Discrete- and Continuous-Time Convolutions

Signals and Systems: Experiment 3

Yuzhou Li

yuzhouli@hust.edu.cn
School of Electronic Information and Communications

April 16, 2024

Overview

1. Discrete-Time Convolution
2. Continuous-Time Convolution

Overview

1. Discrete-Time Convolution

2. Continuous-Time Convolution

Discrete-Time Convolution-1

Discrete-time convolution is defined as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \tag{1}$$

Discrete-Time Convolution-1

Discrete-time convolution is defined as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \tag{1}$$

Assume that, the length of $x[n]$ is N and non-zero when $N_1 \leq n \leq N_2$, and the length of $h[n]$ is M and non-zero when $N_3 \leq n \leq N_4$

Discrete-Time Convolution-1

Discrete-time convolution is defined as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \tag{1}$$

Assume that, the length of $x[n]$ is N and non-zero when $N_1 \leq n \leq N_2$, and the length of $h[n]$ is M and non-zero when $N_3 \leq n \leq N_4$

$$\left.\begin{array}{c} N_1 \leq k \leq N_2 \\ N_3 \leq n-k \leq N_4 \end{array}\right\} \Rightarrow N_1 + N_3 \leq n \leq N_2 + N_4 \tag{2}$$

Discrete-Time Convolution-1

Discrete-time convolution is defined as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \tag{1}$$

Assume that, the length of $x[n]$ is N and non-zero when $N_1 \leq n \leq N_2$, and the length of $h[n]$ is M and non-zero when $N_3 \leq n \leq N_4$

$$\left.\begin{matrix} N_1 \leq k \leq N_2 \\ N_3 \leq n-k \leq N_4 \end{matrix}\right\} \Rightarrow N_1 + N_3 \leq n \leq N_2 + N_4 \tag{2}$$

Length of $y[n]$ is

$$L = (N_2 + N_4) - (N_1 + N_3) + 1 = (N_2 - N_1 + 1) + (N_4 - N_3 + 1) - 1 = N + M - 1 \tag{3}$$

Example 1: Standard Discrete-Time Convolution

MATLAB functions: conv(x,h), stem(L,y)

Signal 1:

$$x[n] = 1, \quad 0 \leq n \leq 4 \tag{4}$$

Signal 2:

$$h[n] = 1.1^n, \quad 0 \leq n \leq 6 \tag{5}$$

(1) Plot $x[n]$ and $h[n]$;
(2) Plot $x[n] * h[n]$.

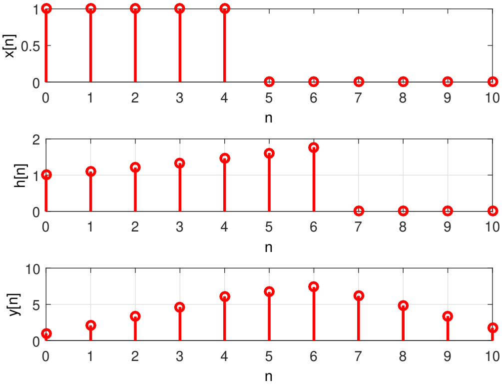

Figure 1.1: Standard discrete convolution.

Application 1: Polynomial Multiplication

Two polynomials are given by

$$
\begin{aligned}
A(x) &= a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n \\
B(x) &= b_0 + b_1 x + b_2 x^2 + \cdots + b_{m-1} x^{m-1} + b_m x^m
\end{aligned} \tag{6}
$$

Application 1: Polynomial Multiplication

Two polynomials are given by

$$\begin{aligned} A(x) &= a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n \\ B(x) &= b_0 + b_1 x + b_2 x^2 + \cdots + b_{m-1} x^{m-1} + b_m x^m \end{aligned} \tag{6}$$

Polynomial multiplication yields

$$C(x) = A(x) \cdot B(x) = \sum_{j=0}^{m+n} c_j x^j \tag{7}$$

where

$$c_j = \sum_{k=-\infty}^{\infty} a_k b_{j-k} = \sum_{\max\{0,j-m\}}^{\min\{n,j\}} a_k b_{j-k} \Rightarrow \boldsymbol{c} = \boldsymbol{a} * \boldsymbol{b} \tag{8}$$

Application 1: Polynomial Multiplication

Two polynomials are given by

$$\begin{aligned} A(x) &= a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n \\ B(x) &= b_0 + b_1 x + b_2 x^2 + \cdots + b_{m-1} x^{m-1} + b_m x^m \end{aligned} \tag{6}$$

Polynomial multiplication yields

$$C(x) = A(x) \cdot B(x) = \sum_{j=0}^{m+n} c_j x^j \tag{7}$$

where

$$c_j = \sum_{k=-\infty}^{\infty} a_k b_{j-k} = \sum_{\max\{0, j-m\}}^{\min\{n, j\}} a_k b_{j-k} \Rightarrow \boldsymbol{c} = \boldsymbol{a} * \boldsymbol{b} \tag{8}$$

Convolution theory: Replacing x by e^{-jw}, $A\left(e^{jw}\right)$ and $B\left(e^{jw}\right)$ are the DTFT of $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively. That is to say, time-domain convolution equals frequency-domain multiplication.

Application 2: Big Number Multiplication

Two numbers are given by

$$\begin{aligned} 789 &= 7 \times 10^2 + 8 \times 10^1 + 9 \times 10^0 \\ 345 &= 3 \times 10^2 + 4 \times 10^1 + 5 \times 10^0 \end{aligned} \tag{9}$$

Application 2: Big Number Multiplication

Two numbers are given by

$$\begin{aligned} 789 &= 7 \times 10^2 + 8 \times 10^1 + 9 \times 10^0 \\ 345 &= 3 \times 10^2 + 4 \times 10^1 + 5 \times 10^0 \end{aligned} \tag{9}$$

Mathematical numbers can be represented by

$$\begin{aligned} A = a_n a_{n-1} \cdots a_0 = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n = \sum_{i=0}^{n} a_j x_j \\ B = b_m b_{m-1} \cdots b_0 = b_0 + b_1 x + b_2 x^2 + \cdots + b_{m-1} x^{m-1} + b_m x^m = \sum_{i=0}^{m} b_j x_j \end{aligned} \tag{10}$$

Application 2: Big Number Multiplication

Two numbers are given by

$$
\begin{aligned}
789 &= 7 \times 10^2 + 8 \times 10^1 + 9 \times 10^0 \\
345 &= 3 \times 10^2 + 4 \times 10^1 + 5 \times 10^0
\end{aligned}
\tag{9}
$$

Mathematical numbers can be represented by

$$
\begin{aligned}
A = a_n a_{n-1} \cdots a_0 = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n = \sum_{i=0}^{n} a_j x_j \\
B = b_m b_{m-1} \cdots b_0 = b_0 + b_1 x + b_2 x^2 + \cdots + b_{m-1} x^{m-1} + b_m x^m = \sum_{i=0}^{m} b_j x_j
\end{aligned}
\tag{10}
$$

Multiplication yields

$$
C = \sum_{j=0}^{n+m} c_j x^j = \sum_{j=0}^{n+m} \left[\sum_{\max\{0,j-m\}}^{\min\{n,j\}} a_k b_{j-k} \right] x^j \tag{11}
$$

Application 2: Big Number Multiplication

Two numbers are given by

$$\begin{aligned} 789 &= 7 \times 10^2 + 8 \times 10^1 + 9 \times 10^0 \\ 345 &= 3 \times 10^2 + 4 \times 10^1 + 5 \times 10^0 \end{aligned} \tag{9}$$

Mathematical numbers can be represented by

$$\begin{aligned} A = a_n a_{n-1} \cdots a_0 = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n = \sum_{i=0}^{n} a_j x_j \\ B = b_m b_{m-1} \cdots b_0 = b_0 + b_1 x + b_2 x^2 + \cdots + b_{m-1} x^{m-1} + b_m x^m = \sum_{i=0}^{m} b_j x_j \end{aligned} \tag{10}$$

Multiplication yields

$$C = \sum_{j=0}^{n+m} c_j x^j = \sum_{j=0}^{n+m} \left[\sum_{\max\{0,j-m\}}^{\min\{n,j\}} a_k b_{j-k} \right] x^j \tag{11}$$

Time complexity: Exhaustive multiplication $O\left(n^2\right) \rightarrow$ FFT $O\left(n \log_2\left(n\right)\right)$.

Overview

Continuous-Time Convolution

Continuous-time convolution is defined as

$$f(t) = \int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau \tag{12}$$

Continuous-Time Convolution

Continuous-time convolution is defined as

$$f(t) = \int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau \tag{12}$$

Sample the signal by a sampling interval T

$$f(nT) = \sum_{m=-\infty}^{\infty} \int_{mT}^{mT+T} f_1(\tau)\, f_2(nT-\tau)\, d\tau \tag{13}$$

Continuous-Time Convolution

Continuous-time convolution is defined as

$$f(t) = \int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau \tag{12}$$

Sample the signal by a sampling interval T

$$f(nT) = \sum_{m=-\infty}^{\infty} \int_{mT}^{mT+T} f_1(\tau)\, f_2(nT-\tau)\, d\tau \tag{13}$$

If T is small enough, we have

$$f(nT) \approx T \sum_{m=-\infty}^{\infty} f_1(mT)\, f_2(nT-mT) = T \sum_{m=-\infty}^{\infty} f_1(mT)\, f_2((n-m)\,T) = \boldsymbol{f}_1 * \boldsymbol{f}_2 \tag{14}$$

Continuous-Time Convolution

Continuous-time convolution is defined as

$$f(t) = \int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau \tag{12}$$

Sample the signal by a sampling interval T

$$f(nT) = \sum_{m=-\infty}^{\infty} \int_{mT}^{mT+T} f_1(\tau)\, f_2(nT-\tau)\, d\tau \tag{13}$$

If T is small enough, we have

$$f(nT) \approx T \sum_{m=-\infty}^{\infty} f_1(mT)\, f_2(nT-mT) = T \sum_{m=-\infty}^{\infty} f_1(mT)\, f_2((n-m)\,T) = \boldsymbol{f}_1 * \boldsymbol{f}_2 \tag{14}$$

If $f_1(t), t \in [t_1, t_2]$ and $f_2(t), t \in [t_3, t_4]$, then $f(t), t \in [t_1 + t_3, t_2 + t_4]$.

Extension: Numerical Integral

One of numerical methods for integral

$$F\left(a, b\right) = \int_a^b f\left(t\right) dt \approx \frac{b-a}{N} \sum_{n=0}^{N-1} f\left(a + n\frac{b-a}{N}\right) \tag{15}$$

Advanced numerical methods include interpolation and fitting: fit().

Problem 1: Non-Standard Discrete-Time Convolution

MATLAB functions: conv(x,h), stem(L,y)

Signal 1:

$$x[n] = \{1, 4, 3, 5, 1, 2, 3, 2\}, -4 \leq n \leq 3 \tag{16}$$

Signal 2:

$$h[n] = \{3, 2, 4, 1, 3, 2\}, -3 \leq n \leq 2 \tag{17}$$

(1) Plot $x[n]$ and $h[n]$;

(2) Plot $x[n] * h[n]$.

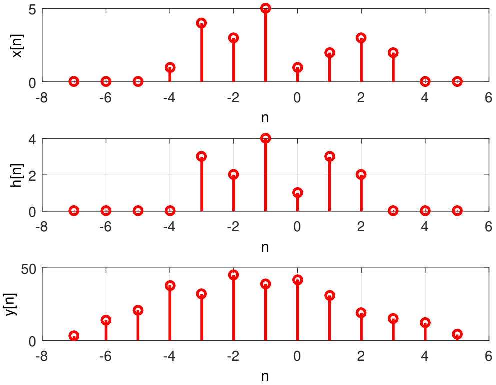

Figure 2.1: Non-standard discrete convolution.

Problem 2: Convolution of Two Continuous-Time Signals

Signal 1

$$x(t) = \begin{cases} 1, & -\frac{1}{2} \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases} \tag{18}$$

Signal 2

$$h(t) = \begin{cases} \frac{t}{2}, & 0 \leq t \leq 2 \\ 0, & \text{otherwise} \end{cases} \tag{19}$$

(1) Plot $x(t)$ and $h(t)$, $t \in [-1, 4]$;

(2) Plot $x(t) * h(t)$.

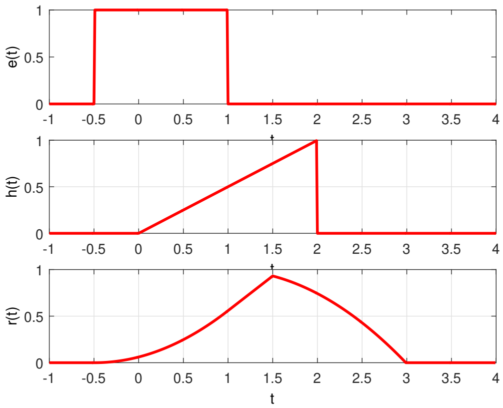

Figure 2.2: Continuous convolution.

Problem 3: Convolution and Convolution Theory

Triangular pulse:

$$f(t) = \begin{cases} E\left(1 - \frac{2|t|}{\tau}\right), & t \le \frac{\tau}{2} \\ 0, & t > \frac{\tau}{2} \end{cases} \tag{20}$$

with $E = 1$ and $\tau = 1$. For $t \in [-1, 1]$ and $w \in [-50, 50]$:

(1) Determine $g(t)$ that satisfies $f(t) = g(t) * g(t)$;

(2) Plot $f(t)$ and $g(t)$;

(3) Plot $F(w), G(w)$, and $G_e(w) = G(w) \cdot G(w)$;

(4) Plot $F_e(w) = \mathrm{FT}\{g(t) * g(t)\}$;

(5) Compare $F(w)$, $G_e(w)$, and $F_e(w)$.

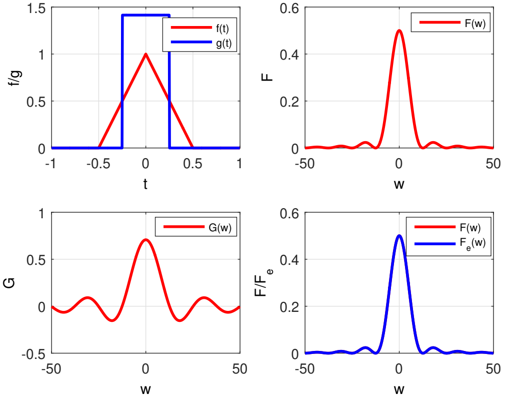

Figure 2.3: Verification of the convolution property.

Thank You!