4. Arguments using 'if'

We write $(\phi \to \psi)$ for 'If ϕ then ψ', where ϕ and ψ are statements.

We begin with the introduction rule. Here is an example.

Exp 1.4-1

Write p for the statement that if x is real then $x^2+1 \geqslant 2x$. We prove p as follows. Assume x is real. Then $x-1$ is real, so

$$0 \leqslant (x-1)^2 = x^2-2x+1 = (x^2+1)-2x.$$

So

$$2x \leqslant x^2+1. \qquad \square$$

It may help to arrange this proof in a diagram:

(1.14)

$$\boxed{\begin{array}{l}\text{Assume } x \text{ is real} \\ \text{Then } x-1 \text{ is real, so} \\ 0 \leq (x-1)^2 = x^2 - 2x + 1 = (x^2+1) - 2x. \\ \text{So, } 2x \leq x^2 + 1.\end{array}}$$

So, if x is real then $x^2+1 \geq 2x$.

We have two proofs here. The larger proof consists of the whole of (1.14), and its conclusion is

(1.15) If x is real then $x^2+1 \geq 2x$.

the smaller proof is inside the box, and its conclusion is

(1.16) $$2x \leq x^2 + 1.$$

The smaller proof assumes that x is real. But the larger proof does not assume anything about x. We say that in the larger proof the assumption 'x is real' is discharged – it is no longer needed, because the assumption has been put as the 'if' part of the conclusion.

In the natural deduction calculus we have a notation for discharging assumptions. Every assumption of a derivation D is written somewhere in D, perhaps in several places. We shall discharge occurrences of assumptions. We do it by writing a line through the text of the assumption. We call this line a dandah.

Thus, if ϕ is an assumption written somewhere in D, then we discharge ϕ by writing a dandah through it: $\cancel{\phi}$. In the rule $(\to I)$ below, and in similar rules later, the $\cancel{\phi}$ means that in forming the derivation we are allowed to discharge any occurrences of the assumption ϕ written in D.

(..., see exp 1.4.4)

NATURAL DEDUCTION RULE $(\to I)$

Suppose

$$\begin{array}{c} \mathcal{D} \\ \psi \end{array}$$

is a derivation of ψ, and ϕ is a statement. Then the following is a derivation of $(\phi \to \psi)$:

$$\dfrac{\begin{array}{c} \cancel{\phi} \\ \mathcal{D} \\ \psi \end{array}}{(\phi \to \psi)} \ (\to I).$$

Its undischarged assumptions are those of $\mathcal{D}$, except possibly ϕ.

We can also express $(\rightarrow I)$ as a sequent rule:

SEQUENT RULE $(\rightarrow I)$

If the sequent $(\Gamma \cup \{\phi\} \vdash \psi)$ is correct, then so is the sequent $(\Gamma \vdash (\phi \rightarrow \psi))$.

Exp 1.4.3

A proof of the sequent $\vdash (\phi \to (\psi \to (\phi \land \psi)))$:

$$
2\;\dfrac{1\;\dfrac{\dfrac{\cancel{\phi}^{\,2} \qquad\qquad \cancel{\psi}^{\,1}}{(\phi \land \psi)}\,(\land I)}{(\psi \to (\phi \land \psi))}\,(\to I)}{(\phi \to (\psi \to (\phi \land \psi)))}\,(\to I)
$$

Exp 1.4.4

Here we prove the sequent $\vdash (\phi \to (\phi \to \phi))$.

$$
\begin{array}{c}
\cancel{\phi}\ ? \\
\textcircled{1}\ \dfrac{\qquad\qquad}{(\phi \to \phi)}\ (\to I) \\
\textcircled{2}\ \dfrac{\qquad\qquad\qquad}{(\phi \to (\phi \to \phi))}\ (\to I)
\end{array}
$$

NATURAL DEDUCTION RULE $(\to E)$

If

$$\begin{array}{ccc} \mathcal{D} & \text{and} & \mathcal{D}' \\ \phi & & (\phi \to \psi) \end{array}$$

are derivations of ϕ and $(\phi \to \psi)$, respectively, then

$$\dfrac{\begin{array}{cc} \mathcal{D} & \mathcal{D}' \\ \phi & (\phi \to \psi) \end{array}}{\psi} \ (\to E)$$

is a derivation of ψ. Its undischarged assumptions are those of $\mathcal{D}$ together with those of $\mathcal{D}'$.

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In sequent terms:

SEQUENT RULE ($\rightarrow$E)

If $(\Gamma \vdash \phi)$ and $(\Delta \vdash (\phi \rightarrow \psi))$ are both correct sequents, then the sequent $(\Gamma \cup \Delta \vdash \psi)$ is correct.

We can combine $(\to E)$ with other rules to make various derivations.

<u>Exp 1.4.5</u>

A derivation to prove the sequent

$$\{(\phi \to \psi), (\psi \to \chi)\} \vdash (\phi \to \chi).$$

$$\textcircled{1}\,\dfrac{\dfrac{\dfrac{\cancel{\phi}^{\,\textcircled{1}} \qquad (\phi \to \psi)}{\psi}\,(\to E) \qquad (\psi \to \chi)}{\chi}\,(\to E)}{(\phi \to \chi)}\,(\to I)$$