# HOMEWORK

# 数理方程与特殊函数

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## 练习十七

1. 证明：

   (1) $\dfrac{d}{dx}[xJ_0(x)J_1(x)] = x[J_0^2(x) - J_1^2(x)]$；

   (2) $\displaystyle\int x^2 J_1(x)dx = 2xJ_1(x) - x^2 J_0(x) + c$;

   (3) $J_2(x) - J_0(x) = 2J_0''(x)$;

   (4) $\displaystyle\int x^n J_0(x)dx = x^n J_1(x) + (n-1)x^{n-1}J_0(x) - (n-1)^2 \int x^{n-2}J_0(x)dx.$

   解：

   (1)

$$
\begin{aligned}
\frac{d}{dx}[xJ_0(x)J_1(x)] &= \frac{d}{dx}[xJ_1(x)]J_0(x) + \frac{d}{dx}[J_0(x)]xJ_1(x) \\
&= xJ_0^2(x) - xJ_1^2(x) \\
&= x[J_0^2(x) - J_1^2(x)]
\end{aligned}
$$

   (2)

$$
\begin{aligned}
\int x^2 J_1(x)dx &= -\int x^2 d(J_0(x)) \\
&= -(x^2 J_0(x) - \int J_0(x)dx^2) \\
&= -x^2 J_0(x) + 2\int J_0(x)dx \\
&= 2xJ_1(x) - x^2 J_0(x) + c
\end{aligned}
$$

   (3)

$$
J_2(x) - J_0(x) = -2J_1'(x) = 2J_0''(x)
$$

   (4)

$$
\begin{aligned}
\int x^n J_0(x)dx &= \int x^{n-1} d[xJ_1(x)] \\
&= x^n J_1(x) - (n-1)\int x^{n-1}J_0(x)dx \\
&= x^n J_1(x) - (n-1)\big(\int x^{n-2} d[xJ_1(x)]\big) \\
&= x^n J_1(x) + (n-1)x^{n-1}J_0(x) - (n-1)^2 \int x^{n-2}J_0(x)dx
\end{aligned}
$$

2. 计算积分:

(1) $\int_0^3 (3-x)J_0(\frac{\mu_2^{(0)}}{3}x)dx$

(2) $\int_0^R r(R^2-r^2)J_0(\frac{\mu_m^{(0)}}{R}r)dr.$

解：

(1) 设 $u=\frac{\mu_2^{(0)}}{3}x$，则 $x=\frac{3}{\mu_2^{(0)}}u$ 和 $dx=\frac{3}{\mu_2^{(0)}}du$ 代入原积分，得到：

$$\int_0^3 (3-x)J_0\left(\frac{\mu_2^{(0)}}{3}x\right)dx=\int_0^{\mu_2^{(0)}}\left(3-\frac{3}{\mu_2^{(0)}}u\right)J_0(u)\frac{3}{\mu_2^{(0)}}du.$$

简化后得到：

$$\frac{9}{\mu_2^{(0)}}\int_0^{\mu_2^{(0)}}\left(1-\frac{u}{\mu_2^{(0)}}\right)J_0(u)du=-\frac{9}{\mu_2^{(0)}}J_1(\mu_2^{(0)}).$$

(2) 设 $u=\frac{\mu_m^{(0)}}{R}r$，则 $r=\frac{R}{\mu_m^{(0)}}u$ 和 $dr=\frac{R}{\mu_m^{(0)}}du$。
代入原积分，得到：

$$\int_0^R r(R^2-r^2)J_0\left(\frac{\mu_m^{(0)}}{R}r\right)dr=\int_0^{\mu_m^{(0)}}\frac{R}{\mu_m^{(0)}}u\left(R^2-\left(\frac{R}{\mu_m^{(0)}}u\right)^2\right)J_0(u)\frac{R}{\mu_m^{(0)}}du.$$

简化后得到：

$$\frac{R^3}{{\mu_m^{(0)}}^2}\int_0^{\mu_m^{(0)}}u\left(1-\frac{u^2}{{\mu_m^{(0)}}^2}\right)J_0(u)du=\frac{2R^4J_2(\mu_m^{(0)})}{\mu_m^{(0)2}}.$$