Three-sided matching

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January 30, 2018

Abstract

In this work we will discuss the three-sided matching problem by presenting some existence results, then discussing the implementation of the three-sided matching. The first part is based on the original paper of Gale and Shapley on the two-sided stable matching (1962) [1], the paper of Danilov on the stable three-sided matching (2003) [2], and the paper of Lahiri which gives a weaker condition for three-sided systems (2004) [3]. This will be followed by the C++ implementation of the different algorithms.

Contents

1 Introduction

A common issue in education system is how to assign students to schools the best possible way, by giving the best students their first choices. If we look further into this problem, we can draw the similarity with another problem that may concern real estate agents, clearing houses, or dating websites, on how to match people in a way that the set of all matching remains stable. Furthermore, we may seek to extend this approach to a larger number number of combinations, such as three-sided matching, in order to describe more realistic economic issues.

We will first introduce the original matching problem for marriage and school admission. Then we will present the existence condition for three-sided system that may be generalized to multi-sided systems, and provide the existence of a weaker condition. Finally, we will go through the implementation of the discussed algorithms in C++.

2 Two-sided matching

In this section, we introduce the original stable matching problem [1] and define the stable marriage and deferred acceptance algorithms.

2.1 Stable assignments problem

A set of n applicants has to be assigned within a set of m colleges, with q_i the quota of the i^{th} college. Each applicant ranks the colleges in order of preference, and similarly each college ranks the applicants. The non-acceptable choices are eliminated from the rankings. Therefore, we expect to determine a stable assignment of applicants to colleges.

Definition 1. *An assignment of applicants to colleges will be called unstable if there are two applicants a and b, who are assigned to colleges A and B, respectively, although b prefers A to B and A prefers b to a.*

Definition 2. *A stable assignment is called optimal if every applicant is at least as well off under it as under any other stable assignment.*

2.2 Existence of stable marriage

An other way to look into the assignment problem is to consider a special case where the numbers of applicants and colleges are the same, which can relate then to a marriage problem. We have n men and n women. We are looking for a set of stable marriages, which, in this case, is the same as stable assignment.

Theorem 1. *There always exists a stable set of marriages.*

2.3 Deferred acceptance

We now can extend to a general assignment problem : All students apply to their first choice college. A college with a quota of q places on its waiting list the q applicants that rank highest. The rest of the applicants apply then to their second choice. The procedures terminates when every applicant is either on a waiting list or rejected by every college. Each college admits the students on the waiting list and it is a stable assignment.

Theorem 2. *Every applicant is at least as well off under the assignment given by the deferred acceptance procedure as it would be under any other stable assignment.*

3 Three-sided matching

In this section, we will look into the three-sided version of the stable marriages problem. Such situation may arise in economics when we ought to match sets of manufacturers, vendors and buyers. Therefore, the existence of a stable matching appears to have a great interest, in the sense that it may show the existence of a stable market structure. We will look into this problem as a stable marriage between Males, Females and children.

3.1 Three-sided systems

We call the three-sided system the following collection of data. There are three (disjoint) finite sets M, W and C, whose elements represent men, women and children. A family is a triple (m, w, c), and a matching is a bundle of monogamous families.

3.2 Existence of three-sided matching

The existence of a stable matching has been looked into by [2] conceding an important hypothesis, which states that there exists a priority order in the choice of families.

Theorem 3. *Suppose that preferences are such that every man is interested in the first place in women, and every woman is interested in the first place in men. Then there exist stable three-sided matchings.*

3.3 Multi-sided matching

The generalization of this existence can be done considering a binary tree that orders the priorities, so we can consider as many sides matching as we can define the induced order relations.

Proposition 1. *Suppose that the preferences of all agents are subordinated to the priority schemes corresponding to a given family formation binary tree. Then there exist stable multi-sided matchings.*

3.4 Weaker condition for three-sided systems

Although the previous characterization provides an existence condition, we may look for a weaker one [3] for a better representation of the market agents. We have no reciprocity of the preferential treatment, we focus on one major actor in order to class linearly his preferences and we finally compare his preferences with the others side preferences.

Proposition 2. *The market is said to satisfy Discrimination Property (DP) if:*

- *For all $m \in M$ there exists a linear order P_m on V such that for all $(v, v') \in V^2$ with $v \neq v'$ and $(b, b') \in B^2$, $vP_m v'$ implies $(v, b) >_m (v', b')$.*
- *There exists a function $\beta : V^* \times M^* \to B^*$ such that:*
  - *for all $(m, m1) \in (M^*)^2$ and $(v, v1) \in (V^*)^2$ with $m \neq m1$ and $v \neq v1$, $\beta(v, m) \neq \beta(v1, m1)$;*
  - *for all $m \in M^*$, $v \in V^*$ and $b \in B^*$, $(m, \beta(v, m)) >_v (m, b)$.*

Theorem 4. *Suppose the market satisfies Discrimination Property. Then there exists a stable matching.*

Here, M can be interpreted as a non-empty finite set of workers, V can be interpreted as a non-empty set of firms and B can be interpreted as a non-empty finite set of techniques. The Discrimination Property characterization can lead to to an extension of the two-sided matching of Gale and Shapley.

4 Implementation of matching algorithms

In this section, we present the code, the inputs and the outputs of the different matching algorithms we implemented using C++.

4.1 Implementation of two-sided matching

We first implement the stable marriages algorithm [1] for a set of n men and n women, and we get n stable matching couples.

```
begin
    assign each person to be free;
    while some man m is free do
    w := first woman on m's list to whom m has not yet proposed;
    if w is free then
        assign m and w to be engaged {to each other};
    else
        if w prefers m to her fiance m' then
            assign m and w to be engaged and m' to be free;
        else
            w rejects m {and m remains free};
    output the stable matching consisting of the n engaged pairs;
end;
```

Men	C_1	C_2	C_3	C_4
1	2	4	1	3
2	3	1	4	2
3	2	3	1	4
4	4	1	3	2

Table 1: Men preferences (1)

Women	C_1	C_2	C_3	C_4
1	2	1	4	3
2	4	3	1	2
3	1	4	3	2
4	2	1	4	3

Table 2: Women preferences (1)

Women	Men
1	4
2	3
3	2
4	1

Table 3: Stable marriages (1)

Men	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
1	5	7	1	2	6	8	4	3
2	2	3	7	5	4	1	8	6
3	8	5	1	4	6	2	3	7
4	3	2	7	4	1	6	8	5
5	7	2	5	1	3	6	8	4
6	1	6	7	5	8	4	2	3
7	2	5	7	6	3	4	8	1
8	3	8	4	5	7	2	6	1

Table 4: Men preferences (2)

Women	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
1	5	3	7	6	1	2	8	4
2	8	6	3	5	7	2	1	4
3	1	5	6	2	4	8	7	3
4	8	7	3	2	4	1	5	6
5	6	4	7	3	8	1	2	5
6	2	8	5	3	4	6	7	1
7	7	5	2	1	8	6	4	3
8	7	4	1	5	2	3	6	8

Table 5: Women preferences (2)

Women	Men
1	6
2	7
3	2
4	8
5	1
6	4
7	5
8	3

Table 6: Stable marriages (2)

4.2 Implementation of student admission matching

Using the previous strategy, we now implement the deferred acceptance algorithm [1] for n students, m schools, with p_j choices for the j^{th} student and q_i students selected in the i^{th} school.

```
begin
    assign each student to be free;
    for each students rank of choices R do
        for each school Sc do
            for each student St do
               C := choice of a school
               for the student St at a given rank R;
               if the choice C is the same as the school S
                   and St is free
                   and St in the school Sc admission list then
                   assign the student St to the school Sc;
               else
                   Sc rejects St {and St remains free};
    output the final admission list;
end;
```

Schools	C_1	C_2	C_3	C_4
1	1	4	0	0
2	2	1	3	0
3	3	4	2	0

Table 7: Schools preferences

Students	C_1	C_2	C_3	C_4
1	3	2	1	0
2	1	3	2	0
3	2	3	0	0
4	1	3	0	0

Table 8: Students preferences

Students	Schools
1	2
2	3
3	2
4	1

Table 9: Deferred acceptance

4.3 Implementation of three-sided matching

We proceed to the implementation of the three-sided matching with preference for men choices [2], using the two-sided matching to pair men and women, then parents with children. We also verify that the matchings are stable.

```
begin
    assign each person to be free;
    while some man m is free do
    w := first woman on m's list to whom m has not yet proposed;
    if w is free then
        assign m and w to be engaged {to each other};
    else
        if w prefers m to her fiance m' then
            assign m and w to be engaged and m' to be free;
        else
            w rejects m {and m remains free};
    output the stable matching consisting of the n engaged pairs;
end;

begin
    while some kid k is free do
    m := first dad on k list;
    w := women assigned to m;
    if m is free then
        assign family(m,w,k);
    else
        if m prefers k to his current kid k' then
            assign family (m,w,k);
        else
            m rejects k {and k remains free};
    output the stable matching consisting of the n families;
end;
```

Men	C_1	C_2	C_3
1	2	3	1
2	3	1	2
3	2	3	1

Table 10: Men preferences on women

Women	C_1	C_2	C_3
1	2	1	3
2	3	1	2
3	1	3	2

Table 11: Women preferences on men

Men	C_1	C_2	C_3
1	2	3	1
2	1	3	2
3	1	2	3

Table 12: Men preferences on kids

Kids	C_1	C_2	C_3
1	3	1	2
2	2	1	3
3	3	2	1

Table 13: Kids preferences on men

Women	C_1	C_2	C_3
1	2	3	1
2	1	3	2
3	1	2	3

Table 14: Women preferences on kids

Kids	C_1	C_2	C_3
1	3	1	2
2	2	1	3
3	3	2	1

Table 15: Kids preferences on women

Men	Women	Kids	Stability
1	3	2	OK
2	1	3	OK
3	2	1	OK

Table 16: Matching families

5 Conclusion

We presented existence results for two-sided and three sided systems, and implementations of each case. The three-sided stable matching is based on the assumption of the induced order used by Danilov [2]. Lahiri presents a weaker condition [3] known as the Discrimination Property. Although, it does not offer a practical way to construct the matching procedure, unlike the previous ones. Nonetheless, both approaches can be seen as an extension of the two-sided matching [1] with specific preference procedure.

The three-sided matching was implemented using the stable marriage algorithm, and the matching obtained was verified to be stable. Therefore, we conclude that the stable marriage offers a reliable way to build stable matching for three-sided systems.

References

[1] D. Gale and L. S. Shapley. College admission and the stability of marriage. *The American Mathematical Monthly*, 69(1):9–15, 1962.

[2] V. I. Danilov. Existence of stable matchings in some three-sided systems. *Mathematical Social Sciences*, 46:145–148, 2003.

[3] S. Lahiri. Stable matchings for three-sided systems : A comment. 2004.