HOMEWORK

数理方程与特殊函数

王翎羽　U202213806　提高 2201 班

2024 年 3 月 29 日

练习十

1. 设一无限长的弦作自由振动，弦的初始位移为 $\varphi(x)$ ，初始速度为 $-k\varphi^{'}(x)$ (k 为常数), 求此振动在时刻 t 在 x 处的位移 $u(x,t)$, 即求如下定解问题的解：

$$\begin{cases} u_{tt}=a^2u_{xx}, \quad -\infty<x<+\infty, t>0 \\ u(x,0)=\varphi(x), \quad u_t(x,0)=-k\varphi^{'}(x) \end{cases}$$

解：该问题是一个齐次方程问题。下面给出达朗贝尔解法:

令 $\xi=x-at, \eta=x+at$, 那么 $x=\dfrac{\xi+\eta}{2}, t=\dfrac{\xi-+\eta}{2a}$. 设 $u(x,t)=u(\dfrac{\xi+\eta}{2},\dfrac{\xi-\eta}{2a})=\overline{u}(\xi,\eta)$.

求导得 $u_x=\overline{u}_\xi+\overline{u}_\eta, \quad u_{xx}=\overline{u}_{\xi\xi}+2\overline{u}_{\xi\eta}+\overline{u}_{\eta\eta}$. 同理得 $u_{tt}=a^2(\overline{u}_{\xi\xi}-2\overline{u}_{\xi\eta}+\overline{u}_{\eta\eta})$.
代入方程中则有 $\overline{u}_{\xi\eta}=0$, 那么可得方程的通解 $\overline{u}(\xi,\eta)=f(\xi)+g(\eta)$, 其中 f 和 g 都是具二阶连续导数的任意函数. 那么原方程的通解为: $u(x,t)=f(x-at)+g(x+at)$. 代入初值条件，两边再同时积分可得 $a(-f(x)+g(x))+c=\displaystyle\int_{x_0}^{x}-k\varphi^{'}(\alpha)d\alpha$. 又有 $f(x+g(x)=\varphi(x)$. 那么

$$\begin{cases} f(x)=\dfrac{1}{2}\varphi(x)-\dfrac{1}{2a}\displaystyle\int_{x_0}^{x}-k\varphi^{'}(\alpha)d\alpha+\dfrac{c}{2a}, \\ g(x)=\dfrac{1}{2}\varphi(x)+\dfrac{1}{2a}\displaystyle\int_{x_0}^{x}-k\varphi^{'}(\alpha)d\alpha-\dfrac{c}{2a} \end{cases}$$

由 $u(x,t)=f(x-at)+g(x+at)$, 所以:

$$u(x,t)=\frac{\varphi(x-at)+\varphi(x+at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}-k\varphi^{'}(\alpha)d\alpha$$

$$=\frac{1}{2}\big(1-\frac{k}{a}\big)\varphi(x+at)+\frac{1}{2}\big(1+\frac{k}{a}\big)\varphi(x-at).$$

2. 求解定解问题：

$$\begin{cases} u_{tt}=a^2u_{xx}, \quad -\infty<x<+\infty, t>0 \\ u(x,0)=\sin x, \quad u_t(x,0)=x^2. \end{cases}$$

解：由达朗贝尔公式,

$$u(x,t)=\frac{\sin(x-at)+\sin(x+at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}\alpha^2d\alpha$$

$$=\sin x\cos at+\frac{1}{6}\big[2at\cdot(3x^2+a^2t^2)\big].$$

$$=\sin x\cos at+x^2t+\frac{1}{3}a^2t^3.$$

3. 求解定解问题：

$$\begin{cases} u_{tt} = a^2 u_{xx} + at + x, & -\infty < x < +\infty, t > 0 \\ u(x,0) = x, \quad u_t(x,0) = \sin x. \end{cases}$$

解：由达朗贝尔公式,

$$u(x,t) = \frac{(x-at)+(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \sin\alpha d\alpha + \frac{1}{2a}\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} (a\tau + \xi) d\xi d\tau$$

$$= x + \frac{\sin x \sin at}{a} + \frac{1}{2a}\int_0^t 2a^2\tau(t-\tau) + \frac{1}{2}(2x)[2a(t-\tau)]d\tau.$$
$$= x + \frac{\sin x \sin at}{a} + \frac{1}{6}at^3 + \frac{1}{2}xt^2.$$

练习十一

1. 用行波法求解下列定解问题：

$$\begin{cases} u_{tt} = a^2 u_{xx}, & -at < x < 0, \quad t > 0 \\ u|_{x=0} = \phi(t), \quad u|_{x+at=0} = \psi(t), \end{cases}$$

其中已知函数 ϕ, Φ 满足相容性条件 $\varphi(0) = \psi(0)$.

解: 设 $u(x,t) = f(x-at)+g(x+at)$. 由边界条件可知,$f(-at)+g(at) = \phi(t)$,$f(-2at)+g(0) = \psi(t)$. 让 $f(x-at)$ 和 $g(x+at)$ 用 $\phi(t)$ 和 $\psi(t)$ 来表示. 则有 $f(-at) = \psi(\frac{t}{2}) - g(0)$.$g(at) = \psi(t) - \phi(\frac{t}{2}) + g(0)$.

$g\big[a\big(t+\frac{x}{a}\big)\big] = \psi(t+\frac{x}{a}) - \phi(\frac{1}{2}(t+\frac{x}{a})) + g(0)$,$f\big[-a\big(t-\frac{x}{a}\big)\big] = \phi(\frac{1}{2}(t-\frac{x}{a})) - g(0)$.

所以 $u(x,t) = \psi(t+\frac{x}{a}) - \phi(\frac{1}{2}(t+\frac{x}{a})) + \phi(\frac{1}{2}(t-\frac{x}{a}))$

2. 求解以下三维波动方程的 $Cauchy$ 问题：

$$\begin{cases} u_{tt} = a^2(u_{xx} + u_{yy} + u_{zz}), & -\infty < x,y,z < +\infty, \quad t > 0 \\ u(x,y,z,0) = yz, \quad u_t(x,y,z,0) = xz. \end{cases}$$

解：由公式可得:

$$u(x,y,z,t) = \frac{\partial}{\partial t}\left(\frac{t}{4\pi}\int\int \varphi(M+atw)dw\right) + \frac{t}{4\pi}\int\int \psi(M+atw)dw$$

$$= \frac{\partial}{\partial t}\left(\frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi}(y+\sin\theta\sin\varphi)(z+\cos\theta)\sin\theta d\theta d\varphi\right)$$
$$+ \frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi}(x+\sin\theta\cos\varphi)(z+\cos\theta)\sin\theta d\theta d\varphi$$
$$= \frac{\partial}{\partial t}(\frac{t}{4\pi}\cdot 4\pi yz) + \frac{t}{4\pi}(4\pi yz)$$
$$= yz + xzt$$

3. 求解以下二维波动方程的 $Cauchy$ 问题：

$$\begin{cases} u_{tt} = a^2(u_{xx} + u_{yy}), & -\infty < x,y < +\infty, \quad t > 0 \\ u(x,y,0) = x^2(x+y), \quad u_t(x,y,0) = 0. \end{cases}$$

解：由公式可得:

$$
\begin{aligned}
u(x,y,t) &= \frac{1}{2\pi a}\frac{\partial}{\partial t}\left[\int\int \frac{\varphi(\xi,\eta)d\sigma}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}}\right] \\
&+ \frac{1}{2\pi a}\int\int \frac{\varphi(\xi,\eta)d\sigma}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}} \\
&= \frac{1}{2\pi a}\frac{\partial}{\partial t}\left[\int_0^{at}\int_0^{2\pi} \frac{(x+\rho\cos\theta)^2(x+\rho\cos\theta+y+\rho\sin\theta)\rho d\theta d\rho}{\sqrt{(at)^2-\rho^2}}\right] \\
&= \frac{1}{2\pi a}\frac{\partial}{\partial t}\left[\int_0^{at} \frac{2\pi\rho x^2(x+y)}{\sqrt{(at)^2-\rho^2}} + \frac{\pi\rho^3(3x+y)}{\sqrt{(at)^2-\rho^2}}\right] d\rho \\
&= \cdots \\
&= x^2(x+y)+a^2t^2(3x+y)
\end{aligned}
$$