule B : If the sequent $(\{\phi\} \vdash \psi)$ is correct, then so is the sequent $(\{\psi\} \vdash \phi)$.

Possible sequent rule C : If the sequents $(\Gamma \vdash \psi)$ and $(\Delta \vdash \psi)$ are both correct, then so is the sequent $((\Gamma \cap \Delta) \vdash \psi)$.

Exercise 4 : Express the following using $\wedge$ between statements :

1. The real number r is positive but not an integer.
2. v is a nonzero vector.
3. ϕ if and only if ψ. [Here ϕ and ψ stand for statements.]

Exercise 5 : Write out derivations that prove the following sequents :

1. $\{\phi, \psi, \chi\} \vdash (\phi \wedge (\psi \wedge \chi))$.
2. $\{\phi, \psi\} \vdash ((\phi \wedge \psi) \wedge (\phi \wedge \psi))$.
3. $\{((\phi \wedge \psi) \wedge \chi)\} \vdash (\phi \wedge (\psi \wedge \chi))$.
4. $\{\phi, (\psi \wedge \chi)\} \vdash (\chi \wedge \phi)$.
5. $\{(\phi \wedge (\psi \wedge \chi))\} \vdash ((\chi \wedge \phi) \wedge \psi)$.

Exercise 6 : Show that $\{\phi_1, \phi_2\} \vdash \psi$ if and only if $\{(\phi_1 \wedge \phi_2)\} \vdash \psi$.

Exercise 7 : Write the following using $\rightarrow$ between statements :

1. f is continuous if f is differentiable.
2. Supposing x is positive, x has a square root.
3. $ab/b = a$ provided $a \neq 0$.

Exercise 8 : In the following two derivations, the names of the rules are missing, and so are the dandahs and step numbers for the assumptions that are discharged. Write out the derivations, including these missing pieces.

1. A proof of $\vdash ((\phi \wedge \psi) \rightarrow (\psi \wedge \phi))$

$$\dfrac{\dfrac{\dfrac{(\phi \wedge \psi)}{\psi} \qquad \dfrac{(\phi \wedge \psi)}{\phi}}{(\psi \wedge \phi)}}{((\phi \wedge \psi) \rightarrow (\psi \wedge \phi))}$$

2. A proof of $\vdash ((\psi \to \chi) \to ((\phi \to \psi) \to (\phi \to \chi)))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\phi \qquad (\phi \to \psi)}{\psi} \qquad (\psi \to \chi)}{\chi}}{(\phi \to \chi)}}{((\phi \to \psi) \to (\phi \to \chi))}}{((\psi \to \chi) \to ((\phi \to \psi) \to (\phi \to \chi)))}$$

Exercise 9 : Each of the following derivations proves a sequent. Write out the sequent that it proves.

1.

$$1\ \dfrac{\dfrac{\cancel{\phi}^{1}}{(\psi \to \phi)}\,(\to I)}{(\phi \to (\psi \to \phi))}\,(\to I).$$

2.

$$\dfrac{\dfrac{\phi}{(\psi \to \phi)}\,(\to I)}{(\phi \to (\psi \to \phi))}\,(\to I).$$

3.

$$1\ \dfrac{\dfrac{\cancel{\psi}^{1} \qquad \dfrac{(\phi \wedge \psi)}{\phi}\,(\wedge E)}{(\psi \wedge \phi)}\,(\wedge I).}{(\psi \to (\psi \wedge \phi))}\,(\to I)$$

4.

$$1\ \dfrac{\cancel{\phi}^{1}}{(\phi \to \phi)}\,(\to I).$$

Exercise 10 : Write out derivations to prove each of the following sequents.

1. $\vdash (\phi \to (\psi \to \psi))$.
2. $\vdash ((\phi \to (\theta \to \psi)) \to (\theta \to (\phi \to \psi)))$.
3. $\{(\phi \to \psi), (\phi \to \chi)\} \vdash (\phi \to (\psi \wedge \chi))$
4. $\{(\phi \to \psi), ((\phi \wedge \psi) \to \chi)\} \vdash (\phi \to \chi)$.
5. $\{(\phi \to (\psi \to \chi))\} \vdash ((\phi \wedge \psi) \to \chi)$.
6. $\vdash ((\phi \to \psi) \to ((\psi \to \theta) \to (\phi \to \theta)))$.
7. $\vdash ((\phi \to (\psi \wedge \theta)) \to ((\phi \to \theta) \wedge (\phi \to \psi)))$.

Exercise 11 : Show that $\{\phi\} \vdash \psi$ if and only if $\vdash (\phi \to \psi)$. [Prove the directions $\Rightarrow$ and $\Leftarrow$ separately.]

Exercise 12 : Let ϕ and ψ be statements and Γ a set of statements. Consider the two sequents

(a) $\Gamma \cup \{\phi\} \vdash \psi$.

(b) $\Gamma \vdash (\phi \to \psi)$.

Show that if D_1 is a derivation proving (a), then D_1 can be used to construct a derivation D_1' proving (b). Show also that if D_2 is a derivation proving (b), then D_2 can be used to construct a derivation D_2' proving (a). (Together, these show that (a) has a proof by derivation if and only if (b) has a proof by derivation. The previous exercise is the special case where Γ is empty.)

Exercise 13 : Give derivations to prove the following sequents :

1. $\{\phi, (\phi \leftrightarrow \psi)\} \vdash \psi$.
2. $\vdash (\phi \leftrightarrow \phi)$.
3. $\{(\phi \leftrightarrow \psi), (\psi \leftrightarrow \chi)\} \vdash (\phi \leftrightarrow \chi)$.
4. $\vdash ((\phi \leftrightarrow (\psi \leftrightarrow \chi)) \leftrightarrow ((\phi \leftrightarrow \psi) \leftrightarrow \chi))$.
5. $\{(\phi \leftrightarrow (\psi \leftrightarrow \psi))\} \vdash \phi$.

Exercise 14 : Let S be any set of statements, and let $\sim$ be the relation on S defined by : for all $\phi, \psi \in S$,

$$\phi \sim \psi \text{ if and only if } \vdash (\phi \leftrightarrow \psi).$$

Show that $\sim$ is an equivalence relation on S.

Exercise 15 : Show that if we have a derivation D of ψ with no undischarged assumptions, then we can use it to construct, for any statement ϕ, a derivation of $((\phi \leftrightarrow \psi) \leftrightarrow \phi)$ with no undischarged assumptions.

Exercise 16 : Find natural deduction proofs for the following sequents :

1. $\vdash (\neg(\phi \wedge (\neg\phi)))$.
2. $\vdash ((\phi \wedge \psi) \to (\neg(\phi \to (\neg\psi))))$.
3. $\{((\neg(\phi \wedge \psi)) \wedge \phi)\} \vdash (\neg\psi)$.
4. $\{(\phi \to \psi)\} \vdash (\neg(\phi \wedge (\neg\psi)))$.
5. $\{((\neg\psi) \to (\neg\phi))\} \vdash (\phi \to \psi)$.
6. $\vdash ((\neg(\phi \to \psi)) \to \phi)$.
7. $\vdash (\phi \to ((\neg\phi) \to \psi))$.
8. $\{(\neg(\phi \leftrightarrow \psi))\} \vdash ((\neg\phi) \leftrightarrow \psi)$. (Hard.)

Exercise 17 : Give natural deduction proofs of the following sequents :

1. $\vdash (\phi \to (\phi \vee \psi))$.
2. $\{(\neg(\phi \vee \psi))\} \vdash ((\neg\phi) \wedge (\neg\psi))$.
3. $\vdash ((\phi \to \psi) \to ((\neg\phi) \vee \psi))$.
4. $\{(\phi \vee \psi)\} \vdash (\psi \vee \phi)$.
5. $\{(\phi \vee \psi), (\phi \to \chi), (\psi \to \chi)\} \vdash \chi$.
6. $\{(\phi \vee \psi), (\neg\phi)\} \vdash \psi$.
7. $\{((\neg\phi) \wedge (\neg\psi))\} \vdash (\neg(\phi \vee \psi))$.
8. $\{(\phi \wedge \psi)\} \vdash (\neg((\neg\phi) \vee (\neg\psi)))$.

Exercise 18 : Give natural deduction proofs of the following sequents :

1. $\{(\phi \wedge (\psi \vee \chi))\} \vdash ((\phi \wedge \psi) \vee (\phi \wedge \chi))$
2. $\vdash ((\phi \to (\psi \wedge \chi)) \to ((\phi \to \chi) \wedge (\phi \to \psi)))$
3. $\vdash ((\psi \to \chi) \to ((\phi \to \psi) \to (\phi \to \chi)))$
4. $\{(\phi \to (\psi \wedge \chi))\} \vdash ((\phi \to \psi) \wedge (\phi \to \chi))$
5. $\{(\phi \to \psi), (\chi \to \theta), (\phi \vee \chi)\} \vdash (\psi \vee \theta)$
6. $\{(\neg(\phi \wedge \psi))\} \vdash ((\phi \to (\neg\psi)) \wedge (\psi \to (\neg\phi)))$
7. $\vdash (((\phi \to \psi) \wedge ((\neg\chi) \to (\neg\psi))) \to (\phi \to \chi))$.