MATH 200
Chapter 3 Summary and Test Review

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This study guide summarizes the main topics of Chapter 3, topics that you should expect will be represented on the test. But please remember that merely remembering these facts is not sufficient preparation for the test; you must work lots of exercises for practice. Please see the Exercise list on the MATH 200 web page.

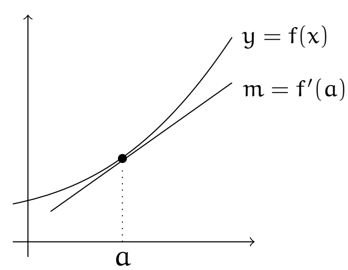
The Derivative and Its Interpretations

Chapter 3 deals entirely with a mathematical concept called a *derivative* of a function.

> The **derivative** of a function $f(x)$ is another function $f'(x)$ defined as
>
> $$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{z\to x} \frac{f(z)-f(x)}{z-x}. \end{aligned}$$
>
> **Notation:** The derivative of $y = f(x)$ can be denoted in various ways, including $f'(x) = y' = \dfrac{dy}{dx} = \dfrac{d}{dx}[f(x)]$.

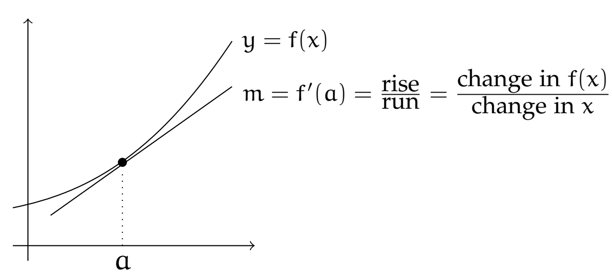
Because of its limit definition, the derivative has the following basic interpretations (see Section 3.6):

$f'(a) =$ **Slope** of the line tangent to the graph of $y = f(x)$ at the point $x = a$.



$f'(a) =$ **Velocity** at time $t = a$ of an object whose position on a straight line at time t is $f(t)$.

$f'(a) =$ **Instantaneous rate of change** of the quantity $f(x)$ with respect to x at $x = a$.



It is important to realize that, although we rarely use the limit definition to compute derivatives, the above interpretations come from the limit definition of $f'(x)$. In other words, it is because of its limit definition that the derivative has a *meaning*. Since we have short-cut rules for computing most derivatives, it is easy to lose sight of the importance of the limit. But without it, the derivative would have no meaning, and there would be no purpose in doing calculus.

Derivative Rules

The following short-cut rules allow us to compute derivatives of certain functions without a limit.
You are expected to remember and internalize each rule on this page.

Derivative of a Constant:	$\frac{d}{dx}\left[\, c \,\right]$	$=$	0	
Derivative of Identity:	$\frac{d}{dx}\left[\, x \,\right]$	$=$	1	
Constant Multiple Rule:	$\frac{d}{dx}\left[\, cf(x) \,\right]$	$=$	$cf'(x)$	
Sum/Difference Rule:	$\frac{d}{dx}\left[f(x) \pm g(x)\right]$	$=$	$f'(x) \pm g'(x)$	
Product Rule:	$\frac{d}{dx}\left[f(x)g(x)\right]$	$=$	$f'(x)g(x) + f(x)g'(x)$	
Quotient Rule:	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$	$=$	$\frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$	
Inverse Rule:	$\frac{d}{dx}\left[\, f^{-1}(x) \,\right]$	$=$	$\frac{1}{f'\left(f^{-1}(x)\right)}$	(Not often used.)
Chain Rule:	$\frac{d}{dx}\left[\, f\big(g(x)\big) \,\right]$	$=$	$f'(g(x))\, g'(x)$	

Additional rules are listed on the left of the following table. The functions listed here can be composed with a second function $g(x)$, and the chain rule can be applied to find the derivative of the composition. For each rule on the left, the corresponding derivative of the composition is indicated on the right.

	Rule	**Chain Rule Generalization**
Power Rule:	$\frac{d}{dx}\left[\, x^n \,\right] = nx^{n-1}$	$\frac{d}{dx}\left[\, \big(g(x)\big)^n \,\right] = n\big(g(x)\big)^{n-1} g'(x)$
Natural Exp. Rule:	$\frac{d}{dx}\left[\, e^x \,\right] = e^x$	$\frac{d}{dx}\left[\, e^{g(x)} \,\right] = e^{g(x)} g'(x)$
Natural Log Rule:	$\frac{d}{dx}\left[\, \ln(x) \,\right] = \frac{1}{x}$	$\frac{d}{dx}\left[\, \ln\big(g(x)\big) \,\right] = \frac{1}{g(x)} g'(x)$
Trig Rules	$\frac{d}{dx}\left[\sin(x)\right] = \cos(x)$	$\frac{d}{dx}\left[\sin\big(g(x)\big)\right] = \cos\big(g(x)\big)\, g'(x)$
	$\frac{d}{dx}\left[\cos(x)\right] = -\sin(x)$	$\frac{d}{dx}\left[\cos\big(g(x)\big)\right] = -\sin\big(g(x)\big)\, g'(x)$
	$\frac{d}{dx}\left[\tan(x)\right] = \sec^2(x)$	$\frac{d}{dx}\left[\tan\big(g(x)\big)\right] = \sec^2\big(g(x)\big)\, g'(x)$
	$\frac{d}{dx}\left[\cot(x)\right] = -\csc^2(x)$	$\frac{d}{dx}\left[\cot\big(g(x)\big)\right] = -\csc^2\big(g(x)\big)\, g'(x)$
	$\frac{d}{dx}\left[\sec(x)\right] = \sec(x)\tan(x)$	$\frac{d}{dx}\left[\sec\big(g(x)\big)\right] = \sec\big(g(x)\big)\tan\big(g(x)\big)\, g'(x)$
	$\frac{d}{dx}\left[\csc(x)\right] = -\csc(x)\cot(x)$	$\frac{d}{dx}\left[\csc\big(g(x)\big)\right] = -\csc\big(g(x)\big)\cot\big(g(x)\big)\, g'(x)$
Inverse Trig Rules	$\frac{d}{dx}\left[\sin^{-1}(x)\right] = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\left[\sin^{-1}\big(g(x)\big)\right] = \frac{1}{\sqrt{1-(g(x))^2}}\, g'(x)$
	$\frac{d}{dx}\left[\cos^{-1}(x)\right] = \frac{-1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\left[\cos^{-1}\big(g(x)\big)\right] = \frac{-1}{\sqrt{1-(g(x))^2}}\, g'(x)$
	$\frac{d}{dx}\left[\tan^{-1}(x)\right] = \frac{1}{1+x^2}$	$\frac{d}{dx}\left[\tan^{-1}\big(g(x)\big)\right] = \frac{1}{1+(g(x))^2}\, g'(x)$
	$\frac{d}{dx}\left[\sec^{-1}(x)\right] = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\frac{d}{dx}\left[\tan^{-1}\big(g(x)\big)\right] = \frac{1}{\lvert g(x)\rvert\sqrt{(g(x))^2-1}}\, g'(x)$

Fundamental Theorem of Calculus, Riemann Sums, Substitution Integration Methods

104003 Differential and Integral Calculus I
Technion International School of Engineering 2010-11
Tutorial Summary – February 27, 2011 – Kayla Jacobs

Indefinite vs. Definite Integrals

> A calc student upset as could be
> That his antiderivative didn't agree
> With the one in the book
> E'en aft one more look.
> Oh! Seems he forgot to write the "+ C".
> *-Anonymous*

- **Indefinite integral:** $\int f(x)\,dx$

  The **function** F(x) that answers question:
  "What function, when differentiated, gives f(x)?"

- **Definite integral:** $\int_a^b f(x)\,dx$
  - The **number** that represents the area under the curve f(x) between $x=a$ and $x=b$
  - a and b are called the **limits of integration**.
  - Forget the +c. Now we're calculating actual values .

Fundamental Theorem of Calculus (Relationship between definite & indefinite integrals)

If $F(x) \coloneqq \int_a^x f(t)\,dt$ and f is continuous, then F is differentiable and $F'(x) = f(x)$.

Important Corollary: For any function F whose derivative is f (i.e., $F'(x) = f(x)$),

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

This lets you easily calculate definite integrals!

Definite Integral Properties

- $\int_a^a f(x)\,dx = 0$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ whether or not $c \in [a, b]$

Area in [a,b] Bounded by Curve f(x)

Case 1: Curve entirely above x-axis. *Really easy!* Area = $\int_a^b f(x)\,dx$

Case 2: Curve entirely below x-axis. *Easy!* Area = $\left|\int_a^b f(x)\,dx\right|$ = - $\int_a^b f(x)\,dx$

Case 3: Curve sometimes below, sometimes above x-axis. *Sort of easy!* Break up into sections.

Average Value

The average value of function f(x) in region [a,b] is:

$$\text{average} = \frac{\int_a^b f(x)\,dx}{b-a}$$

Riemann Sum

Let **[a,b]** = closed interval in the domain of function *f*

Partition [a,b] into n subdivisions: { $[x_0,x_1]$, $[x_1,x_2]$, $[x_2,x_3]$, … , $[x_{n-1},x_n]$} where **a** = $x_0 < x_1 < \ldots < x_{n-1} < x_n$ = **b**

The **Riemann sum** of function f over interval [a,b] is:

$$S = \sum_{i=1}^{n} f(y_i) \cdot (x_i - x_{i-1})$$

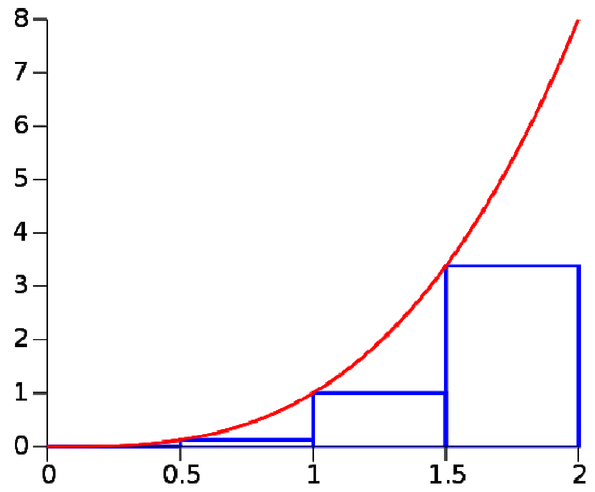
where y_i is *any* value between x_{i-1} and x_i. Note $(x_i - x_{i-1})$ is the length of the i^{th} subdivision $[x_{i-1}, x_i]$.

If for all *i*:	**then…**
$y_i = x_{i-1}$	S = **Left** Riemann sum
$y_i = x_i$	S = **Right** Riemann sum
$y_i = (x_i + x_{i-1})/2$	S = **Middle** Riemann sum
$f(yi) = (f(x_{i-1}) + f(x_i))/2$	S = **Trapezoidal** Riemann sum
$f(y_i)$ = maximum of f over $[x_{i-1}, x_i]$	S = **Upper** Riemann sum
$f(y_i)$ = minimum of f over $[x_{i-1}, x_i]$	S = **Lower** Riemann sum

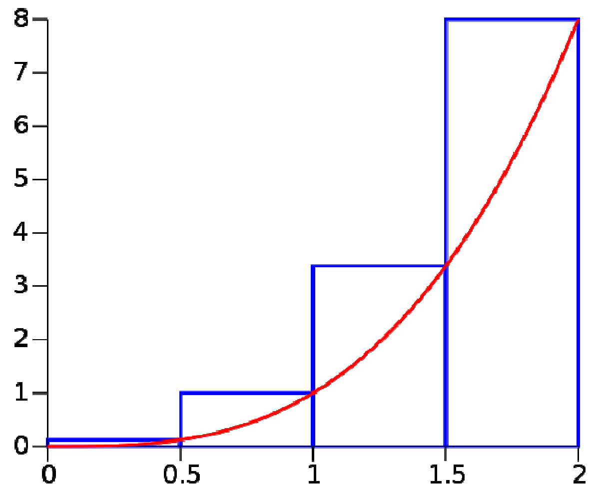
As $n \to \infty$, ***S* converges to the value of the definite integral** of *f* over [*a*,*b*]: $\lim_{n\to\infty} S = \int_a^b f(x)\,dx$

Ex: Riemann sum methods of *f*(*x*) = x^3 over interval [*a*, *b*] = [0, 2] using 4 equal subdivisions of 0.5 each:

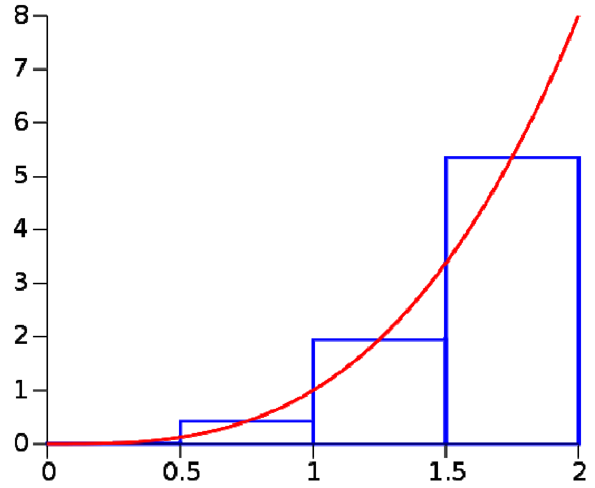
(1) Left Riemann sum:



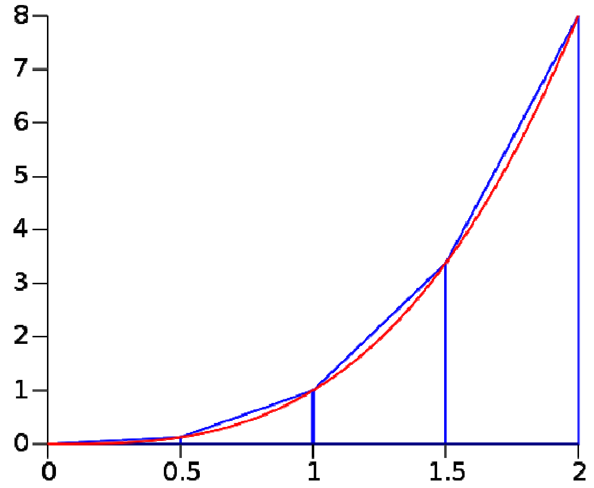
(2) Right Riemann sum:



(4) Middle Riemann sum:



(3) Trapezoidal Riemann sum:



Linear algebra explained in four pages

Excerpt from the NO BULLSHIT GUIDE TO LINEAR ALGEBRA by Ivan Savov

Abstract—**This document will review the fundamental ideas of linear algebra. We will learn about matrices, matrix operations, linear transformations and discuss both the theoretical and computational aspects of linear algebra. The tools of linear algebra open the gateway to the study of more advanced mathematics. A lot of *knowledge buzz* awaits you if you choose to follow the path of *understanding*, instead of trying to memorize a bunch of formulas.**

I. INTRODUCTION

Linear algebra is the math of vectors and matrices. Let n be a positive integer and let $\mathbb{R}$ denote the set of real numbers, then $\mathbb{R}^n$ is the set of all n-tuples of real numbers. A vector $\vec{v} \in \mathbb{R}^n$ is an n-tuple of real numbers. The notation "$\in S$" is read "element of S." For example, consider a vector that has three components:

$$\vec{v} = (v_1, v_2, v_3) \ \in \ (\mathbb{R}, \mathbb{R}, \mathbb{R}) \equiv \mathbb{R}^3.$$

A matrix $A \in \mathbb{R}^{m \times n}$ is a rectangular array of real numbers with m rows and n columns. For example, a 3×2 matrix looks like this:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \in \begin{bmatrix} \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} \end{bmatrix} \equiv \mathbb{R}^{3 \times 2}.$$

The purpose of this document is to introduce you to the mathematical operations that we can perform on vectors and matrices and to give you a feel of the power of linear algebra. Many problems in science, business, and technology can be described in terms of vectors and matrices so it is important that you understand how to work with these.

Prerequisites

The only prerequisite for this tutorial is a basic understanding of high school math concepts[1] like numbers, variables, equations, and the fundamental arithmetic operations on real numbers: addition (denoted $+$), subtraction (denoted $-$), multiplication (denoted implicitly), and division (fractions).

You should also be familiar with *functions* that take real numbers as inputs and give real numbers as outputs, $f : \mathbb{R} \to \mathbb{R}$. Recall that, by definition, the *inverse function* f^{-1} *undoes* the effect of f. If you are given $f(x)$ and you want to find x, you can use the inverse function as follows: $f^{-1}(f(x)) = x$. For example, the function $f(x) = \ln(x)$ has the inverse $f^{-1}(x) = e^x$, and the inverse of $g(x) = \sqrt{x}$ is $g^{-1}(x) = x^2$.

II. DEFINITIONS

A. *Vector operations*

We now define the math operations for vectors. The operations we can perform on vectors $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ are: addition, subtraction, scaling, norm (length), dot product, and cross product:

$$\begin{aligned} \vec{u} + \vec{v} &= (u_1 + v_1, u_2 + v_2, u_3 + v_3) \\ \vec{u} - \vec{v} &= (u_1 - v_1, u_2 - v_2, u_3 - v_3) \\ \alpha\vec{u} &= (\alpha u_1, \alpha u_2, \alpha u_3) \\ ||\vec{u}|| &= \sqrt{u_1^2 + u_2^2 + u_3^2} \\ \vec{u} \cdot \vec{v} &= u_1 v_1 + u_2 v_2 + u_3 v_3 \\ \vec{u} \times \vec{v} &= (u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1) \end{aligned}$$

The dot product and the cross product of two vectors can also be described in terms of the angle θ between the two vectors. The formula for the dot product of the vectors is $\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta$. We say two vectors $\vec{u}$ and $\vec{v}$ are *orthogonal* if the angle between them is $90°$. The dot product of orthogonal vectors is zero: $\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos(90°) = 0$.

The *norm* of the cross product is given by $\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin\theta$. The cross product is not commutative: $\vec{u} \times \vec{v} \neq \vec{v} \times \vec{u}$, in fact $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$.

B. *Matrix operations*

We denote by A the matrix as a whole and refer to its entries as a_{ij}. The mathematical operations defined for matrices are the following:

- addition (denoted $+$)

$$C = A + B \qquad \Leftrightarrow \qquad c_{ij} = a_{ij} + b_{ij}.$$

- subtraction (the inverse of addition)
- matrix product. The product of matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times \ell}$ is another matrix $C \in \mathbb{R}^{m \times \ell}$ given by the formula

$$C = AB \qquad \Leftrightarrow \qquad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj},$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} \end{bmatrix}$$

- matrix inverse (denoted A^{-1})
- matrix transpose (denoted $^{\mathsf{T}}$):

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{bmatrix}^{\mathsf{T}} = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix}.$$

- matrix trace: $\text{Tr}[A] \equiv \sum_{i=1}^{n} a_{ii}$
- determinant (denoted $\det(A)$ or $|A|$)

Note that the matrix product is not a commutative operation: $AB \neq BA$.

C. *Matrix-vector product*

The matrix-vector product is an important special case of the matrix-matrix product. The product of a 3×2 matrix A and the 2×1 column vector $\vec{x}$ results in a 3×1 vector $\vec{y}$ given by:

$$\begin{aligned} \vec{y} = A\vec{x} \qquad \Leftrightarrow \qquad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \\ a_{31}x_1 + a_{32}x_2 \end{bmatrix} \\ &= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \qquad \text{(C)} \\ &= \begin{bmatrix} (a_{11}, a_{12}) \cdot \vec{x} \\ (a_{21}, a_{22}) \cdot \vec{x} \\ (a_{31}, a_{32}) \cdot \vec{x} \end{bmatrix}. \qquad \text{(R)} \end{aligned}$$

There are two[2] fundamentally different yet equivalent ways to interpret the matrix-vector product. In the column picture, **(C)**, the multiplication of the matrix A by the vector $\vec{x}$ produces a **linear combination of the columns of the matrix**: $\vec{y} = A\vec{x} = x_1 A_{[:,1]} + x_2 A_{[:,2]}$, where $A_{[:,1]}$ and $A_{[:,2]}$ are the first and second columns of the matrix A.

In the row picture, **(R)**, multiplication of the matrix A by the vector $\vec{x}$ produces a column vector with coefficients equal to the **dot products of rows of the matrix** with the vector $\vec{x}$.

D. *Linear transformations*

The matrix-vector product is used to define the notion of a *linear transformation*, which is one of the key notions in the study of linear algebra. Multiplication by a matrix $A \in \mathbb{R}^{m \times n}$ can be thought of as computing a *linear transformation* T_A that takes n-vectors as inputs and produces m-vectors as outputs:

$$T_A : \mathbb{R}^n \to \mathbb{R}^m.$$

[1] A good textbook to (re)learn high school math is minireference.com

[2] For more info see the video of Prof. Strang's MIT lecture: bit.ly/1OvmKcL

Instead of writing $\vec{y} = T_A(\vec{x})$ for the linear transformation T_A applied to the vector $\vec{x}$, we simply write $\vec{y} = A\vec{x}$. Applying the linear transformation T_A to the vector $\vec{x}$ corresponds to the product of the matrix A and the column vector $\vec{x}$. We say T_A is *represented by* the matrix A.

You can think of linear transformations as "vector functions" and describe their properties in analogy with the regular functions you are familiar with:

$$\begin{aligned}
\text{function } f:\mathbb{R}\to\mathbb{R} \;&\Leftrightarrow\; \text{linear transformation } T_A:\mathbb{R}^n\to\mathbb{R}^m \\
\text{input } x\in\mathbb{R} \;&\Leftrightarrow\; \text{input } \vec{x}\in\mathbb{R}^n \\
\text{output } f(x) \;&\Leftrightarrow\; \text{output } T_A(\vec{x}) = A\vec{x}\in\mathbb{R}^m \\
g\circ f = g(f(x)) \;&\Leftrightarrow\; T_B(T_A(\vec{x})) = BA\vec{x} \\
\text{function inverse } f^{-1} \;&\Leftrightarrow\; \text{matrix inverse } A^{-1} \\
\text{zeros of } f \;&\Leftrightarrow\; \mathcal{N}(A) \equiv \text{null space of } A \\
\text{range of } f \;&\Leftrightarrow\; \mathcal{C}(A) \equiv \text{column space of } A = \text{range of } T_A
\end{aligned}$$

Note that the combined effect of applying the transformation T_A followed by T_B on the input vector $\vec{x}$ is equivalent to the matrix product $BA\vec{x}$.

E. Fundamental vector spaces

A *vector space* consists of a set of vectors and all linear combinations of these vectors. For example the vector space $\mathcal{S} = \text{span}\{\vec{v}_1, \vec{v}_2\}$ consists of all vectors of the form $\vec{v} = \alpha\vec{v}_1 + \beta\vec{v}_2$, where α and β are real numbers. We now define three fundamental vector spaces associated with a matrix A.

The *column space* of a matrix A is the set of vectors that can be produced as linear combinations of the columns of the matrix A:

$$\mathcal{C}(A) \equiv \{\vec{y}\in\mathbb{R}^m \mid \vec{y} = A\vec{x} \text{ for some } \vec{x}\in\mathbb{R}^n\}.$$

The column space is the *range* of the linear transformation T_A (the set of possible outputs). You can convince yourself of this fact by reviewing the definition of the matrix-vector product in the column picture **(C)**. The vector $A\vec{x}$ contains x_1 times the 1[st] column of A, x_2 times the 2[nd] column of A, etc. Varying over all possible inputs $\vec{x}$, we obtain all possible linear combinations of the columns of A, hence the name "column space."

The *null space* $\mathcal{N}(A)$ of a matrix $A\in\mathbb{R}^{m\times n}$ consists of all the vectors that the matrix A sends to the zero vector:

$$\mathcal{N}(A) \equiv \left\{\vec{x}\in\mathbb{R}^n \mid A\vec{x} = \vec{0}\right\}.$$

The vectors in the null space are *orthogonal* to all the rows of the matrix. We can see this from the row picture **(R)**: the output vectors is $\vec{0}$ if and only if the input vector $\vec{x}$ is orthogonal to all the rows of A.

The *row space* of a matrix A, denoted $\mathcal{R}(A)$, is the set of linear combinations of the rows of A. The row space $\mathcal{R}(A)$ is the orthogonal complement of the null space $\mathcal{N}(A)$. This means that for all vectors $\vec{v}\in\mathcal{R}(A)$ and all vectors $\vec{w}\in\mathcal{N}(A)$, we have $\vec{v}\cdot\vec{w} = 0$. Together, the null space and the row space form the domain of the transformation T_A, $\mathbb{R}^n = \mathcal{N}(A)\oplus\mathcal{R}(A)$, where $\oplus$ stands for *orthogonal direct sum*.

F. Matrix inverse

By definition, the inverse matrix A^{-1} *undoes* the effects of the matrix A. The cumulative effect of applying A^{-1} after A is the identity matrix $\mathbb{1}$:

$$A^{-1}A = \mathbb{1} \equiv \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}.$$

The identity matrix (ones on the diagonal and zeros everywhere else) corresponds to the identity transformation: $T_{\mathbb{1}}(\vec{x}) = \mathbb{1}\vec{x} = \vec{x}$, for all $\vec{x}$.

The matrix inverse is useful for solving matrix equations. Whenever we want to get rid of the matrix A in some matrix equation, we can "hit" A with its inverse A^{-1} to make it disappear. For example, to solve for the matrix X in the equation $XA = B$, multiply both sides of the equation by A^{-1} from the right: $X = BA^{-1}$. To solve for X in $ABCXD = E$, multiply both sides of the equation by D^{-1} on the right and by A^{-1}, B^{-1} and C^{-1} (in that order) from the left: $X = C^{-1}B^{-1}A^{-1}ED^{-1}$.

III. Computational linear algebra

Okay, I hear what you are saying "Dude, enough with the theory talk, let's see some calculations." In this section we'll look at one of the fundamental algorithms of linear algebra called Gauss–Jordan elimination.

A. Solving systems of equations

Suppose we're asked to solve the following system of equations:

$$\begin{aligned} 1x_1 + 2x_2 &= 5, \\ 3x_1 + 9x_2 &= 21. \end{aligned} \tag{1}$$

Without a knowledge of linear algebra, we could use substitution, elimination, or subtraction to find the values of the two unknowns x_1 and x_2.

Gauss–Jordan elimination is a systematic procedure for solving systems of equations based the following *row operations*:

α) Adding a multiple of one row to another row
β) Swapping two rows
γ) Multiplying a row by a constant

These row operations allow us to simplify the system of equations without changing their solution.

To illustrate the Gauss–Jordan elimination procedure, we'll now show the sequence of row operations required to solve the system of linear equations described above. We start by constructing an *augmented matrix* as follows:

$$\left[\begin{array}{cc|c} \mathbf{1} & 2 & 5 \\ 3 & 9 & 21 \end{array}\right].$$

The first column in the augmented matrix corresponds to the coefficients of the variable x_1, the second column corresponds to the coefficients of x_2, and the third column contains the constants from the right-hand side.

The Gauss-Jordan elimination procedure consists of two phases. During the first phase, we proceed left-to-right by choosing a row with a leading one in the leftmost column (called a *pivot*) and systematically subtracting that row from all rows below it to get zeros below in the entire column. In the second phase, we start with the rightmost pivot and use it to eliminate all the numbers above it in the same column. Let's see this in action.

1) The first step is to use the pivot in the first column to eliminate the variable x_1 in the second row. We do this by subtracting three times the first row from the second row, denoted $R_2 \leftarrow R_2 - 3R_1$,
$$\left[\begin{array}{cc|c} \mathbf{1} & 2 & 5 \\ 0 & 3 & 6 \end{array}\right].$$
2) Next, we create a pivot in the second row using $R_2 \leftarrow \frac{1}{3}R_2$:
$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & \mathbf{1} & 2 \end{array}\right].$$
3) We now start the backward phase and eliminate the second variable from the first row. We do this by subtracting two times the second row from the first row $R_1 \leftarrow R_1 - 2R_2$:
$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right].$$
The matrix is now in *reduced row echelon form* (RREF), which is its "simplest" form it could be in. The solutions are: $x_1 = 1$, $x_2 = 2$.

B. Systems of equations as matrix equations

We will now discuss another approach for solving the system of equations. Using the definition of the matrix-vector product, we can express this system of equations (1) as a matrix equation:

$$\begin{bmatrix} 1 & 2 \\ 3 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 21 \end{bmatrix}.$$

This matrix equation had the form $A\vec{x} = \vec{b}$, where A is a 2×2 matrix, $\vec{x}$ is the vector of unknowns, and $\vec{b}$ is a vector of constants. We can solve for $\vec{x}$ by multiplying both sides of the equation by the matrix inverse A^{-1}:

$$A^{-1}A\vec{x} = \mathbb{1}\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1}\vec{b} = \begin{bmatrix} 3 & -\frac{2}{3} \\ -1 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 5 \\ 21 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

But how did we know what the inverse matrix A^{-1} is?

IV. Computing the inverse of a matrix

In this section we'll look at several different approaches for computing the inverse of a matrix. The matrix inverse is *unique* so no matter which method we use to find the inverse, we'll always obtain the same answer.

A. Using row operations

One approach for computing the inverse is to use the Gauss–Jordan elimination procedure. Start by creating an array containing the entries of the matrix A on the left side and the identity matrix on the right side:

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 9 & 0 & 1 \end{array}\right].$$

Now we perform the Gauss-Jordan elimination procedure on this array.

1) The first row operation is to subtract three times the first row from the second row: $R_2 \leftarrow R_2 - 3R_1$. We obtain:
$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 3 & -3 & 1 \end{array}\right].$$
2) The second row operation is divide the second row by 3: $R_2 \leftarrow \frac{1}{3}R_2$
$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & \frac{1}{3} \end{array}\right].$$
3) The third row operation is $R_1 \leftarrow R_1 - 2R_2$
$$\left[\begin{array}{cc|cc} 1 & 0 & 3 & -\frac{2}{3} \\ 0 & 1 & -1 & \frac{1}{3} \end{array}\right].$$

The array is now in reduced row echelon form (RREF). The inverse matrix appears on the right side of the array.

Observe that the sequence of row operations we used to solve the specific system of equations in $A\vec{x} = \vec{b}$ in the previous section are the same as the row operations we used in this section to find the inverse matrix. Indeed, in both cases the combined effect of the three row operations is to "undo" the effects of A. The right side of the 2×4 array is simply a convenient way to record this sequence of operations and thus obtain A^{-1}.

B. Using elementary matrices

Every row operation we perform on a matrix is equivalent to a left-multiplication by an *elementary matrix*. There are three types of elementary matrices in correspondence with the three types of row operations:

$$\mathcal{R}_\alpha : R_1 \leftarrow R_1 + mR_2 \quad \Leftrightarrow \quad E_\alpha = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}$$

$$\mathcal{R}_\beta : R_1 \leftrightarrow R_2 \quad \Leftrightarrow \quad E_\beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\mathcal{R}_\gamma : R_1 \leftarrow mR_1 \quad \Leftrightarrow \quad E_\gamma = \begin{bmatrix} m & 0 \\ 0 & 1 \end{bmatrix}$$

Let's revisit the row operations we used to find A^{-1} in the above section representing each row operation as an elementary matrix multiplication.

1) The first row operation $R_2 \leftarrow R_2 - 3R_1$ corresponds to a multiplication by the elementary matrix E_1:
$$E_1 A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}.$$
2) The second row operation $R_2 \leftarrow \frac{1}{3}R_2$ corresponds to a matrix E_2:
$$E_2(E_1 A) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$
3) The final step, $R_1 \leftarrow R_1 - 2R_2$, corresponds to the matrix E_3:
$$E_3(E_2 E_1 A) = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that $E_3E_2E_1A = \mathbb{1}$, so the product $E_3E_2E_1$ must be equal to A^{-1}:

$$A^{-1} = E_3E_2E_1 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -\frac{2}{3} \\ -1 & \frac{1}{3} \end{bmatrix}.$$

The elementary matrix approach teaches us that every invertible matrix can be decomposed as the product of elementary matrices. Since we know $A^{-1} = E_3E_2E_1$ then $A = (A^{-1})^{-1} = (E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1}$.

C. Using a computer

The last (and most practical) approach for finding the inverse of a matrix is to use a computer algebra system like the one at `live.sympy.org`.

```
>>> A = Matrix( [[1,2],[3,9]] )              # define A
    [1, 2]
    [3, 9]

>>> A.inv()                   # calls the inv method on A
    [ 3, -2/3]
    [-1,  1/3]
```

You can use `sympy` to "check" your answers on homework problems.

V. Other topics

We'll now discuss a number of other important topics of linear algebra.

A. Basis

Intuitively, a basis is any set of vectors that can be used as a coordinate system for a vector space. You are certainly familiar with the standard basis for the xy-plane that is made up of two orthogonal axes: the x-axis and the y-axis. A vector $\vec{v}$ can be described as a coordinate pair (v_x, v_y) with respect to these axes, or equivalently as $\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath}$, where $\hat{\imath} \equiv (1,0)$ and $\hat{\jmath} \equiv (0,1)$ are unit vectors that point along the x-axis and y-axis respectively. However, other coordinate systems are also possible.

Definition (Basis)**.** *A basis for a n-dimensional vector space $\mathcal{S}$ is any set of n linearly independent vectors that are part of $\mathcal{S}$.*

Any set of two linearly independent vectors $\{\hat{e}_1, \hat{e}_2\}$ can serve as a basis for $\mathbb{R}^2$. We can write any vector $\vec{v} \in \mathbb{R}^2$ as a linear combination of these basis vectors $\vec{v} = v_1\hat{e}_1 + v_2\hat{e}_2$.

Note the *same* vector $\vec{v}$ corresponds to different coordinate pairs depending on the basis used: $\vec{v} = (v_x, v_y)$ in the standard basis $B_s \equiv \{\hat{\imath}, \hat{\jmath}\}$, and $\vec{v} = (v_1, v_2)$ in the basis $B_e \equiv \{\hat{e}_1, \hat{e}_2\}$. Therefore, it is important to keep in mind the basis with respect to which the coefficients are taken, and if necessary specify the basis as a subscript, e.g., $(v_x, v_y)_{B_s}$ or $(v_1, v_2)_{B_e}$.

Converting a coordinate vector from the basis B_e to the basis B_s is performed as a multiplication by a *change of basis* matrix:

$$\begin{bmatrix} \vec{v} \end{bmatrix}_{B_s} = {}_{B_s}\begin{bmatrix} & \mathbb{1} & \end{bmatrix}_{B_e} \begin{bmatrix} \vec{v} \end{bmatrix}_{B_e} \quad \Leftrightarrow \quad \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \hat{\imath}\cdot\hat{e}_1 & \hat{\imath}\cdot\hat{e}_2 \\ \hat{\jmath}\cdot\hat{e}_1 & \hat{\jmath}\cdot\hat{e}_2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Note the change of basis matrix is actually an identity transformation. The vector $\vec{v}$ remains unchanged—it is simply expressed with respect to a new coordinate system. The change of basis from the B_s-basis to the B_e-basis is accomplished using the inverse matrix: ${}_{B_e}[\mathbb{1}]_{B_s} = ({}_{B_s}[\mathbb{1}]_{B_e})^{-1}$.

B. Matrix representations of linear transformations

Bases play an important role in the representation of linear transformations $T\colon \mathbb{R}^n \to \mathbb{R}^m$. To fully describe the matrix that corresponds to some linear transformation T, it is sufficient to know the effects of T to the n vectors of the standard basis for the input space. For a linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$, the matrix representation corresponds to

$$M_T = \begin{bmatrix} | & | \\ T(\hat{\imath}) & T(\hat{\jmath}) \\ | & | \end{bmatrix} \in \mathbb{R}^{2\times 2}.$$

As a first example, consider the transformation Π_x which projects vectors onto the x-axis. For any vector $\vec{v} = (v_x, v_y)$, we have $\Pi_x(\vec{v}) = (v_x, 0)$. The matrix representation of Π_x is

$$M_{\Pi_x} = \begin{bmatrix} \Pi_x\!\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) & \Pi_x\!\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

As a second example, let's find the matrix representation of R_θ, the counterclockwise rotation by the angle θ:

$$M_{R_\theta} = \begin{bmatrix} R_\theta\!\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) & R_\theta\!\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

The first column of M_{R_θ} shows that R_θ maps the vector $\hat{\imath} \equiv 1\angle 0$ to the vector $1\angle\theta = (\cos\theta, \sin\theta)^\mathsf{T}$. The second column shows that R_θ maps the vector $\hat{\jmath} = 1\angle\frac{\pi}{2}$ to the vector $1\angle(\frac{\pi}{2} + \theta) = (-\sin\theta, \cos\theta)^\mathsf{T}$.

C. Dimension and bases for vector spaces

The *dimension* of a vector space is defined as the number of vectors in a basis for that vector space. Consider the following vector space $\mathcal{S} = \text{span}\{(1,0,0),(0,1,0),(1,1,0)\}$. Seeing that the space is described by three vectors, we might think that $\mathcal{S}$ is 3-dimensional. This is not the case, however, since the three vectors are not linearly independent so they don't form a basis for $\mathcal{S}$. Two vectors are sufficient to describe any vector in $\mathcal{S}$; we can write $\mathcal{S} = \text{span}\{(1,0,0),(0,1,0)\}$, and we see these two vectors are linearly independent so they form a basis and $\dim(\mathcal{S}) = 2$.

There is a general procedure for finding a basis for a vector space. Suppose you are given a description of a vector space in terms of m vectors $\mathcal{V} = \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ and you are asked to find a basis for $\mathcal{V}$ and the dimension of $\mathcal{V}$. To find a basis for $\mathcal{V}$, you must find a set of linearly independent vectors that span $\mathcal{V}$. We can use the Gauss–Jordan elimination procedure to accomplish this task. Write the vectors $\vec{v}_i$ as the rows of a matrix M. The vector space $\mathcal{V}$ corresponds to the row space of the matrix M. Next, use row operations to find the reduced row echelon form (RREF) of the matrix M. Since row operations do not change the row space of the matrix, the row space of reduced row echelon form of the matrix M is the same as the row space of the original set of vectors. The nonzero rows in the RREF of the matrix form a basis for vector space $\mathcal{V}$ and the numbers of nonzero rows is the dimension of $\mathcal{V}$.

D. Row space, columns space, and rank of a matrix

Recall the fundamental vector spaces for matrices that we defined in Section II-E: the column space $\mathcal{C}(A)$, the null space $\mathcal{N}(A)$, and the row space $\mathcal{R}(A)$. A standard linear algebra exam question is to give you a certain matrix A and ask you to find the dimension and a basis for each of its fundamental spaces.

In the previous section we described a procedure based on Gauss–Jordan elimination which can be used "distill" a set of linearly independent vectors which form a basis for the row space $\mathcal{R}(A)$. We will now illustrate this procedure with an example, and also show how to use the RREF of the matrix A to find bases for $\mathcal{C}(A)$ and $\mathcal{N}(A)$.

Consider the following matrix and its reduced row echelon form:

$$A = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 2 & 6 & 7 & 6 \\ 3 & 9 & 9 & 10 \end{bmatrix} \qquad \text{rref}(A) = \begin{bmatrix} \mathbf{1} & 3 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix}.$$

The reduced row echelon form of the matrix A contains three pivots. The locations of the pivots will play an important role in the following steps.

The vectors $\{(1,3,0,0),(0,0,1,0),(0,0,0,1)\}$ form a basis for $\mathcal{R}(A)$.

To find a basis for the column space $\mathcal{C}(A)$ of the matrix A we need to find which of the columns of A are linearly independent. We can do this by identifying the columns which contain the leading ones in rref(A). The corresponding columns in the original matrix form a basis for the column space of A. Looking at rref(A) we see the first, third, and fourth columns of the matrix are linearly independent so the vectors $\{(1,2,3)^{\mathsf{T}},(3,7,9)^{\mathsf{T}},(3,6,10)^{\mathsf{T}}\}$ form a basis for $\mathcal{C}(A)$.

Now let's find a basis for the null space, $\mathcal{N}(A) \equiv \{\vec{x} \in \mathbb{R}^4 \mid A\vec{x} = \vec{0}\}$. The second column does not contain a pivot, therefore it corresponds to a *free variable*, which we will denote s. We are looking for a vector with three unknowns and one free variable $(x_1, s, x_3, x_4)^{\mathsf{T}}$ that obeys the conditions:

$$\begin{bmatrix} \mathbf{1} & 3 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix} \begin{bmatrix} x_1 \\ s \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \Rightarrow \qquad \begin{aligned} 1x_1 + 3s &= 0 \\ 1x_3 &= 0 \\ 1x_4 &= 0 \end{aligned}$$

Let's express the unknowns x_1, x_3, and x_4 in terms of the free variable s. We immediately see that $x_3 = 0$ and $x_4 = 0$, and we can write $x_1 = -3s$. Therefore, any vector of the form $(-3s, s, 0, 0)$, for any $s \in \mathbb{R}$, is in the null space of A. We write $\mathcal{N}(A) = \text{span}\{(-3,1,0,0)^{\mathsf{T}}\}$.

Observe that the $\dim(\mathcal{C}(A)) = \dim(\mathcal{R}(A)) = 3$, this is known as the *rank* of the matrix A. Also, $\dim(\mathcal{R}(A)) + \dim(\mathcal{N}(A)) = 3 + 1 = 4$, which is the dimension of the input space of the linear transformation T_A.

E. Invertible matrix theorem

There is an important distinction between matrices that are invertible and those that are not as formalized by the following theorem.

Theorem. *For an $n \times n$ matrix A, the following statements are equivalent:*

1) *A is invertible*
2) *The RREF of A is the $n \times n$ identity matrix*
3) *The rank of the matrix is n*
4) *The row space of A is $\mathbb{R}^n$*
5) *The column space of A is $\mathbb{R}^n$*
6) *A doesn't have a null space (only the zero vector $\mathcal{N}(A) = \{\vec{0}\}$)*
7) *The determinant of A is nonzero* $\det(A) \neq 0$

For a given matrix A, the above statements are either all true or all false.

An invertible matrix A corresponds to a linear transformation T_A which maps the n-dimensional input vector space $\mathbb{R}^n$ to the n-dimensional output vector space $\mathbb{R}^n$ such that there exists an inverse transformation T_A^{-1} that can faithfully undo the effects of T_A.

On the other hand, an $n \times n$ matrix B that is not invertible maps the input vector space $\mathbb{R}^n$ to a subspace $\mathcal{C}(B) \subsetneq \mathbb{R}^n$ and has a nonempty null space. Once T_B sends a vector $\vec{w} \in \mathcal{N}(B)$ to the zero vector, there is no T_B^{-1} that can undo this operation.

F. Determinants

The determinant of a matrix, denoted $\det(A)$ or $|A|$, is a special way to combine the entries of a matrix that serves to check if a matrix is invertible or not. The determinant formulas for 2×2 and 3×3 matrices are

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}, \qquad \text{and}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

If the $|A| = 0$ then A is not invertible. If $|A| \neq 0$ then A is invertible.

G. Eigenvalues and eigenvectors

The set of eigenvectors of a matrix is a special set of input vectors for which the action of the matrix is described as a simple *scaling*. When a matrix is multiplied by one of its eigenvectors the output is the same eigenvector multiplied by a constant $A\vec{e}_\lambda = \lambda\vec{e}_\lambda$. The constant λ is called an *eigenvalue* of A.

To find the eigenvalues of a matrix we start from the eigenvalue equation $A\vec{e}_\lambda = \lambda\vec{e}_\lambda$, insert the identity $\mathbb{1}$, and rewrite it as a null-space problem:

$$A\vec{e}_\lambda = \lambda\mathbb{1}\vec{e}_\lambda \qquad \Rightarrow \qquad (A - \lambda\mathbb{1})\,\vec{e}_\lambda = \vec{0}.$$

This equation will have a solution whenever $|A - \lambda\mathbb{1}| = 0$. The eigenvalues of $A \in \mathbb{R}^{n\times n}$, denoted $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, are the roots of the *characteristic polynomial* $p(\lambda) = |A - \lambda\mathbb{1}|$. The *eigenvectors* associated with the eigenvalue λ_i are the vectors in the null space of the matrix $(A - \lambda_i\mathbb{1})$.

Certain matrices can be written entirely in terms of their eigenvectors and their eigenvalues. Consider the matrix Λ that has the eigenvalues of the matrix A on the diagonal, and the matrix Q constructed from the eigenvectors of A as columns:

$$\Lambda = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}, \quad Q = \begin{bmatrix} | & & | \\ \vec{e}_{\lambda_1} & \cdots & \vec{e}_{\lambda_n} \\ | & & | \end{bmatrix}, \quad \text{then} \quad A = Q\Lambda Q^{-1}.$$

Matrices that can be written this way are called diagonalizable.

The decomposition of a matrix into its eigenvalues and eigenvectors gives valuable insights into the properties of the matrix. Google's original `PageRank` algorithm for ranking webpages by "importance" can be formalized as an eigenvector calculation on the matrix of web hyperlinks.

VI. Textbook plug

Comps Study Guide for Multivariable Calculus

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June, 2017

This Study Guide was written to help you prepare for the multivariable calculus portion of the Comprehensive and Honors Qualifying Examination in Mathematics. It is based on the *Syllabus for the Comprehensive Examination in Multivariable Calculus* (*Math 211*) available on the Department website.

Each topic from the syllabus is accompanied by a brief discussion and examples from old exams. When reading this guide, you should focus on three things:

- *Understand the ideas.* If you study problems and solutions without understanding the underlying ideas, you will not be prepared for the exam.
- *Understand the strategy of each problem.* Most solutions in this guide are short—the hardest part is often knowing where to start. Focus on this rather than falling into the trap of memorizing solutions.
- *Understand the value of scratchwork.* Brainstorm possible solution methods and draw pictures when relevant to help you identity a good approach to the problem.

The final section of the guide has some further suggestions for how to prepare for the exam.

1 Elementary Vector Analysis

Most of multivariable calculus takes place in $\mathbb{R}^2$ and $\mathbb{R}^3$. You should be familiar with the Cartesian coordinates $(x, y) \in \mathbb{R}^2$ and $(x, y, z) \in \mathbb{R}^3$.

Vectors. A vector $\mathbf{v}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ is often represented by a directed line segment. In term of coordinates, we write $\mathbf{v} = \langle a_1, a_2\rangle$ in $\mathbb{R}^2$ and $\mathbf{v} = \langle a_1, a_2, a_3\rangle$ in $\mathbb{R}^3$. Know:

- Addition and scalar multiplication of vectors.
- The standard basis vectors $\mathbf{i} = \langle 1, 0\rangle$, $\mathbf{j} = \langle 0, 1\rangle$ in $\mathbb{R}^2$ and $\mathbf{i} = \langle 1, 0, 0\rangle$, $\mathbf{j} = \langle 0, 1, 0\rangle$, $\mathbf{k} = \langle 0, 0, 1\rangle$ in $\mathbb{R}^3$.
- A vector $\mathbf{v}$ has length $|\mathbf{v}|$, sometimes denoted $||\mathbf{v}||$.
- Nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if each is a constant multiple of the other.
- A point P in $\mathbb{R}^2$ or $\mathbb{R}^3$ gives a vector from the origin to P, called the position vector of P. This allows us to regards points as vectors and vice versa.

Also know the formula for $|\mathbf{v}|$ and how it relates to the distance formula for the distance between two points in $\mathbb{R}^2$ or $\mathbb{R}^3$. See $\boxed{\mathbf{12}}$ for a problem that uses the distance formula and $\boxed{\mathbf{20}}$ for a problem that uses the length of a vector. Note that vectors are sometimes written $\vec{v}$ instead of $\mathbf{v}$.

Dot Product. In $\mathbb{R}^2$, the dot product of $\mathbf{u} = \langle a_1, a_2\rangle$ and $\mathbf{v} = \langle b_1, b_2\rangle$ is $\mathbf{u} \cdot \mathbf{v} = a_1b_1 + a_2b_2$, and similarly, in $\mathbb{R}^3$, the dot product of $\mathbf{u} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{v} = \langle b_1, b_2, b_3\rangle$ is $\mathbf{u} \cdot \mathbf{v} = a_1b_1 + a_2b_2 + a_3b_3$. Know:

- Linearity properties of dot product.

- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.
- $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.
- $\mathbf{u} \cdot \mathbf{v} = 0$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are perpendicular (orthogonal).
- $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$.

Dot product is sometimes called the scalar product. See $\boxed{\mathbf{3}}$ and $\boxed{\mathbf{4}}$ for problems that use dot product.

Cross Product. Given $\mathbf{u} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{v} = \langle b_1, b_2, b_3 \rangle$ in $\mathbb{R}^3$, their cross product is

$$\mathbf{u} \times \mathbf{v} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}.$$

Know:

- Linearity properties of cross product.
- $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.
- $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.
- $\mathbf{u} \times \mathbf{v} = 0$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are parallel.
- $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

Cross product is sometimes called the vector product. See $\boxed{\mathbf{6}}$ for a problem that uses cross product.

Lines and Planes. Know:

- In $\mathbb{R}^2$ or $\mathbb{R}^3$, a point $\mathbf{r}_0$ and a nonzero vector $\mathbf{v}$ determine the line parametrized by

  $$\mathbf{r}(t) = \mathbf{r}_0 + t\,\mathbf{v}.$$

  The vector $\mathbf{v}$ is called a direction vector of the line. Be sure you know how to write out the parametric equations of a line for the coordinates $(x, y) \in \mathbb{R}^2$ or $(x, y, z) \in \mathbb{R}^3$.

- A plane in $\mathbb{R}^3$ is defined by an equation of the form $ax + by + cz = d$ where $\langle a, b, c \rangle \neq \langle 0, 0, 0 \rangle$. A more geometric way to write the equation uses a nonzero vector $\mathbf{n}$ perpendicular to the plane and point (x_0, y_0, z_0) in the plane. Then:

  $$\begin{aligned} (x, y, z) \text{ is in the plane} &\iff \mathbf{n} \text{ is perpendicular to the vector from } (x, y, z) \text{ to } (x_0, y_0, z_0) \\ &\iff \mathbf{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0. \end{aligned}$$

  The vector $\mathbf{n}$ is called a normal vector to the plane. For a plane defined by $ax + by + cz = d$, a normal vector is given by $\mathbf{n} = \langle a, b, c \rangle$.

Here is a problem that uses lines and planes.

$\boxed{1}$ (January 2017) Find an equation of the form $ax + by + cz = d$ for the plane passing through the point $(-2, -1, 4)$ that is perpendicular to the line with parametric equations $x = 2t,\ y = 3t - 1,\ z = 5 - t$.

Solution. Since the line is perpendicular to the plane, its direction vector is a normal vector to the plane. Be sure you can draw picture of this. Writing the line as

$$\mathbf{r}(t) = (2t, 3t-1, 5-t) = (0, -1, 5) + t\,\langle 2, 3, -1\rangle,$$

we see that $\langle 2, 3, -1\rangle$ is a normal vector to the plane. Hence the equation of the plane can be written

$$2x + 3y - z = d$$

for some $d \in \mathbb{R}$. Rereading the problems shows that there is further information, namely that the plane passes through $(-2, -1, 4)$. Thus this point satisfies the above equation, i.e.,

$$2(-2) + 3(-1) - (4) = d.$$

This implies $d = -11$ and the equation is $2x + 3y - z = -11$.

Comment. Drawing a picture of a line perpendicular to a plane can help clarify the geometry of the problem and lead you to the right solution. A good strategy is to draw pictures first, rather than immediately jumping into formulas and equations.

Tangent Vector to a Parametrized Curve. Given a curve parametrization $\mathbf{r}(t) = (x(t), y(t))$ in the plane, the tangent vector to the curve at the point $\mathbf{r}(t)$ is

$$\mathbf{r}'(t) = \langle x'(t), y'(t)\rangle.$$

The situation is similar on $\mathbb{R}^3$. Here is a typical problem.

$\boxed{2}$ (March 2017) Find parametric equations for the line that is tangent to the curve given by

$$x = t^2 - 2t - 1, \quad y = t^4 - 4t^2 + 2$$

at the point $(-2, -1)$.

Solution. The tangent vector to $\mathbf{r}(t) = (t^2 - 2t - 1, t^4 - 4t^2 + 2)$ is $\mathbf{r}'(t) = \langle 2t - 2, 4t^3 - 8t\rangle$. Since we want the tangent line at $(-2, -1)$, we need to find $t \in \mathbb{R}$ such that $\mathbf{r}(t) = (-2, -1)$. Be sure you understand this. To solve $(t^2 - 2t - 1, t^4 - 4t^2 + 2) = (-2, -1)$, we begin with the x-coordinate:

$$t^2 - 2t - 1 = -2 \;\Rightarrow\; t^2 - 2t + 1 = 0 \;\Rightarrow\; (t-1)^2 = 0 \;\Rightarrow\; t = 1.$$

Then one computes that $\mathbf{r}(1) = (1^2 - 2\cdot 1 - 1, 1^4 - 4\cdot 1^2 + 2) = (1 - 2 - 1, 1 - 4 + 2) = (-2, -1)$ and $\mathbf{r}'(1) = \langle 2\cdot 1 - 2, 4\cdot 1^3 - 8\cdot 1\rangle = \langle 2 - 2, 4 - 8\rangle = \langle 0, -4\rangle$. Since the tangent line goes through $\mathbf{r}(1)$ with direction vector $\mathbf{r}'(1)$, the tangent line is parametrized by

$$\mathbf{r}(1) + t\,\mathbf{r}'(1) = (-2, -1) + t\,\langle 0, -4\rangle = (-2, -4t - 1), \text{ i.e., } x = -2,\; y = -4t - 1.$$

2 Functions of Several Variables

Partial Derivatives. Know:

- The definition of partial derivative of a function $f(x, y)$ or $f(x, y, z)$.

- The standard notation for the partial derivatives: $\dfrac{\partial f}{\partial x} = f_x(x,y)$, $\dfrac{\partial f}{\partial y} = f_y(x,y)$, $\dfrac{\partial^2 f}{\partial^2 x} = f_{xx}(x,y)$, $\dfrac{\partial^2 f}{\partial x \partial y} = f_{yx}(x,y)$, $\dfrac{\partial^2 f}{\partial^2 y} = f_{yy}(x,y)$ for $f(x,y)$, and similarly for $f(x,y,z)$.
- The rate of change interpretation of a partial derivative.
- How to compute partial derivatives using the standard rules of differentiation.

Directional Derivatives. Know:

- The definition of a unit vector $\mathbf{u}$ and how to rescale a nonzero vector to make it a unit vector.
- The definition of the directional derivative $D_{\mathbf{u}}f(a,b)$ of $f(x,y)$ in the direction of the unit vector $\mathbf{u}$ at the point (a,b), and similarly for $f(x,y,z)$.
- The rate of change interpretation of a directional derivative.

Also know the theorem (stated below) that computes the directional derivative using the gradient when the function is differentiable.

The Gradient. The gradient of $f(x,y)$ at (a,b) is the vector

$$\nabla f(a,b) = \frac{\partial f}{\partial x}(a,b)\mathbf{i} + \frac{\partial f}{\partial y}(a,b)\mathbf{j} = \left\langle \frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b) \right\rangle,$$

and similarly for $f(x,y,z)$. Know:

- $\nabla f(a,b)$ is perpendicular to the level curve $f(x,y) = f(a,b)$ at the point (a,b). Similarly, $\nabla f(a,b,c)$ is perpendicular to the level surface $f(x,y,z) = f(a,b,c)$ at (a,b,c).
- If $f(a,b)$ is differentiable at (a,b) and $\mathbf{u}$ is a unit vector, then

  $$D_{\mathbf{u}}f(a,b) = \nabla f(a,b) \cdot \mathbf{u},$$

  and similarly for $f(x,y,z)$.
- When $\nabla f(a,b) \neq \mathbf{0}$, the unit vector $\nabla f(a,b)/|\nabla f(a,b)|$ gives the direction in which $f(x,y)$ is increasing most rapidly. Furthermore, the maximum rate of increase is $|\nabla f(a,b)|$. Similar results hold for $f(x,y,z)$.

Here are two problems that feature the gradient. See also $\boxed{\mathbf{6}}$ and $\boxed{\mathbf{20}}$.

$\boxed{\mathbf{3}}$ (January 2015) Find the directional derivative of the function $f(x,y,z) = x\sqrt{yz+1}$ at the point $(2,1,3)$ in the direction of the vector $\langle 2,-1,2\rangle$.

Solution. The unit vector in the direction of $\langle 2,-1,2\rangle$ is

$$\mathbf{u} = \frac{\langle 2,-1,2\rangle}{|\langle 2,-1,2\rangle|} = \frac{\langle 2,-1,2\rangle}{\sqrt{2^2+(-1)^2+2^2}} = \frac{\langle 2,-1,2\rangle}{3} = \left\langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \right\rangle.$$

The directional derivative of $f = x\sqrt{yz+1}$ in the direction of $\mathbf{u}$ is therefore

$$\begin{aligned} D_{\mathbf{u}} f(x,y,z) &= \nabla f(x,y,z) \cdot \mathbf{u} \\ &= \left\langle \frac{\partial}{\partial x} x\sqrt{yz+1}, \frac{\partial}{\partial y} x\sqrt{yz+1}, \frac{\partial}{\partial z} x\sqrt{yz+1} \right\rangle \cdot \mathbf{u} \\ &= \left\langle \sqrt{yz+1}, \frac{xz}{2\sqrt{yz+1}}, \frac{xy}{2\sqrt{yz+1}} \right\rangle \cdot \left\langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \right\rangle \\ &= \frac{2}{3}\sqrt{yz+1} - \frac{xz}{6\sqrt{yz+1}} + \frac{xy}{3\sqrt{yz+1}}. \end{aligned}$$

Then setting $(x,y,z) = (2,1,3)$ gives

$$D_{\mathbf{u}} f(2,1,3) = \frac{2}{3}\sqrt{1 \cdot 3 + 1} - \frac{2 \cdot 3}{6\sqrt{1 \cdot 3+1}} + \frac{2 \cdot 1}{3\sqrt{1 \cdot 3 + 1}} = \frac{4}{3} - \frac{1}{2} + \frac{1}{3} = \frac{7}{6}.$$

$\boxed{4}$ (January 2011) Let $f(x,y)$ be differentiable on $\mathbb{R}^2$. Suppose that $f_x(0,0) = 2$ and that the directional derivative of f at $(0,0)$ in the direction $\mathbf{u} = \frac{1}{\sqrt{2}}(1,1)$ is $5/\sqrt{2}$. Determine the value of $f_y(0,0)$.

Solution. By the formula for the directional derivative,

$$D_{\mathbf{u}} f(0,0) = \nabla f(0,0) \cdot \mathbf{u} = f_x(0,0)\,\frac{1}{\sqrt{2}} + f_y(0,0)\,\frac{1}{\sqrt{2}}$$

since $\mathbf{u} = \frac{1}{\sqrt{2}}(1,1)$. From the given conditions we know $D_{\mathbf{u}} f(0,0) = \frac{5}{\sqrt{2}}$ and $f_x(0,0) = 2$. Substituting these numbers into the above equation yields

$$\frac{5}{\sqrt{2}} = 2\,\frac{1}{\sqrt{2}} + f_y(0,0)\,\frac{1}{\sqrt{2}},$$

from which we conclude that $f_y(0,0) = 3$.

Here is a problem you should do yourself.

$\boxed{5}$ (March 2009) The temperature at the point (x,y,z) is

$$T(x,y,z) = \frac{1}{\pi}\sin(\pi xy) + \ln(z^2+1) + 60.$$

(a) Find a vector pointing in the direction in which the temperature increases most rapidly at the point $(2,-1,1)$. ANSWER: $\langle -1, 2, 1 \rangle$

(b) Let $\vec{v} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$. (Notice that $\vec{v}$ is **not** a unit vector.) What is the rate of change of the temperature at the point $(2,-1,1)$ in the direction of $\vec{v}$? ANSWER: $\frac{7}{3}$

The Tangent Plane to a Surface. Tangent planes arise in two situations:

- If $f(x,y)$ is differentiable at (x_0,y_0), then the tangent plane to the graph $z = f(x,y)$ at the point $(x_0, y_0, f(x_0,y_0))$ is defined by

(1) $$z - f(x_0,y_0) = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

- If $F(x,y,z)$ is differentiable at (x_0,y_0,z_0), then (x_0,y_0,z_0) lies on the level surface $F(x,y,z) = F(x_0,y_0,z_0)$, and the equation of the tangent plane to the surface at this point is defined by

  $$\nabla F(x_0,y_0,z_0)\cdot(x-x_0,y-y_0,z-z_0)=0, \tag{2}$$

  provided that the gradient $\nabla F(x_0,y_0,z_0)$ is nonzero. Written out, this is the equation

  $$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0.$$

The two situations are related since the graph $z=f(x,y)$ is the level surface $F(x,y,z)=f(x,y)-z=0$. Since $\nabla F = f_x\,\mathbf{i}+f_y\,\mathbf{j}-\mathbf{k}$, equation (2) reduces to equation (1) in this case.

Here are two problems that involve tangent planes.

$\boxed{6}$ (March 2007) Let $F(x,y,z)=xy^2z^3$.
(a) Find the equation of the tangent plane to the level surface $F(x,y,z)=1$ at the point $(1,1,1)$.
(b) Compute $\nabla F(1,1,1)\times\vec{v}$, where $\vec{v}=(2,-1,3)$.

Solution. (a) Since we need the gradient for part (b) and the gradient is normal to the tangent plane, it make sense to start with the gradient:

$$\nabla F=\left\langle \frac{\partial}{\partial x}(xy^2z^3),\frac{\partial}{\partial y}(xy^2z^3),\frac{\partial}{\partial z}(xy^2z^3)\right\rangle=\langle y^2z^3,2xyz^3,3xy^2z^2\rangle.$$

Then $\nabla F(1,1,1)=\langle 1,2,3\rangle$. Since the given point is $(1,1,1)$, the equation of the tangent plane is

$$1\cdot(x-1)+2\cdot(y-1)+3\cdot(z-1)=0 \;\Rightarrow\; x+2y+3z=6.$$

(b) Be sure you can do this straightforward computation. ANSWER: $9\,\mathbf{i}+3\,\mathbf{j}-5\,\mathbf{k}$

$\boxed{7}$ (January 2014) Suppose the plane $z=2x-y-1$ is tangent to the graph of $z=f(x,y)$ at $P=(5,3)$. Determine $f(5,3)$, $\dfrac{\partial f}{\partial x}(5,3)$ and $\dfrac{\partial f}{\partial y}(5,3)$.

Solution. In general, the tangent plane to the surface $z=f(x,y)$ at the point $(x_0,y_0,f(x_0,y_0))$ is defined by the equation

$$z-f(x_0,y_0)=\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0)+\frac{\partial f}{\partial y}(x_0,y_0,)(y-y_0),$$

which for $(x_0,y_0)=(5,3)$ reduces to

$$\begin{aligned} z&=\frac{\partial f}{\partial x}(5,3)(x-5)+\frac{\partial f}{\partial y}(5,3)(y-3)+f(5,3)\\ &=\frac{\partial f}{\partial x}(5,3)\,x+\frac{\partial f}{\partial y}(5,3)\,y+\left(f(5,3)-5\frac{\partial f}{\partial x}(5,3)-3\frac{\partial f}{\partial y}(5,3)\right). \end{aligned}$$

However, the problem tells us that the tangent plane at $(5,3,f(5,3))$ is

$$z=2x-y-1.$$

Comparing coefficients of x and y, we obtain

$$\frac{\partial f}{\partial x}(5,3)=2 \qquad \frac{\partial f}{\partial y}(5,3)=-1.$$

Then comparing constant terms gives

$$-1 = f(5,3) - 5\frac{\partial f}{\partial x}(5,3) - 3\frac{\partial f}{\partial y}(5,3) = f(5,3) - 5 \cdot 2 - 3 \cdot (-1) = f(5,3) - 7,$$

so that $f(5,3) = 6$. Or, more simply, substitute $x = 5$ and $y = 3$ into the given tangent plane equation to obtain $z = 2 \cdot 5 - 3 - 1 = 6$.

3 Maxima and Minima of Functions of Several Variables

Finding Critical Points. In two dimensions, (a,b) is a critical point of $f(x,y)$ provided

$$f_x(a,b) = f_y(a,b) = 0.$$

Also know the definition in three dimensions. Here is a problem involving critical points.

$\boxed{8}$ (January 2011) Let $f(x,y) = 4xy - x^4 - y^4$. Find the critical points of $f(x,y)$.

Solution. We need to solve the equations

$$\frac{\partial}{\partial x}\left(4xy - x^4 - y^4\right) = 4y - 4x^3 = 0 \;\Rightarrow\; y = x^3$$
$$\frac{\partial}{\partial y}\left(4xy - x^4 - y^4\right) = 4x - 4y^3 = 0 \;\Rightarrow\; x = y^3.$$

Substituting the first equation into the second gives

$$x = (x^3)^3 = x^9 \;\Rightarrow\; x - x^9 = 0 \;\Rightarrow\; x(1 - x^8) = 0.$$

Factoring further, we obtain

$$0 = x(1-x^8) = x(1-x^4)(1+x^4) = x(1-x^2)(1+x^2)(1+x^4) = x(1-x)(1+x)(1+x^2)(1+x^4).$$

The last two factors never vanish, so $x = 0, \pm 1$. Since $y = x^3$, we get three critical points

$$(0,0),\ (1,1),\ (-1,-1).$$

Comment. A common mistake is canceling a factor from an equation such as $x = x^9$. Here, canceling x would give $1 = x^8$, which loses the critical point $(0,0)$.

The Second Derivative Test for Local Maxima/Minima and Saddle Points. Know the definitions of local maximum and local minimum, and the fact that local maxima and minima occur at critical points when the function is differentiable. Also know the definition of saddle point.

For a suitably nice function $f(x,y)$, the second derivative test goes as follows. Let (a,b) be a critical point of f, and define

$$D = \det\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}.$$

Then:

If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then f has a local minimum at (a,b).
If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then f has a local maximum at (a,b).
If $D(a,b) < 0$, then f has a saddle point at (a,b).

The second derivative test is inconclusive in all other cases. Here is a typical problem.

$\boxed{9}$ (March 2011) Let $f(x,y) = xy^2 - 2x^2 - y^2$. Find all critical points of f, and classify them as local maxima, local minima, and saddle points.

Solution. To find the critical points, we need to solve the equations

$$f_x(x,y) = y^2 - 4x = 0, \quad f_y(x,y) = 2xy - 2y = 0,$$

which are equivalent to

$$x = \tfrac{1}{4}y^2, \quad y(x-1) = 0.$$

The second equation gives $y = 0$ or $x = 1$. We pursue each separately:

$$y = 0 \;\Rightarrow\; x = \tfrac{1}{4}0^2 = 0, \text{ giving } (x,y) = (0,0)$$
$$x = 1 \;\Rightarrow\; 1 = \tfrac{1}{4}y^2 \;\Rightarrow\; y^2 = 4 \;\Rightarrow\; y = \pm 2, \text{ giving } (x,y) = (1,\pm 2).$$

To classify the critical points $(0,0)$, $(1,\pm 2)$, we compute the second partials:

$$f_{xx}(x,y) = -4, \quad f_{xy}(x,y) = 2y, \quad f_{yy}(x,y) = 2x - 2,$$

so that

$$D = \det\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \det\begin{pmatrix} -4 & 2y \\ 2y & 2x-2 \end{pmatrix} = -4(2x-2) - (2y)^2 = -8x + 8 - 4y^2.$$

Thus

$$D(0,0) = 8 > 0, f_{xx}(0,0) = -4 < 0 \;\Rightarrow\; \text{local maximum at } (0,0)$$
$$D(1,\pm 2) = -8 + 8 - 4(\pm 2)^2 = -16 < 0 \;\Rightarrow\; \text{saddle point at } (1,\pm 2).$$

Here is a similar problem you should do yourself for practice.

$\boxed{10}$ (March 2006) Locate the critical points of $f(x,y) = (x+y)^3 + 6(x^2+y^2)$ and determine the type (local maximum, local minimum, saddle point) of each critical point.

Answer: local minimum at $(0,0)$, saddle point at $(-1,-1)$

The Method of Lagrange Multipliers. In a constrained optimization problem, you want to find the maximum or minimum of a function subject to a constraint. Such problems occur in two and three dimensions and use the method of Lagrange multipliers. We assume that the function and constraint are differentiable.

- To maximize or minimize $f(x,y)$ subject to the constraint $g(x,y) = 0$, solve

$$\nabla f(x,y) = \lambda \nabla g(x,y), \quad g(x,y) = 0,$$

or equivalently,

$$f_x(x,y) = \lambda g_x(x,y), \quad f_y(x,y) = \lambda g_y(x,y), \quad g(x,y) = 0.$$

- To maximize or minimize $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$, solve

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z), \quad g(x, y, z) = 0,$$

or equivalently,

$$f_x(x, y, z) = \lambda g_x(x, y, z), \quad f_y(x, y, z) = \lambda g_y(x, y, z), \quad f_z(x, y, z) = \lambda g_z(x, y, z), \quad g(x, y, z) = 0.$$

It is customary to call λ the Lagrange multiplier. Here are two problems that use Lagrange multipliers.

$\boxed{11}$ (January 2009) Let $f(x, y) = 2x + 5y^2$. Find the maximum and minimum values of $f(x, y)$ on the curve $x^2 + 5y^4 = 9$.

Solution. By Lagrange multipliers, we need to solve the equations

$$2 = \lambda \cdot 2x, \quad 10y = \lambda \cdot 20y^3, \quad x^2 + 5y^4 = 9.$$

The first equation tells us that $\lambda x = 1$ (so $\lambda \neq 0$), while the second implies

$$y = 2\lambda y^3 \;\Rightarrow\; y - 2\lambda y^3 = 0 \;\Rightarrow\; y(1 - 2\lambda y^2) = 0 \;\Rightarrow\; y = 0 \text{ or } 2\lambda y^2 = 1.$$

We pursue the two possibilities for y separately:
$y = 0$: The constraint implies $x^2 + 9 \cdot 0^2 = 9$, so that $x = \pm 3$. This gives the points $(\pm 3, 0)$.
$2\lambda y^2 = 1$: Here, there are two ways to proceed:

- (Systematic) Write x, y in terms of λ and substitute into $x^2 + 5y^4 = 9$. Since $x = \frac{1}{\lambda},\ y^2 = \frac{1}{2\lambda}$ (the constraint involves $y^4 = (y^2)^2$), we have

$$\left(\frac{1}{\lambda}\right)^2 + 5\left(\frac{1}{2\lambda}\right)^2 = 9 \;\Rightarrow\; \frac{1}{\lambda^2} + \frac{5}{4\lambda^2} = \frac{9}{4\lambda^2} = 9 \;\Rightarrow\; 4\lambda^2 = 1 \;\Rightarrow\; \lambda = \pm\frac{1}{2}.$$

When $\lambda = \frac{1}{2}$, we get $x = 2,\ y^2 = 1$, giving the points $(2, \pm 1)$. When $\lambda = -\frac{1}{2}$, we get $x = -2,\ y^2 = -1$, which has no solutions over $\mathbb{R}$.

- (Clever) Since $\lambda = 1/x$, $2\lambda y^2 = 1$ implies $2y^2/x = 1$, so that $y^2 = x/2$. Substituting into the constraint gives

$$x^2 + 5\left(\frac{x}{2}\right)^2 = 9 \;\Rightarrow\; \frac{9x^2}{4} = 9 \;\Rightarrow\; x^2 = 4 \Rightarrow\; x = \pm 2.$$

When $x = 2$, we get $y^2 = 2/2 = 1$, giving the points $(2, \pm 1)$. When $x = -2$, we get $y^2 = (-2)/2 = -1$, which has no solutions over $\mathbb{R}$.

It follows that the maximum and minimum of $f = 2x + 5y^2$ occur among $(\pm 3, 0), (2, \pm 1)$. Since

$$f(\pm 3, 0) = 2 \cdot (\pm 3) + 5 \cdot 0^2 = \pm 6, \quad f(2, \pm 1) = 2 \cdot 2 + 5 \cdot (\pm 1)^2 = 9,$$

we see that the maximum value is 9 and the minimum value is -6.

Comment. This problem illustrates that solving Lagrange multiplier equations sometimes requires discipline and attention to detail.

Here is slightly different problem.

$\boxed{12}$ (January 2010) Find the point on the plane $2x - y + 2z = 16$ that is nearest the origin.

Solution. This problem initially looks confusing since it does not state explicitly the function to be minimized. The key phrase is "nearest the origin", which means minimize the distance between the $(0, 0, 0)$ and a point (x, y, z) on the plane. Hence the function to minimize is

$$\sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = \sqrt{x^2 + y^2 + z^2}.$$

But minimizing a square root is equivalent to minimizing the quantity under the square root symbol, namely $x^2 + y^2 + z^2$.

Hence we need to minimize $x^2 + y^2 + z^2$ subject to the constraint $2x - y + 2z = 16$. The respective gradients are $\langle 2x, 2y, 2z\rangle$ and $\langle 2, -1, 2\rangle$. So we need to solve

$$2x = \lambda \cdot 2, \quad 2y = \lambda \cdot (-1), \quad 2z = \lambda \cdot 2,\ 2x - y + 2z = 16.$$

The first three equations imply $x = \lambda$, $y = -\frac{1}{2}\lambda$ and $z = \lambda$. Substituting into the constraint gives

$$2\lambda - (-\tfrac{1}{2}\lambda) + 2\lambda = 16 \ \Rightarrow\ \tfrac{9}{2}\lambda = 16 \ \Rightarrow\ \lambda = \tfrac{2}{9} \cdot 16 = \tfrac{32}{9}.$$

Hence $x = \frac{32}{9}$, $y = -\frac{1}{2} \cdot \frac{32}{9} = -\frac{16}{9}$, and $z = \frac{32}{9}$. This unique point must be point on the plane closest to the origin, which gives the minimum distance

$$\sqrt{\left(\tfrac{32}{9}\right)^2 + \left(-\tfrac{16}{9}\right)^2 + \left(\tfrac{32}{9}\right)^2} = \sqrt{\left(\tfrac{16}{9}\right)^2 (4 + 1 + 4)} = \tfrac{16}{9} \cdot \sqrt{9} = \tfrac{16}{3}.$$

Comment. The last line used the common factor $(\frac{16}{9})^2$ to compute a complicated looking square root. This is a good illustration of why algebra is such a powerful tool in mathematics.

Here is a problem for you to do.

$\boxed{13}$ (March 2016) Find the absolute maximum value of the function

$$f(x, y) = x - 2y + 3$$

on the domain D given by the circle

$$D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 5\}.$$

ANSWER: 8

Absolute Minima and Maxima. A problem may ask for the maximum and maximum values (also called extreme values) of a differentiable function $f(x, y)$ on a closed and bounded region in the plane. Extreme values are known to exist in this situation. They can occur in one of two places:

- In the interior of the region, where they occur among the critical points of f.
- On the boundary of the region, where you use Lagrange multipliers. The constraint is the defining equation of the boundary.

Note that when you find the critical points in the interior, you do *not* need to apply the second derivative test (which would only tell you about local maxima or minima). Here is a typical problem.

$\boxed{14}$ (January 2016) Find the points at which the absolute maximum and minimum of the function $f(x, y) = xy - 1$ on the disk $x^2 + y^2 \leq 2$ occur. State all points where the extrema occur as well as the maximum and minimum values.

Solution. The first step is to find the critical points of f in the interior of the disk. This is easy, since $f_x = y = 0$ and $f_y = x = 0$ imply that $(x, y) = (0, 0)$.

We next use Lagrange multipliers to find where the maximum or minimum values of f can occur on the boundary $x^2 + y^2 = 2$. Writing this as $g(x, y) = x^2 + y^2 - 2 = 0$, Lagrange multipliers gives the equations

$$\nabla f(x, y) = \lambda \nabla g(x, y), \quad g(x, y) = 0,$$

where can be written as

$$y = \lambda \cdot 2x, \quad x = \lambda \cdot 2y, \quad x^2 + y^2 - 2 = 0.$$

The first two equations imply $y = 4\lambda^2 y$, so $y(4\lambda^2 - 1) = 0$. So there are two cases to pursue:

$y = 0$: This implies $x = \lambda \cdot 2 \cdot 0 = 0$, which doesn't satisfy $g(x, y) = 0$. So no solutions here.

$4\lambda^2 - 1 = 0$: This implies $2\lambda = \pm 1$. It follows that $y = \pm x$. Substituting into the constraint gives $2x^2 = 2$, so $x = \pm 1$. Hence we get the four boundary points $(\pm 1, \pm 1)$.

Thus there are five points on the disk where extrema could possibly occur: $(0, 0)$, $(\pm 1, \pm 1)$. Since

$$f(0, 0) = -1, \quad f(1, 1) = f(-1, -1) = 0, \quad f(1, -1) = f(-1, 1) = -2,$$

we conclude that the absolute maximum occurs at $(1, 1)$ and $(-1, -1)$ and has a value of 0, while the absolute minimum occurs at $(1, -1)$ and $(-1, 1)$ and has a value of -2.

Here is a problem that combines several types of questions about maxima and minima.

$\boxed{15}$ (March 2007) Consider $f(x, y) = 2x^2 + 3y^2$ on the closed disk $x^2 + y^2 \leq 1$.

(a) Find the critical points of f in the interior of the disk and classify them using the 2nd derivative test. ANSWER: local minimum at $(0, 0)$

(b) Find the minimum and maximum values of $f(x, y)$ on the circle $x^2 + y^2 = 1$ using the method of Lagrange multipliers. ANSWER: minimum value 2, maximum value 3

(c) What are the minimum and maximum values of $f(x, y)$ on $x^2 + y^2 \leq 1$? ANSWER: minimum value 0, maximum value 3

4 Double Integrals

Given a function $f(x, y)$ on a region R in the plane, one can define the double integral $\iint_R f(x, y)\, dA$.

Iterated Integrals. When the region R has a nice description in Cartesian coordinates, the double integral can be expressed as an iterated integral in two ways:

- The first way is

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

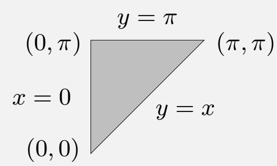
when R consists of all points (x, y) where $a \leq x \leq b$, and for x in this range, $g_1(x) \leq y \leq g_2(x)$. So $y = g_2(x)$ is the top of R, $y = g_1(x)$ is the bottom, and $x = a$, $x = b$ are the sides. When doing the inner integral $\displaystyle\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy$, you should treat x as a constant.

- The second way is
$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$
when R consists of all points (x,y) where $c \le y \le d$, and for y in this range, $h_1(y) \le x \le h_2(y)$. From the point of view of someone on the y-axis, $x = h_2(y)$ is the "top" of R, $x = h_1(y)$ is the "bottom", and $y = c$, $y = d$ are the "sides". When doing the inner integral $\displaystyle\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx$, treat y as a constant.

Some double integrals can be expressed as iterated integrals in both ways. Here is an example.

16 (March 2014) Evaluate $\displaystyle\int_0^\pi \int_x^\pi \frac{\sin(y)}{y}\,dy\,dx$.

Solution. The inner integral $\displaystyle\int_x^\pi \frac{\sin(y)}{y}\,dy$ is impossible by the standard techniques of integration. Because of this, we change the order of integration. To do so, the first step is to understand region of integration, which is the following triangle:



Be sure you know how the limits of integration give this triangle. Then changing the order of integration gives

$$\int_0^\pi \int_x^\pi \frac{\sin(y)}{y}\,dy\,dx = \int_0^\pi \int_0^y \frac{\sin(y)}{y}\,dx\,dy = \int_0^\pi \left(\frac{\sin(y)}{y}\, x\Big|_{x=0}^{y} \right) dy = \int_0^\pi \frac{\sin(y)}{y}\, y\,dy$$
$$= \int_0^\pi \sin(y)\,dy = -\cos(y)\Big|_0^\pi = -\cos(\pi) - (-\cos(0)) = -(-1) - (-1) = 2.$$

Be sure you know how to figure out the new limits of integration.

This solution requires that you remember some basic calculus, including the integral of $\sin(y)$ and the values of trig functions such as $\cos(\pi)$ and $\cos(0)$. Here is a similar problem you should do yourself.

17 (March 2010) Evaluate $\displaystyle\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3+1}\,dx\,dy$. ANSWER: $\frac{52}{9}$

Comments. This problem has two new features:

- The region of integration has $x = \sqrt{y}$ as one of its boundary curves. When you change of the order of integration, the inverse function $y = x^2$ appears:
$$\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3+1}\,dx\,dy = \int_0^2 \int_0^{x^2} \sqrt{x^3+1}\,dy\,dx.$$

It is *essential* that you be able to draw the region of integration and see how the limits change when you change the order of integration.

- Another step in the solution involves recognizing that $\int_0^2 x^2\sqrt{x^3+1}\,dx$ can be done by the substitution method (also called u-substitution). Again this is part of basic calculus that you need to know for the exam.

Polar Coordinates. Be familiar with polar coordinates (r,θ) in the plane and how to convert between Cartesian and polar coordinates. When the region R in a double integral $\iint_R f(x,y)\,dA$ has a nice description in polar coordinates, the integral can be expressed as the iterated integral

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r,\theta)\,r\,dr\,d\theta,$$

where R consists of all points with polar coordinates (r,θ) such that $\alpha \le \theta \le \beta$, and for θ in this range, $h_1(\theta) \le r \le h_2(\theta)$. When doing the inner integral $\int_{h_1(\theta)}^{h_2(\theta)} f(r,\theta)\,r\,dr$, you should treat θ as a constant.

$\boxed{18}$ (January 2013) Evaluate $\iint_R (x+y)\,dy\,dx$, where R is the top half of the circle of radius 2 centered at the origin.

Solution. The region R is a semicircle of radius 2 with polar description $0 \le \theta \le \pi$ and $0 \le r \le 2$. Since $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we obtain

$$\begin{aligned}
\iint_R (x+y)\,dy\,dx &= \int_0^\pi \int_0^2 (r\cos(\theta) + r\sin(\theta))\,r\,dr\,d\theta \\
&= \int_0^\pi \int_0^2 (\cos(\theta)+\sin(\theta))\,r^2\,dr\,d\theta \\
&= \int_0^\pi \left((\cos(\theta)+\sin(\theta))\,\frac{r^3}{3}\Big|_{r=0}^{2} \right) d\theta \\
&= \frac{8}{3}\int_0^\pi \cos(\theta)+\sin(\theta)\,d\theta = \frac{8}{3}\big(\sin(\theta)-\cos(\theta)\big)\Big|_0^\pi \\
&= \frac{8}{3}\big(\sin(\pi)-\cos(\pi)\big) - \frac{8}{3}\big(\sin(0)-\cos(0)\big) \\
&= \frac{8}{3}\big(0-(-1)\big) - \frac{8}{3}\big(0-1\big) = \frac{16}{3}.
\end{aligned}$$

Comment. In Cartesian coordinates, $\iint_R (x+y)\,dy\,dx = \int_{-2}^2 \int_0^{\sqrt{4-x^2}} (x+y)\,dy\,dx$. For practice, you should do this integral, which will require a u-substitution. Be aware that given a double integral such as $\iint_R (x+y)\,dy\,dx$, you may be asked to express it as an iterated integral in both Cartesian and polar coordinates.

The above problem is good reminder that you need to be able to convert between Cartesian and polar coordinates. Also remember that for double integrals, $dA = dx\,dy$ or $dy\,dx$ in Cartesian coordinates and $dA = r\,dr\,d\theta$ in polar coordinates.

Finding Area and Volume. The basic area interpretation of the double integral is $\iint_R 1\,dA = \text{Area}(R)$. See $\boxed{\mathbf{25}}$ for a problem that uses this. For volumes, there are two situations to consider:

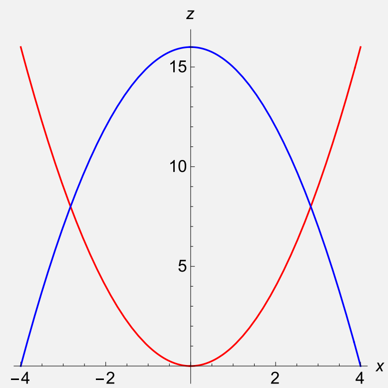
- When $f(x,y) \geq 0$ on R, $\iint_R f(x,y)\,dA$ is the volume under the surface $z = f(x,y)$ for $(x,y) \in R$.
- More generally, suppose that a 3-dimensional region V in $\mathbb{R}^3$ consists of all points (x,y,z) such that $(x,y) \in R$ and $f_1(x,y) \leq z \leq f_2(x,y)$. Thus $z = f_2(x,y)$ is the top of V, $z = f_1(x,y)$ is the bottom, and the sides lie over the boundary of R. In this case, the volume of V is

$$\text{Vol}(V) = \iint_R \big(f_2(x,y) - f_1(x,y)\big)\,dA.$$

Here is an example.

$\boxed{\mathbf{19}}$ (March 2014) Find the volume of the region bounded by the two paraboloids $z = x^2 + y^2$ and $z = 16 - x^2 - y^2$.

Solution. The paraboloid $z = x^2 + y^2$ starts at the origin and opens up, while $z = 16 - x^2 - y^2$ starts at $(0,0,16)$ and opens down. Visualizing this in 3-dimensions is not easy. However, the graphs are symmetric about the z-axis, which means that the cross-sections where $y = 0$ give useful information. The cross-sections are $z = x^2$ and $z = 16 - x^2$, which are easy to draw in the (x,z)-plane:



Be sure you understand the importance of pictures like this. The 3-dimensional region we want is trapped between the two surfaces. The top is $z = 16 - x^2 - y^2$ and the bottom is $z = x^2 + y^2$. To find the region R in the plane, we consider where the top and bottom meet, which is where

$$16 - x^2 - y^2 = x^2 + y^2 \;\Leftrightarrow\; 16 = 2(x^2 + y^2) \;\Leftrightarrow\; x^2 + y^2 = 8.$$

Thus R is the region where $x^2 + y^2 \leq 8$, the circle of radius $\sqrt{8}$ centered at the origin. We compute:

$$\begin{aligned}\text{volume} &= \iint_R (16 - x^2 - y^2) - (x^2 + y^2)\,dA = \iint_R 16 - 2(x^2 + y^2)\,dA\\ &= \int_0^{2\pi}\int_0^{\sqrt{8}} (16 - 2r^2)\,r\,dr\,d\theta = 2\pi \int_0^{\sqrt{8}} (16 - 2r^2)\,r\,dr,\end{aligned}$$

where the last equality follows since the inner integral is independent of θ and $\int_0^{2\pi} d\theta = 2\pi$. We now continue to the final answer:

$$= 2\pi \int_0^{\sqrt{8}} 16r - 2r^3\, dr = 2\pi\left(16\frac{r^2}{2} - 2\frac{r^4}{4}\right)\Big|_0^{\sqrt{8}}$$
$$= 2\pi\left(16 \cdot \frac{8}{2} - 2 \cdot \frac{64}{4}\right) = 2\pi(64 - 32) = 2\pi \cdot 32 = 64\pi.$$

Comment. Polar coordinates work nicely because the region R has a nice polar description and the function $16 - 2(x^2 + y^2)$ converts nicely to polar coordinates. This problem also can be done using a triple integral. Do you see why cylindrical coordinates would be the best choice among the possible options for triple integral coordinate systems?

5 Triple Integrals

Given a function $f(x, y, z)$ on a region R in $\mathbb{R}^3$, one can define the triple integral $\iiint_R f(x, y)\, dV$.

Cartesian, Cylindrical and Spherical Coordinates. You need to be able to work with triple integrals in three coordinate systems:

- Cartesian coordinates x, y, z, where $dV = dx\, dy\, dz$ or $dz\, dy\, dx$. Other orders are possible.
- Cylindrical coordinates r, θ, z, where $dV = r\, dz\, dr\, d\theta$.
- Spherical coordinates ρ, ϕ, θ, where $dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$.

Be sure you know the geometric meaning of these coordinate systems and how to convert between them.

Here is a triple integral problem that uses gradients and lengths of vectors.

20 (March 2006) Let $F(x, y, z) = x^2 + y^2 + z^2$.

(a) Compute the gradient vector ∇F.

(b) Compute $\iiint_R ||\nabla F||\, dV$, where R is the region $\{(x, y, z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 1\}$ and $||v||$ denotes the length of the vector v.

Solution. (a) $\nabla F = \frac{\partial}{\partial x}(x^2+y^2+z^2)\,\mathbf{i} + \frac{\partial}{\partial y}(x^2+y^2+z^2)\,\mathbf{j} + \frac{\partial}{\partial z}(x^2+y^2+z^2)\,\mathbf{k} = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$.

(b) $||\nabla F|| = ||2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}|| = \sqrt{(2x)^2 + (2y)^2 + (2z)^2} = 2\sqrt{x^2 + y^2 + z^2}$. Hence

$$\iiint_R ||\nabla F||\, dV = \iiint_R 2\sqrt{x^2 + y^2 + z^2}\, dV.$$

The region R (a sphere of radius 1) and function $2\sqrt{x^2 + y^2 + z^2}$ suggest spherical coordinates. Then

$$\iiint_R 2\sqrt{x^2 + y^2 + z^2}\, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^1 2\rho \cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = 2\pi \int_0^{\pi}\int_0^1 2\rho^3 \sin\phi\, d\rho\, d\phi$$
$$= 4\pi \int_0^{\pi} \left(\frac{\rho^4}{4}\sin\phi\right)\Big|_{\rho=0}^1 d\phi = \pi \int_0^{\pi} \sin\phi\, d\phi$$
$$= \pi\big(-\cos\phi\big)\Big|_0^{\pi} = \pi\big(-\cos\pi - (-\cos 0)\big) = \pi(-(-1) + 1) = 2\pi.$$

Comment. Be sure you understand the limits of integration.

Finding Volume. The basic volume interpretation of the triple integral is $\iiint_R 1\,dV = \mathrm{Vol}(R)$. You may be asked to express a volume in all three coordinate systems and evaluate one of them. Here is an example.

$\boxed{\mathbf{21}}$ (January 2011) Let V be the region in $\mathbb{R}^3$ inside the sphere $x^2+y^2+z^2=1$ and above the plane $z=0$.

(a) Express the volume of V in cartesian, cylindrical and spherical coordinates.
(b) Evaluate one of the integrals found in part (a).

Solution. (a) We are working with a hemisphere whose projection onto the xy-plane is $x^2+y^2\le 1$. The answer for cartesian coordinates is:

$$\mathrm{Vol}(V) = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} 1\,dz\,dy\,dx.$$

For cylindrical coordinates, the region in the xy-plane is described by $0\le r\le 1$, with no restriction on θ. The top half of the sphere is $z=\sqrt{1-r^2}$, so the integral becomes:

$$\mathrm{Vol}(V) = \int_0^{2\pi}\int_0^1\int_0^{\sqrt{1-r^2}} r\,dz\,dr\,d\theta.$$

For spherical coordinates, the hemisphere has radius 1, so $0\le\rho\le 1$. There is no restriction on θ, and being above the plane $z=0$ means that $0\le\phi\le\pi/2$. Therefore the integral is:

$$\mathrm{Vol}(V) = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

(b) It makes sense to use cylindrical coordinates or spherical coordinates since they have simpler limits of integration. Here are solutions for both.

For cylindrical coordinates:

$$\begin{aligned}
\mathrm{Vol}(V) &= \int_0^{2\pi}\int_0^1\int_0^{\sqrt{1-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r\sqrt{1-r^2}\,dr\,d\theta \\
&= 2\pi\int_0^1 r\sqrt{1-r^2}\,dr \quad u=1-r^2,\ du=-2r\,dr,\ \text{so } -\frac{1}{2}\,du = r\,dr \\
&= 2\pi\int_1^0 \sqrt{u}\Big(-\frac{1}{2}du\Big) = 2\pi\cdot\frac{1}{2}\int_0^1 u^{1/2}\,du\,d\theta = \pi\cdot\frac{2}{3}u^{3/2}\Big|_0^1 = \frac{2\pi}{3}.
\end{aligned}$$

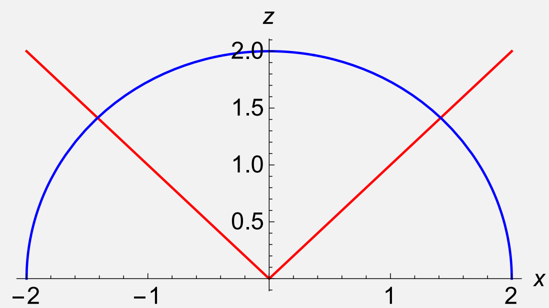
For spherical coordinates:

$$\begin{aligned}
\mathrm{Vol}(V) &= \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 2\pi\int_0^{\pi/2}\int_0^1 \rho^2\sin\phi\,d\rho\,d\phi \\
&= 2\pi\int_0^{\pi/2} \frac{1}{3}\rho^3\sin\phi\Big|_{\rho=0}^1 d\phi = 2\pi\int_0^{\pi/2}\frac{1}{3}\sin\phi\,d\phi \\
&= \frac{2\pi}{3}\big(-\cos\phi\big)\Big|_0^{\pi/2} = \frac{2\pi}{3}\Big(-\cos\frac{\pi}{2} - (-\cos 0)\Big) = \frac{2\pi}{3}.
\end{aligned}$$

Sometimes you are simply asked for the volume, leaving it to you to pick the best coordinate system.

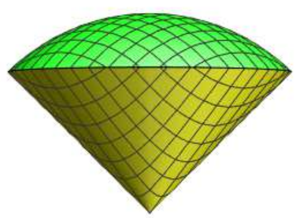
$\boxed{\mathbf{22}}$ (January 2008) Find the volume of the region that is inside the sphere $x^2 + y^2 + z^2 = 4$ and above the cone $z = \sqrt{x^2 + y^2}$. ANSWER: $\frac{16\pi}{3}(1 - \frac{\sqrt{2}}{2})$

Comment. Be sure you understand why drawing a picture is the best place to start. Similar to $\boxed{\mathbf{19}}$, the symmetry about the z-axis means that the cross-section with $y = 0$ gives you good information. This means graphing $x^2 + z^2 = 4$ and $z = \sqrt{x^2} = |x|$:



This problem can be done in either cylindrical or spherical coordinates, though one of these has very simple limits of integration. Do you see how above picture implies that $0 \leq \phi \leq \pi/4$ when you use spherical coordinates?

Here is a 3-dimensional picture of the region in $\boxed{\mathbf{22}}$:



If you can draw something like this, great, but keep in mind that if you understand how cross-sections work, it is often not essential to make a 3-dimensional drawing.