On the characterization of renormalizable Lorenz-like Maps

Tarek Frahi

Supervised by
Pr. Antonio Falco

Friday, September 15th, 2017

Abstract

In this work, we investigate the renormalizability of Lorenz maps using combinatorial properties of rotation numbers and kneading pairs, to demonstrate that among the topologically expansive Lorenz maps, only the ones with single rational rotation number are renormalizable. The main theorem states an equivalence between the renormalizability of Lorenz map and the existence of a single rational rotation number, which also defines the kneading pair of the Lorenz map. Using two results, from Falco A. and Alsedà Ll. and from Cui H. and Ding Y, we prove the two implications of the theorem.

Contents

Introduction

The characterization of small geometric properties of dynamical systems using only combinatorial properties is an active and important area of interest, because the study of the topological dynamics is easier with the symbolic representations of the maps instead of the maps themselves. More attention has been paid recently to understanding the Lorenz maps because they arise while studying geometric properties of differential equations, such as is three dimensional flows.

The idea is to study the relation between renormalizability and the combinatorial properties such as the rotation number and the kneading pair. Therefore, the question is to know if only the Lorenz maps with rational rotation numbers are renormalizable.

Falco A. and Alsedà Ll. [1] have proven that a symbolically expansive Lorenz map with a single rotation number and a kneading pair defined by the $\odot -$ *product* from this rotation number, is renormalizable. On the other hand, Cui H. and Ding Y. [2] have proven that for a topologically expansive Lorenz map to be renormalizable, the rotation interval must be reduced to a rational rotation number. Thus, in order to have a characterization of the renormalizability of a topologically expansive Lorenz map, we have to demonstrate the equivalence between the two statements.

After introducing the main frame of the Lorenz and Lorenz-like class of maps, we give a characterization of the Lorenz maps twist periodic orbit, and the set of all kneading pairs. Then, we define the $\odot -$ *product* that operates on this set of kneading pairs of Lorenz maps. Next, we recall the main results on renormalization and combinatorial properties. Finally, we present the main theorem, which states that there is an equivalence between the renormalizability of a topologically expansive Lorenz map and the existence of single rational rotation number which also characterize the kneading pair of the Lorenz map.

1 Lorenz and Lorenz-like maps

Definition 1. *A* Lorenz map *is a map* $f : [0,1] \to [0,1]$ *which has the following three properties:*

1. f *is differentiable and monotonic for* $x \neq c$, $c \in (0,1)$
2. $\lim_{x \nearrow c} f(x) = 1$, $\lim_{x \searrow c} f(x) = 0$ *and* $f(c) = 0$
3. *there exists* $\varepsilon > 0$ *such that* $f'(x) \geq 1 + \varepsilon$ *for* $x \neq c$

Definition 2. *A Lorenz map* $f : [0,1] \to [0,1]$ *is said to be* renormalizable *if there exists an interval* $[u,v] \subset [0,1]$ *such that* $c \in [u,v]$, *and integers* $l, r > 1$, *such that the map* $g : [u,v] \to [u,v]$ *defined by*

$$g(x) = \begin{cases} f^l(x) & \text{if } x \in [u,c) \\ f^r(x) & \text{if } x \in (c,v] \end{cases}$$

is itself a Lorenz map. If f *is not renormalizable, it said to be* prime. *A renormalization* g *of* f *is said to be* minimal *if for any other renormalization* $(f^{l'}, f^{r'})$ *we have* $l' \geq l$ *and* $r' \geq r$.

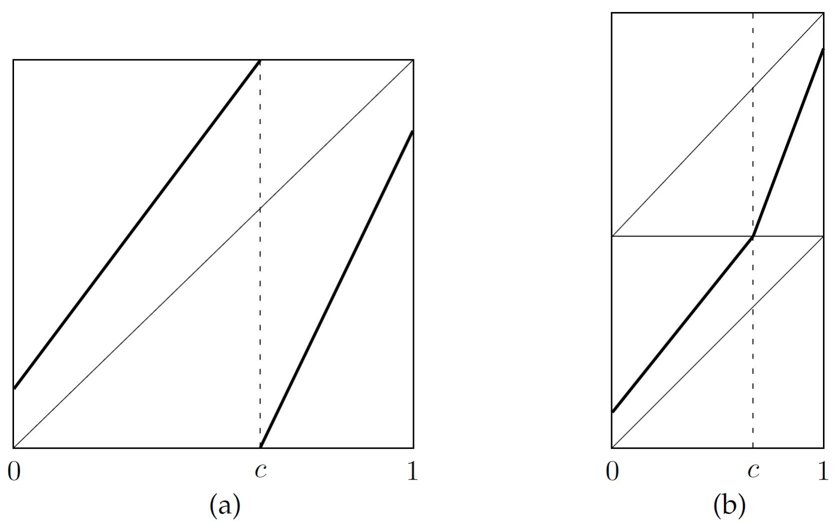

Figure 1: A Lorenz map (a) and its representation as a degree one lifting (b)

Definition 3. *If* f *is a Lorenz map then the map* $F : \mathbb{R} \to \mathbb{R}$ *defined by*

$$F(x)|_{[0,1)} = \begin{cases} f(x) & \text{if } x \in [0,c) \\ f(x)+1 & \text{if } x \in [c,1) \end{cases}$$

and $F(x) = F(x - E(x)) + E(x)$, *where* $E(\cdot)$ *denotes the integer part function is the lifting of a (discontinuous) circle map of degree one.*

This map allows us to use the whole theory of rotation numbers for degree one circle maps in the study of the Lorenz maps. Given a map $F : \mathbb{R} \to \mathbb{R}$ we denote $\lim_{y \downarrow x} F(y)$ by $F(x^+)$ and $\lim_{y \uparrow x} F(y)$ by $F(x^-)$, if they exist.

Definition 4. *We say that $F \in L$ the class of* Lorenz-like maps *if:*

1. $F(x + 1) = F(x) + 1$ *for each* $x \in \mathbb{R}$
2. $F|_{(0,1)}$ *is bounded, continuous, non–decreasing and* $F(1^-) \geq F(0^+) + 1$

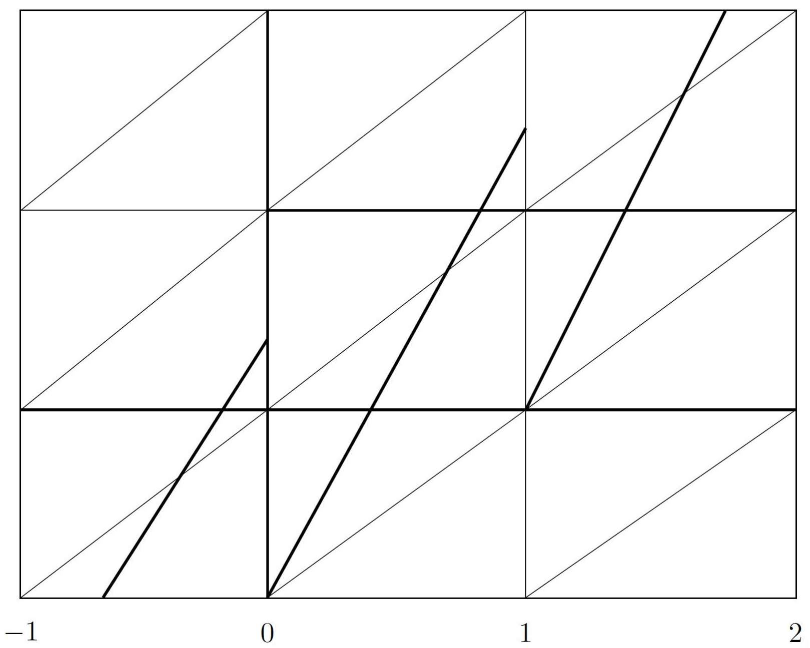

Figure 2: A Lorenz-like map

A *Lorenz map* will be a map F from L such that $F(0^+) \in [0, 1)$ and $F(1^-) \leq F(1^+) + 1$. The class of all Lorenz maps will be denoted by L_m. As it has been said before, the class of Lorenz-like maps extends the (interval) Lorenz maps and the topologically expansive Lorenz maps.

Definition 5. *If X is a compact metric space we say that a continuous mapping $f : X \to X$ is topologically expansive if there exists $\varepsilon > 0$ such that any two distinct forward orbits $(x_0, x_1, \ldots)$ and $(y_0, y_1, \ldots)$ satisfy $|x_i - y_i| \geq \varepsilon$ for some i. We use the definition that a sequence $(x_0, x_1, \ldots)$ is a forward orbit if $f(x_i) = x_{i+1}$ for each $i \geq 0$, and $x_{i+1} = 0$ or 1 if $x_i = c$.*

2 Kneading pair for Lorenz-like map

Definition 6. *For $F \in L, x \in \mathbb{R}$ and $i \geq 1$, let $d_i = E(F^i(x)) - E(F^{i-1}(x))$. Then the* reduced itinerary of x*, denoted by $I_F(x)$, is defined as*

$$I_F(x) = \begin{cases} d_1 d_2 \dots & \text{if } D(F^i(x)) \neq 0 \text{ for all } i \geq 1 \\ d_1 d_2 \dots d_n & \text{if } D(F^i(x)) \neq 0 \text{ for } i \in \{1, \dots, n-1\} \text{ and } D(F^n(x)) = 0 \end{cases}$$

where $D(z)$ denotes the function $z - E(z)$.

Note that, since $F \in L$ we have $I_F(x) = I_F(x+k)$ for all $k \in \mathbb{Z}$, and that $I_F(0)$ is the empty sequence.

Definition 7. *A map $F \in L_m$ is* symbolically expansive *if the map $I(\cdot)$ from $(0,1)$ to $AD_{0,1}$ is injective. Such a condition is guaranteed by usual topological conditions like F being topologically expansive or having negative Schwarzian derivative.*

Definition 8. *Let $\alpha = \alpha_1 \alpha_2 \dots$ be a finite or infinite sequence of integers. We say that α is* admissible *if it is either finite (or empty), or infinite and there exists $k \in \mathbb{N}$ such that $|\alpha_i| \leq k$ for all $i \geq 1$. The set of admissible sequences of a maps from L (resp. L_m) will be denoted by AD (resp. $AD_{0,1}$).*

Notice that any reduced itinerary is an admissible sequence. The cardinality of an admissible sequence α will be denoted by $|\alpha|$ (if α is infinite we write $|\alpha| = \infty$). We write $\alpha\beta$ to denote the concatenation of α and β.

Definition 9. *Let α and β be two admissible sequences such that $\alpha \neq \beta$. We say that $\alpha < \beta$ if either there exists n such that $\alpha_i = \beta_i$ for $i = 1, 2, \dots, n-1$ and $\alpha_n < \beta_n$, or $|\alpha| = n-1$ and $\beta = \alpha_1 \alpha_2 \dots \alpha_{n-1} \beta_n \beta_{n+1} \dots$.*

Now we have endowed the set of admissible sequences with a total ordering. In order to define the kneading pair of a map $F \in L$ we introduce the following notation.

Definition 10. *For a point $x \in \mathbb{R}$ we define the sequences $I_F(x^+)$ and $I_F(x^-)$ as follows. For each $n \geq 0$ there exists $\delta(n) > 0$ such that $d(F^{n-1}(y))$ takes constant value for each $y \in (x, x+\delta(n))$ (respectively, $y \in (x-\delta(n), x)$). Denote this value by $d(F^{n-1}(x^+))$ (respectively, $d(F^{n-1}(x^-))$). Then we set*

$$I_F(x^+) = d(x^+) d(F(x^+)) \dots \text{ and } I_F(x^-) = d(x^-) d(F(x^-)) \dots$$

Clearly, $I_F(x^+)$ and $I_F(x^-)$ are infinite admissible sequences and, $I_F(x^+) = I_F((x+k)^+)$ and $I_F(x^-) = I_F((x+k)^-)$ for all $k \in \mathbb{Z}$. Moreover, for each $x \in \mathbb{R}$ we have $I_F(x^-) \leq I_F(x) \leq I_F(x^+)$ and if $|I_F(x)| = \infty$ then $I_F(x^+) = I_F(x^-) = I_F(x)$. We note that for each $x \in \mathbb{R}$ we have $I_F(x^\pm) = d(x^\pm) I_F((F(x^\pm))^\pm)$. Consequently, for each $n \in \mathbb{N}$ we have

$$I_F(x^\pm) = d(x^\pm) d(F(x^\pm)) \cdots d(F^{n-1}(x^\pm)) I_F((F^n(x^\pm))^\pm).$$

Definition 11. *Let $F \in L$. The pair $(I_F(0^+), I_F(0^-))$ will be called* the kneading pair of F *and will be denoted by KP. Clearly, for each $F \in L$ we have $KP \in AD \times AD$.*

Definition 12. *Let S be the shift operator which acts on the set of admissible sequences of length greater than one as follows: $S(\alpha) = \alpha_2\alpha_3\ldots$ if $\alpha = \alpha_1\alpha_2\alpha_3\ldots$ We will write S^k for the k^{th} iterate of S, only defined for admissible sequences of length greater than k.*

For each $x \in \mathbb{R}$ we have $S^n(I_F(x)) = I_F(F^n(x))$ if $|I_F(x)| > n$.

Definition 13. *Let $\alpha, \beta, \nu \in AD$ be such that α and β are infinite.*

1. *We will say that ν is* quasidominated *by (α, β) if and only if for all $n < |\alpha|$*
$$\alpha \leq S^n(\nu) \leq \beta$$
2. *We will say that ν is* dominated *by (α, β) if and only if ν is quasidominated by (α, β) and the above inequalities are strict.*
3. *We will say that ν is quasidominated (respectively dominated) by F if it is quasidominated (respectively dominated) by $(I_F(0^+), I_F(0^-))$.*

To characterize the pairs of admissible sequences that can occur as a kneading pair of a map from L we will define a set $E \subset AD \times AD$ which turns to be the set of all kneading pairs of maps from L.

Definition 14. *Let $\alpha = \alpha_1\alpha_2\ldots$ be an admissible sequence. We will denote by α' the sequence $(\alpha_1 + 1)\alpha_2\ldots$. Now, let E^* be the set of all pairs $(\nu_1, \nu_2) \in AD \times AD$ such that $|\nu_i| = \infty$ for $i = 1, 2$ and the following conditions hold:*

1. *$\nu_1' < \nu_2$ $(c1)$*
2. *ν_i is quasidominated by (ν_1, ν_2) for all $i \in \{1, 2\}$ $(c2)$*

The condition $(c2)$ states that ν_1 and ν_2 are respectively minimal and maximal according to the following definition.

Definition 15. *For $\alpha \in AD$ we say that α is* minimal *(respectively,* maximal*) if and only if $\alpha \leq S^n(\alpha)$ (respectively, $\alpha \geq S^n(\alpha)$) for all $n < |\alpha|$.*

The above set contains the kneading pairs of maps from L with non-degenerate rotation interval. To deal with some special kneading pairs associated to maps with degenerate rotation interval we introduce the following sets.

Definition 16. *For $a \in \mathbb{R}$ we set $\varepsilon_i(a) = E(ia) - E((i-1)a)$ and $\delta_i = \bar{E}(ia) - \bar{E}((i-1)a)$ where $\bar{E} : \mathbb{R} \to \mathbb{Z}$ is defined as $E(x)$ when $x \notin \mathbb{Z}$ and $x - 1$ otherwise. We set*
$$\bar{I}_\varepsilon(a) = \varepsilon_1(a)\varepsilon_2(a)\ldots\varepsilon_n(a)\ldots \text{ and } \bar{I}_\delta(a) = \delta_1(a)\delta_2(a)\ldots\delta_n(a)\ldots$$
Set ${\bar{I}_\varepsilon}^*(a) = (\bar{I}_\varepsilon(a))'$ and let ${\bar{I}_\delta}^*(a)$ denote the sequence which satisfies $({\bar{I}_\delta}^*(a))' = \bar{I}_\delta(a)$. Now, for each $a \in \mathbb{R}$, we set
$$E_a = \begin{cases} \{(\bar{I}_\varepsilon(a), {\bar{I}_\varepsilon}^*(a)), ({\bar{I}_\delta}^*(a), \bar{I}_\delta(a)), (\bar{I}_\varepsilon(a), {\bar{I}_\varepsilon}^*(a))\} & \text{if } a \in \mathbb{Q} \\ \{({\bar{I}_\delta}^*(a), \bar{I}_\delta(a))\} & \text{otherwise} \end{cases}$$
We note that $E_a \cap E^* = \emptyset$ for each $a \in \mathbb{R}$, because condition $(c1)$ does not hold for any of the elements of E_a. Finally we denote by E the set $E^* \cup (\cup_{a \in \mathbb{R}} E_a)$.

3 Rotation interval and twist periodic orbits

The notion of *rotation number* was introduced by Poincaré for homeomorphisms of the circle of degree one. This notion will be used to characterize the set of periods of circle maps of degree one.

Definition 17. *Let $F \in L$. For $x \in \mathbb{R}$ we define the* rotation number of x *as*

$$\rho_F(x) = \limsup_{n\to\infty} \frac{F^n(x) - x}{n}$$

Definition 18. *We denote by R_F the set of all rotation numbers of F. It follows that $R_F = [a(F), b(F)]$ with*

$$a(F) = \inf_{x\in\mathbb{R}} \liminf_{n\to\infty} \frac{F^n(x) - x}{n} \text{ and } b(F) = \sup_{x\in\mathbb{R}} \limsup_{n\to\infty} \frac{F^n(x) - x}{n}$$

Therefore, in what follows the set R_F will be called the rotation interval of F.

Definition 19. *Let $F \in L$ and let $x \in \mathbb{R}$. We shall say that x is* periodic (mod.1) of period q with rotation number p/q *for a map $F \in L$ if $F^q(x) - x = p$ and $F^i(x) - x \notin \mathbb{Z}$ for $i = 1, \ldots, q-1$.*

A periodic (mod. 1) point of period 1 will be called *fixed (mod.1).* The set $\{F^n(x) : n \in \mathbb{Z}^+\} + \mathbb{Z}$ will be called the *(mod.1) orbit of x by F.* Each point from an orbit (mod.1) P has the same rotation number.

Definition 20. *If x is a periodic (mod.1) point of F of period q with rotation number p/q then its (mod.1) orbit is called a* periodic (mod.1) orbit of F of period q with rotation number p/q. *If P is a (mod.1) orbit of F then we denote by P_i the set $P \cap [i, i+1)$ for all $i \in \mathbb{Z}$. Obviously $P_i = i + P_0$ and, if P has period q, then $Card(P_i) = q$ for all $i \in \mathbb{Z}$.*

4 The $\odot-$product

The main goal of this section is to construct a product on the set of admissible sequences for the class of Lorenz maps by using the symbolic properties of the twist periodic orbits.

Definition 21. *The set of all admissible sequences of maps from L_m, denoted by $AD_{0,1}$, is the set $\Sigma_2 = \{0,1\}^{\mathbb{N}}$ union the set of finite sequences in the symbols $\{0,1\}$. We take $(E_\varepsilon)^* = (\Sigma_2 \cap E_\varepsilon) \setminus \{1^\infty\}$ and $E_\delta^* = (\Sigma_2 \cap E_\delta) \setminus \{0^\infty\}$. We have that $E_\varepsilon^* \cup \{1^\infty\}$ (respectively, $E_\delta^* \cup \{0^\infty\}$) are all the minimal (resp. maximal) sequences in Σ_2. We endow Σ_2 with the topology defined by the metric*

$$d\big((d_1 d_2 \dots), (t_1 t_2 \dots)\big) = \sum_{i=1}^{\infty} \frac{|d_i - t_i|}{2^i}.$$

This topology is compatible with the order topology given by the lexicographical order in Σ_2. With this topology Σ_2 is a compact metric space.

Definition 22. *Let $S : \Sigma_2 \to \Sigma_2$ denote the usual shift transformation restricted to Σ_2. Clearly, S is continuous. Set*

$$E_{0,1} = (E_\varepsilon^* \times E_\delta^*) \cap E = (E \cap (\Sigma_2 \times \Sigma_2)) \setminus \{(\alpha, \beta) \colon \alpha = 1^\infty \text{ or } \beta = 0^\infty\} \subset \Sigma_2 \times \Sigma_2$$

and consider $E_{0,1}$ endowed with the product topology of $\Sigma_2 \times \Sigma_2$ given by

$$d^* \left((\alpha, \beta), (\nu, \kappa) \right) = \max \left\{ d(\alpha, \nu), d(\beta, \kappa) \right\}.$$

Definition 23. *Given $d \in \{0,1\}$ we will denote $1-d$ by $\hat{d}$. Then, for $a \in \mathbb{Q}^*$ and $\alpha_1 \alpha_2 \dots \in AD_{0,1}$, we define*

$$a \odot (\alpha_1 \alpha_2 \cdots) = \begin{cases} \alpha_1 r(a) \alpha_1 \hat{\alpha}_1 r(a) \alpha_2 \hat{\alpha}_2 \cdots & \text{if } |\alpha| = \infty \\ \alpha_1 r(a) \alpha_1 \hat{\alpha}_1 r(a) \alpha_2 \hat{\alpha}_2 \cdots \hat{\alpha}_{n-1} r(a) \alpha_n & \text{if } |\alpha| = n \end{cases}$$

and $\overline{a \odot \alpha} = \hat{\alpha} S(a \odot \alpha)$.

5 Results and main theorem

5.1 Preliminaries

Theorem 1. *For $F \in L$ we have $KP(F) \in E$. Conversely, for each $(\nu_1, \nu_2) \in E$, there exists $F \in L$ such that $KP(F) = (\nu_1, \nu_2)$. (see [1])*

Theorem 2. *Let $F \in L_m$. Then each renormalization of F is periodic. (see [2])*

Theorem 3. *Let F be an expanding Lorenz map. Then the minimal renormalization of F is periodic if and only if the rotation interval of F is degenerated to be a rational point. (see [2])*

Theorem 4. *Let $F \in L_m$ be topologically expansive and renormalizable. Then there exists $p/q \in \mathbb{Q}$, with $(p, q) = 1$, such that $R_F = \{p/q\}$. (see [1])*

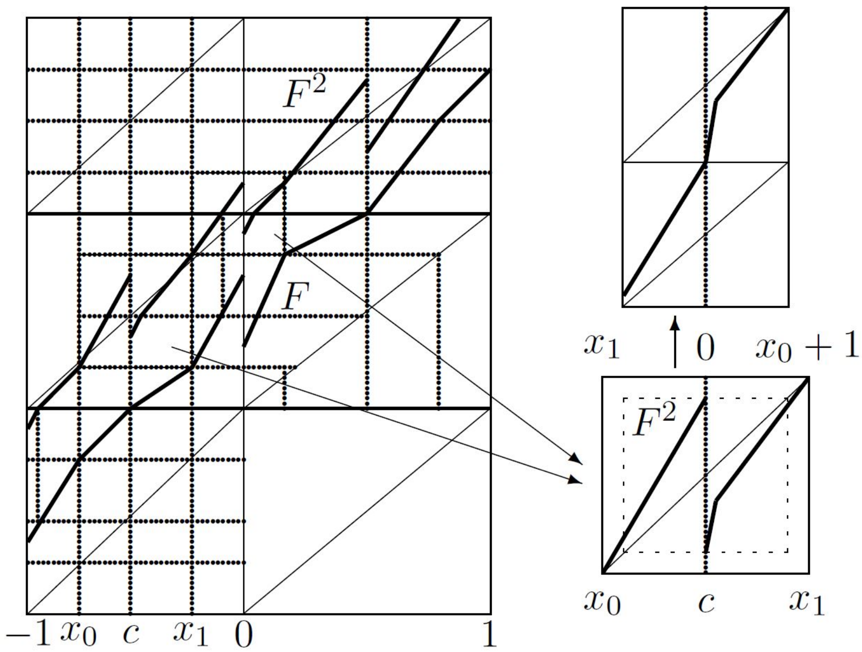

Figure 3: An example of renormalization with $p/q = 1/2$

Theorem 5. *Let $F \in L_m$ symbolically expansive such that $R_F = \{a\}$ and $K(F) = a \odot (\alpha, \beta)$ with $(\alpha, \beta) \in E_{0,1}$. Then F is renormalizable. (see [1])*

5.2 Main theorem

Theorem. *Let $F \in L_m$ be topologically expansive. Then the following statements are equivalent:*

1. *F is renormalizable*

2. *There exists $p/q \in \mathbb{Q}$, with $(p,q) = 1$, such that $R_F = \{p/q\}$ and $K(F) = p/q \odot (\alpha, \beta)$ for some $(\alpha, \beta) \in E_{0,1}$*

5.3 Proof of the main theorem

Lemma 1. *Let $F \in L_m$ renormalizable and topologically expansive. Then there exist $p/q \in \mathbb{Q}$ with $(p,q) = 1$, such that $R_F = \{p/q\}$ and $K(F) = p/q \odot (\alpha, \beta)$ for some $(\alpha, \beta) \in E_{0,1}$.*

Proof of Lemma 1. *Let $F \in L_m$. Then (thm1) F is periodic. So $R_F = a$ and $K(F) = a \odot (\alpha, \beta)$ for some $(\alpha, \beta) \in E_{0,1}$. Let F be topologically expansive in addition. Then there exist $p/q \in \mathbb{Q}$, with $(p,q) = 1$, such that $a = p/q$.*

Lemma 2. *Let $F \in L_m$ topologically expansive such that there exist $p/q \in \mathbb{Q}$ with $(p,q) = 1$, with $R_F = \{p/q\}$ and $K(F) = p/q \odot (\alpha, \beta)$ for some $(\alpha, \beta) \in E_{0,1}$. Then F is renormalizable.*

Proof of Lemma 2. *Let $F \in L_m$ topologically expansive. Then F is symbolically expansive, so F is renormalizable.*

Proof of main Theorem. *These two lemma prove the two implications of the main theorem.*

Conclusion

We have demonstrated that the only topologically expansive Lorenz maps that are renormalizable are the one with single rational rotation number. This characterization can offer an easier way for the study of renormalizability of Lorenz maps with their symbolic representation. It can also be a useful tool while using the unimodal Lorenz maps theory.

References

[1] Alsedà Ll., Falcó A., *On the topological dynamics and phase-locking renormalization of lorenz-like maps*, Annales de l'Institut Fourier, Grenoble (2003), 859-883.

[2] Cui H., Ding Y., *Renormalization and conjugacy of piecewise linear Lorenz maps*, Advances in Mathematics Volume 271, (2015), 235-272.

[3] Alsedà Ll., Falcó A., *A characterization of the kneading pair for bimodal degree one circle maps*, Annales de l'Institut Fourier, Grenoble (1997), 273-304.

[4] Hubbard J. H., Sparrow C., *The Classification of Topologically Expansive Lorenz Maps*, Communications on Pure and Applied Mathematics Vol XLIII (1990), 431-443.

[5] Falcó A., *Bifurcations and symbolic dynamics for bimodal degree one circle maps: The Arnold tongues and the Devil's staircase*, Ph.D. Thesis, Universitat Autònoma de Barcelona (1995).

[6] Alsedà Ll., Llibre J. and Misiurewicz M.,*Combinatorial dynamics and entropy in dimension one*, Advanced Series in Nonlinear Dynamics, World Scientific, Singapore (1993).

[7] Alsedà Ll., Llibre J., Misiurewicz M., Tresser Ch., *Periods and entropy for Lorentz-like maps*, Annales de l'Institut Fourier, Grenoble (1989), 929-952.

[8] Alsedà Ll., Mañosas F., *Kneading theory and rotation interval for a class of circle maps of degree one*, Nonlinearity (1990), 413-452.

[9] Alsedà Ll., Mañosas F., *Kneading theory for a family of circle maps with one discontonuity*, Acta Mathematica Universitatis Comenianae Vol LXV (1996), 11-22.

[10] Glendinning P., *Topological conjugation of Lorenz maps by β-transformations*, Mathematical Proceedings of the Cambridge Philosophical Society (1990), 401-413.

[11] Glendinning P., Sparrow C., *Prime and renormalizable kneading invariants and the dynamics of expanding Lorenz maps*, Physica D (1993), 22–50.

[12] Lorenz E. N., *Deterministic non-periodic flow*, Journal of the Atmospherical Sciences (1963), 130-141.

[13] Martens M., *The periodic points of renormalization*, Ann. of Math. (1998), 543-584.

[14] Martens M., de Melo W., Universal models for Lorenz maps, Ergodic Theory and Dynamical Systems (2001), 833-860.

[15] Milnor J. Thurston P., *On iterated maps on the interval I, II* "Dynamical Systems", Lecture Notes in Math, Springer (1988).

[16] Misiurewicz M., *Rotation intervals for a class of maps of the real line into itself*, Ergodic Theory and Dynamical Systems (1986), 117-132.

[17] Poincaré H., *Sur les courbes définies par les équations differentielles*, Oeuvres Complètes, vol 1, Gauthiers-Villars, Paris, (1952), 137-158.

[18] Rhodes F., Thompson C. L., *Rotation numbers for monotone functions on the circle*, Journal of London Mathematical Society (1986), 360-368.