Chapter 4 : Interpolation

part of polinomial approximatin

general idea : given $f(x), x \leq b$, how do you approxmiate $f(x)$ by a polinomial $P_n(x)$ **?**

Answer : You find some points of $f(x)$ and you use these points to find $P_n(x)$ where $P_n(x) \approx f(x)$ for $a \leq x \leq b$

Given the following points $(x_0, y_0), (x_1, y_1), \ldots., (x_n, y_n)$ where $x_0 < x_1 < \ldots. < x_n$, then $\exists!$ Polimomial $P_n(x)$ of degree **at most** n that passes through these points.

Newton Polynomial

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \ldots. + a_n(x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

where a_n is the value of $f[x_0, \ldots. x_n]$

Larange

1. for 2 points : $f(x) = \frac{(x-x_1)}{x_0-x_1} f_0 + \frac{(x-x_0)}{(x_1-x_0)} f_1$
2. for 3 points : $\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f_2$

Divided Difference

The general form of the divided difference is

$$f[x_0, x_1 \ldots., x_n] = \frac{f[x_0, \ldots\ldots x_{n-1}] - f[x_1, \ldots., x_n]}{x_0 - x_n}$$

and

$$f[x] = f(x)$$

so

$$f[x_0, x_1] = \frac{f_0 - f_1}{x_0 - x_1}$$

Uniform Partition(Equally Spaced Nodes)

- divide the range into n points
- Find $h = \frac{x_0 - x_n}{n}$
- Find $x_n = x_0 + n * h$
- Find $y(x)$ for all x
- Solve like before

Interpolation Error

Interpolation Error is a type of **truncation error.** The general idea is that you have a set of points $(x_0, y_0)\ldots(x_n, y_n)$ that you use to approximate $f(x)$. however, it is not the complete function, so

$f(x) = P_n(x) + E_n(x)$ it is similar to the error in taylor series, given by :

$$E_n(x) = \frac{(x - x_0)(x - x_1)\dots(x - x_n)f^{(n+1)}(c)}{(n+1)!}$$

We cannot find the true error, but we can fin the upper bound of the error

Uniform :

1- $|E_1(x)| \leq \frac{h^2 M_2}{8}$

2- $|E_2(x)| \leq \frac{h^3 M_3}{9\sqrt{3}}$

3- $|E_3(x)| \leq \frac{h^4 M_4}{24}$

Where M_k is the max of $f^{n+1}(x)$

Non-Uniform

- Use the general formula, then find the max of $Q(x) = (x - x_0)\dots.(x - x_n)$
- Find the max of $M_k = f^{n+1}(x)$
- calculate $|E(x)| = \frac{MAX(Q(x)) * MAX(f^{(n+1)}(x))}{(n+1)!}$

Proofs

Proof that the upper bound of the uniform of error is given by $|E_1(x)| \leq \frac{h^2 M_2}{8}$

proof

$$E_1(x) \frac{(x - x_0)(x - x_1)f''(x)}{2}$$