Chapter 1 : Review of Calculus, Errors and Rounding

Review of Calculus

Continuity

A function f(x) is continuous on [a,b] if f(x) ∈ C[a,b] and f(x) ∈ C`[a,b], that is, f(x) and its derivative are continous on [a,b].

In laymans terms, a function is continous over [a,b] if there are no points where there are problems, such as division by zero, or the function is not a piecewise function that does not cover every point in [a,b]

Increasing and Decreasing Functions

A function is classified as increasing or decreasing depending on its derivative.

- if f`(x) > 0 then the function is increasing
- if f`(x) < 0 then the function is decreasing

if f`(x) = 0, then that is called an inflection or critical point.

if f(x) is continuous on [a,b] then f has an absolute **upper bound** = M and an absolute **lower bound** = m, such that :

$$m \leq f(x) \leq M \forall x \in [a, b]$$

Initial Value Theorem

Lets say that f ∈ [a,b], L is between f(a),f(b) ∃ c ∈ [a,b] such that f(c) = L

Bolzano Theorem

let us assume that :

1. f ∈ C[a,b]
2. f(a).f(b) < 0

Then ∃ c in (a,b) such that f(c) = 0.

x = c is a root(a zero) for f(x), and x = c is a solution for the equasion.

Mean Value Theorem

if f ∈ C[a,b] and f`∈ C(a,b), then ∃ c ∈ (a,b) such that :

$$f‘(c) = \frac{f(b) - f(a)}{b - a}$$

alternatively, this can be expressed as

$$|f(b) - f(a)| = |f`(c)||(b - a)|$$

Recall that :

$$|x \pm y| \leq |x| + |y|$$

Rolle's Theorem

Rolle's Theorem is a special case of the MVT, which states :

> If f is continuos and differentiable on (a,b) and if f(a) ≤ f(b) , then ∃ c ∈ (a,b) such that f`(c) = 0 .

In simple terms, if f is generally decreasing on an open interval, then there is a critical point in that interval.

Taylor Series

The Taylor Series/ Taylor Expansion is a way to express a function that isn't very "nice" as a series of polynomials that can be used to express the function. In more technical terms, the taylor series can be formally defined as follows :

> Given f(x) and a center x = a, then the taylor expantion series for f(x) about x = a is given by :

$$f(x) = f(a) + f`(a)(x - a) + \frac{f``(a)(x-a)}{2!} + \ldots + \frac{f^n(a)(x-a)^n}{n!}$$

Which is used to approximate f(x) when x is close to a.

The error of this approximation is given by :

$$E(x) = \frac{f^{n+1}(c)(x-a)^{n+1}}{(n+1)!}$$

where c is a number between x and a.

Errors and Rounding

Significant Digits

A digit is said to be significant when a change in its value changes the number. To find out the number of significant digits in a number, start counting from the the first non-zero number on the left. For example,

```
0.0012341
```

has `5` significant digits, and

```
1200
```

has `4` significant digits.

Sources of Errors

There are 2 sources of errors :

1. Round-off Error : an error when a number is approximated by another number.
2. Truncation Error : an error when a function is approximated by another function.

The reason we have round-off error is because computers use **finite digit arithematic.** There are only a finite number of digits a computer can handle at a time. Numbers in computers are represented in **floating point representation.** Normalized floating point representation is a special form of this, where all the **significant digits** are to the right of the decimal point.

Floating Point Representation

FPR is a way of representing numbers such that we reduce the number of uneeded digits/zeros. Basically, lets say we have the number

> 0.0023141

then we can represent this as

> $2.3141 * 10^{-3}$

It is called floating point because we are moving the decimal point around. Normalized floating point is simply rewriting the number as

> $0.dddddddd * 10^{n}$

where the first digit is non-zero.

Round-off

There are 2 types of round-off :

1. Chopping --> simple cut off after a certain amount of significant digits.

2. Rounding --> estimating using the last significant digit, such that if it is ≥ 5, then we add one, otherwise, we add zero.

Propagation of Error

given that

$$p = \hat{p} + E_p$$

and

$$q = \hat{q} + E_q$$

then

$$p + q = \hat{p} + \hat{q} + E_p + E_q$$

That is to say, error may be increased in operations.

Order Of Approximation

Let us say that we are approximating e^h using the taylor expantion :

$$e^h = 1 + h + \frac{h^2}{2}$$

Then we know that the error is given by

$$E = \frac{f^3(c)h^3}{6} \rightarrow \frac{e^c h^3}{6}$$

so we can say

$$E \approx ch^3$$

we can express this error by saying :

$$O(h^3)$$

That is to say, the error is about some constant between `0` and h times h^3.

Formally, we say :

$$f(h) = p(h) + O(h^n)$$

then

$$f(h) \approx p(h)$$

with

$$E \approx C * h^n$$

Determening The Order Of Approximation In Operations.

Say that we have

$$f(h) = p(h) + O(h^n)$$

$$g(h) = k(h) + O(h^m)$$

Then

$$f(h) + g(h) = p(h) + k(h) + O(h^r)$$

where

1. r = min [n,m]
2. $h^l + O(h^r) = O(h^r)$, if $l \geq r$
