Report

3D Computer Vision

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1 Introduction

This report provides answers to theoretical questions in the first exercise and the result of the developed program.

2 Theory

1. Properties of Rotation Matrices

(a) Let us consider the rotation matrix around axis x

$$) = [\qquad (1)$$

Prove that rows and columns are orthogonal by proving the

$$\frac{\quad}{\quad} \qquad (2)$$

To calculate , calculate and using M:

$$A_M = (-1)^{i+j} M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\phi & -sin\phi \\ 0 & sin\phi & cos\phi \end{bmatrix}$$

$$R_x^{\ -1} = \frac{1}{\det(R_x)} A_M^T = 1 * \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\phi & sin\phi \\ 0 & -sin\phi & cos\phi \end{bmatrix}$$

$$R_x^{\ T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\phi & sin\phi \\ 0 & -sin\phi & cos\phi \end{bmatrix}$$

$R_x^{\ -1} = R_x^{\ T}$ from this follows that rows and columns of rotation matrix around x axis is orthogonal.

Prove that length of rows and columns are equal to 1.

For the 1st row, $\vec{a} = (1,0,0)$:

$\|\vec{a}\| = \sqrt{1^2 + 0^2 + 0^2} = \sqrt{1} = 1$

For the 2nd raw, $\vec{a} = (0, cos\phi, -sin\phi)$:

$\|\vec{a}\| = \sqrt{0^2 + cos^2\phi + (-sin\phi)^2} = \sqrt{cos^2\phi + sin^2\phi} = \sqrt{1} = 1$

For the 3rd raw, $\vec{a} = (0, sin\phi, cos\phi)$:

$\|\vec{a}\| = \sqrt{0^2 + sin^2\phi + cos^2\phi} = \sqrt{sin^2\phi + cos^2\phi} = \sqrt{1} = 1$

For the 1st column, $\vec{a} = (1,0,0)$:

$\|\vec{a}\| = \sqrt{1^2 + 0^2 + 0^2} = \sqrt{1} = 1$

For the 2nd column, $\vec{a} = (0, cos\phi, sin\phi)$:

$\|\vec{a}\| = \sqrt{0^2 + cos^2\phi + sin^2\phi} = \sqrt{cos^2\phi + sin^2\phi} = \sqrt{1} = 1$

For the 3rd column, $\vec{a} = (0, -sin\phi, cos\phi)$:

$\|\vec{a}\| = \sqrt{0^2 + (-sin\phi)^2 + cos^2\phi} = \sqrt{sin^2\phi + cos^2\phi} = \sqrt{1} = 1$

So length of each of raw and each column is equal to 1.

Since rows and columns of matrix R_x are orthogonal and their length is equal to 1, they are orthonormal.

Let us consider the rotation matrix around axis y R_y

$$R_y(\theta) = \begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{bmatrix} \qquad (3)$$

Prove that rows and columns are orthogonal:

$$\begin{aligned} det(R_y) &= cos\theta * (1 * cos\theta - 0) - 0 * (0 * cos\theta - 0 * (-sin\theta)) + sin\theta * (0 - (-sin\theta) * 1) = \\ &= cos\theta * cos\theta - 0 * 0 + sin\theta * sin\theta = cos^2\theta + sin^2\theta = 1 \end{aligned}$$

$$M = \begin{bmatrix} \begin{bmatrix} 1 & -0 \\ 0 & cos\theta \end{bmatrix} & \begin{bmatrix} 0 & -0 \\ -sin\theta & cos\theta \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ -sin\theta & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & sin\theta \\ 0 & cos\theta \end{bmatrix} & \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} & \begin{bmatrix} cos\theta & 0 \\ -sin\theta & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & sin\theta \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} cos\theta & sin\theta \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} cos\theta & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} cos\theta & 0-0 & sin\theta \\ 0-0 & cos^2\theta + sin^2\theta & 0+0 \\ -sin\theta & 0-0 & cos\theta \end{bmatrix} =$$

$$= \begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{bmatrix}$$

$$A_M = (-1)^{i+j} M = \begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{bmatrix}$$

$$R_y{}^{-1} = \frac{1}{\det{(R_y)}} A_M^T = 1 * \begin{bmatrix} cos\theta & 0 & -sin\theta \\ 0 & 1 & 0 \\ sin\theta & 0 & cos\theta \end{bmatrix}$$

$$R_y{}^T = \begin{bmatrix} cos\theta & 0 & -sin\theta \\ 0 & 1 & 0 \\ sin\theta & 0 & cos\theta \end{bmatrix}$$

As we can see $R_y{}^{-1} = R_y{}^T$ from this follows that rows and columns of rotation matrix around y axis is orthogonal.

Prove that length of rows and columns are equal to 1.

For the 1st row, $\vec{a} = (cos\theta, 0, sin\theta)$:

$||\vec{a}|| = \sqrt{cos^2\theta + 0^2 + sin^2\theta} = \sqrt{cos^2\theta + sin^2\theta} = \sqrt{1} = 1$

For the 2nd raw, $\vec{a} = (0,1,0)$:

$||\vec{a}|| = \sqrt{0^2 + 1^2 + 0^2} = \sqrt{1} = 1$

For the 3rd raw, $\vec{a} = (-sin\theta, 0, cos\theta)$:

$||\vec{a}|| = \sqrt{(-sin\theta)^2 + 0^2 + cos^2\theta} = \sqrt{sin^2\theta + cos^2\theta} = \sqrt{1} = 1$

For the 1st column, $\vec{a} = (cos\theta, 0, -sin\theta)$:

$||\vec{a}|| = \sqrt{cos^2\theta + 0^2 + (-sin\theta)^2} = \sqrt{cos^2\theta + sin^2\theta} = \sqrt{1} = 1$

For the 2nd column, $\vec{a} = (0,1,0)$:

$||\vec{a}|| = \sqrt{0^2 + 1^2 + 0^2} = \sqrt{1} = 1$

For the 3rd column, $\vec{a} = (sin\theta, 0, cos\theta)$:

$||\vec{a}|| = \sqrt{sin^2\theta + 0^2 + cos^2\theta} = \sqrt{sin^2\theta + cos^2\theta} = \sqrt{1} = 1$

So length of each of raw and each column is equal to 1.

Since rows and columns of matrix R_y are orthogonal and their length is equal to 1, they are orthonormal.

Let us consider the rotation matrix around axis z R_z

$$R_z(\psi) = \begin{bmatrix} cos\psi & -sin\psi & 0 \\ sin\psi & cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (4)$$

Prove that rows and columns are orthogonal:

$$det(R_z) = cos\psi * (cos\psi — 0) − (−sin\psi) * (sin\psi * 1 − 0 * 0) + 0 * (sin\psi * 0 − 0 * cos\psi)$$
$$= cos\psi * cos\psi − (−sin\psi * sin\psi) + 0 = cos^2\psi + sin^2\psi = 1$$

$$M = \begin{bmatrix} \begin{bmatrix} cos\psi & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} sin\psi & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} sin\psi & cos\psi \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} -sin\psi & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} cos\psi & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} cos\psi & -sin\psi \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} -sin\psi & 0 \\ cos\psi & 0 \end{bmatrix} & \begin{bmatrix} cos\psi & 0 \\ sin\psi & 0 \end{bmatrix} & \begin{bmatrix} cos\psi & -sin\psi \\ sin\psi & cos\psi \end{bmatrix} \end{bmatrix} = \begin{bmatrix} cos\psi & sin\psi & 0-0 \\ -sin\psi & cos\psi & 0+0 \\ -0-0 & 0-0 & cos^2\psi + sin^2\psi \end{bmatrix}$$
$$= \begin{bmatrix} cos\psi & sin\psi & 0 \\ -sin\psi & cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_M = (-1)^{i+j}M = \begin{bmatrix} cos\psi & -sin\psi & 0 \\ sin\psi & cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_z{}^{-1} = \frac{1}{\det(R_z)} A_M^T = 1 * \begin{bmatrix} cos\psi & sin\psi & 0 \\ -sin\psi & cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_z{}^T = \begin{bmatrix} cos\psi & sin\psi & 0 \\ -sin\psi & cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

As we can see $R_z{}^{-1} = R_z{}^T$ from this follows that rows and columns of rotation matrix around z axis is orthogonal.

Prove that length of rows and columns are equal to 1.

For the 1st row, $\vec{a} = (cos\psi, -sin\psi, 0)$:

$\|\vec{a}\| = \sqrt{cos^2\psi + (-sin\psi)^2 + 0^2} = \sqrt{cos^2\psi + sin^2\psi} = \sqrt{1} = 1$

For the 2nd raw, $\vec{a} = (sin\psi, cos\psi, 0)$:

$\|\vec{a}\| = \sqrt{sin^2\psi + cos^2\psi + 0^2} = \sqrt{sin^2\psi + cos^2\psi} = \sqrt{1} = 1$

For the 3rd raw, $\vec{a} = (0,0,1)$:

$\|\vec{a}\| = \sqrt{0^2 + 0^2 + 1^2} = \sqrt{1} = 1$

For the 1st column, $\vec{a} = (cos\psi, sin\psi, 0)$:

$\|\vec{a}\| = \sqrt{cos^2\psi + sin^2\psi + 0^2 +} = \sqrt{cos^2\psi + sin^2\psi} = \sqrt{1} = 1$

For the 2nd column, $\vec{a} = (-sin\psi, cos\psi, 0)$:

$\|\vec{a}\| = \sqrt{(-sin\psi)^2 + cos^2\psi + 0^2} = \sqrt{sin^2\psi + cos^2\psi} = \sqrt{1} = 1$

For the 3rd column, $\vec{a} = (0,0,1)$:

$\|\vec{a}\| = \sqrt{0^2 + 0^2 + 1^2} = \sqrt{1} = 1$

So length of each of raw and each column is equal to 1.

Since rows and columns of matrix R_z are orthogonal and their length is equal to 1, they are orthonormal.

(b)

We have Euler angles for rotation $\boldsymbol{R} = \boldsymbol{RzRyRx}$ (5):

Rotation by ψ around z axis: $\boldsymbol{Rz}$ (4)

Rotation by θ around y axis: $\boldsymbol{Ry}$ (3)

Rotation by ϕ around x axis: $\boldsymbol{Rx}$ (1)

Therefore, let's consider rotation matrices Rz(ψ), Ry(θ) and Rx(φ) and their properties:

$$\mathbf{R}^{-1} = \mathbf{R}^{T} \quad \text{and} \quad \mathbf{det(R) = 1} \qquad (6)$$

We should prove that property $R^{-1} = R^T$ (6) also hold true for R = Rz(ψ)Ry(θ)Rx(φ) as the consecutive execution of these matrices.

1. Firstly, we should multiply rotation matrices Rz(ψ), Ry(θ) and Rx(φ):

$$\boldsymbol{R} = Rz(\psi)\ Ry(\theta)\ Rx(\phi) = \begin{bmatrix} cos\psi & -sin\psi & 0 \\ sin\psi & cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\phi & -sin\phi \\ 0 & sin\phi & cos\phi \end{bmatrix} =$$

$$\begin{bmatrix} cos\psi cos\theta & -sin\psi & cos\psi sin\theta \\ sin\psi cos\theta & cos\psi & sin\psi sin\theta \\ -sin\theta & 0 & cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\phi & -sin\phi \\ 0 & sin\phi & cos\phi \end{bmatrix} =$$

$$\begin{bmatrix} cos\psi cos\theta & -sin\psi cos\phi + cos\psi sin\theta sin\phi & -sin\psi(-sin\phi) + cos\psi sin\theta cos\phi \\ sin\psi cos\theta & cos\psi cos\phi + sin\psi sin\theta sin\phi & cos\psi(-sin\phi) + sin\psi sin\theta cos\phi \\ -sin\theta & cos\theta sin\phi & cos\theta cos\phi \end{bmatrix}$$

The resulting matrix $\mathbf{R} = \begin{bmatrix} cos\psi cos\theta & -sin\psi cos\phi + cos\psi sin\theta sin\phi & -sin\psi(-sin\phi) + cos\psi sin\theta cos\phi \\ sin\psi cos\theta & cos\psi cos\phi + sin\psi sin\theta sin\phi & cos\psi(-sin\phi) + sin\psi sin\theta cos\phi \\ -sin\theta & cos\theta sin\phi & cos\theta cos\phi \end{bmatrix}$

2. Secondly, we need to get a transposed matrix $\mathbf{R}^{T}$**:**

$$\mathbf{R}^{T} = \begin{bmatrix} cos\psi cos\theta & sin\psi cos\theta & -sin\theta \\ -sin\psi cos\phi + cos\psi sin\theta sin\phi & cos\psi cos\phi + sin\psi sin\theta sin\phi & cos\theta sin\phi \\ -sin\psi(-sin\phi) + cos\psi sin\theta cos\phi & cos\psi(-sin\phi) + sin\psi sin\theta cos\phi & cos\theta cos\phi \end{bmatrix}$$

3. The next step is to get inverse matrix $\mathbf{R}^{-1}$ due to the following formula (2)

3.1. The property det(R) = 1 (6)

$$det(R) = \sum_{j=1}^{n}(-1)^{1+j} a_{1j}\bar{M}_j^1 \qquad (7)$$

Dimension of our matrix is 3x3 and this formula will look like:

$$det(R) = \sum_{j=1}^{n}(-1)^{1+j} a_{1j}\bar{M}_j^1 = a_{11}\bar{M}_1^1 + (-1)a_{12}\bar{M}_2^1 + a_{13}\bar{M}_3^1 = a_{11}\bar{M}_1^1 - a_{12}\bar{M}_2^1 + a_{13}\bar{M}_3^1 =$$

$$\underbrace{cos\psi cos\theta \begin{bmatrix} cos\psi cos\phi + sin\psi sin\theta sin\phi & sin\psi sin\theta cos\phi - cos\psi sin\phi \\ cos\theta sin\phi & cos\theta cos\phi \end{bmatrix}}_{1st\ augend} -$$

$$\underbrace{(\ cos\psi sin\theta sin\phi - sin\psi cos\phi) \begin{bmatrix} sin\psi cos\theta & sin\psi sin\theta cos\phi - cos\psi sin\phi \\ -sin\theta & cos\theta cos\phi \end{bmatrix}}_{2nd\ augend}$$

$$+\underbrace{(sin\psi sin\phi + cos\psi sin\theta cos\phi)\begin{bmatrix} sin\psi cos\theta & cos\psi cos\phi + sin\psi sin\theta sin\phi \\ -sin\theta & cos\theta sin\phi \end{bmatrix}}_{3rd\ augend}$$

To make calculations easier and more visual, we will calculate each of the augends separately.
Calculation of the 1st augend:

$$cos\psi cos\theta \begin{bmatrix} cos\psi cos\phi + sin\psi sin\theta sin\phi & sin\psi sin\theta cos\phi - cos\psi sin\phi \\ cos\theta sin\phi & cos\theta cos\phi \end{bmatrix} =$$

$$cos\psi cos\theta * \big((cos\psi cos\phi + sin\psi sin\theta sin\phi)(cos\theta cos\phi) - cos\theta sin\phi(sin\psi sin\theta cos\phi - cos\psi sin\phi)\big) =$$

$$cos\psi cos\theta * \big((cos\theta cos\phi cos\psi cos\phi + cos\theta cos\phi sin\psi sin\theta sin\phi) \\ - (cos\theta sin\phi sin\psi sin\theta cos\phi - cos\theta sin\phi cos\psi sin\phi)\big) =$$

$$cos\psi cos\theta * (cos\theta cos\phi cos\psi cos\phi + cos\theta cos\phi sin\psi sin\theta sin\phi - cos\theta sin\phi sin\psi sin\theta cos\phi + cos\theta sin\phi cos\psi sin\phi) \\ =$$

$$cos\psi cos\theta * (cos\theta cos^2\phi cos\psi + cos\theta cos\phi sin\psi sin\theta sin\phi - cos\theta sin\phi sin\psi sin\theta cos\phi + cos\theta sin^2\phi cos\psi) =$$

$$cos\psi cos\theta cos\theta cos^2\phi cos\psi + cos\psi cos\theta cos\theta cos\phi sin\psi sin\theta sin\phi - cos\psi cos\theta cos\theta sin\phi sin\psi sin\theta cos\phi \\ + cos\psi cos\theta cos\theta sin^2\phi cos\psi =$$

$$cos^2\psi cos^2\theta cos^2\phi + cos\psi cos^2\theta cos\phi sin\psi sin\theta sin\phi - cos\psi cos^2\theta sin\phi sin\psi sin\theta cos\phi + cos^2\psi cos^2\theta sin^2\phi =$$

$$cos^2\psi cos^2\theta(cos^2\phi + sin^2\phi) + cos\psi cos^2\theta cos\phi sin\psi sin\theta sin\phi - cos\psi cos^2\theta sin\phi sin\psi sin\theta cos\phi =$$

$$cos^2\psi cos^2\theta(cos^2\phi + sin^2\phi) + cos\psi cos^2\theta cos\phi sin\psi sin\theta sin\phi - cos\psi cos^2\theta sin\phi sin\psi sin\theta cos\phi =$$

$$cos^2\psi cos^2\theta + cos\psi cos^2\theta cos\phi sin\psi sin\theta sin\phi - cos\psi cos^2\theta sin\phi sin\psi sin\theta cos\phi =$$

$$cos^2\psi cos^2\theta + cos\psi cos^2\theta cos\phi sin\psi sin\theta(sin\phi - sin\phi) = cos^2\psi cos^2\theta + cos\psi cos^2\theta cos\phi sin\psi sin\theta * 0 =$$

$$\underline{cos^2\psi cos^2\theta}$$

(8)

Calculation of the 2nd augend:

$$(\,cos\psi sin\theta sin\phi - sin\psi cos\phi)\begin{bmatrix} sin\psi cos\theta & sin\psi sin\theta cos\phi - cos\psi sin\phi \\ -sin\theta & cos\theta cos\phi \end{bmatrix} =$$

$$(\,cos\psi sin\theta sin\phi - sin\psi cos\phi)(sin\psi cos\theta cos\theta cos\phi - (-sin\theta)(sin\psi sin\theta cos\phi - cos\psi sin\phi)) =$$

$$(\,cos\psi sin\theta sin\phi - sin\psi cos\phi)(sin\psi cos\theta cos\theta cos\phi - (-sin\theta sin\psi sin\theta cos\phi + sin\theta cos\psi sin\phi)) =$$

$$(\,cos\psi sin\theta sin\phi - sin\psi cos\phi)(sin\psi cos\theta cos\theta cos\phi + sin\theta sin\psi sin\theta cos\phi - sin\theta cos\psi sin\phi)) =$$

$$cos\psi sin\theta sin\phi sin\psi cos\theta cos\theta cos\phi + cos\psi sin\theta sin\phi sin\theta sin\psi sin\theta cos\phi - cos\psi sin\theta sin\phi sin\theta cos\psi sin\phi \\ - sin\psi cos\phi sin\psi cos\theta cos\theta cos\phi - sin\psi cos\phi sin\theta sin\psi sin\theta cos\phi + sin\psi cos\phi sin\theta cos\psi sin\phi \\ = cos\psi sin\theta sin\phi sin\psi cos^2\theta cos\phi + cos\psi sin^3\theta sin\phi sin\psi cos\phi - cos^2\psi sin^2\theta sin^2\phi \\ - sin^2\psi cos^2\phi cos^2\theta - sin^2\psi cos^2\phi sin^2\theta + sin\psi cos\phi sin\theta cos\psi sin\phi =$$

$cos\psi sin\theta sin\phi sin\psi cos^2\theta cos\phi + cos\psi sin^3\theta sin\phi sin\psi cos\phi - cos^2\psi sin^2\theta sin^2\phi - sin^2\psi cos^2\phi(cos^2\theta + sin^2\theta) + sin\psi cos\phi sin\theta cos\psi sin\phi =$

$cos\psi sin\theta sin\phi sin\psi cos^2\theta cos\phi + cos\psi sin^3\theta sin\phi sin\psi cos\phi - cos^2\psi sin^2\theta sin^2\phi - sin^2\psi cos^2\phi + sin\psi cos\phi sin\theta cos\psi sin\phi =$

$cos\psi sin\phi sin\psi cos\phi sin\theta(cos^2\theta + sin^2\theta) - cos^2\psi sin^2\theta sin^2\phi - sin^2\psi cos^2\phi + sin\psi cos\phi sin\theta cos\psi sin\phi =$

$cos\psi sin\phi sin\psi cos\phi sin\theta - cos^2\psi sin^2\theta sin^2\phi - sin^2\psi cos^2\phi + sin\psi cos\phi sin\theta cos\psi sin\phi =$

$\underline{2cos\psi sin\phi sin\psi cos\phi sin\theta - cos^2\psi sin^2\theta sin^2\phi - sin^2\psi cos^2\phi \quad (9)}$

Calculation of the 3rd augend:

$(sin\psi sin\phi + cos\psi sin\theta cos\phi)\begin{bmatrix} sin\psi cos\theta & cos\psi cos\phi + sin\psi sin\theta sin\phi \\ -sin\theta & cos\theta sin\phi \end{bmatrix} =$

$(sin\psi sin\phi + cos\psi sin\theta cos\phi)\big(sin\psi cos\theta cos\theta sin\phi - (-sin\theta)(cos\psi cos\phi + sin\psi sin\theta sin\phi)\big) =$

$(sin\psi sin\phi + cos\psi sin\theta cos\phi)(sin\psi cos\theta cos\theta sin\phi - (-sin\theta cos\psi cos\phi - sin\theta sin\psi sin\theta sin\phi)) =$

$(sin\psi sin\phi + cos\psi sin\theta cos\phi)(sin\psi cos\theta cos\theta sin\phi + sin\theta cos\psi cos\phi + sin\theta sin\psi sin\theta sin\phi)=$

$sin\psi sin\phi sin\psi cos\theta cos\theta sin\phi + sin\psi sin\phi sin\theta cos\psi cos\phi + sin\psi sin\phi sin\theta sin\psi sin\theta sin\phi + cos\psi sin\theta cos\phi sin\psi cos\theta cos\theta sin\phi + cos\psi sin\theta cos\phi sin\theta cos\psi cos\phi + cos\psi sin\theta cos\phi sin\theta sin\psi sin\theta sin\phi =$

$sin^2\psi sin^2\phi cos^2\theta + sin\psi sin\phi sin\theta cos\psi cos\phi + sin^2\psi sin^2\phi sin^2\theta + cos\psi sin\theta cos\phi sin\psi cos^2\theta sin\phi + cos^2\psi sin^2\theta cos^2\phi + cos\psi sin^3\theta cos\phi sin\psi sin\phi =$

$sin^2\psi sin^2\phi(cos^2\theta + sin^2\theta) + sin\psi sin\phi sin\theta cos\psi cos\phi + cos\psi sin\theta cos\phi sin\psi cos^2\theta sin\phi + cos^2\psi sin^2\theta cos^2\phi + cos\psi sin^3\theta cos\phi sin\psi sin\phi =$

$sin^2\psi sin^2\phi + sin\psi sin\phi sin\theta cos\psi cos\phi + cos\psi sin\theta cos\phi sin\psi cos^2\theta sin\phi + cos^2\psi sin^2\theta cos^2\phi + cos\psi sin^3\theta cos\phi sin\psi sin\phi =$

$sin^2\psi sin^2\phi + sin\psi sin\phi sin\theta cos\psi cos\phi + cos\psi cos\phi sin\psi sin\phi sin\theta(cos^2\theta + sin^2\theta) + cos^2\psi sin^2\theta cos^2\phi =$

$sin^2\psi sin^2\phi + sin\psi sin\phi sin\theta cos\psi cos\phi + sin\psi sin\phi sin\theta cos\psi cos\phi + cos^2\psi sin^2\theta cos^2\phi =$

$\underline{sin^2\psi sin^2\phi + 2sin\psi sin\phi sin\theta cos\psi cos\phi + cos^2\psi sin^2\theta cos^2\phi \quad (10)}$

Now summarize all augends (8)+(9)+(10):

$cos^2\psi cos^2\theta -(2cos\psi sin\phi sin\psi cos\phi sin\theta - cos^2\psi sin^2\theta sin^2\phi - sin^2\psi cos^2\phi + sin^2\psi sin^2\phi + 2sin\psi sin\phi sin\theta cos\psi cos\phi + cos^2\psi sin^2\theta cos^2\phi=$

$cos^2\psi cos^2\theta - 2cos\psi sin\phi sin\psi cos\phi sin\theta + cos^2\psi sin^2\theta sin^2\phi + sin^2\psi cos^2\phi + sin^2\psi sin^2\phi + 2sin\psi sin\phi sin\theta cos\psi cos\phi + cos^2\psi sin^2\theta cos^2\phi$

$=cos^2\psi cos^2\theta + cos^2\psi sin^2\theta sin^2\phi + sin^2\psi cos^2\phi + sin^2\psi sin^2\phi + cos^2\psi sin^2\theta cos^2\phi=$

$cos^2\psi cos^2\theta + cos^2\psi sin^2\theta sin^2\phi + sin^2\psi cos^2\phi + sin^2\psi sin^2\phi + cos^2\psi sin^2\theta cos^2\phi=$

$cos^2\psi cos^2\theta + sin^2\theta cos^2\psi(sin^2\phi + cos^2\phi) + sin^2\psi cos^2\phi + sin^2\psi sin^2\phi =$

$cos^2\psi cos^2\theta + sin^2\theta cos^2\psi + sin^2\psi cos^2\phi + sin^2\psi sin^2\phi = cos^2\psi(cos^2\theta + sin^2\theta) + sin^2\psi cos^2\phi + sin^2\psi sin^2\phi =$

$cos^2\psi + sin^2\psi cos^2\phi + sin^2\psi sin^2\phi = \underline{cos^2\psi + sin^2\psi(cos^2\phi + sin^2\phi) = cos^2\psi + sin^2\psi = 1}$

3.2 Initially, we need to create a minor matrix M**;**

M1,1

$(cos\psi \cos^2\phi cos\theta + sin\psi sin\theta sin\phi cos\theta cos\phi + cos\psi sin\phi cos\theta sin\phi - sin\psi sin\theta cos\phi cos\theta sin\phi) =$

$= cos\psi cos\theta(\cos^2\phi + \sin^2\phi) + sin\psi sin\theta sin\phi cos\theta cos\phi - sin\psi sin\theta cos\phi cos\theta \sin\phi = \boldsymbol{cos\psi cos\theta}$

M1,2

$(sin\psi \cos^2\theta cos\phi - (-sin\theta)(-cos\psi sin\phi + sin\psi sin\theta cos\phi)(\cos^2\theta + \sin^2\phi) =$

$= sin\psi \cos^2\theta cos\phi - (sin\theta cos\psi sin\phi - sin\theta sin\theta cos\phi) = sin\psi cos\theta(\cos^2\theta + \sin^2\theta) - sin\theta cos\psi sin\phi =$

$= \boldsymbol{sin\psi cos\phi - sin\theta cos\psi sin\phi}$

M1,3

$sin\psi \cos^2\theta\, sin\phi + sin\theta cos\psi cos\phi + sin\psi \sin^2\theta sin\phi = \boldsymbol{sin\psi sin\phi} + \boldsymbol{sin\theta cos\psi cos\phi}$

M 2,1

$-sin\psi \cos^2\phi cos\theta + cos\psi sin\theta sin\phi\, cos\theta cos\phi - sin\psi \sin^2\phi cos\theta - cos\psi sin\theta cos\phi sin\phi =$

$= - sin\psi cos\theta(\cos^2\phi + \sin^2\phi) = - \boldsymbol{sin\psi cos\theta}$

M 2,2

$cos\psi \cos^2\theta cos\phi - (-sin\theta\, sin\psi sin\phi - sin\theta cos\psi sin\theta cos\phi) =$

$= cos\psi \cos^2\theta cos\phi + sin\theta sin\psi sin\phi + \sin^2\theta cos\psi cos\phi = cos\psi cos\phi(\cos^2\theta + \sin^2\theta) +$

$+ sin\theta sin\psi sin\phi = \boldsymbol{cos\psi cos\phi} + \boldsymbol{sin\theta sin\psi sin\phi}$

M 2,3

$cos\psi \cos^2\theta sin\phi - sin\theta sin\psi cos\phi + \sin^2\theta cos\psi\, sin\phi = cos\psi sin\phi(\cos^2\theta + \sin^2\theta) - sin\theta sin\psi cos\phi =$

$= \boldsymbol{cos\psi sin\phi - sin\theta sin\psi cos\phi}$

M 3,1

$(cos\psi sin\theta sin\phi - sin\psi cos\phi)(sin\psi sin\theta cos\phi - cos\psi sin\phi) - (cos\psi cos\phi + sin\psi sin\theta sin\phi)*$

$*(sin\psi sin\phi + cos\psi sin\theta \cos\phi) = cos\psi \sin^2\theta\, sin\phi sin\psi cos\phi - \cos^2\psi\, sin\theta \sin^2\phi - \sin^2\psi \cos^2\phi sin\theta +$
$sin\psi cos\phi\, cos\psi sin\phi - - cos\psi cos\phi sin\psi sin\phi - \cos^2\psi \cos^2\phi sin\theta - \sin^2\phi sin\theta \sin^2\psi -$
$sin\psi \sin^2\theta cos\psi sin\phi cos\phi = - \cos^2\psi\, sin\theta \sin^2\phi - \sin^2\psi sin\theta(\cos^2\phi + \sin^2\phi) - \cos^2\psi \cos^2\phi sin\theta =$

$= - \cos^2\psi\, sin\theta - \sin^2\psi sin\theta = -sin\theta(\cos^2\psi + \sin^2\psi) = -\boldsymbol{sin\theta}$

M 3,2

$cos\psi cos\theta(-cos\psi sin\phi + sin\psi sin\theta cos\phi) - sin\psi \cos\theta(sin\psi sin\phi + cos\psi sin\theta cos\phi) =$

$- cos\theta sin\phi(\cos^2\psi + \sin^2\psi) + cos\psi cos\theta sin\psi sin\theta cos\phi - sin\psi \cos\theta(cos\psi sin\theta cos\phi) =$

$= -\boldsymbol{cos\theta sin\phi}$

M 3,3

$cos\psi cos\theta(cos\psi cos\phi\ +\ sin\psi sin\theta sin\phi) -\ sin\psi cos\theta(-sin\psi cos\phi + cos\psi sin\theta sin\phi) =$

$= \cos^2\psi cos\theta\, cos\phi +\ sin\psi sin\theta sin\phi cos\psi cos\theta - \sin^2\psi cos\theta cos\phi - sin\psi cos\theta cos\psi sin\theta sin\phi =$

$= cos\theta cos\phi$

Let's create minor matrix **M**:

$$M = \begin{bmatrix} cos\psi cos\theta & sin\psi cos\phi -\ sin\theta cos\psi sin\phi & sin\psi sin\phi +\ sin\theta cos\psi cos\phi \\ sin\psi cos\theta & cos\psi cos\phi +\ sin\theta sin\psi sin\phi & cos\psi sin\phi -\ sin\theta sin\psi cos\phi \\ -sin\theta & -\ cos\theta sin\phi & cos\theta cos\phi \end{bmatrix}$$

3.3 Then we need to find the matrix of algebraic complements A_m

$$A_m = \begin{bmatrix} cos\psi cos\theta & sin\psi cos\theta & -sin\theta \\ -sin\psi cos\phi + cos\psi sin\theta sin\phi & cos\psi cos\phi + sin\psi sin\theta sin\phi & cos\theta sin\phi \\ -sin\psi(-sin\phi) + cos\psi sin\theta cos\phi & cos\psi(-sin\phi) + sin\psi sin\theta cos\phi & cos\theta cos\phi \end{bmatrix}$$

3.4 Finally, we should make computation according to formula (2)

$$R^{-1} = 1/1 * \begin{bmatrix} cos\psi cos\theta & sin\psi cos\theta & -sin\theta \\ -sin\psi cos\phi + cos\psi sin\theta sin\phi & cos\psi cos\phi + sin\psi sin\theta sin\phi & cos\theta sin\phi \\ -sin\psi(-sin\phi) + cos\psi sin\theta cos\phi & cos\psi(-sin\phi) + sin\psi sin\theta cos\phi & cos\theta cos\phi \end{bmatrix}$$

Matrixes R^{-1} and R^T are fully equal to each other.

Therefore, we have proved that property $\mathbf{R^{-1} = R^T}$ hold true for R = Rz(ψ)Ry(θ)Rx(φ) as the consecutive execution of these matrices and **det(R) = 1** .

(c) Determinant of a square 3x3 matrix A is the volume of a 3-dimensional parallelepiped, which is formed if we consider the rows of the matrix as vectors forming the edges of this parallelepiped.
Determinant of square 3x3 matrix –

$$\det A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

The determinant of a 3x3 square matrix is the volume of the parallelepiped built on these vectors, as shown in Figure 1.

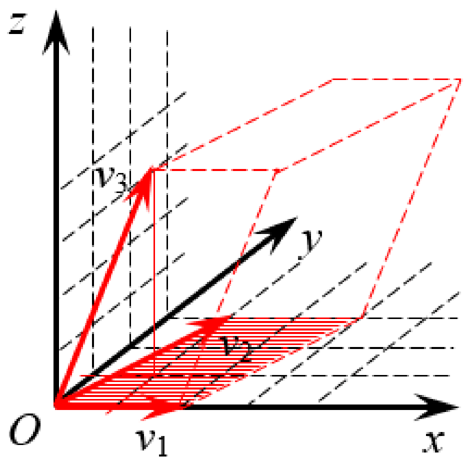

Figure 1: The determinant of a 3x3 matrix is the volume of the parallelepiped built on these vectors [2]

To prove that the rotation matrix must have determinant 1, consider the Euler angles for rotation (5), where the angles of rotation ψ, θ, ϕ are equal around the z, y, x axes respectively.
Rotation by ψ around z axis(4)
Rotation by θ around y axis (3)
Rotation by ϕ around x axis (1)
For rotation along each axis, the determinant is equal $\cos^2 \alpha + \sin^2 \alpha \ = \ 1$
Euler angles for rotation $R \ = \ R_z \, R_y \, R_x$.[1]
Rotations are described by matrix multiplication (5). Three-dimensional rotation are non-commutative, so the order of multiplication affects the final result.
Therefore, the determinant of any rotation matrix is 1.
For comparison, consider the translation matrix:

$$\mathrm{T} = \begin{pmatrix} 1 & 0 & t_1 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix} \qquad (11)$$

where $v \rightarrow= (t_x, t_y)$ - parallel translation vector, $det(T) = 1$.

For comparison, consider the scaling matrix

$$\mathrm{S} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (12)$$

where a and b are the coefficients of stretching along the x and y axes, respectively

$$det(S) = a * b$$

The volume of the parallelepiped built on these vectors changes when the vector is stretched.
Therefore, the volume depends on a and b and only in some cases is equal to 1.
The determinant of the rotation matrix and the translation matrix is always equal to one, because the volume of the parallelepiped built on these vectors does not change.
If the determinant of the rotation matrix and translation matrix is equal to -1, then it is no longer a rotation and translation matrix.
This means that the figures have turned in the opposite direction. Therefore, the determinant of any rotation matrix is equal to 1.

2. Transformation Chain

First of all, we need to define points in the world coordinate system and camera coordinate system. World coordinates and camera coordinates are not match.
The point in the world coordinate system is denoted by W (Xw, Yw, Zw);
The point in the camera coordinate system is denoted by C (Xc, Yc, Zc).

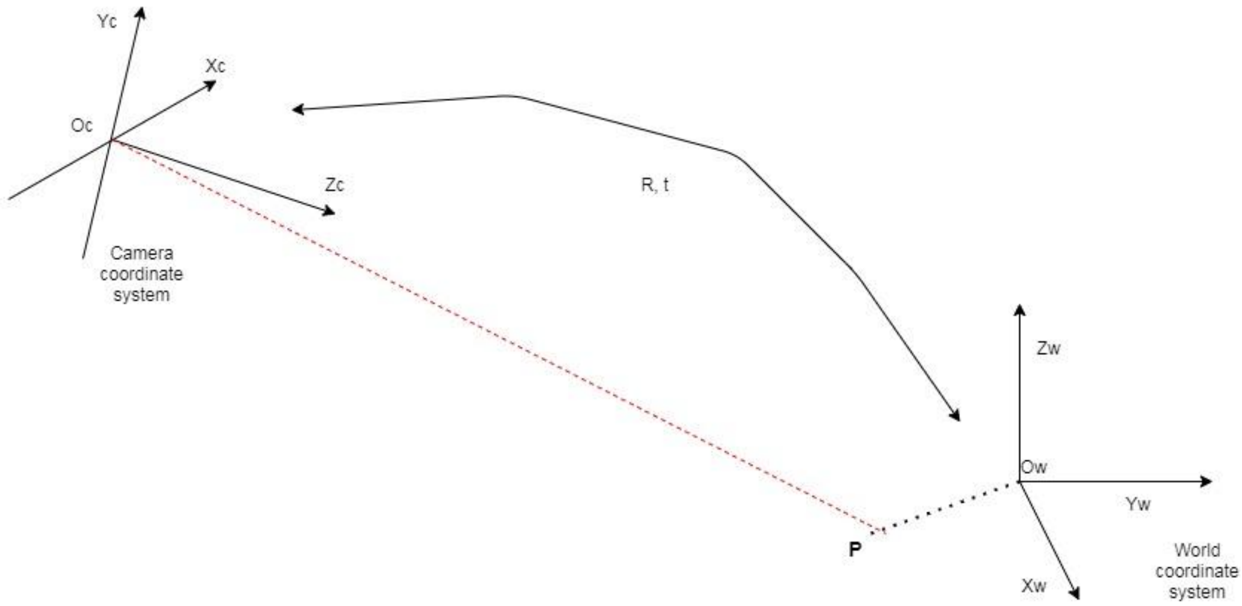

Figure 2: World coordinate system and camera coordinate system [3]

It is important to understand that the camera coordinate system rotates around the world coordinate system.
3D camera rotation is determined using three parameters: roll (around x-axis), pitch (around y-axis) and yaw (around z-axis).
A 3 × 3 matrix is used to calculate the rotation.
The world coordinates and camera coordinates are related to the rotation matrix R and the translation vector t consisting of 3 elements.
Point P will have different coordinate values in the camera coordinate system and in the world coordinate system:
P = (Xw, Yw, Zw) point in world coordinates,
P = (Xc, Yc, Zc) point in camera coordinates,
t = (0c, 0w) translation vector between origins.
These coordinates are related by the following equation:

$$\begin{pmatrix} Xc \\ Yc \\ Zc \end{pmatrix} = t + R \begin{pmatrix} Xw \\ Yw \\ Zw \end{pmatrix} \tag{13}$$

The expression (13) is often written in a more compact form.
We need to translate the 3×1 vector, for this translation we need to append a column at the end of the 3×3 rotation matrix. We'll get to obtain the 3 × 4 Extrinsic Matrix:

$$\begin{pmatrix} Xc \\ Yc \\ Zc \end{pmatrix} = (R|t) \begin{pmatrix} Xw \\ Yw \\ Zw \\ 1 \end{pmatrix} \tag{14}$$

After we get the point in the 3D camera coordinate system.
We can project a point on the image plane to calculate the location of the point in the image. (Figure 3)

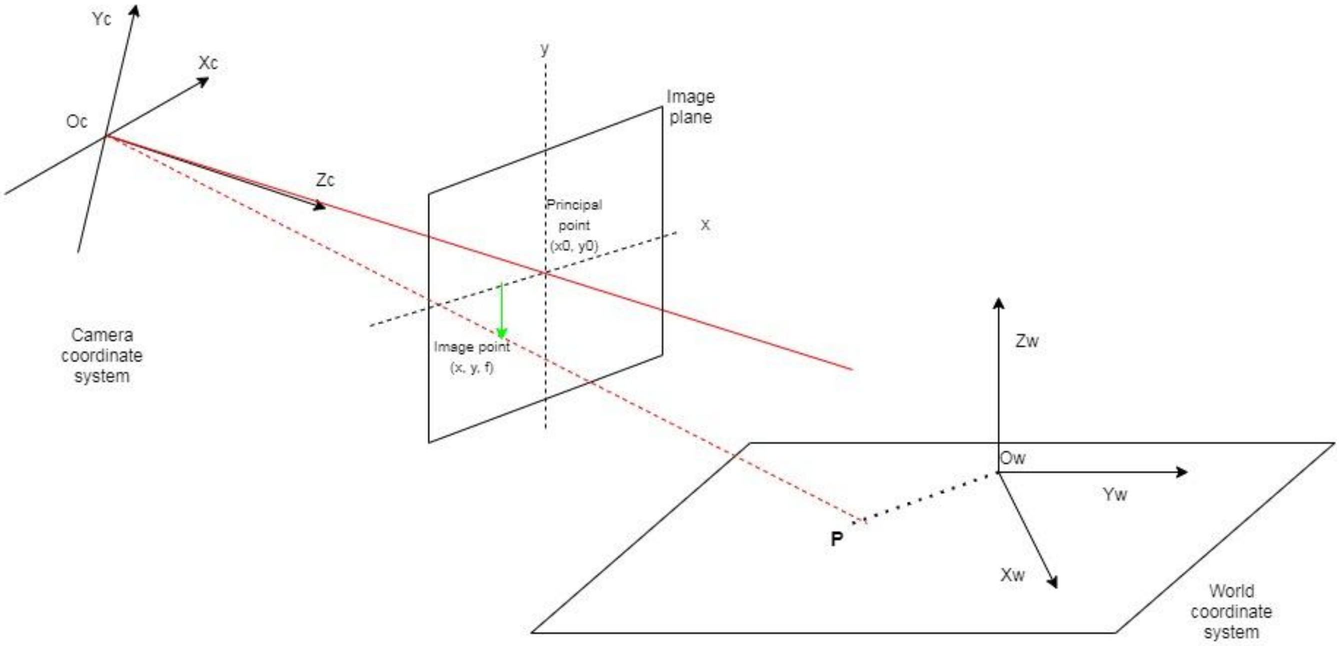

Figure 3: Projection of point P onto the image plane[3]

Transformation uses:
Principal point (x_o, y_o)
Image point (x_i, y_i)

$$x_i = f\frac{Xc}{Zc}$$
$$y_i = f\frac{Yc}{Zc}$$

(15)

These two equations (15) can be rewritten in matrix form as follows:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Xc \\ Yc \\ Zc \end{pmatrix}$$

(16)

Matrix K (17) is the Intrinsic Matrix that contains the internal parameters of the camera. Matrix K shows Focal lengths in pixels:

$$\mathrm{K} = \begin{pmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(17)

Since the pixels in the matrix may not be square, and we can have two different fx and fy focal lengths. The optical center (x_o, y_o) of the camera does not always coincide with the center of the image coordinate system. Moreover, there can be a skew s between the x and y axes.
As a result, the camera matrix can be written like this[1](18):

$$\mathrm{K} = \begin{pmatrix} fx & s & x0 \\ 0 & fy & y0 \\ 0 & 0 & 1 \end{pmatrix}$$

(18)

If we do (u, v) coordinates transformation, then the matrix will look like this:

$$\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = \begin{pmatrix} fx & s & x0 \\ 0 & fy & y0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Xc \\ Yc \\ Zc \end{pmatrix}$$

where

$$u = \frac{u'}{w'} \quad v = \frac{v'}{w'}$$

(19)

Thus, we have projected 3D points from the world coordinate system to the pixel coordinates of the camera.

Homogeneous coordinates allow us to represent infinite quantities using finite numbers. A 3D point (X, Y, Z) in Cartesian coordinates can be written as (X, Y, Z, 1) in homogeneous coordinates. For example, a point at infinity can be represented as (1, 1, 1, 0) in homogeneous coordinates.

3 Implementation

The code of the developed program is in the "main.py" file.
The results of executed program are presented in the folder «results». Images with distortion and without distortion are in the folders " result_without_correction" and " result_with_correction"", respectively

Example is presented below:

1. Image with projected points without correction of the distortion in red:

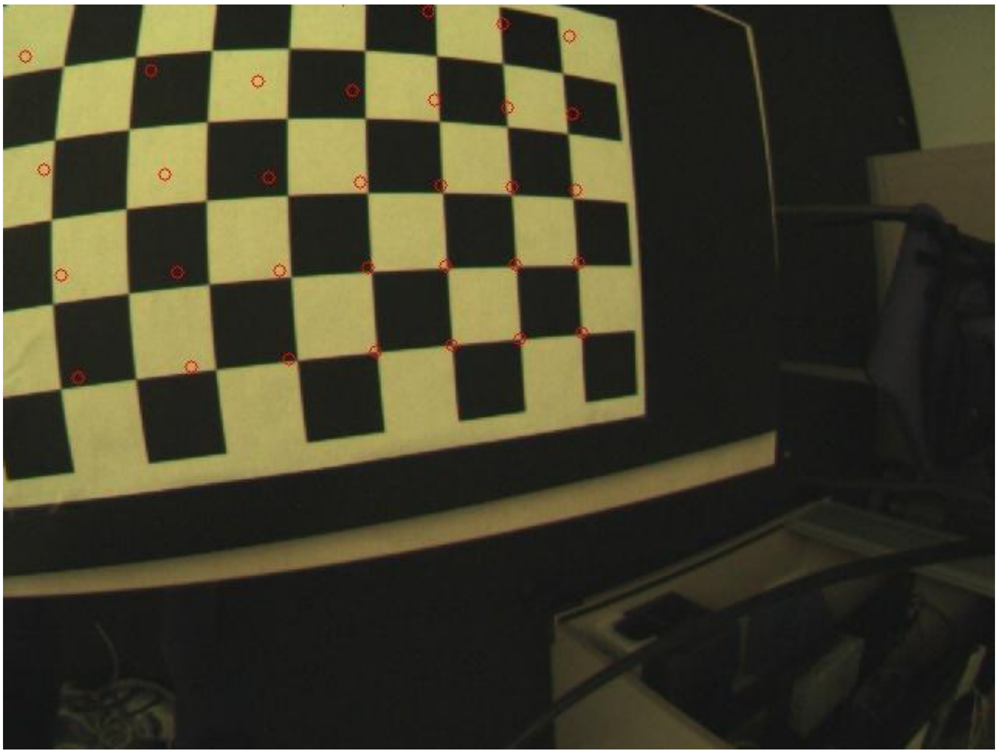

Figure 4: image without correction of the distortion

2. Image with projected points with correction of the radial distortion (k1, k2 and k5) in green:

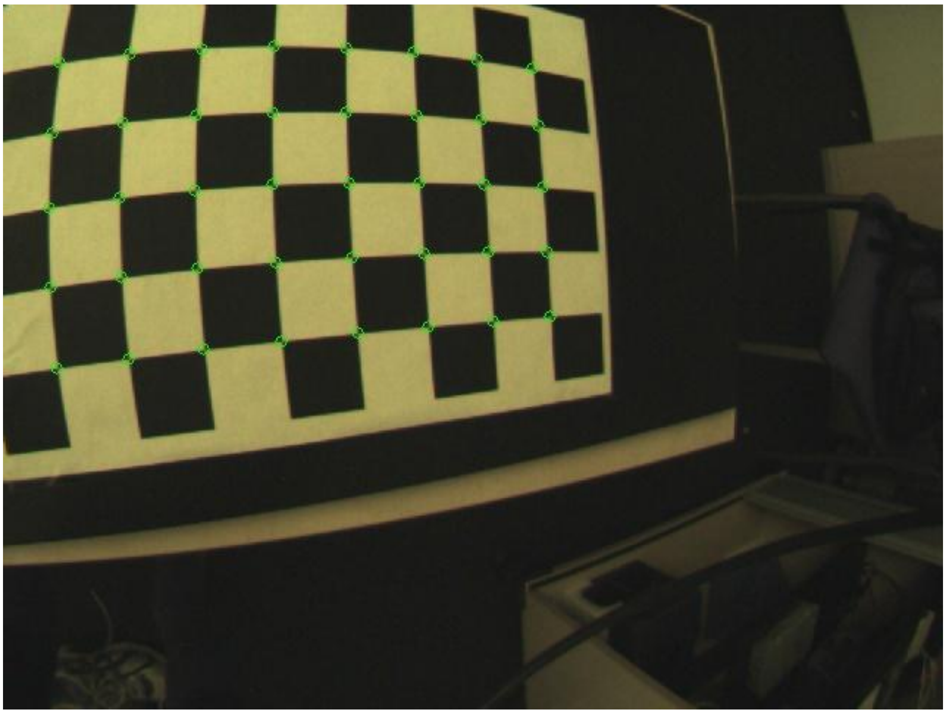

Figure 5: image with correction of the radial distortion

References

1. Prof. Didier Stricker, *Camera Model & Calibration, 3D Computer Vision*, Kaiserslautern University, 44 p.

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