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IF420 – ANALISIS NUMERIK

Pertemuan ke 14 – Fourier Transform

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Capaian Pembelajaran Mingguan Mata Kuliah (Sub-CPMK):

Sub-CPMK 14: Mahasiswa mampu memahami dan menerapkan transformasi Fourier – C3

Reviews

- ODE Boundary Value Problem Statement
- The Shooting Method
- Finite Difference Method
- Numerical Error and Instability

Outlines

- The Basics of Waves
- Discrete Fourier Transform (DFT)
- Fast Fourier Transform (FFT)
- FFT in Python

Motivation

- In this last lesson, we will introduce you the **Fourier method** that named after the **French mathematician** and **physicist Joseph Fourier**, who used this type of method to study the **heat transfer**. The **basic idea** of this method is to express some **complicated functions** as an **infinite sum** of **sine** and **cosine waves**.
- We saw this in the previous lessons, that we can **decompose** a function using the **Taylor series**, which express the function with an **infinite sum** of **polynomials**.
- The **Fourier method** has many applications in engineering and science, such as signal processing, partial differential equations, image processing, and so on.
- The **Fast Fourier Transform** is chosen as one of the 10 algorithms with the **greatest influence** on the development and practice of science and engineering in the 20th century in the January/February 2000 issue of **Computing in Science and Engineering**.

The Basics of Waves

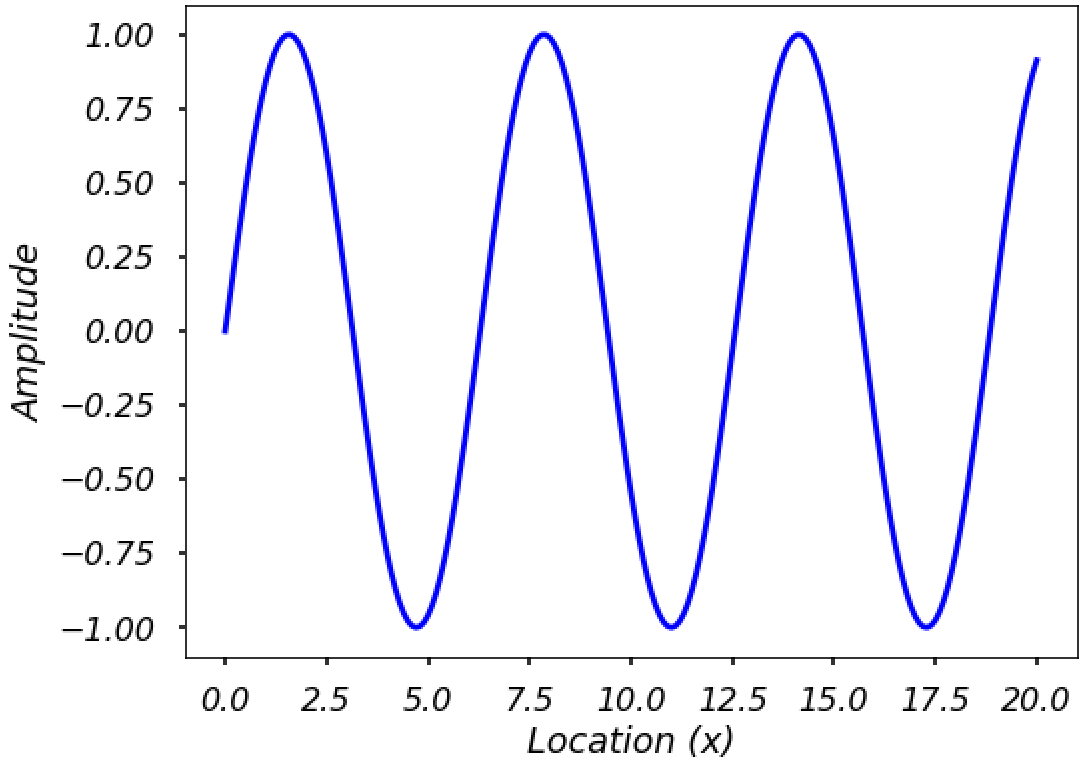
- There are **many types** of **waves** in our life, for example, if you throw a rock into a pond, you can see the waves form and travel in the water. Of course, there are many more examples of waves, some of them are even **difficult to see**, such as sound waves, earthquake waves, microwaves (that we use to cook our food in the kitchen), etc.
- In **physics**, a **wave** is a **disturbance** that **travels** through **space** and **matter** with a **transferring** **energy** from one place to another.
- It is important to study waves in our life to understand how they **form**, **travel,** and so on.
- In this lesson, we will cover a **basic tool** that help us to understand and study the waves - the **Fourier Transform**.
- But before we proceed, let's first get familiar on how do we actually **model** the **waves** and study it.

Model a wave using Mathematical tools

- We can **model** a **single wave** as a **field** with a **function** $F(x,t)$, where x is the **location** of a point in space, while t is the **time**.
- One simplest case is the shape of a **sine wave** change over x.

```python
import matplotlib.pyplot as plt
import numpy as np

plt.style.use('seaborn-poster')
%matplotlib inline
```

```python
x = np.linspace(0, 20, 201)
y = np.sin(x)

plt.figure(figsize = (8, 6))
plt.plot(x, y, 'b')
plt.ylabel('Amplitude')
plt.xlabel('Location (x)')
plt.show()
```

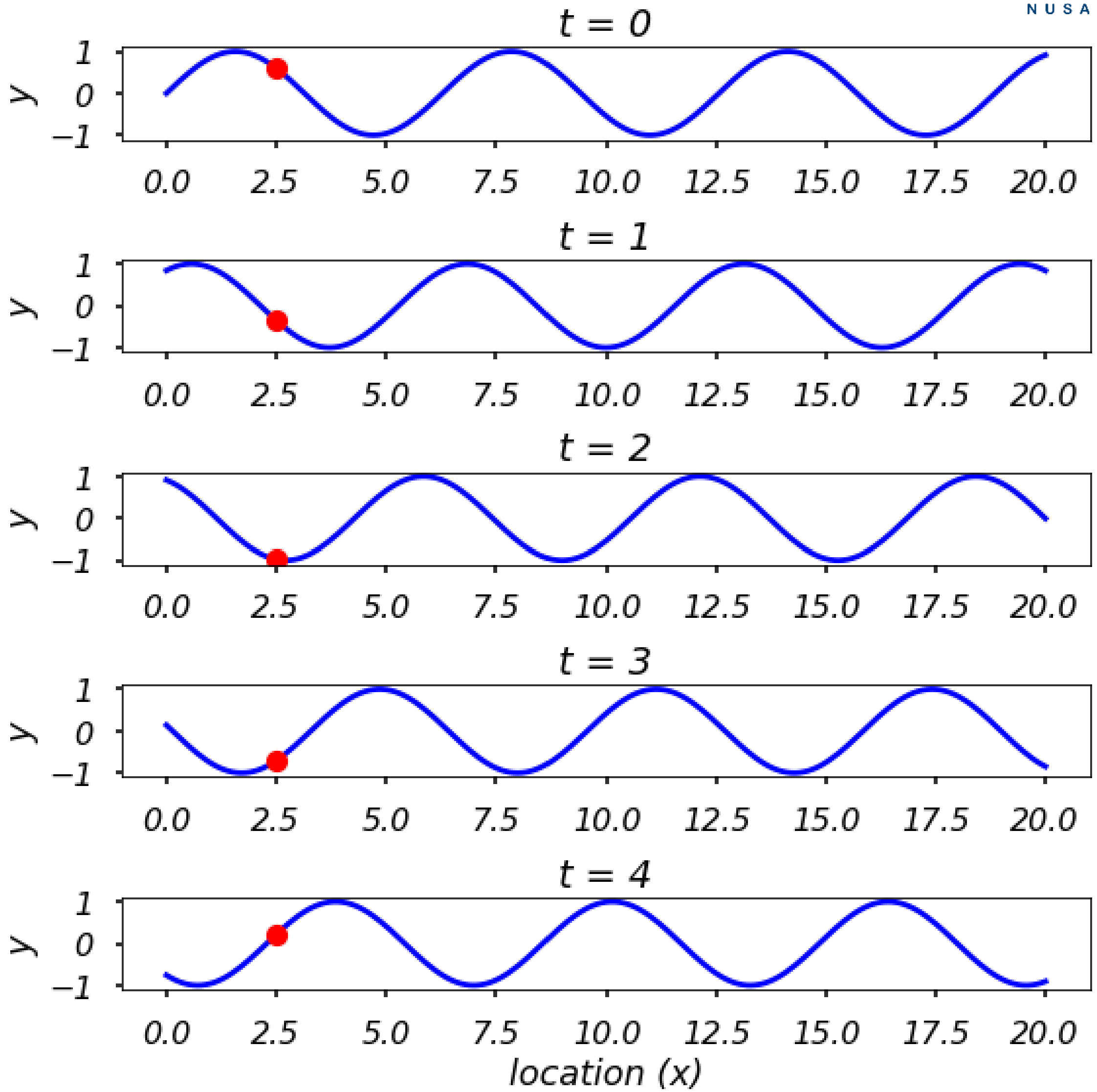


Model a wave using Mathematical tools

- We can think of the **sine** wave that can **change** both in **time** and **space**.
- If we plot the **changes** at various locations, each time snapshot will be a sine wave changes with location.
- See the following figure with a fix point at $x = 2.5$ showing as a **red dot**.
- Of course, you can see the **changes over time** at specific **location** as well, you can plot this by yourself.

Model a wave using Mathematical tools

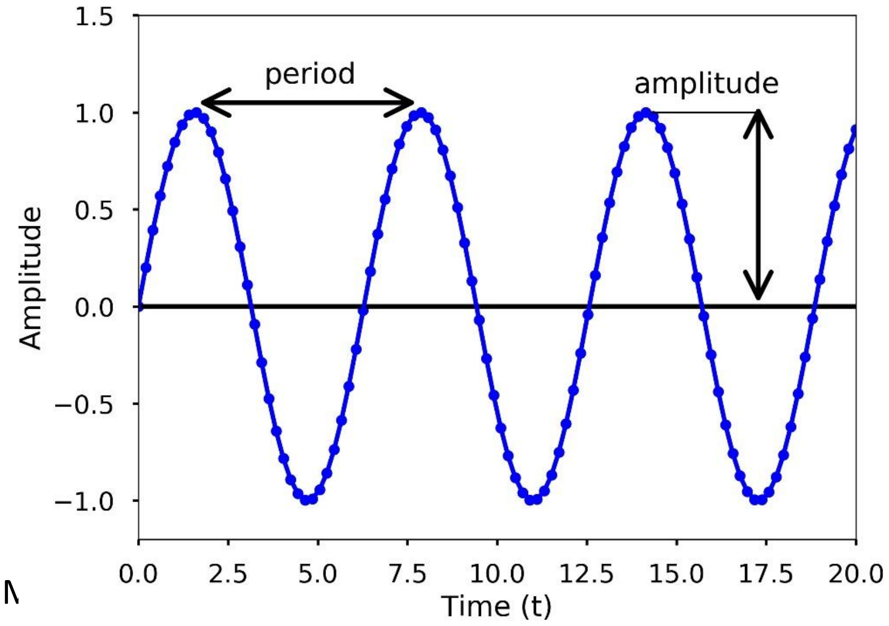
```python
fig = plt.figure(figsize = (8,8))

times = np.arange(5)

n = len(times)

for t in times:
    plt.subplot(n, 1, t+1)
    y = np.sin(x + t)
    plt.plot(x, y, 'b')
    plt.plot(x[25], y [25], 'ro')
    plt.ylim(-1.1, 1.1)
    plt.ylabel('y')
    plt.title(f't = {t}')

plt.xlabel('location (x)')
plt.tight_layout()
plt.show()
```

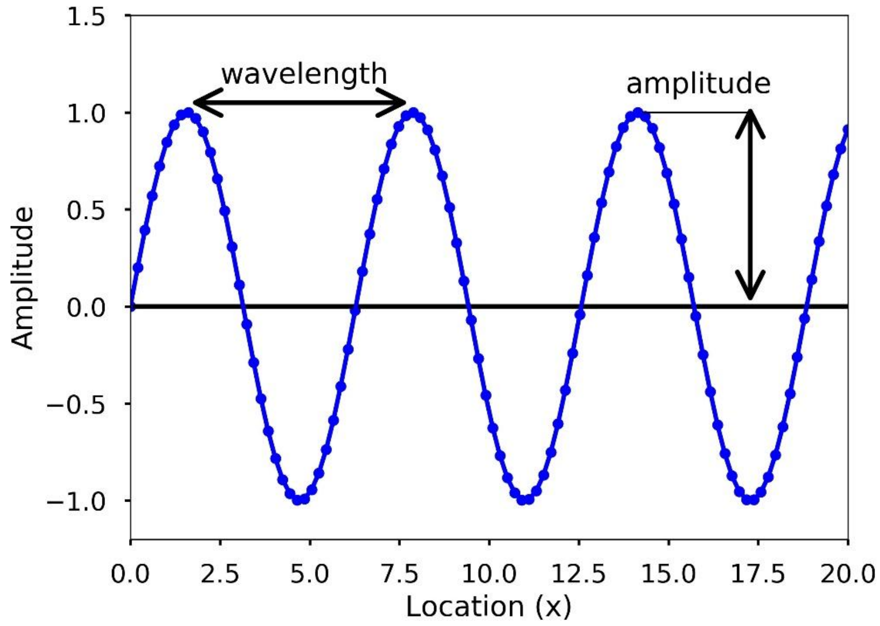


Characteristics of a wave

- We can see wave as a **continuous** entity both in **time** and **space**. But in reality, many times we **discrete** the time and space at **various points**. For example, we can use sensors such as **accelerometers** at **different locations** on the Earth to monitor the earthquakes, which is a **spatial discretization**. Similarly, these sensors usually **record** the **data** at **certain times** which is a **temporal discretization**.

- For a single wave, it has **different characteristics**. See the following two figures.





Characteristics of a wave

- **Amplitude** is used to describe the difference between the **maximum values** to the **baseline value**.
- A **sine** wave is a **periodic signal**, which means it **repeats itself** after certain time, which can be measured by **period**.
- **Period** of a wave is **time** it takes to **finish** a **complete cycle.** In previous figure, we can see that the period can be measured from the **two adjacent peaks**.
- **Wavelength** measures the **distance** between **two** successive **crests** or **troughs** of a wave.
- **Frequency** describes the **number of waves** that pass a **fixed** place in a given amount of **time**. Frequency can be measured by **how many cycles** pass within **1 second**. Therefore, the unit of frequency is **cycles/second**, or more commonly called **Hertz** (abbreviated **Hz**).

Characteristics of a wave

- **Frequency** is **different** from **period**, but they are related to each other. **Frequency** refers to how often something **happens** while **period** refers to the **time** it takes to complete something, mathematically,

$$period = \frac{1}{frequency}$$

- From the previous two figures, we can also see that **blue dots** on the sine waves, these are the **discretization points** we did both in **time** and **space**. Therefore, only at these dots, we have **sampled** the value of the wave.
- Usually when we **record** a **wave**, we need to specify how **often we sample** the wave in time, this is called **sampling**.
- The rate is called **sampling rate**, with the unit **Hz**. For example, if we sample a wave at 2 Hz, it means that every second we sample two data points.

Characteristics of a wave

- Since we understand more about the basics about a wave, now let's see a **sine wave** more carefully.
- A **sine wave** can be represented by the following equation:

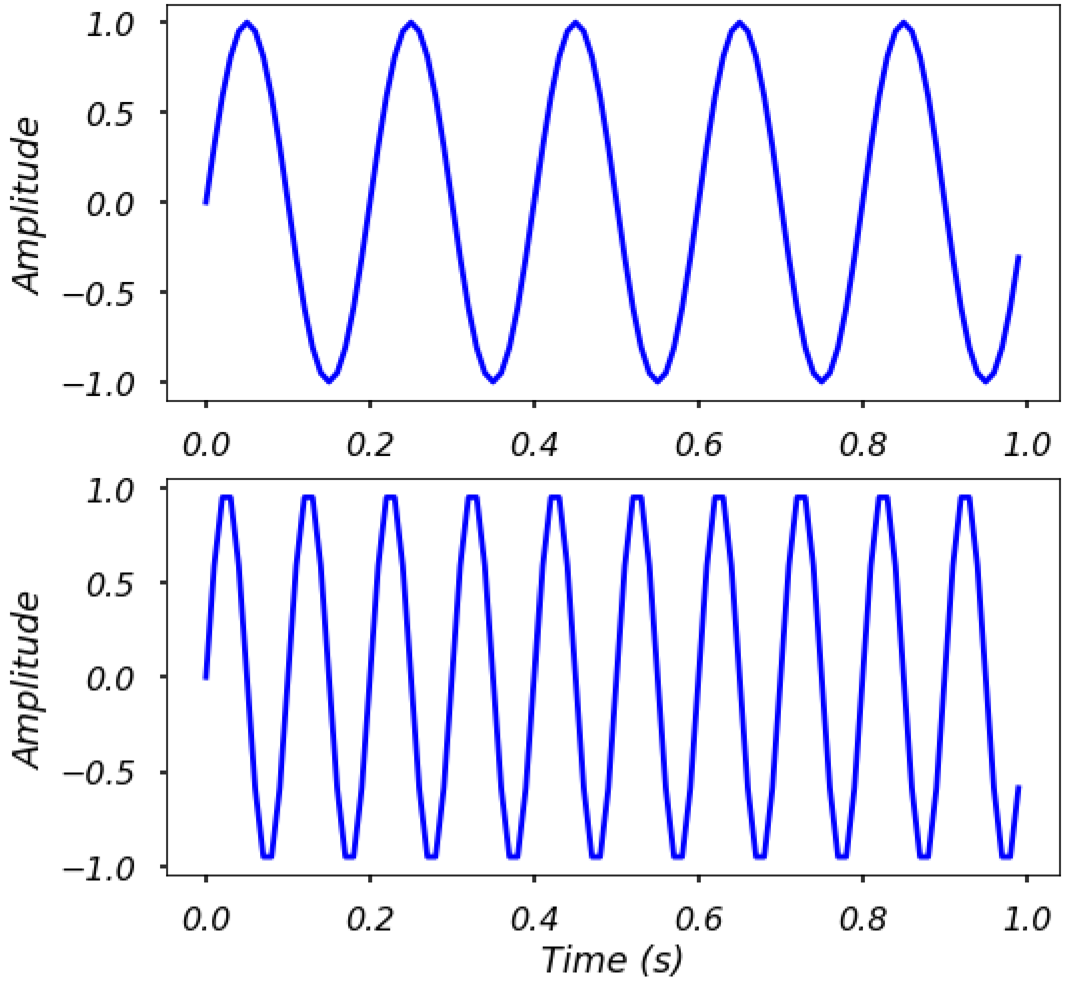
$$y(t) = A\sin(\omega t + \phi)$$

where A is the **amplitude** of the wave, ω is the **angular frequency**, which specifies how many cycles occur in a second, in **radians per second**. ϕ is the **phase** of the signal. If T is the **period** of the wave, and f is the **frequency** of the wave, then ω has the following relationship to them:

$$\omega = \frac{2\pi}{T} = 2\pi f$$

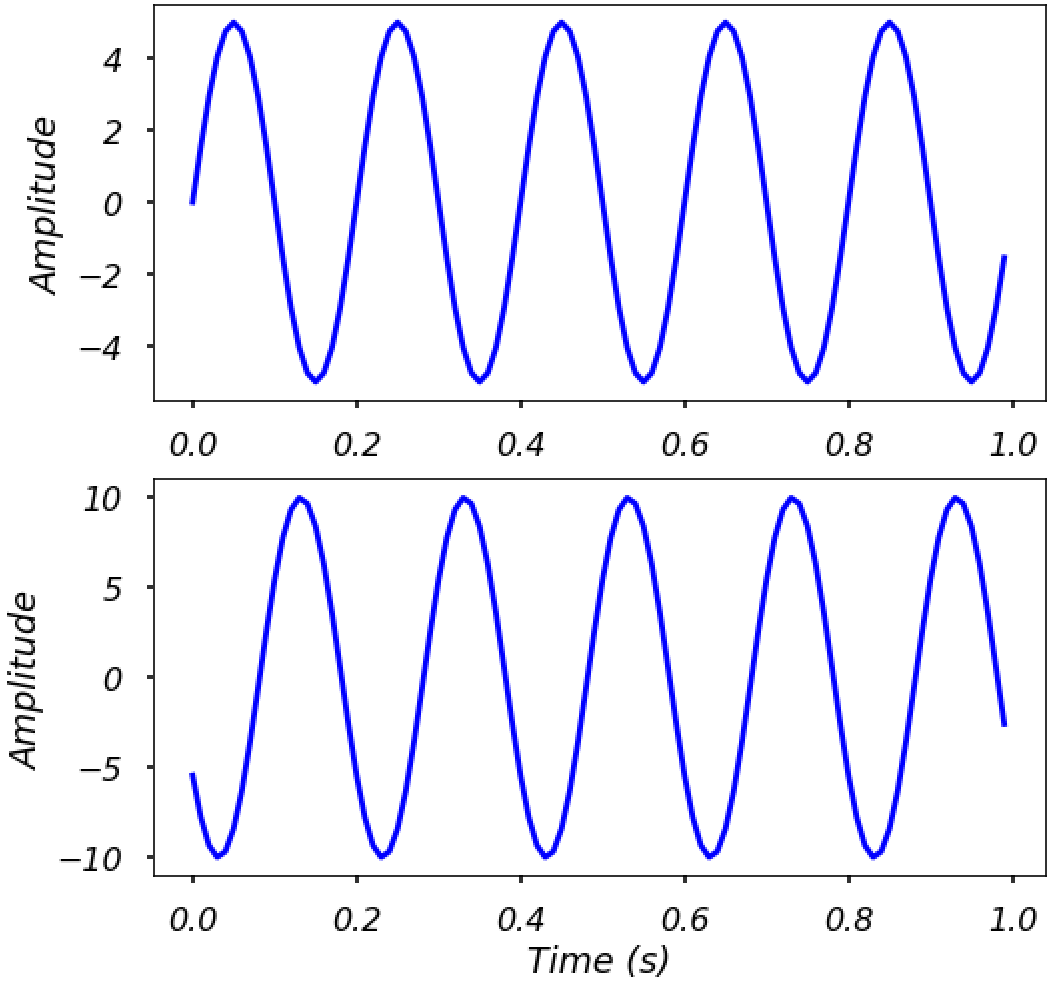
- **Example**: Generate two sine waves with time between 0 and 1 seconds and frequency is 5 Hz and 10 Hz, all sampled at 100 Hz. Plot the two waves and see the difference. Count how many cycles in the 1 second.

```
# sampling rate
sr = 100.0
# sampling interval
ts = 1.0/sr
t = np.arange(0,1,ts)

# frequency of the signal
freq = 5
y = np.sin(2*np.pi*freq*t)

plt.figure(figsize = (8, 8))
plt.subplot(211)
plt.plot(t, y, 'b')
plt.ylabel('Amplitude')

freq = 10
y = np.sin(2*np.pi*freq*t)

plt.subplot(212)
plt.plot(t, y, 'b')
plt.ylabel('Amplitude')

plt.xlabel('Time (s)')
plt.show()
```



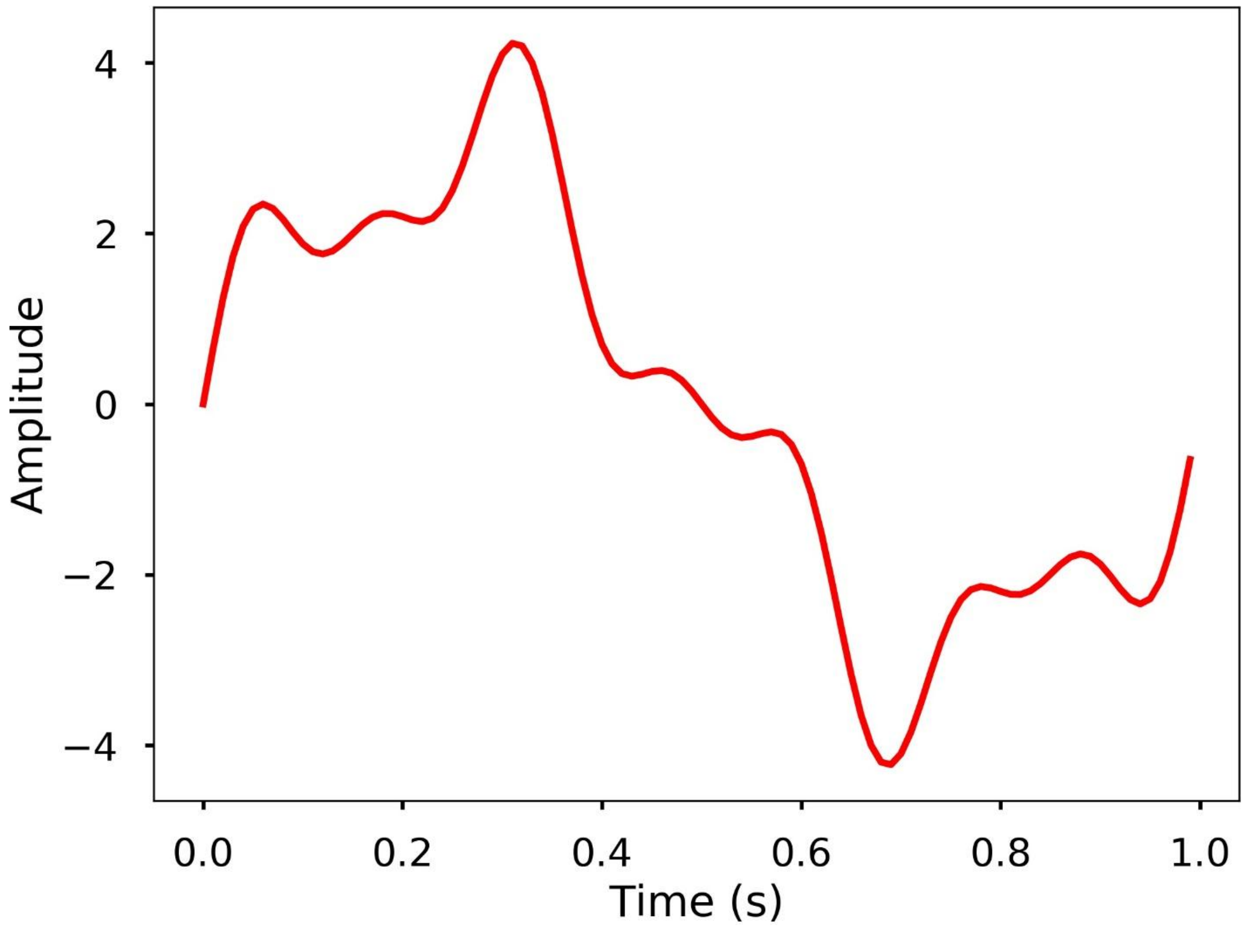
- **Example**: Generate two sine waves with time between 0 and 1 seconds. Both waves have frequency 5 Hz and sampled at 100 Hz, but the phase at 0 and 10, respectively. Also, the amplitude of the two waves are 5 and 10. Plot the two waves and see the difference.

```python
# frequency of the signal
freq = 5
y = 5*np.sin(2*np.pi*freq*t)

plt.figure(figsize = (8, 8))
plt.subplot(211)
plt.plot(t, y, 'b')
plt.ylabel('Amplitude')

y = 10*np.sin(2*np.pi*freq*t + 10)

plt.subplot(212)
plt.plot(t, y, 'b')
plt.ylabel('Amplitude')

plt.xlabel('Time (s)')
plt.show()
```



Discrete Fourier Transform (DFT)

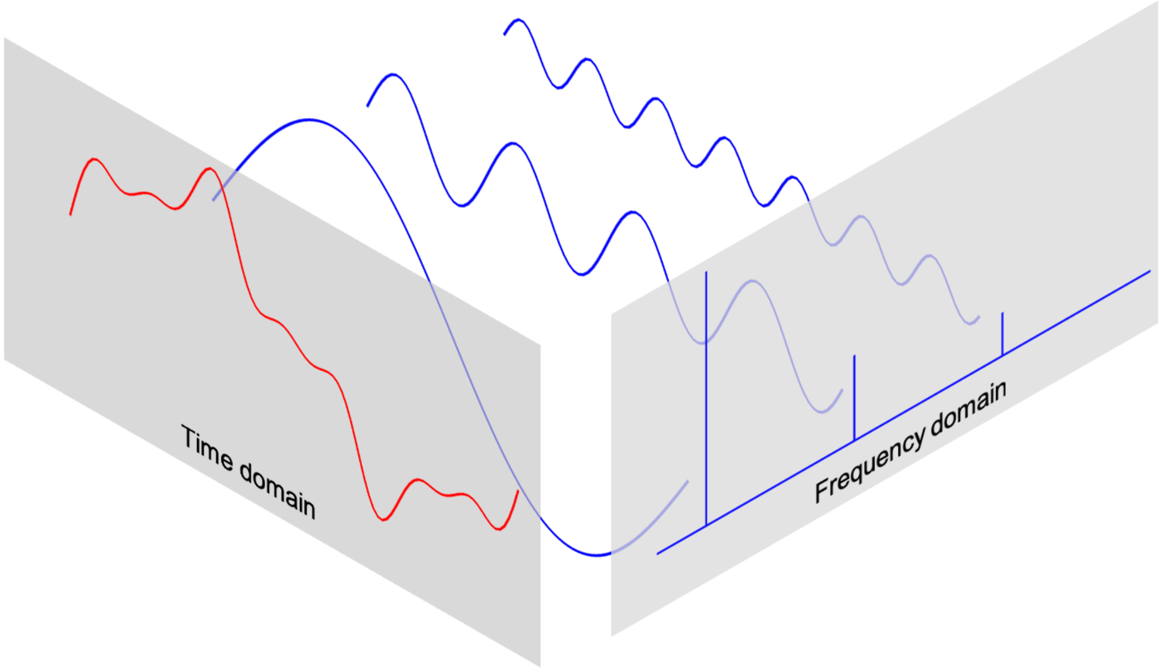
- From the previous section, we learned how we can easily **characterize** a **wave** with **period/frequency**, **amplitude**, **phase**.
- But these are easy for **simple periodic signal**, such as **sine** or **cosine** waves.
- For **complicated waves**, it is **not easy** to **characterize** like that.
- For example, the following is a relatively more complicate waves, and it is hard to say what's the frequency, amplitude of the wave, right?



Discrete Fourier Transform (DFT)

- There are more **complicated cases** in real world, it would be great if we have a method that we can use to **analyze** the **characteristics** of the wave.
- The **Fourier Transform** can be used for this purpose, which **decomposes** any signal into a **sum** of simple **sine** and **cosine waves** that we can easily measure the **frequency**, **amplitude**, and **phase**.
- The **Fourier transform** can be applied to **continuous** or **discrete** **waves.**
- In this section, we focus on the **Discrete Fourier Transform** (DFT).

- Using the **DFT**, we can compose the previous signal to a series of **sinusoids** and each of them will have a **different frequency**. The following 3D figure shows the idea behind the DFT, that the complicated signal is actually the results of the **sum** of **3 different sine waves**.
- The **time domain signal**, which is the previous signal we saw can be transformed into a figure in the **frequency domain** called **DFT amplitude spectrum**, where the signal **frequencies** are showing as **vertical bars**. The **height** of the bar after normalization is the **amplitude** of the signal in the **time domain**. You can see that the **3 vertical bars** are corresponding the **3 frequencies** of the **sine wave**, which are also plotted in the figure.



- In this section, we will learn how to use DFT to **compute** and **plot** the **DFT amplitude spectrum**.

- The **DFT** can **transform** a sequence of **evenly spaced signal** to the information about the **frequency** of all the **sine waves** that needed to be **sum** to the **time domain signal**. It is defined as:

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-i2\pi kn/N} = \sum_{n=0}^{N-1} x_n \left[\cos\left(\frac{2\pi kn}{N}\right) - i \cdot \sin\left(\frac{2\pi kn}{N}\right)\right]$$

  where

  - N = number of samples
  - n = current sample
  - k = current frequency, where $k \in [0, N-1]$
  - x_n = the sine value at sample n
  - X_k = the DFT which include information of both amplitude and phase
- Also, the last expression in the above equation derived from the **Euler's formula**, which links the **trigonometric functions** to the **complex exponential function**:

$$e^{i \cdot x} = \cos x + i \cdot \sin x$$

Discrete Fourier Transform (DFT)

- Due to the **nature** of the transform, $X_0 = \sum_{n=0}^{N-1} x_n$.
- If N is an **odd** number, the elements $X_1, X_2, \ldots, X_{(N-1)/2}$ contain the **positive frequency** terms and the elements $X_{(N+1)/2}, \ldots, X_{N-1}$ contain the **negative frequency** terms, in order of **decreasingly negative frequency**.
- While if N is **even**, the elements $X_1, X_2, \ldots, X_{N/2-1}$ contain the **positive frequency** terms, and the elements $X_{N/2}, \ldots, X_{N-1}$ contain the **negative frequency** terms, in order of **decreasingly negative frequency**.
- In the case that our input signal x is a **real-valued** sequence, the DFT output X_n for **positive frequencies** is the **conjugate** of the values X_n for **negative frequencies**, the **spectrum** will be **symmetric**. Therefore, usually we only plot the DFT corresponding to the **positive frequencies**.

Discrete Fourier Transform (DFT)

- Note that the X_k is a **complex number** that encodes both the **amplitude** and **phase** information of a **complex sinusoidal component** $e^{i \cdot 2\pi kn/N}$ of function x_n.
- The **amplitude** and **phase** of the signal can be calculated as:

$$amp = \frac{|X_k|}{N} = \frac{\sqrt{Re(X_k)^2 + Im(X_k)^2}}{N}$$

$$phase = atan2\left(Im(X_k), Re(X_k)\right)$$

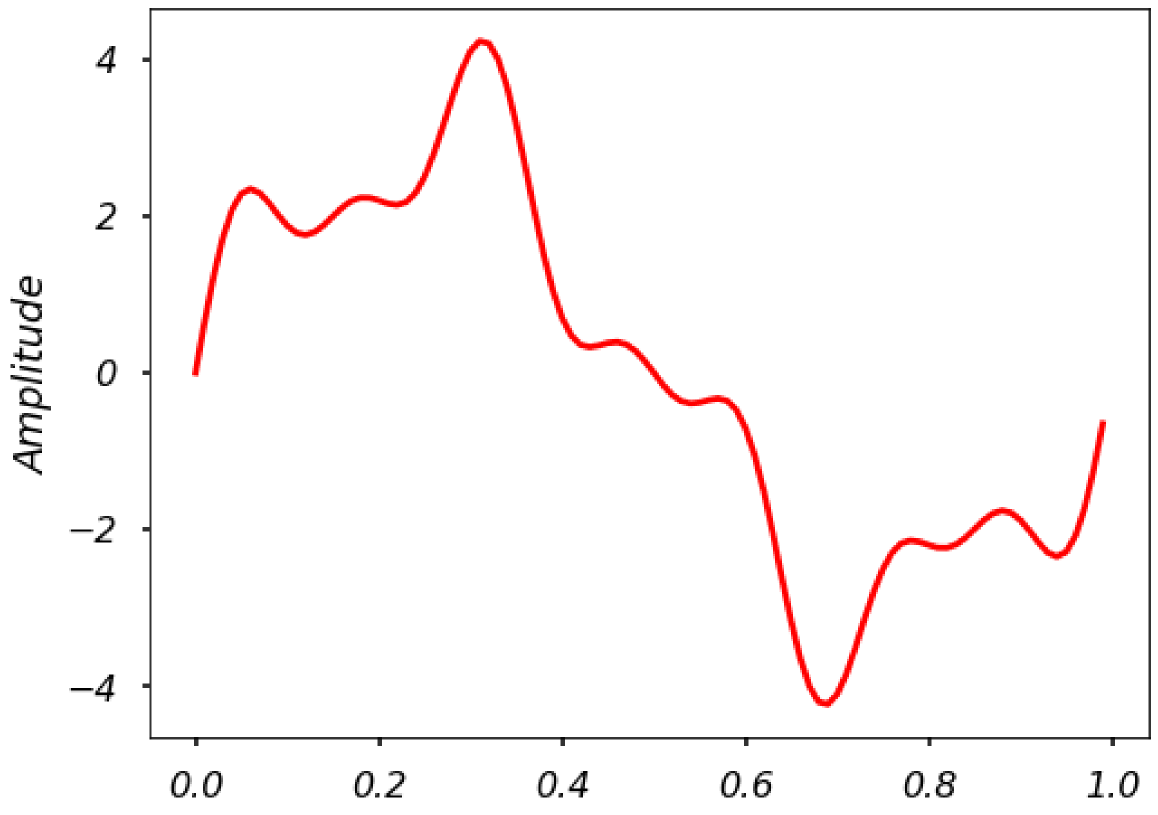
where $Im(X_k)$ and $Re(X_k)$ are the **imagery** and **real** part of the complex number, $atan2$ is the two-argument form of the **arctan** function.

Discrete Fourier Transform (DFT)

- The **amplitudes** returned by **DFT equal** to the **amplitudes** of the **signals** fed into the DFT if we **normalize** it by the **number of sample points**.
- Note that doing this will **divide** the **power** between the **positive** and **negative** sides.
- If the input signal is **real-valued** sequence as we described previously, the spectrum of the **positive** and **negative frequencies** will be **symmetric**, therefore, we will only look at **one side** of the **DFT result**, and instead of divide by N, we divide by $N/2$ to get the amplitude corresponding to the time domain signal.
- Now that we have the basic knowledge of DFT, let's see how we can use it.

- **Example**: Generate 3 sine waves with frequencies 1 Hz, 4 Hz, and 7 Hz, amplitudes 3, 1, and 0.5, and phase all zeros. Add this 3 sine waves together with a sampling rate 100 Hz, you will see that it is the same signal we just shown at the beginning of the section.

```
# sampling rate
sr = 100
# sampling interval
ts = 1.0/sr
t = np.arange(0,1,ts)

freq = 1.
x = 3*np.sin(2*np.pi*freq*t)

freq = 4
x += np.sin(2*np.pi*freq*t)

freq = 7
x += 0.5* np.sin(2*np.pi*freq*t)

plt.figure(figsize = (8, 6))
plt.plot(t, x, 'r')
plt.ylabel('Amplitude')

plt.show()
```

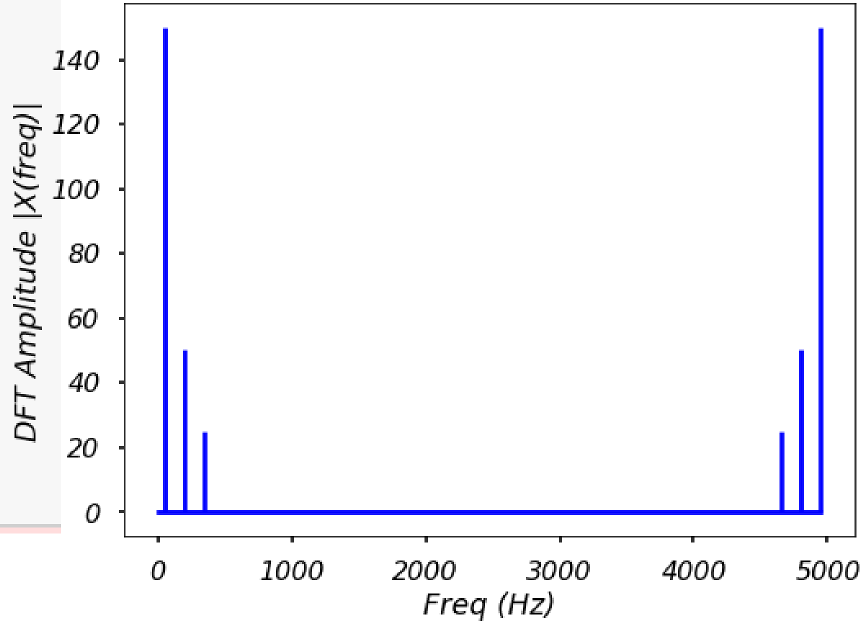


Discrete Fourier Transform (DFT)

- **Example**: Write a function **DFT(x)** which takes in one argument, x - the input 1 dimensional real-valued signal. The function will calculate the DFT of the signal and return the DFT values. Apply this function to the signal we generated before and plot the result.

```python
def DFT(x):
    """
    Function to calculate the
    discrete Fourier Transform
    of a 1D real-valued signal x
    """

    N = len(x)
    n = np.arange(N)
    k = n.reshape((N, 1))
    e = np.exp(-2j * np.pi * k * n / N)

    X = np.dot(e, x)

    return X
```

```python
X = DFT(x)

# calculate the frequency
N = len(X)
n = np.arange(N)
T = N/sr
freq = n/T

plt.figure(figsize = (8, 6))
plt.stem(freq, abs(X), 'b', \
         markerfmt=" ", basefmt="-b")
plt.xlabel('Freq (Hz)')
plt.ylabel('DFT Amplitude |X(freq)|')
plt.show()
```

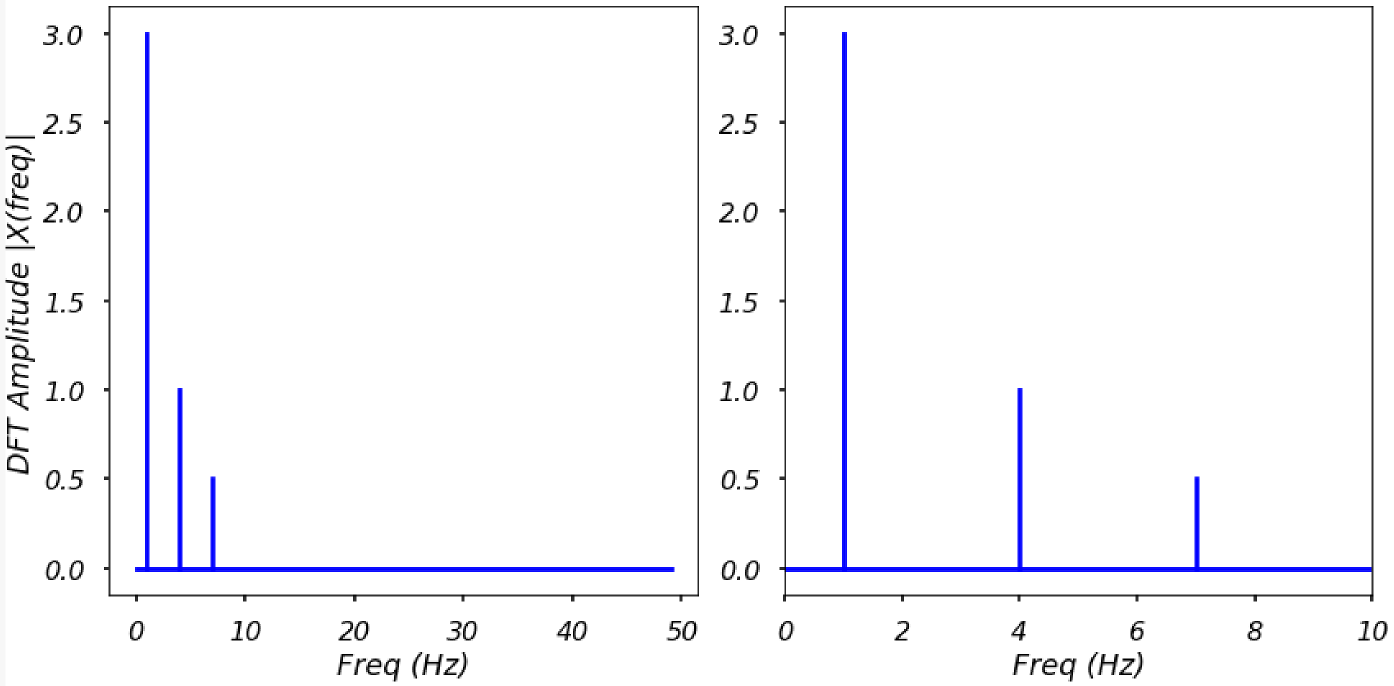


Discrete Fourier Transform (DFT)

- We can see from here that the output of the DFT is **symmetric** at **half** of the **sampling rate** (you can try different sampling rate to test).
- This half of the sampling rate is called **Nyquist** **frequency** or the **folding** **frequency**, it is named after the electronic engineer, **Harry Nyquist**.
- He and **Claude Shannon** have introduced the **Nyquist-Shannon** **sampling theorem**, which states that a **signal sampled at a rate** can be **fully reconstructed** if it contains only **frequency components below half** that sampling frequency, thus the **highest frequency output** from the DFT is **half** the **sampling rate**.

Discrete Fourier Transform (DFT)

```
n_oneside = N//2
# get the one side frequency
f_oneside = freq[:n_oneside]

# normalize the amplitude
X_oneside =X[:n_oneside]/n_oneside

plt.figure(figsize = (12, 6))
plt.subplot(121)
plt.stem(f_oneside, abs(X_oneside), 'b', \
         markerfmt=" ", basefmt="-b")
plt.xlabel('Freq (Hz)')
plt.ylabel('DFT Amplitude |X(freq)|')

plt.subplot(122)
plt.stem(f_oneside, abs(X_oneside), 'b', \
         markerfmt=" ", basefmt="-b")
plt.xlabel('Freq (Hz)')
plt.xlim(0, 10)
plt.tight_layout()
plt.show()
```



- We can see by plotting the first half of the DFT results, 3 clear **peaks** at frequency **1 Hz, 4 Hz**, and **7 Hz**, with amplitude **3, 1, 0.5** as expected.
- This is how we can use the DFT to analyze an **arbitrary signal** by decomposing it to **simple sine waves**.

The Inverse DFT

- Of course, we can do the **inverse transform** of the DFT easily.

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k \cdot e^{i \cdot 2\pi kn/N}$$

- **Try It**: We implement the DFT previously, can you implement the **inverse Discrete Fourier Transform** in Python similarly?

The Limit of DFT

- The main issue with the previous DFT implementation is that it is **not efficient** if we have a **signal** with **many data points**. It may take a **long time** to compute the DFT if the signal is **large**.
- **Example**: Write a function to generate a simple signal with different sampling rate, and see the difference of the computing time by varying the sampling rate.

```python
def gen_sig(sr):
    '''
    function to generate
    a simple 1D signal with
    different sampling rate
    '''
    ts = 1.0/sr
    t = np.arange(0,1,ts)

    freq = 1.
    x = 3*np.sin(2*np.pi*freq*t)
    return x
```

```python
# sampling rate =2000
sr = 2000
%timeit DFT(gen_sig(sr))
```

```
710 ms ± 75.8 ms per loop (mean ± std. dev. of 7 runs, 1 loop each)
```

```python
# sampling rate 5000
sr = 5000
%timeit DFT(gen_sig(sr))
```

```
4.2 s ± 249 ms per loop (mean ± std. dev. of 7 runs, 1 loop each)
```

The Limit of DFT

- We can see that, with the number of **data points increasing**, we can get a lot of **computation time** with this DFT.
- Luckily, the **Fast Fourier Transform** (FFT) was popularized by **Cooley** and **Tukey** in their 1965 paper that solve this problem efficiently, which will be the topic for the next section.

Fast Fourier Transform (FFT)

- The **Fast Fourier Transform** (FFT) is an **efficient algorithm** to calculate the **DFT** of a sequence. It is described first in **Cooley** and **Tukey's** classic paper in 1965, but the idea actually can be traced back to **Gauss**'s unpublished work in 1805.
- It is a **divide and conquer** algorithm that **recursively** breaks the DFT into **smaller** DFTs to bring down the computation. As a result, it successfully reduces the **complexity** of the DFT from $O(n^2)$ to $O(n \log n)$, where n is the size of the data.
- This reduction in computation time is **significant** especially for data with **large** N, therefore, making FFT widely been used in engineering, science, and mathematics. The FFT algorithm is the **Top 10 algorithm** of 20th century by the journal **Computing in Science & Engineering**.
- In this section, we will introduce you how does the FFT reduce the computation time?
- The content of this section is heavily based on this great tutorial put together by **Jake VanderPlas**.

Symmetries in the DFT

- The answer to how FFT **speed-up** the computing of DFT lies in the **exploitation** of the **symmetries** in the **DFT**.
- Let's take a look of the **symmetries** in the DFT.
- From the definition of the DFT equation

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-i2\pi kn/N}$$

- We can calculate the

$$X_{k+N} = \sum_{n=0}^{N-1} x_n \cdot e^{-i2\pi(k+N)n/N} = \sum_{n=0}^{N-1} x_n \cdot e^{-i2\pi n} \cdot e^{-i2\pi kn/N}$$

Symmetries in the DFT

- Note that, $e^{-i2\pi n} = 1$, therefore, we have

$$X_{k+N} = \sum_{n=0}^{N-1} x_n \cdot e^{-i2\pi kn/N} = X_k$$

- With a little extension, we can have

$$X_{k+i\cdot N} = X_k, \qquad for\ any\ integer\ i$$

- This means that within the DFT, we clearly have **some** **symmetries** that we can use to **reduce** the **computation**.

Tricks in FFT

- Since we know there are symmetries in the DFT, we can consider to use it to **reduce** the computation, because if we need to calculate both X_k and X_{k+N}, we only need to do this once.
- This is exactly the idea behind the FFT. **Cooley** and **Tukey** showed that we can calculate DFT more **efficiently** if we continue to divide the problem into **smaller** ones.
- Let's first divide the whole series into **two parts**, i.e. the **even number part** and the **odd number part**:

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-i2\pi kn/N} = \sum_{m=0}^{N/2-1} x_{2m} \cdot e^{-i2\pi k(2m)/N} + \sum_{m=0}^{N/2-1} x_{2m+1} \cdot e^{-i2\pi k(2m+1)/N}$$

$$= \sum_{m=0}^{N/2-1} x_{2m} \cdot e^{-i2\pi km/(\frac{N}{2})} + e^{-i2\pi k/N} \sum_{m=0}^{N/2-1} x_{2m+1} \cdot e^{-i2\pi km/(\frac{N}{2})}$$

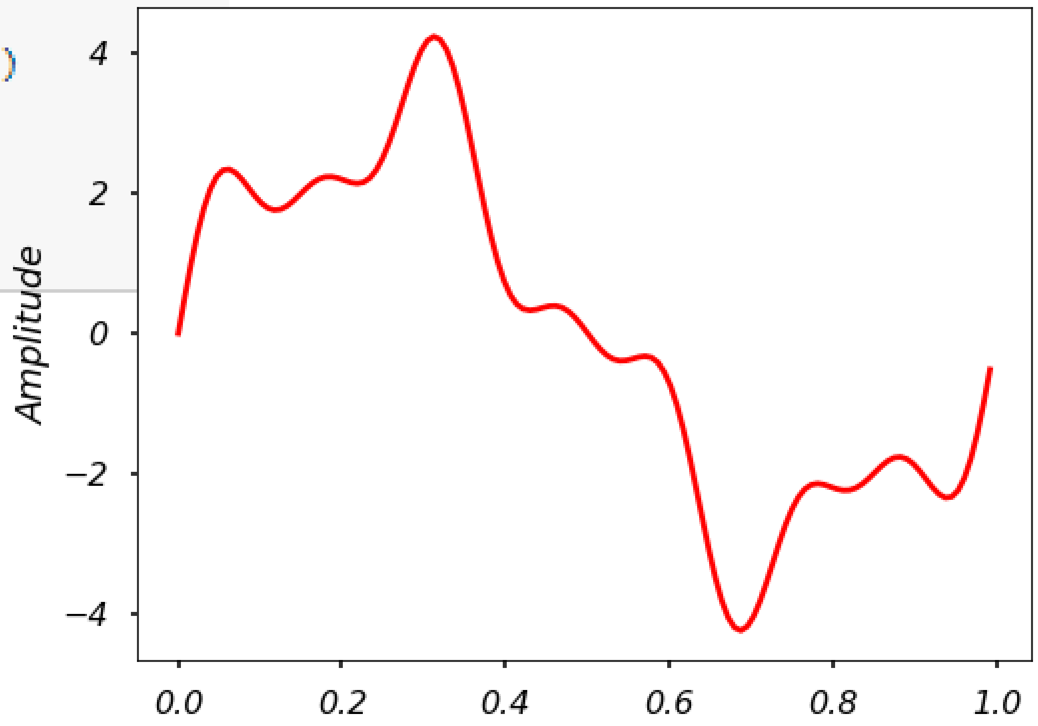
Tricks in FFT

- We can see that, the **two smaller terms** which only have **half of the size** ($N/2$) in the previous equation are **two smaller DFTs**. For each term, the $0 \leq m \leq N/2$, but $0 \leq k \leq N$, therefore, we can see that half of the values will be the **same** due to the **symmetry** properties we described above. Thus, we only need to **calculate half** of the **fields** in each term.
- Of course, we don't need to stop here, we can continue to **divide each term into half** with the **even** and **odd values** until it reaches the last two numbers, then the calculation will be really simple.
- This is how FFT works using this **recursive** approach.
- Let's see a quick and dirty implementation of the FFT. Note that, the **input signal** to FFT should have a **length of power of 2**. If the length is not, usually we need to fill up **zeros** to the **next power of 2** size.

```python
import matplotlib.pyplot as plt
import numpy as np

plt.style.use('seaborn-poster')
%matplotlib inline
```

```python
def FFT(x):
    """
    A recursive implementation of
    the 1D Cooley-Tukey FFT, the
    input should have a length of
    power of 2.
    """
    N = len(x)

    if N == 1:
        return x
    else:
        X_even = FFT(x[::2])
        X_odd = FFT(x[1::2])
        factor = \
          np.exp(-2j*np.pi*np.arange(N)/ N)

        X = np.concatenate(\
            [X_even+factor[:int(N/2)]*X_odd,
             X_even+factor[int(N/2):]*X_odd])
        return X
```

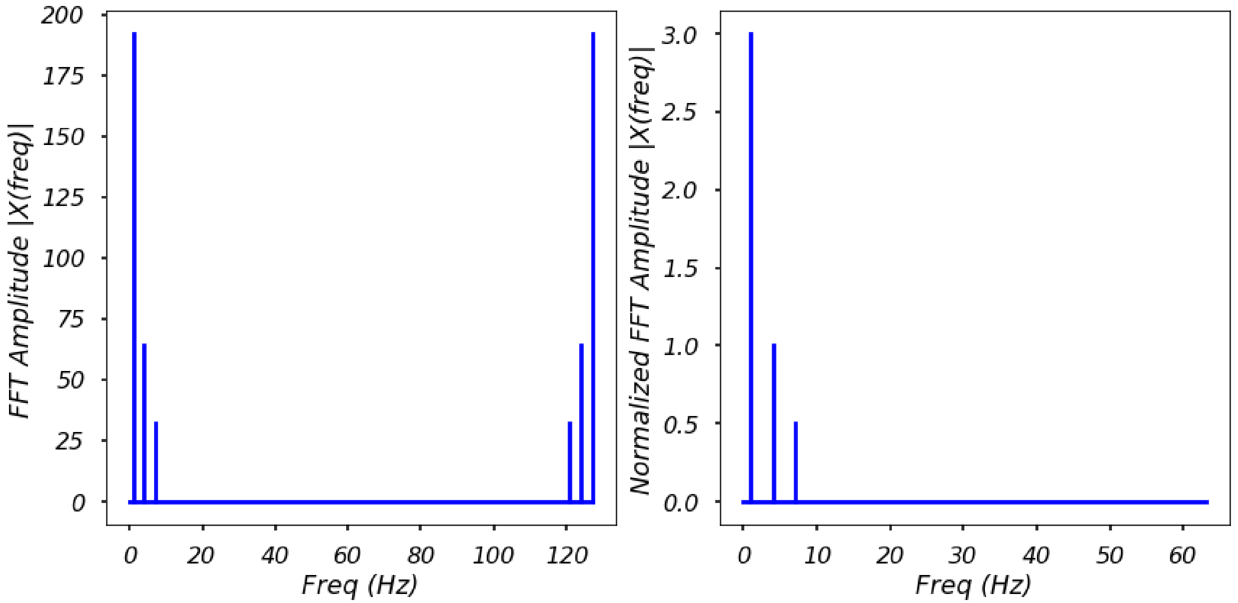
```python
# sampling rate
sr = 128
# sampling interval
ts = 1.0/sr
t = np.arange(0,1,ts)

freq = 1.
x = 3*np.sin(2*np.pi*freq*t)

freq = 4
x += np.sin(2*np.pi*freq*t)

freq = 7
x += 0.5* np.sin(2*np.pi*freq*t)

plt.figure(figsize = (8, 6))
plt.plot(t, x, 'r')
plt.ylabel('Amplitude')

plt.show()
```



- **Example**: Use the FFT function to calculate the Fourier transform of the previous example signal. Plot the amplitude spectrum for both the two-sided and one-side frequencies.

```
X=FFT(x)

# calculate the frequency
N = len(X)
n = np.arange(N)
T = N/sr
freq = n/T

plt.figure(figsize = (12, 6))
plt.subplot(121)
plt.stem(freq, abs(X), 'b', \
         markerfmt=" ", basefmt="-b")
plt.xlabel('Freq (Hz)')
plt.ylabel('FFT Amplitude |X(freq)|')

# Get the one-sided specturm
n_oneside = N//2
# get the one side frequency
f_oneside = freq[:n_oneside]

# normalize the amplitude
X_oneside =X[:n_oneside]/n_oneside
```

```
plt.subplot(122)
plt.stem(f_oneside, abs(X_oneside), 'b', \
         markerfmt=" ", basefmt="-b")
plt.xlabel('Freq (Hz)')
plt.ylabel('Normalized FFT Amplitude |X(freq)|')
plt.tight_layout()
plt.show()
```



Tricks in FFT

- **Example**: Generate a simple signal for length 2048, and time how long it will run the FFT and compare the speed with the DFT.

```python
def gen_sig(sr):
    '''
    function to generate
    a simple 1D signal with
    different sampling rate
    '''
    ts = 1.0/sr
    t = np.arange(0,1,ts)

    freq = 1.
    x = 3*np.sin(2*np.pi*freq*t)
    return x
```

```python
# sampling rate =2048
sr = 2048
%timeit FFT(gen_sig(sr))
```

```
59.7 ms ± 3.07 ms per loop (mean ± std. dev. of 7 runs, 10 loops each)
```

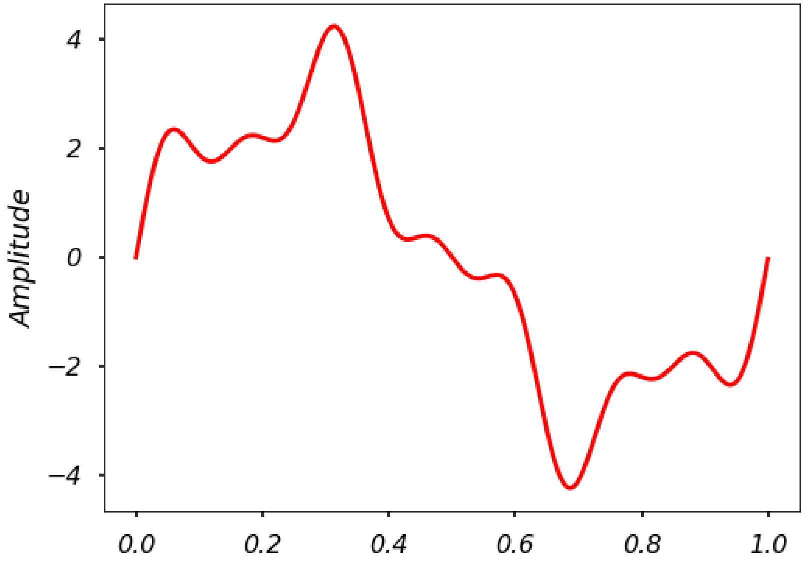
- We can see that, for a signal with length 2048 (about 2000), this implementation of **FFT** uses 59.7 ms instead of 710 ms using **DFT**. Note that, there are also a lot of ways to optimize the FFT implementation which will make it faster.
- In the next section, we will take a look of the **Python built-in FFT functions**, which will be much faster.

FFT in Python

- In Python, there are very mature **FFT functions**, both in **numpy** and **scipy**.
- In this section, we will take a look of **both packages** and see how we can easily use them in our work.
- Let's first generate the signal as before.

```
# sampling rate
sr = 2000
# sampling interval
ts = 1.0/sr
t = np.arange(0,1,ts)

freq = 1.
x = 3*np.sin(2*np.pi*freq*t)

freq = 4
x += np.sin(2*np.pi*freq*t)

freq = 7
x += 0.5* np.sin(2*np.pi*freq*t)

plt.figure(figsize = (8, 6))
plt.plot(t, x, 'r')
plt.ylabel('Amplitude')

plt.show()
```

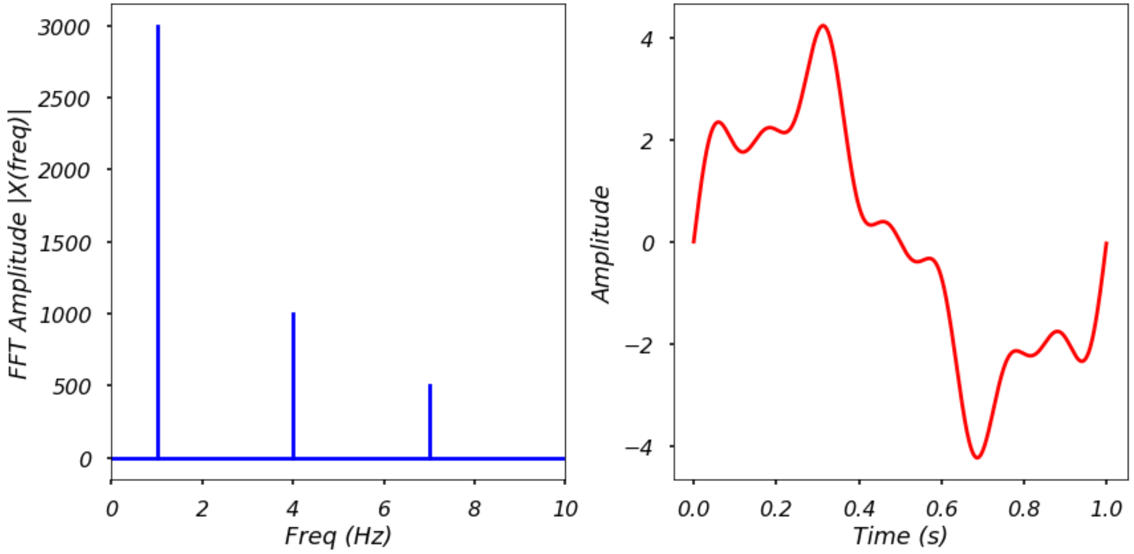
```
import matplotlib.pyplot as plt
import numpy as np

plt.style.use('seaborn-poster')
%matplotlib inline
```



FFT in Numpy

```
from numpy.fft import fft, ifft

X = fft(x)
N = len(X)
n = np.arange(N)
T = N/sr
freq = n/T

plt.figure(figsize = (12, 6))
plt.subplot(121)

plt.stem(freq, np.abs(X), 'b', \
         markerfmt=" ", basefmt="-b")
plt.xlabel('Freq (Hz)')
plt.ylabel('FFT Amplitude |X(freq)|')
plt.xlim(0, 10)

plt.subplot(122)
plt.plot(t, ifft(X), 'r')
plt.xlabel('Time (s)')
plt.ylabel('Amplitude')
plt.tight_layout()
plt.show()
```

- **Example**: Use **fft** and **ifft** function from **numpy** to calculate the FFT amplitude spectrum and **inverse** FFT to obtain the original signal. Plot both results. Time the **fft** function using this 2000 length signal.



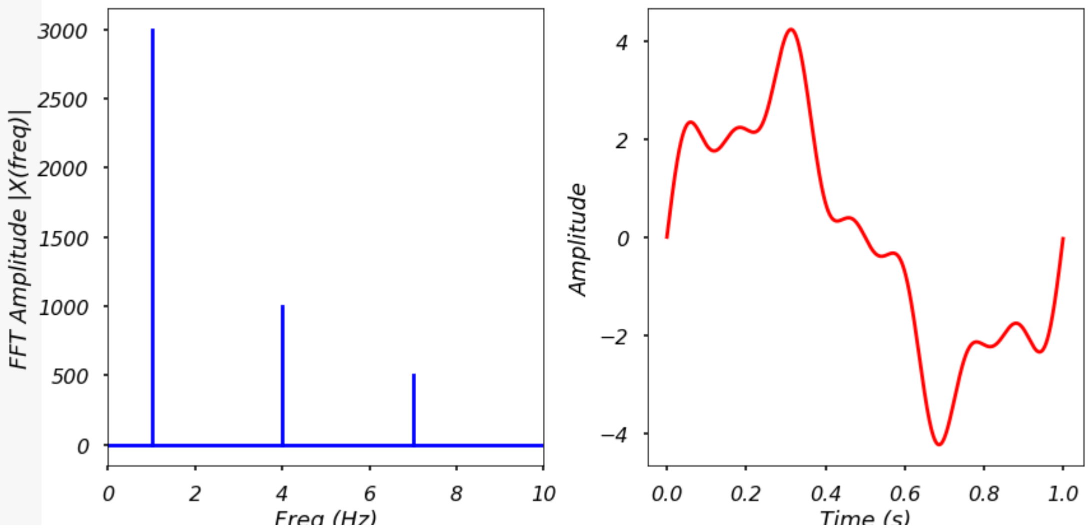

```
%timeit fft(x)
```

```
84.1 µs ± 9.29 µs per loop (mean ± std. dev. of 7 runs, 10000 loops each)
```

FFT in Scipy

```
from scipy.fftpack import fft, ifft

X = fft(x)

plt.figure(figsize = (12, 6))
plt.subplot(121)

plt.stem(freq, np.abs(X), 'b', \
         markerfmt=" ", basefmt="-b")
plt.xlabel('Freq (Hz)')
plt.ylabel('FFT Amplitude |X(freq)|')
plt.xlim(0, 10)

plt.subplot(122)
plt.plot(t, ifft(X), 'r')
plt.xlabel('Time (s)')
plt.ylabel('Amplitude')
plt.tight_layout()
plt.show()
```

- **Example**: Use **fft** and **ifft** function from **scipy** to calculate the FFT amplitude spectrum and **inverse** FFT to obtain the original signal. Plot both results. Time the fft function using this 2000 length signal.




```
%timeit fft(x)
```

```
47 µs ± 6.61 µs per loop (mean ± std. dev. of 7 runs, 10000 loops each)
```

Filtering a signal using FFT

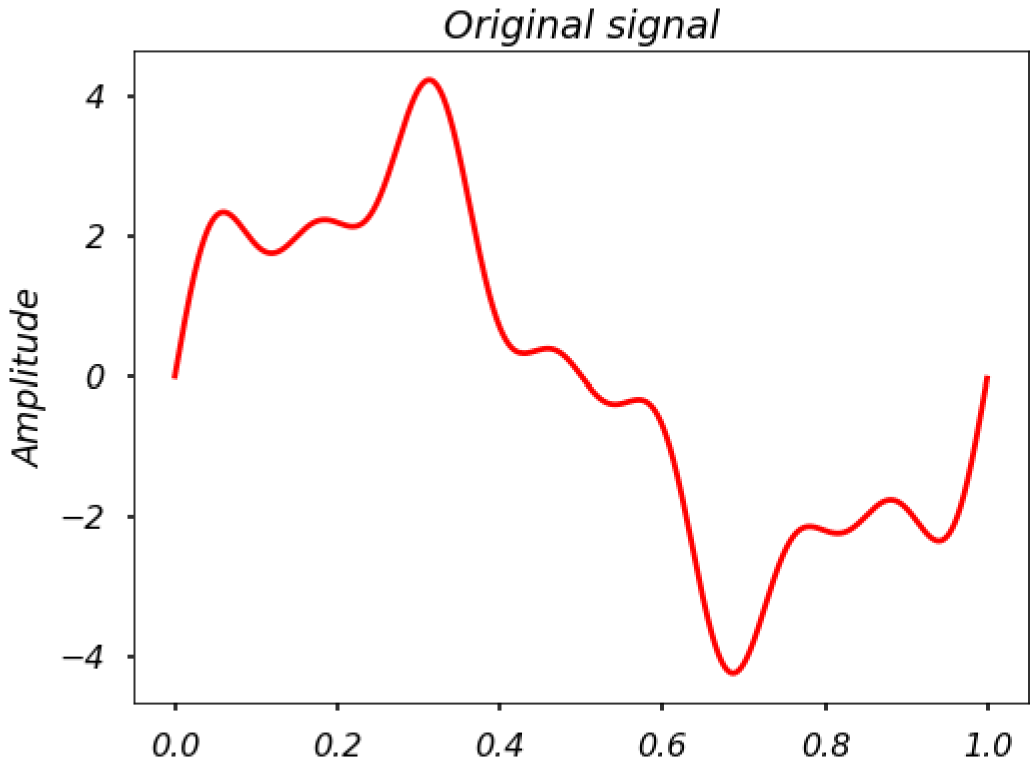
- **Filtering** is a process in **signal processing** to **remove** some **unwanted** part of the signal within **certain frequency** range.
- There are **low-pass filter**, which tries to **remove all** the **signal** above certain **cut-off** frequency, and **high-pass filter**, which does the opposite.
- Combining **low-pass** and **high-pass filter**, we will have **band-pass filter**, which means we only keep the signals within a **pair of frequencies**.
- Using FFT, we can easily do this. Let us play with the following example to illustrate the basics of **filtering**.
- **Note**: we just want to show the idea of filtering using very basic operations, in reality, the **filtering process** are much more **sophisticated**.

Filtering a signal using FFT

- **Example**: We can use the signal we generated at the beginning of this section (the mixed sine waves with 1, 4, and 7 Hz), and high-pass filter this signal at 6 Hz. Plot the filtered signal and the FFT amplitude before and after the filtering.

```
from scipy.fftpack import fftfreq
```

```
plt.figure(figsize = (8, 6))
plt.plot(t, x, 'r')
plt.ylabel('Amplitude')
plt.title('Original signal')
plt.show()
```



```python
# FFT the signal
sig_fft = fft(x)
# copy the FFT results
sig_fft_filtered = sig_fft.copy()

# obtain the frequencies using scipy function
freq = fftfreq(len(x), d=1./2000)

# define the cut-off frequency
cut_off = 6

# high-pass filter by assign zeros to the
# FFT amplitudes where the absolute
# frequencies smaller than the cut-off
sig_fft_filtered[np.abs(freq) < cut_off] = 0

# get the filtered signal in time domain
filtered = ifft(sig_fft_filtered)

# plot the filtered signal
plt.figure(figsize = (12, 6))
plt.plot(t, filtered)
plt.xlabel('Time (s)')
plt.ylabel('Amplitude')
plt.show()
```

```python
# plot the FFT amplitude before and after
plt.figure(figsize = (12, 6))
plt.subplot(121)
plt.stem(freq, np.abs(sig_fft), 'b', \
         markerfmt=" ", basefmt="-b")
plt.title('Before filtering')
plt.xlim(0, 10)
plt.xlabel('Frequency (Hz)')
plt.ylabel('FFT Amplitude')
plt.subplot(122)
plt.stem(freq, np.abs(sig_fft_filtered), 'b', \
         markerfmt=" ", basefmt="-b")
plt.title('After filtering')
plt.xlim(0, 10)
plt.xlabel('Frequency (Hz)')
plt.ylabel('FFT Amplitude')
plt.tight_layout()
plt.show()
```

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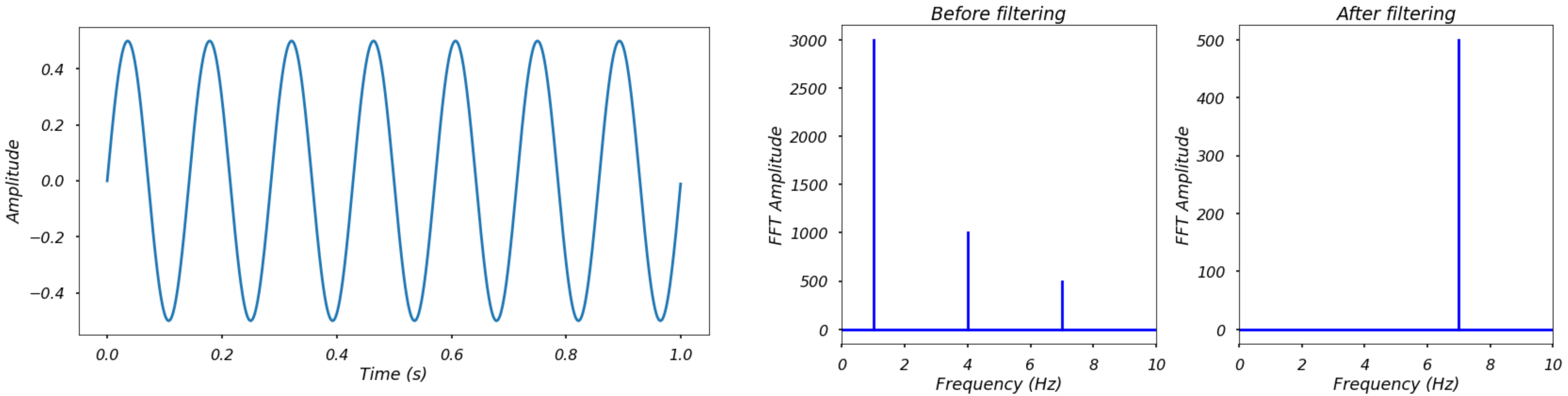
Filtering a signal using FFT

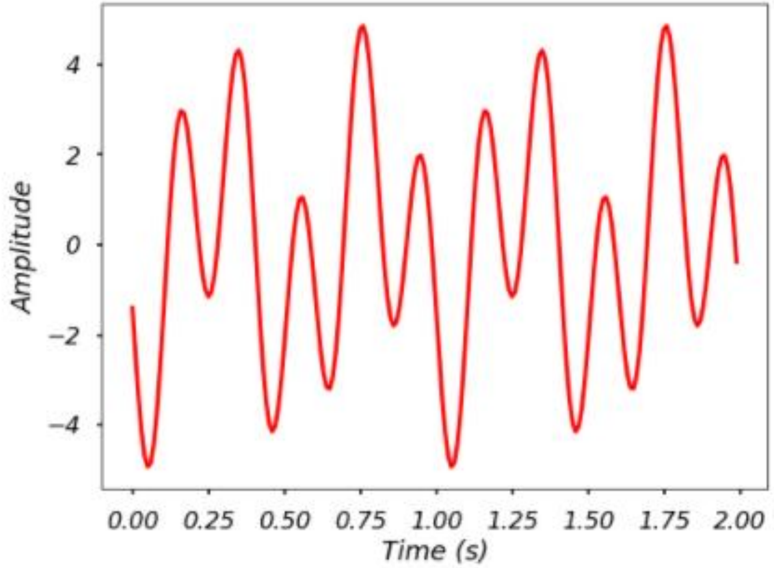


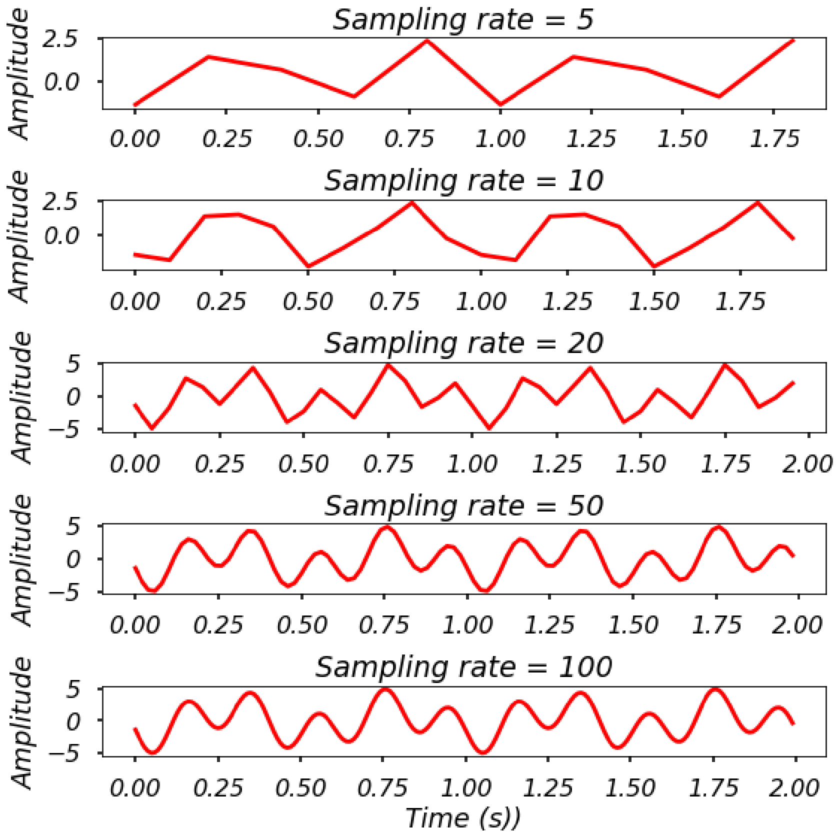
- From the above example, by assigning any absolute frequencies' FFT amplitude to **zero**, and returning back to **time domain signal**, we achieve a very basic **high-pass filter** in a few steps. You can try to implement a simple **low-pass** or **band-pass filter** by yourself.
- Therefore, FFT can help us get the signal we are **interested in** and **remove** the ones that are **unwanted**.

Practice

1. Generate two signals, signal 1 is a sine wave with 5 Hz, amplitude 3 and phase shift 3, signal 2 is a sine wave with 2 Hz, amplitude 2 and phase shift -2. Add this sine waves together with a sampling rate 100 Hz and plot the signal for 2 seconds.

2. Sample the signal you generated in problem 1 using a sampling rate 5, 10, 20, 50, and 100 Hz, and see the differences between different sampling rates.





Next Week's Outline

- Final-term Exam – Good Luck ^
- Don't forget to complete the SoloLearn Assignment (https://www.sololearn.com/learn/courses/python-intermediate) and submit the certificate of completion via eLearning UMN before the due date.
- Thank you.

References

- Kong, Qingkai; Siauw, Timmy, and Bayen, Alexandre. 2020. Python Programming and Numerical Methods: A Guide for Engineers and Scientists. Academic Press. https://www.elsevier.com/books/python-programming-and-numerical-methods/kong/978-0-12-819549-9
- Other online and offline references

Visi

Menjadi Program Studi Strata Satu Informatika **unggulan** yang menghasilkan lulusan **berwawasan internasional** yang **kompeten** di bidang Ilmu Komputer (*Computer Science*), **berjiwa wirausaha** dan **berbudi pekerti luhur**.

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