Continuous-Time Fourier Transform

Signals and Systems: Experiment 2

Yuzhou Li

yuzhouli@hust.edu.cn
School of Electronic Information and Communications

April 9, 2024

Overview

Overview

Definitions of CTFT and ICTFT

Continuous-time Fourier transform (CTFT) is given by

$$F(w) = \int_{-\infty}^{\infty} f(t)\, e^{-jwt} dt \tag{1}$$

Definitions of CTFT and ICTFT

Continuous-time Fourier transform (CTFT) is given by

$$F(w) = \int_{-\infty}^{\infty} f(t)\, e^{-jwt} dt \tag{1}$$

Inverse continuous-time Fourier transform (ICTFT) is defined as

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(w)\, e^{jwt} dw \tag{2}$$

Example 1: Symbolic Functions for CTFT and ICTFT

Symbolic functions in MATLAB: fourier() and ifourier().

```
syms t;    % define a symbol t

FT0 = fourier(cos(t))    % calculate the FT of cos t

f1 = dirac(t);    % calculate the FT of δ(t)
FT1 = fourier(f1)

f2 = heaviside(t);    % calculate the FT of u(t)
FT2 = fourier(f2)

syms t0;    % calculate the FT of u(t − t0)
FT3 = fourier(heaviside(t - t0))
```

Overview

Loop Calculation for CTFT-1

Consider the main-value interval $[t_1, t_2]$

$$F(w) = \int_{-\infty}^{\infty} f(t)\, e^{-jwt} dt = \int_{t_1}^{t_2} f(t)\, e^{-jwt} dt \tag{3}$$

Loop Calculation for CTFT-1

Consider the main-value interval $[t_1, t_2]$

$$F(w) = \int_{-\infty}^{\infty} f(t)\, e^{-jwt} dt = \int_{t_1}^{t_2} f(t)\, e^{-jwt} dt \tag{3}$$

Define the interval length $T = t_2 - t_1$ and let N be the time-domain smapling number, then the smapling interval $\Delta t = \frac{T}{N}$

$$F(w) = \frac{T}{N} \sum_{n=0}^{N-1} f(t_1 + n\Delta t)\, e^{-jw(t_1 + n\Delta t)} \tag{4}$$

Loop Calculation for CTFT-1

Consider the main-value interval $[t_1, t_2]$

$$F(w) = \int_{-\infty}^{\infty} f(t)\, e^{-jwt} dt = \int_{t_1}^{t_2} f(t)\, e^{-jwt} dt \tag{3}$$

Define the interval length $T = t_2 - t_1$ and let N be the time-domain smapling number, then the smapling interval $\Delta t = \frac{T}{N}$

$$F(w) = \frac{T}{N} \sum_{n=0}^{N-1} f(t_1 + n\Delta t)\, e^{-jw(t_1 + n\Delta t)} \tag{4}$$

Consider $w \in [w_1, w_2]$ and K frequency-domain samples: $\Omega = w_2 - w_1$ and $\Delta w = \frac{\Omega}{K}$

$$F(w_1 + k\Delta w) = \frac{T}{N} \sum_{n=0}^{N-1} f(t_1 + n\Delta t)\, e^{-j(w_1 + k\Delta w)(t_1 + n\Delta t)} \tag{5}$$

Loop Calculation for CTFT-2

Formula for CTFT calculation in MATLAB

$$F\left(w_1 + k\Delta w\right) = \frac{T}{N}\sum_{n=0}^{N-1} f\left(t_1 + n\Delta t\right) e^{-j(w_1+k\Delta w)(t_1+n\Delta t)}$$

Pseudocode for loop calculation

for $k = 1, \cdots, K$
    for $n = 1, \cdots, N$
        $F[k, n] = F[k, n-1] + \frac{T}{N} f\left(t_1 + n\Delta t\right) e^{-j(w_1+k\Delta w)(t_1+n\Delta t)}$
    end
end

Loop Calculation for ICTFT-1

Inverse continuous-time Fourier transform is given by

$$f\left(t\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F\left(w\right)e^{jwt}dw=\frac{\Omega}{2\pi K}\sum_{k=0}^{K-1}F\left(w_1+k\Delta w\right)e^{j\left(w_1+k\Delta w\right)t} \tag{6}$$

Loop Calculation for ICTFT-1

Inverse continuous-time Fourier transform is given by

$$f\left(t\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F\left(w\right)e^{jwt}dw=\frac{\Omega}{2\pi K}\sum_{k=0}^{K-1}F\left(w_1+k\Delta w\right)e^{j\left(w_1+k\Delta w\right)t} \tag{6}$$

Discretize the time-domain signal

$$f\left(t_1+n\Delta t\right)=\frac{\Omega}{2\pi K}\sum_{k=0}^{K-1}F\left(w_1+k\Delta w\right)e^{j\left(w_1+k\Delta w\right)\left(t_1+n\Delta t\right)} \tag{7}$$

Loop Calculation for ICTFT-2

Formula for ICTFT calculation in MATLAB

$$f\left(t_1 + n\Delta t\right) = \frac{\Omega}{2\pi K}\sum_{k=0}^{K-1} F\left(w_1 + k\Delta w\right) e^{j(w_1+k\Delta w)(t_1+n\Delta t)}$$

Pseudocode for loop calculation

for $n = 1, \cdots, N$
    for $k = 1, \cdots, K$
        $f[n,k] = f[n,k-1] + \frac{\Omega}{2\pi K} F\left(w_1 + k\Delta w\right) e^{j(w_1+k\Delta w)(t_1+n\Delta t)}$
    end
end

Example 1: Loop calculation for CTFT and ICTFT

Rectangular pulse

$$f\left(t\right) = \begin{cases} 1, & |x| < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases} \tag{8}$$

(1) Plot $f\left(t\right), t \in [-1, 1]$;

(2) Plot $F\left(w\right), w \in [-8\pi, 8\pi]$;

(3) Recover $f\left(t\right)$ from $F\left(w\right)$.

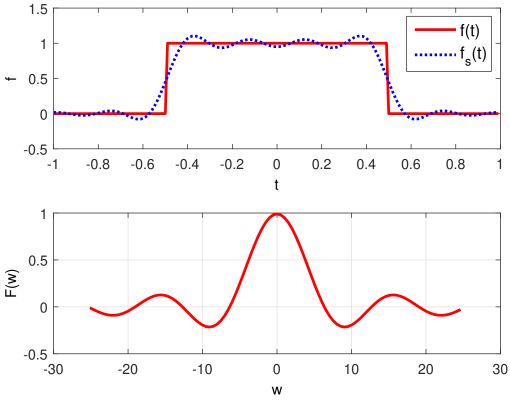

Figure 2.1: Waveform and spectrum

Overview

Vector Product for CTFT

Formula for CTFT calculation in MATLAB

$$F\left(w_1 + k\Delta w\right) = \frac{T}{N}\sum_{n=0}^{N-1} f\left(t_1 + n\Delta t\right) e^{-j(w_1+k\Delta w)(t_1+n\Delta t)}$$

Vector Product for CTFT

Formula for CTFT calculation in MATLAB

$$F\left(w_1 + k\Delta w\right) = \frac{T}{N} \sum_{n=0}^{N-1} f\left(t_1 + n\Delta t\right) e^{-j(w_1 + k\Delta w)(t_1 + n\Delta t)}$$

Vector form for CTFT calculation

$$F(w_1 + k\Delta w) = \frac{T}{N} \begin{bmatrix} e^{-j(w_1+k\Delta w)t_1} & e^{-j(w_1+k\Delta w)(t_1+\Delta t)} & \dots & e^{-j(w_1+k\Delta w)(t_2-\Delta t)} \end{bmatrix} \begin{bmatrix} f(t_1) \\ f(t_1 + \Delta t) \\ \vdots \\ f(t_2 - \Delta t) \end{bmatrix} \tag{9}$$

Vector Product for CTFT

Formula for CTFT calculation in MATLAB

$$F\left(w_1 + k\Delta w\right) = \frac{T}{N}\sum_{n=0}^{N-1} f\left(t_1 + n\Delta t\right) e^{-j(w_1+k\Delta w)(t_1+n\Delta t)}$$

Vector form for CTFT calculation

$$F(w_1 + k\Delta w) = \frac{T}{N}\begin{bmatrix} e^{-j(w_1+k\Delta w)t_1} & e^{-j(w_1+k\Delta w)(t_1+\Delta t)} & \cdots & e^{-j(w_1+k\Delta w)(t_2-\Delta t)} \end{bmatrix} \begin{bmatrix} f(t_1) \\ f(t_1+\Delta t) \\ \vdots \\ f(t_2-\Delta t) \end{bmatrix} \tag{9}$$

Pseudocode for vector product

for $k = 1, \cdots, K$

$\quad F_k = \frac{T}{N}\boldsymbol{a}_k^\top \boldsymbol{f}$

end

Vector Product for ICTFT

Formula for ICTFT calculation in MATLAB

$$f\left(t_1 + n\Delta t\right) = \frac{\Omega}{2\pi K} \sum_{k=0}^{K-1} F\left(w_1 + k\Delta w\right) e^{j(w_1+k\Delta w)(t_1+n\Delta t)}$$

Vector Product for ICTFT

Formula for ICTFT calculation in MATLAB

$$f\left(t_1 + n\Delta t\right) = \frac{\Omega}{2\pi K} \sum_{k=0}^{K-1} F\left(w_1 + k\Delta w\right) e^{j(w_1 + k\Delta w)(t_1 + n\Delta t)}$$

Vector form for ICTFT calculation

$$f(t_1 + n\Delta t) = \frac{\Omega}{2\pi K} \left[e^{jw_1(t_1+n\Delta t)} \quad e^{j(w_1+\Delta w)(t_1+n\Delta t)} \quad \dots \quad e^{j(w_2-\Delta w)(t_1+n\Delta t)} \right] \begin{bmatrix} F(w_1) \\ F(w_1 + \Delta w) \\ \vdots \\ f(w_2 - \Delta w) \end{bmatrix} \tag{10}$$

Vector Product for ICTFT

Formula for ICTFT calculation in MATLAB

$$f\left(t_1+n\Delta t\right)=\frac{\Omega}{2\pi K}\sum_{k=0}^{K-1}F\left(w_1+k\Delta w\right)e^{j(w_1+k\Delta w)(t_1+n\Delta t)}$$

Vector form for ICTFT calculation

$$f(t_1+n\Delta t)=\frac{\Omega}{2\pi K}\begin{bmatrix}e^{jw_1(t_1+n\Delta t)} & e^{j(w_1+\Delta w)(t_1+n\Delta t)} & \cdots & e^{j(w_2-\Delta w)(t_1+n\Delta t)}\end{bmatrix}\begin{bmatrix}F(w_1)\\ F(w_1+\Delta w)\\ \vdots\\ f(w_2-\Delta w)\end{bmatrix} \tag{10}$$

Pseudocode for vector product

for $n=1,\cdots,N$

$\quad f_n=\frac{\Omega}{2\pi K}\boldsymbol{b}_n^\top\boldsymbol{F}$

end

Example 2: Vector Product for CTFT and ICTFT

Rectangular pulse

$$f(t) = \begin{cases} 1, & |x| < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases} \quad (11)$$

(1) Plot $f(t), t \in [-1, 1]$;

(2) Plot $F(w), w \in [-8\pi, 8\pi]$;

(3) Recover $f(t)$ from $F(w)$.

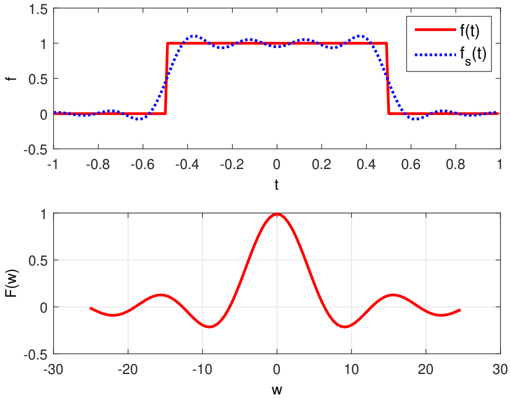

Figure 3.1: Waveform and spectrum

Overview

Matrix Product for CTFT and ICTFT-1

Matrix form for the CTFT

$$\begin{bmatrix} F(w_1) \\ F(w_1+\Delta w) \\ \vdots \\ f(w_2-\Delta w) \end{bmatrix} = \frac{T}{N} \begin{bmatrix} e^{-jw_1 t_1} & e^{-jw_1(t_1+\Delta t)} & \cdots & e^{-jw_1(t_2-\Delta t)} \\ e^{-j(w_1+\Delta w)t_1} & e^{-j(w_1+\Delta w)(t_1+\Delta t)} & \cdots & e^{-j(w_1+\Delta w)(t_2-\Delta t)} \\ \vdots & \vdots & & \vdots \\ e^{-j(w_2-\Delta w)t_1} & e^{-j(w_2-\Delta w)(t_1+\Delta t)} & \cdots & e^{-j(w_2-\Delta w)(t_2-\Delta t)} \end{bmatrix} \begin{bmatrix} f(t_1) \\ f(t_1+\Delta t) \\ \vdots \\ f(t_2-\Delta t) \end{bmatrix} \quad (12)$$

Matrix Product for CTFT and ICTFT-1

Matrix form for the CTFT

$$\begin{bmatrix} F(w_1) \\ F(w_1+\Delta w) \\ \vdots \\ f(w_2-\Delta w) \end{bmatrix} = \frac{T}{N} \begin{bmatrix} e^{-jw_1 t_1} & e^{-jw_1(t_1+\Delta t)} & \cdots & e^{-jw_1(t_2-\Delta t)} \\ e^{-j(w_1+\Delta w)t_1} & e^{-j(w_1+\Delta w)(t_1+\Delta t)} & \cdots & e^{-j(w_1+\Delta w)(t_2-\Delta t)} \\ \vdots & \vdots & & \vdots \\ e^{-j(w_2-\Delta w)t_1} & e^{-j(w_2-\Delta w)(t_1+\Delta t)} & \cdots & e^{-j(w_2-\Delta w)(t_2-\Delta t)} \end{bmatrix} \begin{bmatrix} f(t_1) \\ f(t_1+\Delta t) \\ \vdots \\ f(t_2-\Delta t) \end{bmatrix} \quad (12)$$

Matrix form for the ICTFT

$$\begin{bmatrix} f(t_1) \\ f(t_1+\Delta t) \\ \vdots \\ f(t_2-\Delta t) \end{bmatrix} = \frac{\Omega}{2\pi K} \begin{bmatrix} e^{jw_1 t_1} & e^{j(w_1+\Delta w)t_1} & \cdots & e^{j(w_2-\Delta w)t_1} \\ e^{jw_1(t_1+\Delta t)} & e^{j(w_1+\Delta w)(t_1+\Delta t)} & \cdots & e^{j(w_2-\Delta w)(t_1+\Delta t)} \\ \vdots & \vdots & & \vdots \\ e^{jw_1(t_2-\Delta t)} & e^{j(w_1+\Delta w)(t_2-\Delta t)} & \cdots & e^{j(w_2-\Delta w)(t_2-\Delta t)} \end{bmatrix} \begin{bmatrix} F(w_1) \\ F(w_1+\Delta w) \\ \vdots \\ f(w_2-\Delta w) \end{bmatrix} \quad (13)$$

Matrix Product for CTFT and ICTFT-1

Matrix form for the CTFT

$$\begin{bmatrix} F(w_1) \\ F(w_1+\Delta w) \\ \vdots \\ f(w_2-\Delta w) \end{bmatrix} = \frac{T}{N} \begin{bmatrix} e^{-jw_1 t_1} & e^{-jw_1(t_1+\Delta t)} & \cdots & e^{-jw_1(t_2-\Delta t)} \\ e^{-j(w_1+\Delta w)t_1} & e^{-j(w_1+\Delta w)(t_1+\Delta t)} & \cdots & e^{-j(w_1+\Delta w)(t_2-\Delta t)} \\ \vdots & \vdots & & \vdots \\ e^{-j(w_2-\Delta w)t_1} & e^{-j(w_2-\Delta w)(t_1+\Delta t)} & \cdots & e^{-j(w_2-\Delta w)(t_2-\Delta t)} \end{bmatrix} \begin{bmatrix} f(t_1) \\ f(t_1+\Delta t) \\ \vdots \\ f(t_2-\Delta t) \end{bmatrix} \tag{12}$$

Matrix form for the ICTFT

$$\begin{bmatrix} f(t_1) \\ f(t_1+\Delta t) \\ \vdots \\ f(t_2-\Delta t) \end{bmatrix} = \frac{\Omega}{2\pi K} \begin{bmatrix} e^{jw_1 t_1} & e^{j(w_1+\Delta w)t_1} & \cdots & e^{j(w_2-\Delta w)t_1} \\ e^{jw_1(t_1+\Delta t)} & e^{j(w_1+\Delta w)(t_1+\Delta t)} & \cdots & e^{j(w_2-\Delta w)(t_1+\Delta t)} \\ \vdots & \vdots & & \vdots \\ e^{jw_1(t_2-\Delta t)} & e^{j(w_1+\Delta w)(t_2-\Delta t)} & \cdots & e^{j(w_2-\Delta w)(t_2-\Delta t)} \end{bmatrix} \begin{bmatrix} F(w_1) \\ F(w_1+\Delta w) \\ \vdots \\ f(w_2-\Delta w) \end{bmatrix} \tag{13}$$

Simplified matrix form

$$\boldsymbol{F} = \frac{T}{N}\boldsymbol{U}\boldsymbol{f}, \qquad \boldsymbol{f} = \frac{\Omega}{2\pi K}\boldsymbol{V}\boldsymbol{F} \tag{14}$$

Matrix Product for CTFT and ICTFT-2

How to obtain $\boldsymbol{U}$ and $\boldsymbol{V}$: Kronecker product

$$
\begin{aligned}
&\begin{bmatrix} w_1 \\ w_1 + \Delta w \\ \vdots \\ w_2 - \Delta w \end{bmatrix} \otimes \begin{bmatrix} t_1 & t_1 + \Delta t & \cdots & t_2 - \Delta t \end{bmatrix} \\
&= \begin{bmatrix} w_1 t_1 & w_1(t_1 + \Delta t) & \cdots & w_1(t_2 - \Delta t) \\ (w_1 + \Delta w)t_1 & (w_1 + \Delta w)(t_1 + \Delta t) & \cdots & (w_1 + \Delta w)(t_2 - \Delta t) \\ \vdots & \vdots & & \vdots \\ (w_2 - \Delta w)t_1 & (w_2 - \Delta w)(t_1 + \Delta t) & \cdots & (w_2 - \Delta w)(t_2 - \Delta t) \end{bmatrix}
\end{aligned} \tag{15}
$$

$\otimes$ denotes Kronecker tensor product: kron() in MATLAB.

Example 3: Matrix product for CTFT and ICTFT

Rectangular pulse

$$f\left(t\right)=\begin{cases}1, & |x|<\frac{1}{2}\\ 0, & \text{otherwise}\end{cases} \quad (16)$$

(1) Plot $f\left(t\right), t\in[-1,1]$;

(2) Plot $F\left(w\right), w\in[-8\pi,8\pi]$;

(3) Recover $f\left(t\right)$ from $F\left(w\right)$.

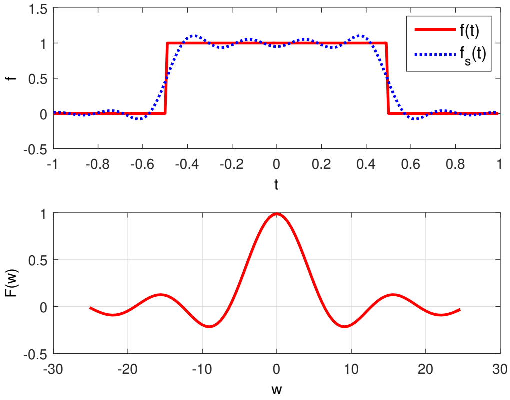

Figure 4.1: Waveform and spectrum

Overview

1. Symbolic Function
2. Loop Calculation
3. Vector Product
4. Matrix Product
5. Experiment Problem

Problem 1: CTFT and ICTFT

Triangular pulse:

$$f(t) = \begin{cases} E\left(1 - \frac{2|t|}{\tau}\right), & t \leq \frac{\tau}{2} \\ 0, & t > \frac{\tau}{2} \end{cases} \tag{17}$$

with $E = 1$ and $\tau = 1$. For $t \in [-1, 1]$ and $w \in [-50, 50]$:

(1) Determine $g(t)$ that satisfies $f(t) = g(t) * g(t)$ and plot $f(t)$ and $g(t)$;

(2) Plot $F(w), G(w)$, and $G_e(w) = G(w) \cdot G(w)$;

(3) Plot $F_e(w) = \mathrm{CTFT}\{g(t) * g(t)\}$;

(4) Compare $F(w)$, $G_e(w)$, and $F_e(w)$;

(5) Compare the time costs of 3 methods in a bar form.

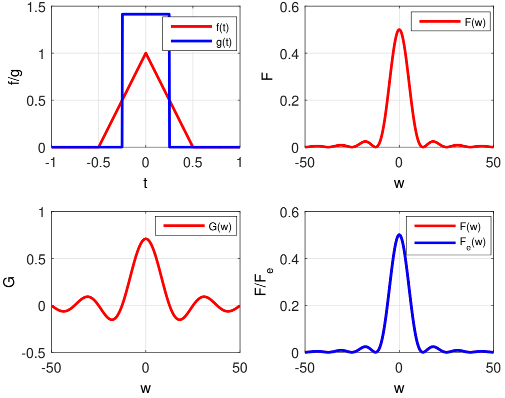

Figure 5.1: Waveform and spectrum.

Thank You!