Available online at www.sciencedirect.com
SCIENCE DIRECT
Mathematical Social Sciences 46 (2003) 145–148
mathematical social sciences
www.elsevier.com/locate/econbase

Existence of stable matchings in some three-sided systems

V.I. Danilov*

Central Economical Mathematical Institute, Nakhimovskii pr. 47, Moscow 117418, Russia

Received 1 March 2002; accepted 1 November 2002

Abstract

We give here some restrictions on preferences sufficient for existence of stable matching in multi-sided systems.

1. Introduction

About 40 years ago, Gale and Shapley (1962) proved the existence of stable marriages. Their work stimulated in turn plenty of investigations focusing on, for instance, algorithms, lattice structures, strategic behaviour or relationships with Tarski's fixed-point theorem. There have been numerous attempts to broaden the scope of this existence theorem. The literature evidences notably enlargements to one-to-many and many-to-many set-ups, to 'roommate' issues and finally to multi-sided matchings. We give another thought here to this last issue.

We call the *three-sided system* the following collection of data. There are three (disjoined) finite sets M, W and C (whose elements represent, somewhat obviously, men, women and children). A *family* is a triple $(w, c, m) \in W \times C \times M$. A matching is a bundle of monogamous families. More formally, a *matching* is a subset $\mu \subset W \times C \times M$ whose projections onto M, W and C are injective. Given a matching μ and a woman w, we denote by $\mu(w)$ the unique pair (c, m) such that $(w, c, m) \in \mu$. In case no such pair exists, we shall say that woman w is single and write $\mu(w) = *$. The notations $\mu(c)$ and $\mu(m)$ are defined analogously.

Assume now that every woman w has some preference (linear order) $\preceq_w$ on the set $C \times M$; analogously for every man m and child c. (For simplicity, we assume that being

*Tel.: +7-95-129-1011; fax: +7-95-310-7015.
E-mail address: danilov@cemi.rssi.ru (V.I. Danilov).

0165-4896/03/$ – see front matter
doi:10.1016/S0165-4896(03)00073-8

in a family is always preferred to being alone.) A matching μ is *non-stable* if there exists a 'blocking' family (w, c, m) such that $(c, m) \succ_w \mu(w)$, $(w, m) \succ_c \mu(c)$ and $(w, c) \succ_m \mu(m)$. In other words, the candidate family is preferred by everyone of its members to the corresponding family from matching μ. The very existence of such a blocking family points to some failure of the proposed matching μ. On the contrary, a matching μ is *stable* if there are no blocking triples.

Alkan (1988) provided an example showing that, in general, stable three-sided matchings fail to exist. This does not preclude the fact that we might be able to exhibit stable matchings in some interesting cases. We describe one such case here.

2. A special case in terms of preferences

Suppose that a man is interested, in the first place, in women; children interested him less. More formally, this means that there exists a (linear) order on the set W such that the natural projection $W \times C \to W$ is compatible with the order $\preceq_m$ on $W \times C$. This order on W is uniquely determined, and we shall denote it by the same symbol $\preceq_m$.

Proposition 1. *Suppose that preferences are such that every man is interested in the first place in women, and every woman is interested in the first place in men. Then there exist stable three-sided matchings.*

It is rather easy to figure out how to prove this assertion and how to construct a stable matching. Firstly, we marry off men and women in a stable fashion; the corresponding matching is denoted as μ_{MW} or simply MW. Then we form a stable matching between the set of 'pre-families' MW and the set C of children. The children's preferences are obviously defined on $M \times W$ and therefore on $MW \subset M \times W$. Now we define the preferences of the 'pre-family' (m, w) on the set C simply as the preferences of the man m:

$$c' \prec_{(m,w)} c \quad \text{if} \quad (w, c') \prec_m (w, c).$$

Denote by μ an arbitrary stable matching between MW and C.

We assert that μ is a stable three-sided matching. Indeed, suppose that there exists a blocking triple (m, w, c). This means that $(w, c) \succ_m \mu(m) = (w', c')$ and similarly for the woman w and the child c. Since matching μ is constructed from 'pre-families' in MW, we conclude that $(m, w') \in MW$. Since the preferences of m have the special form described above, we conclude that $w \succeq_m w'$. Therefore there are two possibilities:

(a) $w \succ_m w'$ or
(b) $w = w'$.

Similarly, denoting by m' the 'husband' of woman w in the matching μ (or in the matching MW), we have the following two possibilities:

(a′) $m \succ_m m'$ or
(b′) $m = m'$.

The case $(a - a')$ contradicts the stability of the matching MW. Therefore either $w = w'$ or $m = m'$. Let us consider the first case. Since $w = w'$ then $(m, w) \in MW$ and $m' = m$.

Denote by c' the child of the 'pre-family' (m, w) in the matching μ. Since the triple (m, w, c) is a blocking, we have $(w, c) \succ_m (w, c')$ and therefore $c \succ_{(m,w)} c'$. The 'pre-family' (m, w) strictly prefers the child c to the child c'. The child c also prefers the pair (m, w) to his 'pre-family' $\mu(c)$. But this contradicts the stability of μ, which is a matching between MW and C. Thus μ is a stable three-sided matching.

3. A multi-sided generalization

The previous simple result has a multi-sided generalization. To be concrete, let us discuss the following five-sided case. There are five groups of individuals: A, B, C, D, and E. A family (or a team) is a five-tuple $(a, b, c, d, e) \in A \times B \times C \times D \times E$. A matching is a subset $\mu \in A \times \cdots \times E$ whose projections onto any of the sets $A, \ldots, E$ are injective. Each agent $a \in A$ has a preference (a linear order) $\preceq_a$ on the set $B \times C \times D \times E$; similarly agents $b \in B$ have preferences on $A \times C \times D \times E$ and so on. Given the preferences, we can speak about stable matchings.

In order to implement this idea of sequential formation of stable matchings, we need to devise a 'scheme' of sequential family formation. This is a planar binary tree whose terminal vertices are A, B, C, D, E. Let us consider an example of such a tree (Fig. 1).

This means that one starts by forming 'pre-families' of the AB and CD type. Then CD type 'pre-families' join with agents of the type E to form CDE type 'pre-families'. Lastly, AB and CDE type 'pre-families' merge. At each step, we deal with two-sided situations and thus construct stable matchings.

However, in order to be able to speak about the 'preferences' of the various acting agents (for example, type A agents with respect to type B agents, or type CDE families with respect to type AB families), we have to impose structural restrictions on the agents' preferences. For instance, we have to assume that the type A agents are interested in the first place in type B agents (and conversely). Agents of type C are interested in the first place in agents of the type D, in the second place in agents of the

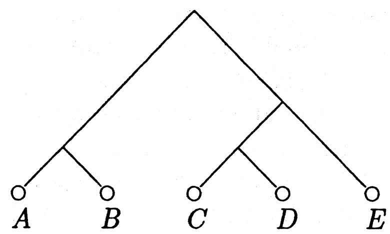

Fig. 1. A sequential family formation scheme.

type C, and in the last place in families of the type AB. We define in a similar fashion a priority of interests for any one of the agents in the tree: to spell it out, it suffices to move along the tree from the corresponding terminal vertex to the root vertex. This path expresses the priority scheme of interest of agents of a given type.

Now, we shall say that a preference (say, for an agent c) on the set $A \times B \times D \times E$ is *subordinated to* the priority scheme $DE(AB)$ if there exist orders on the sets D and $D \times E$ that are compatible with the natural projections

$$A \times B \times D \times E \to D \times E \to D.$$

Proposition 2. *Suppose that the preferences of all agents are subordinated to the priority schemes corresponding to a given family formation binary tree. Then there exist stable multi-matchings.*

The proof is the same as in the three-sided case. Moving along in the tree, we merge pairs of families both sequentially and in a stable fashion. The preferences of every family (for instance, of a family of the type CDE on families of the type AB) are defined as the preferences of one of its members. At each step, there exists a stable (two-sided) matching. We end up with complete families (that is with $ABCDE$ families in this example).

We claim that the resulting (complete) matching $\mu = \mu_{ABCDE}$ is stable. Indeed, suppose that (a, b, c, d, e) is a blocking family. Then a (strictly) prefers (b, c, d, e) to his family $\mu(a) = (b', \dots)$. Similarly, b prefers $(a, c, \dots)$ to $\mu(b) = (a', \dots)$. Given the structural properties of preferences, then either $b' = b$ or b is preferred by a to b'; similarly, either $a = a'$ or a is preferred by b to a'. Both the second relations contradict the stability of the matching μ_{AB} because the pair (a, b) blocks the matching. Therefore $b' = b$ (or $a' = a$). Any one of these equalities gives $(a, b) \in \mu_{AB}$. Similarly, we have that $(c, d) \in \mu_{CD}$, then $((c, d), e) \in \mu_{CDE}$, and lastly $(a, \dots, e) \in \mu_{A \dots E}$. But then the family $(a, \dots, e)$ cannot block the matching μ.

Acknowledgements

The author is grateful to C. Lang and the referee for suggesting improvements to the original version of the paper. This paper was supported by the RFFI grant 00-15-98873.

References

Alkan, A., 1988. Non existence of stable threesome matchings. Math. Soc. Sci. 16, 207–209.
Gale, D., Shapley, L.S., 1962. College admission and the stability of marriage. Am. Math. Monthly 69, 9–14.