Fundamentals of Information Theory

Homework Four

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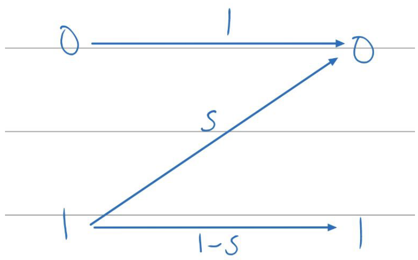
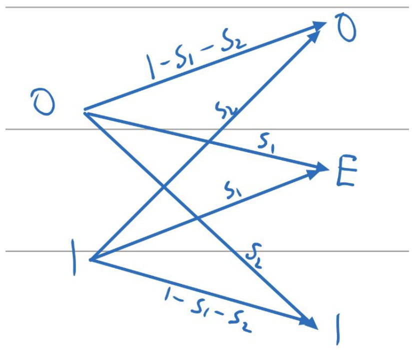
Problem 1 Solutions:

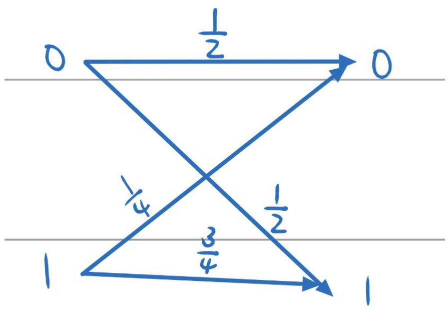
(a)



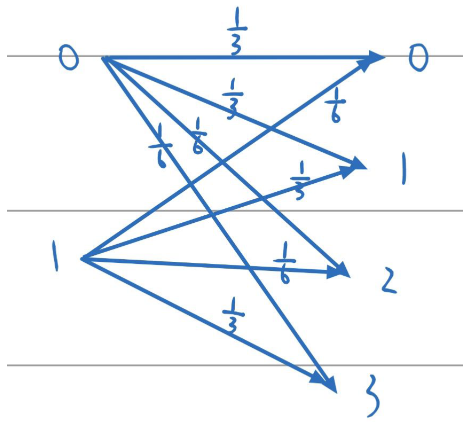
(b)



(c)



(d)



Problem 2 Solutions:

Channel Model

Given the channel is memoryless and the noise Z is additive and independent of the input X, the output Y can be expressed as:

$$Y = X \oplus Z$$

where $\oplus$ denotes the binary addition (XOR).

Channel Matrix

The channel transition probabilities are:

$X \backslash Y$	0	1
0	$\Pr[Z = 0]$	$\Pr[Z = a]$
1	$\Pr[Z = a]$	$\Pr[Z = 0]$

with $\Pr[Z = 0] = \Pr[Z = a] = \frac{1}{2}$.

Channel Capacity Calculation

The channel capacity C is given by:

$$C = \max_{p(x)} I(X;Y)$$

Using the symmetry of the channel and the uniform distribution of Z, the mutual information $I(X;Y)$ is maximized when X is also uniformly distributed. This gives us:

$$I(X;Y) = H(Y) - H(Y|X)$$

Since $Y = X \oplus Z$, and both X and Z are independent and uniformly distributed:

$$H(Y) = 1 \text{ bit (Entropy of a fair coin)}$$

$$H(Y|X) = H(Z) = 1 \text{ bit}$$

$$C = 1 - 1 = 0 \text{ bits}$$

This indicates that the capacity of this channel is 0 bits per channel use, meaning no reliable communication is possible if $a = 0$. If $a \neq 0$, further analysis on the value of a would be required to compute its specific impact on the channel capacity.

Problem 3 Solutions:

(a) Given the transition probabilities shown in the figure:

$X \backslash Y$	0	E	1
0	$1-\alpha-e$	α	e
1	e	α	$1-\alpha-e$

The capacity C of this channel is given by:

$$C = \max_{p(x)} I(X;Y)$$

where $I(X;Y)$ is the mutual information between X and Y. Assume X is distributed Bernoulli(0.5) for symmetry. The capacity can then be computed as:

$$C = 1 - H_b(e) - \alpha \log(2)$$

where $H_b(p)$ is the binary entropy function $-p \log_2(p) - (1-p) \log_2(1-p)$.

(b)

$$C = 1 - H_b(e)$$

(c)

$$C = 1 - \alpha$$