Fundamentals of Information Theory

Homework Two

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Problem 1 Solutions:

If behind No.1 door is car, then the probability of switching to No.2 door winning the car is 0.If behind No.1 door is goat, then the probability of switching to No.2 door winning the car is $\frac{2}{3} \cdot 1 = \frac{2}{3}$.So we should pick No.2 door as the host suggests.

Problem 2 Solutions:

(a) Every coin can be fake, and the fake coin can be lighter or heavier, so there's total 48 possibilities.The uncertainty is $-\log_2 \frac{1}{48} = 5.585$.The max entropy elimination of each weigh occurs when the balance is balance, which is $\log_2 3 = 1.585$.So the minimum number is $1 + [\frac{5.585}{1.585}] = 4$.

(b) The steps are as follows.

Step 1 and 2: Divide the 24 coins into three groups of 8 coins each.Group 1 weighs against group 2 and 3 respectively.Then we know which group has fake coin and whether it is lighter or heavier.

Step 3: Divide the 8 coins into three groups of 3,3,2 coins each.Then we can get to know which group has fake coins.

Step 4: Obviously we can get to know which coin is fake in the group which has only 2 or 3 coins.

Problem 3 Solutions:

(a) $H(X) = H(Y) = \frac{1}{3}\log_2\frac{1}{3} - \frac{2}{3}\log_2\frac{2}{3} = 0.918$ bits,

(b) $H(X|Y) = -\sum_{x\in X}\sum_{y\in Y} p(x,y)\log p(x|y) = \frac{2}{3}$,as the same,$H(Y|X) = \frac{2}{3}$ bits,

(c) $H(X,Y) = -\sum_{x\in X}\sum_{y\in Y} p(x,y)\log[p(x,y)] = 1.585$ bits,

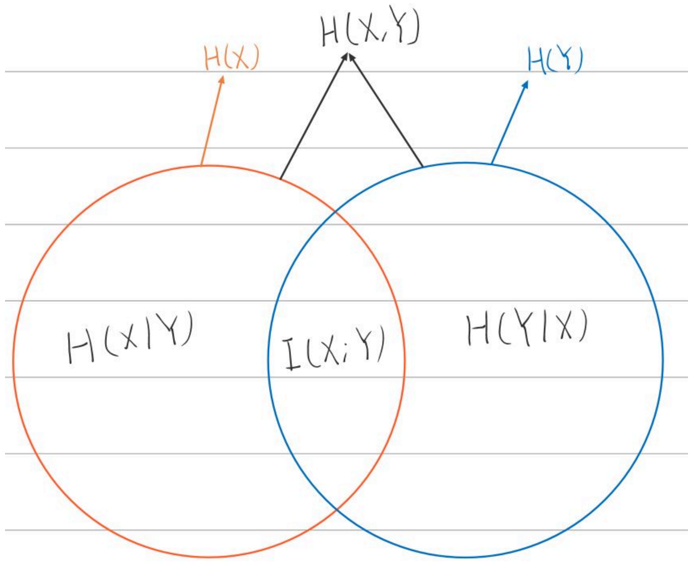
(d) $H(Y) - H(Y|X) = I(X;Y) = 0.918 - \frac{2}{3} = 0.251$ bits,

(e)

$$
\begin{aligned}
I(X;Y) &= D[p(x,y)||p(x)p(y)] \\
&= \sum_{x\in X}\sum_{y\in Y} p(x,y)\log\left[\frac{p(x,y)}{p(x)p(y)}\right] \\
&= \sum_{x\in X}\sum_{y\in Y} p(x,y)\log\left[\frac{p(x|y)}{p(x)}\right] \\
&= \sum_{x\in X}\sum_{y\in Y} p(x,y)\log[p(x|y)] - \sum_{x\in X}\sum_{y\in Y} p(x,y)\log[p(x)] \\
&= \sum_{x\in X}\sum_{y\in Y} p(x,y)\log[p(x|y)] - \sum_{x\in X} p(x)\log[p(x)] \\
&= H(X) - H(X|Y)
\end{aligned}
$$

so $H(Y) - H(Y|X) = I(X;Y) = 0.918 - \dfrac{2}{3} = 0.251$ bits.

(f)



Problem 4 Solutions:

$H(p(x)) = 1.5$ bits, $\quad H(q(x)) = 1.585$ bits.

$D[p(x)||q(x)] = \sum_{x \in X} p(x) \log\left[\dfrac{p(x)}{q(x)}\right] = 0.085$ bits, $D[q(x)||p(x)] = \sum_{x \in X} q(x) \log\left[\dfrac{p(x)}{q(x)}\right] = 0.082$ bits.

So $D[p(x)||q(x)] \neq D[q(x)||p(x)]$.

Problem 5 Solutions:

(a) $H(X) = \sum_{i=1}^{8} p_i(x) \log p_i(x) = 2$ bits.

(b) $1024 \times 3 = 3072$ bits.

(c) $T_{c_2} = \sum_{i=A}^{H} S_i N_i = 2051$ bits, of which S_i denotes for bits required to send symbol i, while N_i denotes for number of symbol i. The average bits used to transmit one symbol is $\dfrac{2051}{1024} = 2.0$ bits.

Problem 6 Solutions:

(a) $\rho = \dfrac{H(X_1) - H(X_2|X_1)}{H(X_1)} = \dfrac{I(X_1;X_2)}{H(X_1)}$.

(b) As we know,$I(X_1;X_2) \geq 0$ and $I(X_1;X_2) \leq \min(H(X_1), H(X_2))$. So $0 \leq \rho \leq 1$.

(c) $\rho = 0 \Leftrightarrow I(X_1;X_2) = 0 \Leftrightarrow X_1$ andX_2 are independent.

(d) $\rho = 1 \Leftrightarrow I(X_1;X_2) = H(X_1) \Leftrightarrow H(X_1) - H(X_1|X_2) = H(X_1) \Leftrightarrow X_1 = X_2$.

Problem 7 Solutions:

(a) By chain rules, $H(X,Y|Z) = H(X|Z) + H(Y|X,Z) \geq H(X|Z)$, and Y is independent of X given Z for equality.

(b) By chain rules, $I(X,Y;Z) = H(X) - H(X|Y,Z)$ and $I(X;Z) = H(X) - H(X|Z)$, $H(X,Y|Z) \geq H(X|Z)$, So $I(X,Y;Z) > I(X:Z)$.The equality condition is the same as last problem.

(c) $H(X,Y,Z) - H(X,Y) = H(Z|X,Y)$,$H(X,Z) - H(X) = H(Z|X)$, as $H(Z|X,Y) \leq H(Z|X)$, $H(X,Y,Z) - H(X,Y) \leq H(X,Z) - H(X)$. The equality condition is $X = Y$.

(d)

$$\begin{aligned}
Right &= I(Z;Y|X) - I(Z;Y) + I(X;Z) \\
&= H(Z|X) - H(Z|X,Y) - (H(Z) - H(Z|Y)) + (H(Z) - H(Z|X)) \\
&= H(Z|Y) - H(Z|X,Y) \\
&= I(Z;X|Y)
\end{aligned}$$

Obviously, $Right = Left$, so the inequality is actually an equality.