Report

3D ComputerVision

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Introduction

This report provides answers to theoretical questions in the second exercise and the result of the developed program.

Theory

1. Homography Definition

(a) Homography is a projective transformation in the image or in the projective space. It is happen when camera or object rotate or move from one plane to another plane. We calculate the transformation of the detected points on the plane into an image. These points are projected through the P matrix.

A homography is a non-singular, line preserving, projective mapping *h*: [1].

There are the two important properties of homography:

1. Non-singular means that you can invert it. Square matrix A (n x n) is called non-singular, if there exists an n-by-n square matrix B such that [2]

2. Line preserving is a transformation in which three points that lie on the line will be points on the line again. The transformed line will be a line, not a curve or something like that. Linear transformation always defined through a square matrix.

We usually term **mapping** as **h** from . For example, we model all points from a three-dimensional space to another three-dimensional space through a four-dimensional vector.

(b) A homography is represented through a square matrix of dimension (n+1) and it has degree of freedom (DOF).

We use the degree of freedom reduced by one because we have the square matrix. We take one last element from our matrix and divide by itself so we get fixed element (one) and other elements. So we have actually which defines this transformation degree of freedom.

2. Line Preservation

One of the most important properties of homography's general definition is a line preserving (a projectivity) (1), which is represents an invertible mapping of points and lines on the projective plane P^2.

Firstly, we should consider 2D points x1, x2, x3 ∈ P^2 in the image. Let us give one point for further consideration $x_{n}=x$. Then, x = (x, y) can be shown as a 3D vector x = (x1, x2, x3), where x = x1 x3 , y = x2x3 and x3 = 0.

This is the homogeneous representation of a point x that lies on the projective plane P2. Definition of projectivity theorem tells us that a mapping from $P^2 \rightarrow P^2$ is a projectivity if and only if there exists a non-singular 3×3 matrix H, such that for any point in P^2 represented by vector x it is true that its mapped point equals Hx.

From this theorem we conclude that in order to mapping each x_i to its corresponding x'_i we should calculate the homography. That is why we need to calculate the 3×3 homography matrix H for each x_i:

$x'_i = Hx_i$ (1)

Due to homogeneous coordinates equation (1) can be shown as:

$$c\begin{pmatrix}u\\v\\1\end{pmatrix} = H\begin{pmatrix}x\\y\\1\end{pmatrix}(2)$$

where c is constant c!=0, $(u\ v\ 1)^T$ defines x', $(x\ y\ 1)^T$ defines x and

$$H = \begin{bmatrix}h_1\ h_2 h_3\\h_4\ h_5\ h_6\\h_7\ h_8\ h_9\end{bmatrix}$$

We achieve the following two equations by dividing the first row of equation (2) by the third row and the second row by the third row and finally we get:

$$-h_1 x - h_2 y - h_3 + (h_7 x + h_8 y + h_9)u = 0\ (3)$$

$$-h_4 x - h_5 y - h_6 + (h_7 x + h_8 y + h_9)u = 0\ (4)$$

Additionally, we can write them down in matrix form:

$A_i h = 0\ (5)$

$$Ai = \begin{bmatrix}-x & -y & -1 & 0 & 0 & 0 & ux & uy & u\\0 & 0 & 0 & -x & -y & -1 & vx & vy & v\end{bmatrix}$$

$$h = (h_1\ h_2 h_3\ h_4\ h_5\ h_6\ h_7\ h_8\ h_9)^T$$

In conclusion, we can see four 2×9 A_i matrices which can be added on top of one another to get a single 8×9 matrix A. The solution space for h is null space of matrix A .

Then we can assume, that a line in a plane can be represented by the following equation: ax+by+c = 0. Then we have to represent this line as the vector $(a,b,c)^T$

Let us get a line l in one image that maps to the line l' in the other image. Let x be a point on l and x' be a point on l'. We know that a point x lies on a line l if and only if $x^T l = 0$ (or equivalently, $l^T x = 0$). Therefore:

$l^T x = 0\ (6)$

$l'^T x' = 0\ (7)$

From (1) we know that $x' =$ Hx. We have to substitute this in (3) and manipulate the result, which is a match between the lines in the images:

$l = H^T l'\ (8)$

And that is why we can represent the following result due to homogeneous coordinates and equation (1) can be shown as:

$$c\begin{pmatrix}u\\v\\1\end{pmatrix}=H^T\begin{pmatrix}x\\y\\1\end{pmatrix}\ (9)$$

where $(u\ v\ 1)^T$ represents l. In this case after the derivation it gives the matrix A_i for a line correspondence:

$$Ai = \begin{bmatrix}-u & 0 & ux & -v & 0 & vx & -1 & 0 & x\\0 & -u & uy & 0 & -v & vy & 0 & -1 & y\end{bmatrix}\ (10)$$

We present the use of both point and line correspondences. For each correspondence i, evaluate the 2×9 matrix A_i . We can add all the matrices A_i on top of each other to get a matrix A for which the zero space is the solution for the homography vector h.

Therefore, we can make conclusion that a homography H preserves property of a line preserving, because both points and line were invertible mapped on the projective plane P^2.

3. Camera Center in World Coordinates

(a)

General transformation of point from the world coordinate system C_w (XS, YS, ZS) to the camera coordinate system C_c (xs, ys, zs):

$$\begin{pmatrix} xs \\ ys \\ zs \\ 1 \end{pmatrix} = [R|t] \begin{pmatrix} XS \\ YS \\ ZS \\ 1 \end{pmatrix} \quad (11)$$

To transform from camera coordinate system to world coordinate system, we write equation (12) using the equation (11):

$$\begin{pmatrix} XS \\ YS \\ ZS \\ 1 \end{pmatrix} = [R|t]^{-1} \begin{pmatrix} xs \\ ys \\ zs \\ 1 \end{pmatrix} \quad (12)$$

As we found in the Exercise 1, the rotation matrix around all axis looks like:

$$R = \begin{pmatrix} cos\psi cos\theta & -sin\psi cos\phi + cos\psi sin\theta sin\phi & -sin\psi(-sin\phi) + cos\psi sin\theta cos\phi \\ sin\psi cos\theta & cos\psi cos\phi + sin\psi sin\theta sin\phi & cos\psi(-sin\phi) + sin\psi sin\theta cos\phi \\ -sin\theta & cos\theta sin\phi & cos\theta cos\phi \end{pmatrix}$$

Make the matrix $[R|t]$:

$$[R|t] = \begin{pmatrix} cos\psi cos\theta & -sin\psi cos\phi + cos\psi sin\theta sin\phi & sin\psi sin\phi + cos\psi sin\theta cos\phi & tx \\ sin\psi cos\theta & cos\psi cos\phi + sin\psi sin\theta sin\phi & -cos\psi sin\phi + sin\psi sin\theta cos\phi & ty \\ -sin\theta & cos\theta sin\phi & cos\theta cos\phi & tz \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

To find inverted matrix $[R|t]^{-1}$, we need to create a minor matrix M:

$$M= \left(\begin{array}{cc}
\begin{pmatrix} cos\psi cos\phi + sin\psi sin\theta sin\phi & -cos\psi sin\phi + sin\psi sin\theta cos\phi & ty \\ cos\theta sin\phi & cos\theta cos\phi & tz \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} sin\psi cos\theta & -cos\psi sin\phi + sin\psi sin\theta cos\phi & ty \\ -sin\theta & cos\theta cos\phi & tz \\ 0 & 0 & 1 \end{pmatrix} \\
\begin{pmatrix} -sin\psi cos\phi + cos\psi sin\theta sin\phi & sin\psi sin\phi + cos\psi sin\theta cos\phi & tx \\ cos\theta sin\phi & cos\theta cos\phi & tz \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} cos\psi cos\theta & sin\psi sin\phi + cos\psi sin\theta cos\phi & tx \\ -sin\theta & cos\theta cos\phi & tz \\ 0 & 0 & 1 \end{pmatrix} \\
\begin{pmatrix} -sin\psi cos\phi + cos\psi sin\theta sin\phi & sin\psi sin\phi + cos\psi sin\theta cos\phi & tx \\ cos\psi cos\phi + sin\psi sin\theta sin\phi & -cos\psi sin\phi + sin\psi sin\theta cos\phi & ty \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} cos\psi cos\theta & sin\psi sin\phi + cos\psi sin\theta cos\phi & tx \\ sin\psi cos\theta & -cos\psi sin\phi + sin\psi sin\theta cos\phi & ty \\ 0 & 0 & 1 \end{pmatrix} \\
\begin{pmatrix} -sin\psi cos\phi + cos\psi sin\theta sin\phi & sin\psi sin\phi + cos\psi sin\theta cos\phi & tx \\ cos\psi cos\phi + sin\psi sin\theta sin\phi & -cos\psi sin\phi + sin\psi sin\theta cos\phi & ty \\ cos\theta sin\phi & cos\theta cos\phi & tz \end{pmatrix} & \begin{pmatrix} cos\psi cos\theta & sin\psi sin\phi + cos\psi sin\theta cos\phi & tx \\ sin\psi cos\theta & -cos\psi sin\phi + sin\psi sin\theta cos\phi & ty \\ -sin\theta & cos\theta cos\phi & tz \end{pmatrix}
\end{array} \right.$$

$$
\left(\begin{matrix}
\begin{pmatrix} sin\psi cos\theta & cos\psi cos\phi + sin\psi sin\theta sin\phi & ty \\ -sin\theta & cos\theta sin\phi & tz \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} sin\psi cos\theta & cos\psi cos\phi + sin\psi sin\theta sin\phi & -cos\psi sin\phi + sin\psi sin\theta cos\phi \\ -sin\theta & cos\theta sin\phi & cos\theta cos\phi \\ 0 & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} cos\psi cos\theta & -sin\psi cos\phi + cos\psi sin\theta sin\phi & tx \\ -sin\theta & cos\theta sin\phi & tz \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} cos\psi cos\theta & -sin\psi cos\phi + cos\psi sin\theta sin\phi & sin\psi sin\phi + cos\psi sin\theta cos\phi \\ -sin\theta & cos\theta sin\phi & cos\theta cos\phi \\ 0 & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} cos\psi cos\theta & -sin\psi cos\phi + cos\psi sin\theta sin\phi & tx \\ sin\psi cos\theta & cos\psi cos\phi + sin\psi sin\theta sin\phi & ty \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} cos\psi cos\theta & -sin\psi cos\phi + cos\psi sin\theta sin\phi & sin\psi sin\phi + cos\psi sin\theta cos\phi \\ sin\psi cos\theta & cos\psi cos\phi + sin\psi sin\theta sin\phi & -cos\psi sin\phi + sin\psi sin\theta cos\phi \\ 0 & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} cos\psi cos\theta & -sin\psi cos\phi + cos\psi sin\theta sin\phi & tx \\ sin\psi cos\theta & cos\psi cos\phi + sin\psi sin\theta sin\phi & ty \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} cos\psi cos\theta & -sin\psi cos\phi + cos\psi sin\theta sin\phi & sin\psi sin\phi + cos\psi sin\theta cos\phi \\ sin\psi cos\theta & cos\psi cos\phi + sin\psi sin\theta sin\phi & -cos\psi sin\phi + sin\psi sin\theta cos\phi \\ -sin\theta & cos\theta sin\phi & cos\theta cos\phi \end{pmatrix}
\end{matrix}\right)
$$

=

$$
= \begin{pmatrix}
cos\theta cos\psi & sin\psi cos\phi - sin\phi sin\theta cos\psi & sin\phi sin\psi + sin\theta cos\phi cos\psi & 0 \\
-sin\psi cos\theta & sin\phi sin\theta sin\psi + cos\phi cos\psi & sin\phi cos\psi - sin\theta sin\psi cos\phi & 0 \\
-sin\theta & -sin\phi cos\theta - sin\phi & cos\phi cos\theta & 0 \\
tx cos\theta cos\psi + ty sin\psi cos\theta - tz sin\theta & -tx sin\phi sin\theta cos\psi + tx sin\psi cos\phi - ty sin\phi sin\theta sin\psi - ty cos\phi cos\psi - tz sin\phi cos\theta & tx sin\phi sin\psi + tx sin\theta cos\phi cos\psi - ty sin\phi cos\psi + ty sin\theta sin\psi cos\phi + tz cos\phi cos\theta & 1
\end{pmatrix}
$$

Find the matrix of algebraic complements A_m:

$$
A_m = \begin{pmatrix}
cos\theta cos\psi & -sin\psi cos\phi + sin\phi sin\theta cos\psi & sin\phi sin\psi + sin\theta cos\phi cos\psi & 0 \\
sin\psi cos\theta & sin\phi sin\theta sin\psi + cos\phi cos\psi & -sin\phi cos\psi + sin\theta sin\psi cos\phi & 0 \\
-sin\theta & sin\phi cos\theta + sin\phi & cos\phi cos\theta & 0 \\
-tx cos\theta cos\psi - ty sin\psi cos\theta + tz sin\theta & -tx sin\phi sin\theta cos\psi + tx sin\psi cos\phi - ty sin\phi sin\theta sin\psi - ty cos\phi cos\psi - tz sin\phi cos\theta & -tx sin\phi sin\psi - tx sin\theta cos\phi cos\psi + ty sin\phi cos\psi - ty sin\theta sin\psi cos\phi - tz cos\phi cos\theta & 1
\end{pmatrix}
$$

Transpose the matrix of algebraic complements A^T:

$$
A^T = \begin{pmatrix}
cos\theta cos\psi & sin\psi cos\theta & -sin\theta & -tx cos\theta cos\psi - ty sin\psi cos\theta + tz sin\theta \\
-sin\psi cos\phi + sin\phi sin\theta cos\psi & sin\phi sin\theta sin\psi + cos\phi cos\psi & sin\phi cos\theta + sin\phi & -tx sin\phi sin\theta cos\psi + tx sin\psi cos\phi - ty sin\phi sin\theta sin\psi - ty cos\phi cos\psi - tz sin\phi cos\theta \\
sin\phi sin\psi + sin\theta cos\phi cos\psi & -sin\phi cos\psi + sin\theta sin\psi cos\phi & cos\phi cos\theta & -tx sin\phi sin\psi - tx sin\theta cos\phi cos\psi + ty sin\phi cos\psi - ty sin\theta sin\psi cos\phi - tz cos\phi cos\theta \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

We get the inverted matrix:

$$
[R|t]^{-1} = \frac{1}{\det([R|t])} * A^T =
$$

$$= \frac{1}{1} * \begin{pmatrix} cos\theta cos\psi & sin\psi cos\theta & -sin\theta & -tx cos\theta cos\psi - ty sin\psi cos\theta + tz sin\theta \\ -sin\psi cos\phi + sin\phi sin\theta cos\psi & sin\phi sin\theta sin\psi + cos\phi cos\psi & sin\phi cos\theta + sin\phi & -tx sin\phi sin\theta cos\psi + tx sin\psi cos\phi - ty sin\phi sin\theta sin\psi - ty cos\phi cos\psi - tz sin\phi cos\theta \\ sin\phi sin\psi + sin\theta cos\phi cos\psi & -sin\phi cos\psi + sin\theta sin\psi cos\phi & cos\phi cos\theta & -tx sin\phi sin\psi - tx sin\theta cos\phi cos\psi + ty sin\phi cos\psi - ty sin\theta sin\psi cos\phi - tz cos\phi cos\theta \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

To get camera center coordinates in the world coordinate system, we need to multiply inverted matrix $[R|t]^{-1}$ by camera coordinates in the camera coordinate system Cc:

$$\begin{pmatrix} XC \\ YC \\ ZC \\ 1 \end{pmatrix} = \begin{pmatrix} cos\theta cos\psi & sin\psi cos\theta & -sin\theta & -tx cos\theta cos\psi - ty sin\psi cos\theta + tz sin\theta \\ -sin\psi cos\phi + sin\phi sin\theta cos\psi & sin\phi sin\theta sin\psi + cos\phi cos\psi & sin\phi cos\theta + sin\phi & -tx sin\phi sin\theta cos\psi + tx sin\psi cos\phi - ty sin\phi sin\theta sin\psi - ty cos\phi cos\psi - tz sin\phi cos\theta \\ sin\phi sin\psi + sin\theta cos\phi cos\psi & -sin\phi cos\psi + sin\theta sin\psi cos\phi & cos\phi cos\theta & -tx sin\phi sin\psi - tx sin\theta cos\phi cos\psi + ty sin\phi cos\psi - ty sin\theta sin\psi cos\phi - tz cos\phi cos\theta \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Camera center coordinate in camera coordinate system is $C_c=(0,0,0)$

$$\begin{pmatrix} XS \\ YS \\ ZS \\ 1 \end{pmatrix} = \begin{pmatrix} cos\theta cos\psi & sin\psi cos\theta & -sin\theta & -tx cos\theta cos\psi - ty sin\psi cos\theta + tz sin\theta \\ -sin\psi cos\phi + sin\phi sin\theta cos\psi & sin\phi sin\theta sin\psi + cos\phi cos\psi & sin\phi cos\theta + sin\phi & -tx sin\phi sin\theta cos\psi + tx sin\psi cos\phi - ty sin\phi sin\theta sin\psi - ty cos\phi cos\psi - tz sin\phi cos\theta \\ sin\phi sin\psi + sin\theta cos\phi cos\psi & -sin\phi cos\psi + sin\theta sin\psi cos\phi & cos\phi cos\theta & -tx sin\phi sin\psi - tx sin\theta cos\phi cos\psi + ty sin\phi cos\psi - ty sin\theta sin\psi cos\phi - tz cos\phi cos\theta \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} cos\theta cos\psi * 0 + sin\psi cos\theta * 0 + (-sin\theta) * 0 + (-tx cos\theta cos\psi - ty sin\psi cos\theta + tz sin\theta) * 1 \\ (-sin\psi cos\phi + sin\phi sin\theta cos\psi) * 0 + (sin\phi sin\theta sin\psi + cos\phi cos\psi) * 0 + (sin\phi cos\theta + sin\phi) * 0 + (-tx sin\phi sin\theta cos\psi + tx sin\psi cos\phi - ty sin\phi sin\theta sin\psi - ty cos\phi cos\psi - tz sin\phi cos\theta) * 1 \\ (sin\phi sin\psi + sin\theta cos\phi cos\psi) * 0 + (-sin\phi cos\psi + sin\theta sin\psi cos\phi) * 0 + cos\phi cos\theta * 0 + (-tx sin\phi sin\psi - tx sin\theta cos\phi cos\psi + ty sin\phi cos\psi - ty sin\theta sin\psi cos\phi - tz cos\phi cos\theta) * 1 \\ 0 * 0 + 0 * 0 + 0 * 0 + 1 * 1 \end{pmatrix}$$

$$= \begin{pmatrix} -tx cos\theta cos\psi - ty sin\psi cos\theta + tz sin\theta \\ -tx sin\phi sin\theta cos\psi + tx sin\psi cos\phi - ty sin\phi sin\theta sin\psi - ty cos\phi cos\psi - tz sin\phi cos\theta \\ -tx sin\phi sin\psi - tx sin\theta cos\phi cos\psi + ty sin\phi cos\psi - ty sin\theta sin\psi cos\phi - tz cos\phi cos\theta \\ 1 \end{pmatrix} = \text{(rid of homogeneous coordinates)} =$$

$$= \begin{pmatrix} -tx cos\theta cos\psi - ty sin\psi cos\theta + tz sin\theta \\ -tx sin\phi sin\theta cos\psi + tx sin\psi cos\phi - ty sin\phi sin\theta sin\psi - ty cos\phi cos\psi - tz sin\phi cos\theta \\ -tx sin\phi sin\psi - tx sin\theta cos\phi cos\psi + ty sin\phi cos\psi - ty sin\theta sin\psi cos\phi - tz cos\phi cos\theta \end{pmatrix} = \text{(represent it in the matrix form)} =$$

$$= \begin{pmatrix} cos\theta cos\psi & sin\psi cos\theta & -sin\theta \\ sin\phi sin\theta cos\psi - sin\psi cos\phi & sin\phi sin\theta sin\psi + cos\phi cos\psi & sin\phi cos\theta \\ sin\phi sin\psi + sin\theta cos\phi cos\psi & -sin\phi cos\psi + sin\theta sin\psi cos\phi & cos\phi cos\theta \end{pmatrix} \begin{pmatrix} -tx \\ -ty \\ -tz \end{pmatrix} = R^T(-t) = -R^T t$$

Thus, we have proved that $\begin{pmatrix} XS \\ YS \\ ZS \\ 1 \end{pmatrix} = -\,R^T t,\qquad C_{w} = \begin{pmatrix} XS \\ YS \\ ZS \\ 1 \end{pmatrix} => \qquad C_{w} = -\,R^T t$

(b)

In this context, illustrate the meaning of $t = -RC_w$, i.e. from where to where does this vector point?

t is a three-dimensional vector of the offset of the origin of the world coordinate system relative to the origin of the camera coordinate system.

$t = -RC_w$ is the vector of the offset origin of the camera coordinate system relative to the origin of the world coordinate system.

Since the world coordinate system moves in the direction opposite to the direction of camera movement, the elements of the translation matrix are inverted (have the opposite sign).

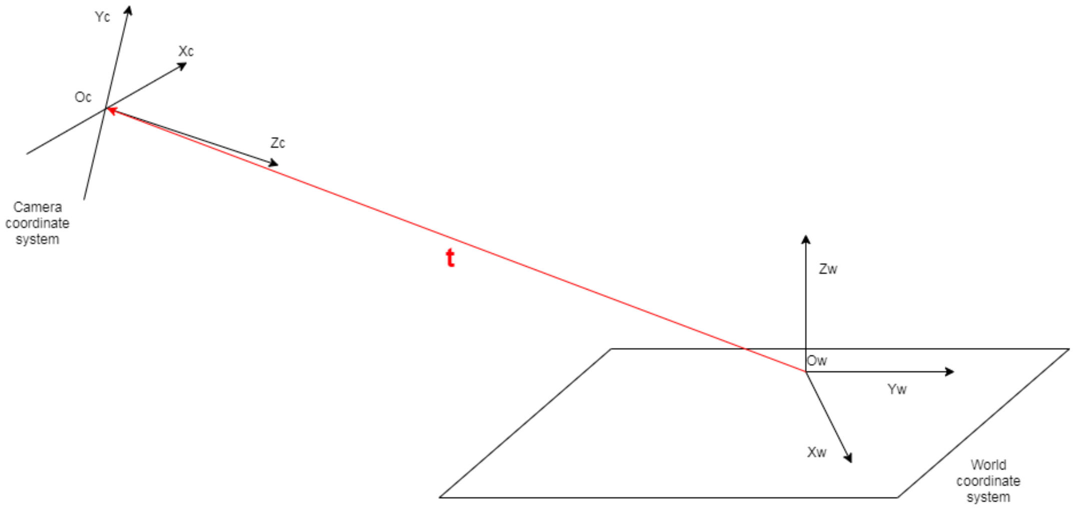

Figure 1: Illustration of the translation vector meaning

Practical Part

The code of the developed program is in the "main.py" file.

1. The rotation matrix computed from H_2 needs correction the properties of the rotation matrix are not fulfilled (determinant of this matrix is not equal to 1 and inverted matrix is not equal to transposed matrix).
2. The third element of the vector t (t_z) can be negative when the camera rotates along the axis against the direction of the z axis.

References

1. Prof. Didier Stricker, 2D Projective Transformations (Homographies), 3D Computer Vision, Kaiserslautern University, 9-12 p.
2. Invertible matrix [Electronic resourse]. Available at:

   https://en.wikipedia.org/wiki/Invertible_matrix (accessed 16.11.2020)
3. Elan Dubrofsky, Robert J. Woodham. Combining Line and Point Correspondences for Homography Estimation [Electronic resourse]. Available at:

   https://www.researchgate.net/publication/220845575_Combining_Line_and_Point_Correspondences_for_Homography_Estimation (accessed 16.11.2020)