Chapter 6 : Numerical Differentiation

Remember :

$$f'(x_0) = \lim_{h->0} \frac{f(x_0+h) - f(x_0)}{h}$$

where h is the step size or bandwidth.

But now, we will use numerical methods to find $f'(x_0)$ or higher order derivatives. These formulas are called **difference formulas** or finite difference formulas.

Difeerence Formulas

There are three types of difference formulas :

1. Central Difference Formulas $\rightarrow$ CDF
2. Forward Difference Formulas $\rightarrow$ FDF
3. Backwards Difference Formulas $\rightarrow$ BDF

There are five formulas we are going to be studying

1. CDF of order $O(h^2)$ for $f'(x_0)$

   $f'(x_0) \approx \frac{f(x_0+h)-f(x_0-h)}{2h}$ with error $E = \frac{-h^2 f'''(c)}{6}$.

   Notation : $f(x_0 + kh) = f_k, k = \pm 1, \pm 2, ..$

   so we can express the above as $f'(x_0) = \frac{f_1 - f_{-1}}{2h}$ with error $E = \frac{h^2 f'''(c)}{6}$

2. FDF of order $O(h^2)$ for f'

   $f'(x_0) \approx \frac{-3f_0+4f_1-f_2}{2h}$ Where error $E = \frac{h^2 f'''(c)}{3}$

3. BDF of order $O(h^2)$ for f'

   $f'(x_0) \approx \frac{3f_0+4f_{-1}+f_{-2}}{2h}$ with error $E = \frac{h^2 f'''(c)}{3}$

4. CDF of order $O(h^4)$ flor f'

   $f'(x_0) \approx \frac{-f_2+8f_1-8f_{-1}+f_{-2}}{12}$ with error $E = \frac{h^4 f^{(5)}(c)}{30}$

5. CDF of order $O(h^2)$ for f''

   $f''(x_0) \approx \frac{f_1-2f_0+f_{-1}}{h^2}$ with error $E = \frac{-h^2 f^{(4)}(c)}{12}$

But how do we derive a difference formula?

Deriving Difference Formulas

There are 3 ways to do this

1. Taylor Series

2. Lagrange Interpolation
3. Newton's Polynomial

Deriving Difference Formulas : Taylor Series

Notice that the numerator always contains x_0 somwhere. Say we apply the taylor series of $f(x)$ about x_0. We get $f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x-x_0)^2 f''(x_0)}{2!} + \ldots$

Now, lets replace x with $x_0 + kh$. Then we get

$$f(x) = f(x_0) + (x_0 + kh - x_0)f'(x_0) + \frac{(x_0+kh-x_0)^2 f''(x_0)}{2!} + \ldots + \frac{(x_0+kh+x_0)^{n+1} f^(n+1)(c)}{(n+1)!}$$

Simplifying, we get

$$f_k = f_0 + khf'(x_0) + \frac{(kh)^2}{2} f''(x_0) + \ldots . + \frac{(kh)^{n+1} f^{(n+1)}(c)}{(n+1)!}$$

let $k = 1$

$$f_1 = f_0 + hf'(x_0) + \frac{(h)^2}{2} f''(x_0) + \ldots . + \frac{(h)^{n+1} f^{(n+1)}(c)}{(n+1)!}$$

let $k = 2$

$$f_2 = f_0 + 2hf'(x_0) + \frac{(2h)^2}{2} f''(x_0) + \ldots . + \frac{(2h)^{n+1} f^{(n+1)}(c)}{(n+1)!}$$

Simplifying results in our above formulas.

Deriving Difference Formulas : Lagrange and Newton Polynomials.

The general idea is to find $P_n(t)$, so we have

$f(t) = P_n(t) + E_n(t)$

Now, deriving this, we get

$f'(t) = P_n'(t) + E_n'(t)$