Random Process Exam For Advanced Class

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1. $X(t) = A\sin(\pi t) + B\cos(\pi t)$, where A and B are Gaussian random variables with mean 0 and variance σ^2.

(1) Find the probability density function (PDF) of $X(1/2)$.

(2) Determine $\mathbb{E}[X(t)]$.

(3) Determine whether $X(t)$ is a stationary process.

2. Let $Y(t) = X(t) - X(t+\tau)$, where $X(t)$ is a wide-sense stationary process with zero mean.

(1) Find $\mathbb{E}[Y(t)]$.

(2) Find $\mathrm{Var}[Y(t)]$.

(3) Repeat parts (1) and (1) for $Y(t) = X(t) - X(t-\tau)$ and compare the results.

3. Prove the following properties:

(1) $R_{X\hat{X}}(\tau) = -\hat{R}_X(\tau)$

(2) $S_{\hat{X}}(f) = S_X(f)$

(3) Find $S_Z(f)$ for $Z = X(t) + j\hat{X(t)}$.

where $\hat{X}$ represents the result of the Hilbert transform of X.

4. Let $X(t)$ be a random process with autocorrelation function $R_X(\tau) = \dfrac{\sin(\pi\tau)}{\pi\tau}$. The process is passed through a linear time-invariant system with impulse response $h_1(t) = A\mathrm{sinc}^2(t)$, the output is $Y(t)$. The output power of $Y(t)$ is 3 milli-Watts.

(1) Find $\mathbb{E}[Y]$.

(2) Find $R_Y(\tau)$.

(3) If the impulse response is changed to $h_2(t) = 2A\mathrm{sinc}^2(t)$, find the output power of $Y_2(t)$.

(4) If the impulse response is changed to $h_3(t) = A\text{sinc}^2(t-1)$, find the output power of $Y_3(t)$.

5. Let $X(t)$ be a random process with autocorrelation function

$$R_X(\tau) = \begin{cases} 1-|\tau|, & |\tau| \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Define $Y(t) = X(t)\cos(w_c t + \theta)$, where $\theta \sim U[-\pi, \pi]$ and θ is independent of $X(t)$.

(1) Find $\mathbb{E}[Y]$.

(2) Find $R_Y(\tau)$.

(3) Sketch $R_Y(\tau)$.

(4) Find $S_Y(f)$.

(5) Find the power of $Y(t)$.

6. Let $Y(t) = \sum_{n=1}^{N(t)} X_n$, where X_n are i.i.d. random variables with $P(X_n = 1) = p$ and $P(X_n = 0) = 1-p$. $N(t)$ is a Poisson process with intensity γ.

(1) Explain why $Y(t)$ is a stationary process with independent increments.

(2) Find the characteristic function $\phi_Y(t)$.

(3) Find $\mathbb{E}[Y]$ and $\text{Var}[Y]$.

7. Green cars arrive according to a Poisson process with rate λ_1, and red cars arrive according to a Poisson process with rate λ_2.

(1) Find the distribution of the total number of cars (green + red) that arrive.

(2) Find the probability that the first car to arrive is red.

8. Data packets are transmitted according to a Poisson process with a rate of 10 packets per second. Let X_k denote the number of data packets transmitted in the k-th hour. Each packet has a probability of 0.8% of being transmitted erroneously.

(1) Find the joint distribution of X_1 and X_2.

(2) Find the probability that at least one packet is transmitted erroneously between 12:00 and 12:10.

9. In Example 4 of Chapter 6.4 (pdf), let $p_{12} = A$.

(1) Calculate the value of A.

(2) Sketch the transition diagram.

(3) Explain why the Markov chain is ergodic.

(4) Find the stationary distribution of the Markov chain.

10. There are 3 white balls and 3 black balls distributed in 2 boxes, with each box containing 3 balls. Let X_n denote the number of white balls in the first box after the n-th swap. In each step, one ball is randomly selected from each box and the two balls are swapped.

(1) Explain why $\{X_n\}$ is a Markov chain.

(2) Compute the transition probability matrix of the Markov chain.