Fundamentals of Information Theory

Data Compression

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Outline

- Efficiency vs. Reliability
- Three key questions about data compression
- What is source coding?
- Get to know some codes
- What do we want from a source code?
- Kraft inequality——constraints on prefix codes
- How to find the optimal code?
- Shannon’s first theorem——Zero-error source coding theorem
- From Theory to Applications: source coding algorithms

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- Kraft inequality——constraints on prefix codes
- How to find the optimal code?
- Shannon’s first theorem——Zero-error source coding theorem
- From Theory to Applications: source coding algorithms

本节学习目标

1. 理解效率与可靠性之间的折衷关系
2. 说出信源编码器与信源译码器各自的目标
3. 写出信源编码效率的评价指标
4. 说出信源编码优化问题
5. 说出什么是non-singular code
6. 说出什么是Uniquely decodable code
7. 说出什么是prefix code
8. 说出以上三种code的优缺点
9. 说出对信源编码的三个要求

重难点：

- 信源编码优化问题
- 认识几种编码类型

01

Efficiency vs. Reliability

Core Problem of Communication Systems

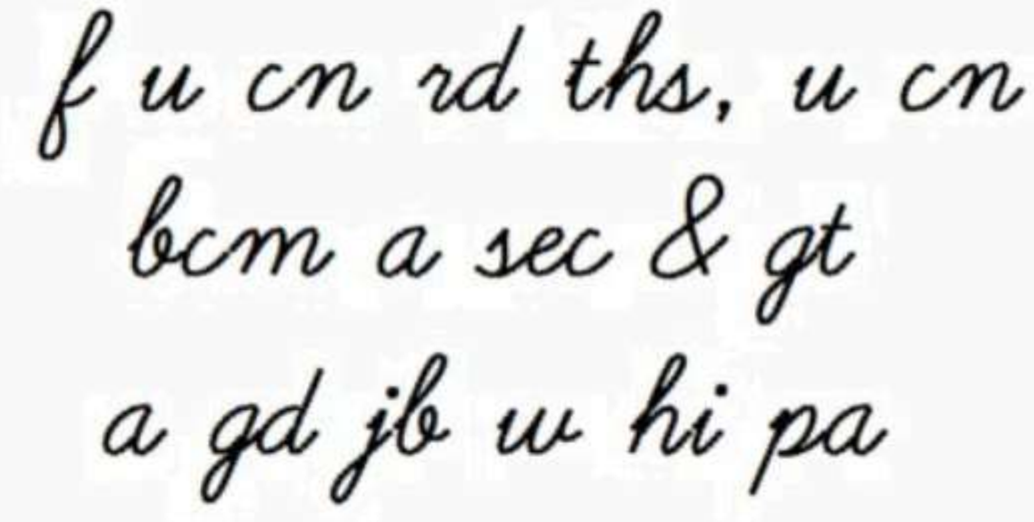
Major challenges for establishing a communication theory

- How much information is transmitted?
  - **Quantify** information carried by symbols

- How to evaluate the performance of communication systems?
  - Transmission **efficiency** in communication systems
  - **Accuracy** of information transmission

- Nature of real-world scenarios?
  - Noise interference
- Core problems: **efficiency vs. reliability**
- Pioneering work by Claude E. Shannon and Norbert Wiener

Efficiency vs. Reliability



- f u cn rd ths, u cn bcm a sec & gt a gd jb w hi pa!

- If you can read this, you can become a secretary and get a good job with high pay!

Efficiency vs. Reliability

- The eternal issues of information theory
  - Lose reliability to achieve higher efficiency
  - Lose efficiency to achieve higher reliability

- Balance between efficiency and reliability
  - **Efficiency**
    - digital case : send as few symbols as possible
    - analog case : reduce the time that the channel or the bandwidth is used
  - **Reliability**
    - digital case : reduce the error probability as small as possible
    - analog case : reduce the noise as much as possible

Source coding vs. Channel coding

- **Source Coding**
  - Core problem: **efficiency**
  - Efficiency: having an average code length that is as small as possible
  - Example: to use shorter code for the English letters which appear frequently, so as to reduce the average code length

- **Channel Coding**
  - Core problem: **reliability**
  - Reliability: to cope with the errors in the transmission
  - Example: to send the same sequence multiple times, so as to recover from the errors in channel

Efficiency vs. Reliability: Redundancy

- 信源有冗余，可进行压缩
- **信源编码：**
- 信源编码是通过尽可能压缩信源冗余度的手段，实现提高通信有效性的目的。

例：中华人民共和国 $\xrightarrow{\text{压缩}}$ 中国　效率最高

显然，压缩后信源的冗余度越低，通信的**有效性**越好。但另一方面，信源冗余度过低，甚至没有冗余度，又会带来通信**可靠性**方面的问题。

Efficiency vs. Reliability: Redundancy

若通信过程中出现错误：

1. 当信源无冗余度

中国 → ×国 美国 法国 德国…？

中国 → 中× 中国 中央 中间…？

2. 当信源存在一定冗余度

中华人民共和国 → ×华人民×和国 $\xrightarrow{\text{恢复}}$ 中华人民共和国

结论：通信有效性（信源编码）与可靠性（信道编码）

往往是一对矛盾。

02

Three key questions about data compression

Question 1: Why do we want to compress data?

- Save storage space
- Improve communication efficiency

Question 2: Can we compress the data?

→ 鸟唱着歌儿，
花飘着清香，
春光明媚，
真是美啊！ → 鸟语花香

- There exists redundancy in the messages we transmit.

Eliminate redundancy → Compress data

- **Basic idea**: keep the same meaning and present it in the shortest manner.

Question 3: Can we compress the data unlimitedly?

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个人电脑 笔记本电脑 计算机科学 文件管理

如果把压缩文件改个后缀再进行压缩，最终得到的文件会是越来越小且可复原的吗?

不太了解计算机相关知识，如果这个问题显得我很呆，请别骂我

关注问题 写回答 邀请回答 好问题 添加评论 分享 ··· 收起

知乎 首页 知乎知学堂 发现 等你来答 恒大海花岛一楼盘被质疑使用海砂

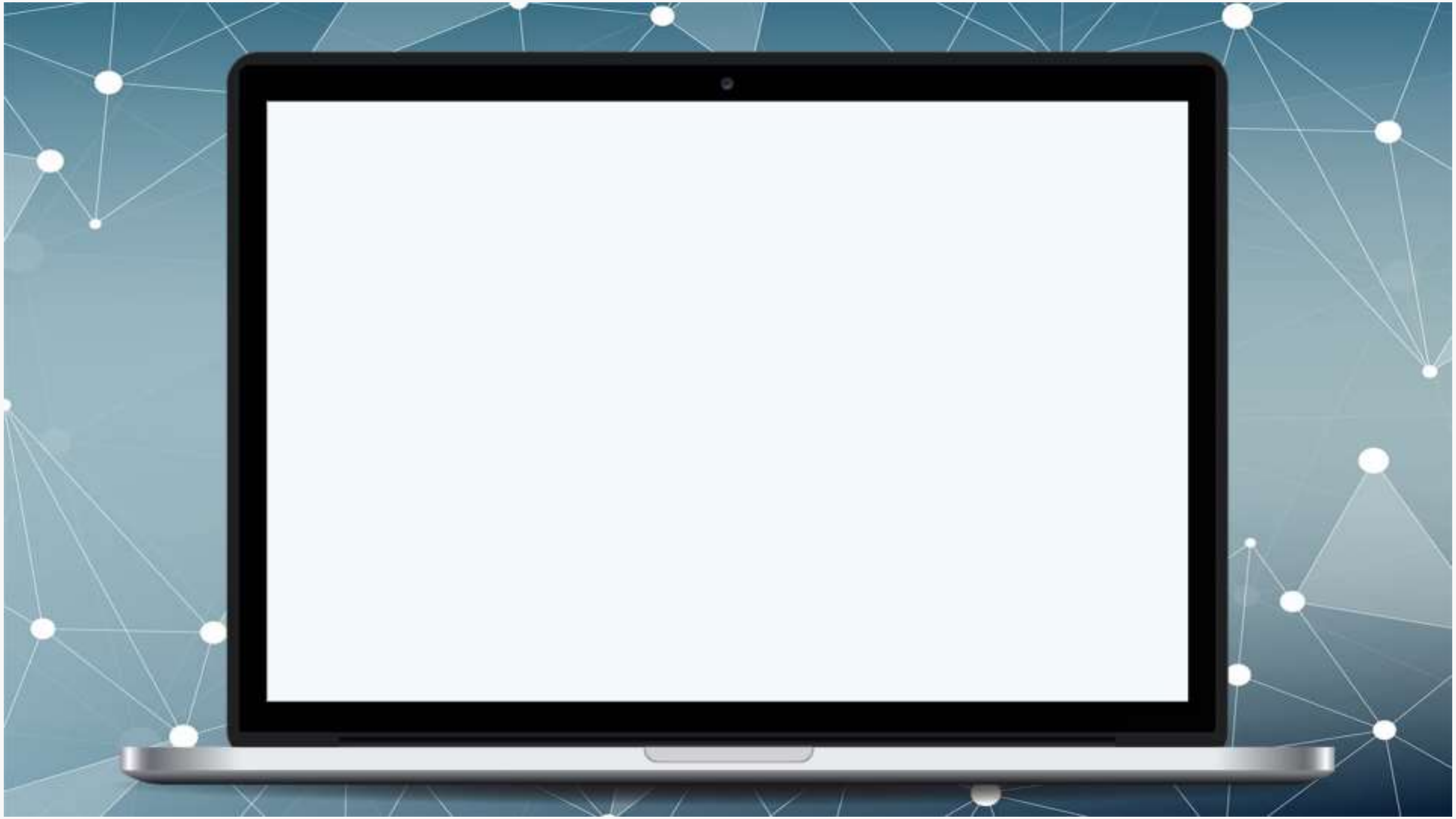
压缩软件 压缩 数据压缩

有没有可能把一个文件反复压缩n次达到压缩后文件仅有原来1/10大小?

会不会原文件被改变或者无法反复压缩?

关注问题 写回答 邀请回答 好问题 添加评论 分享 ··· 收起

Question 3: Can we compress the data unlimitedly?



Key question of this chapter

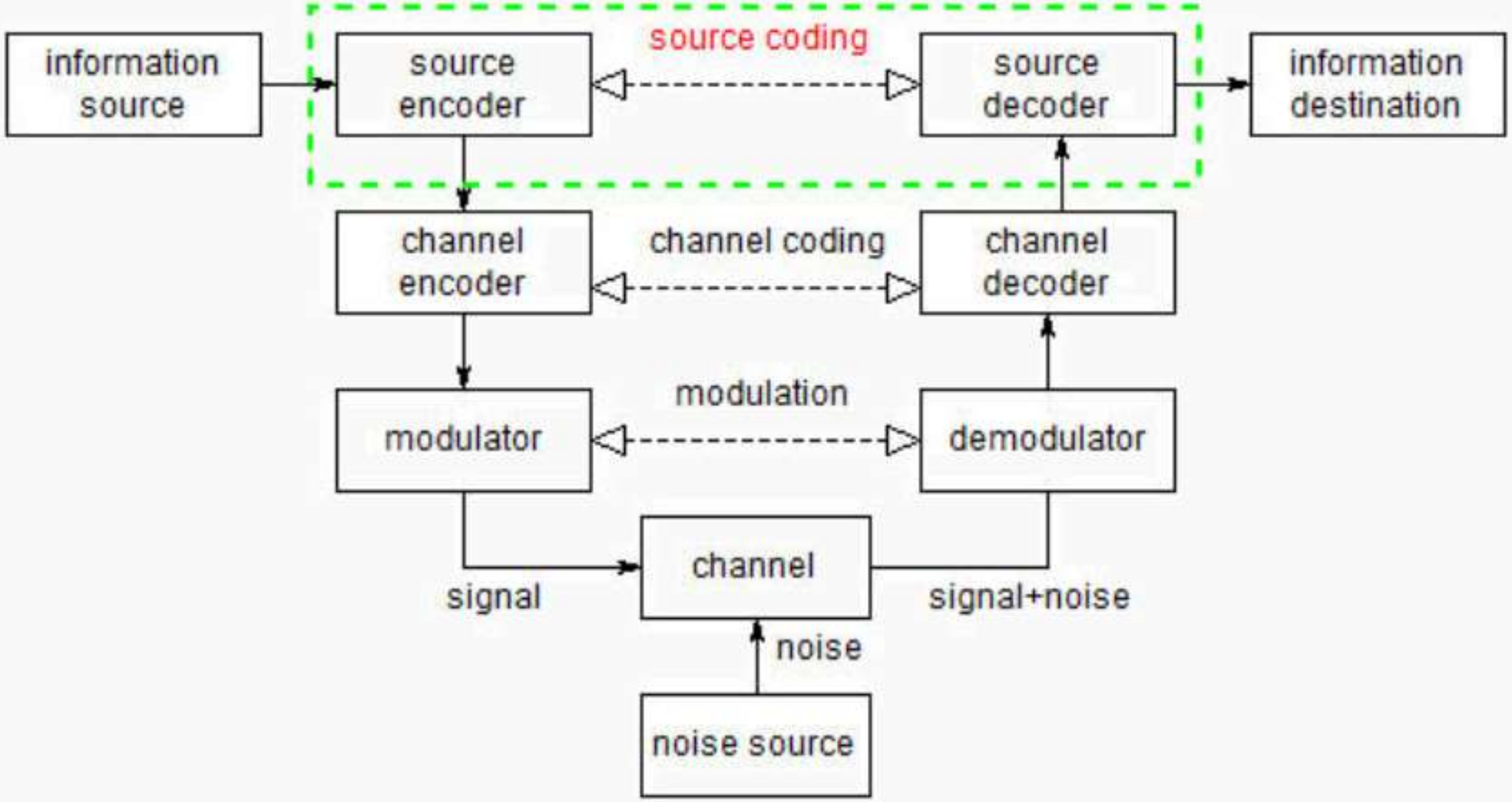
- Data compression has a limit?
- What is the limit?

03

What is Source Coding?

Who will do the job of data compression?

- In communication systems, data compression is done by the **source coding module**.



Overview of source coding

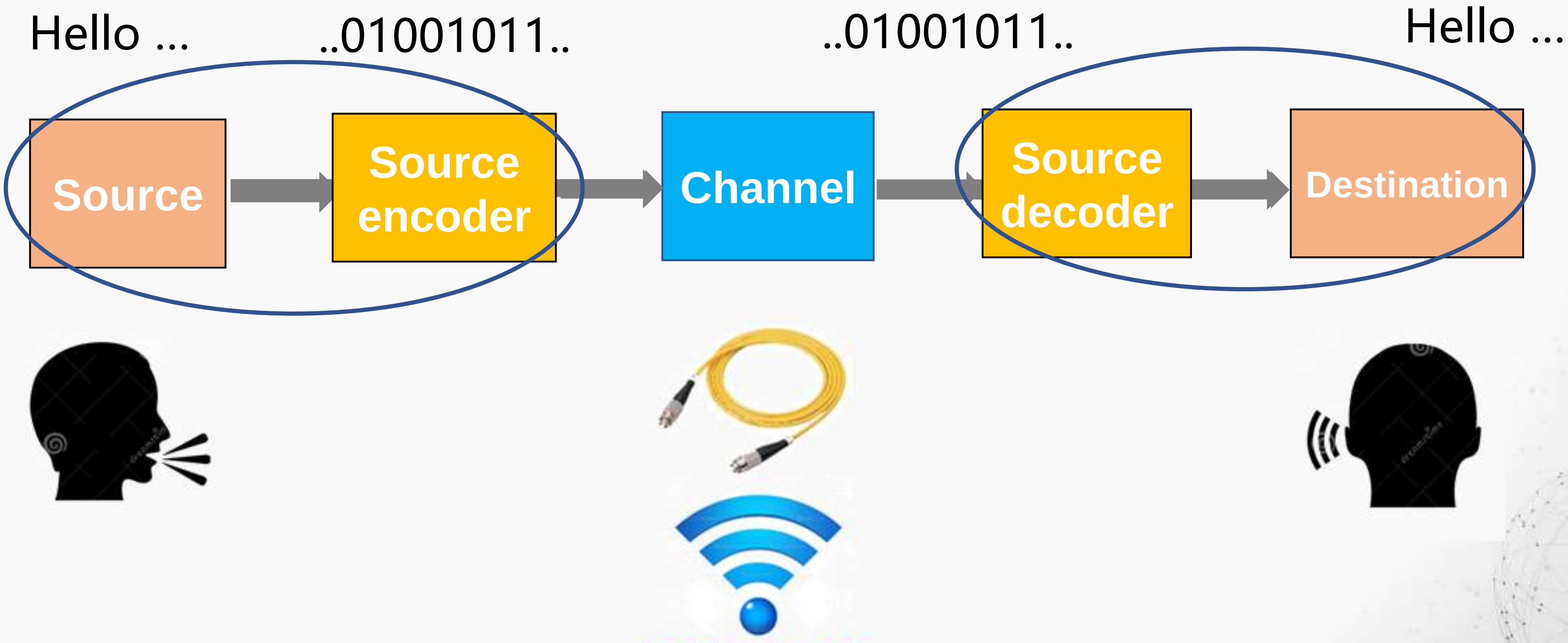
- Theoretical foundation of source codes
  - Zero-error source coding theorem
  - Rate-distortion source coding theorem

- Classification of source codes
  - Discrete source coding: zero-error coding
    - Example: Text compression
  - Continuous source coding: rate-distortion coding
    - Example: video compression

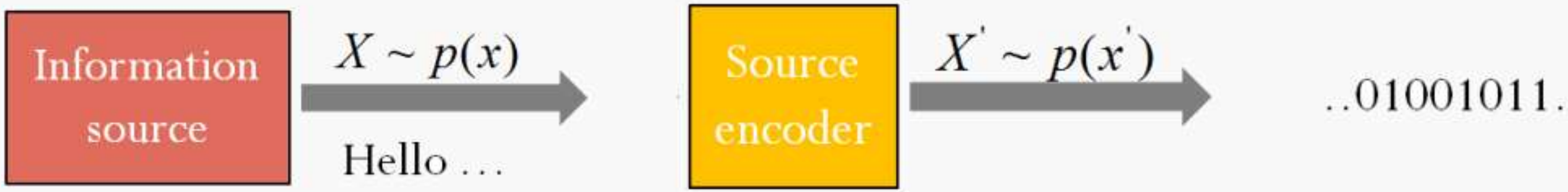
- In this chapter, we focus on the **zero-error coding for discrete information sources**.

What is source coding?

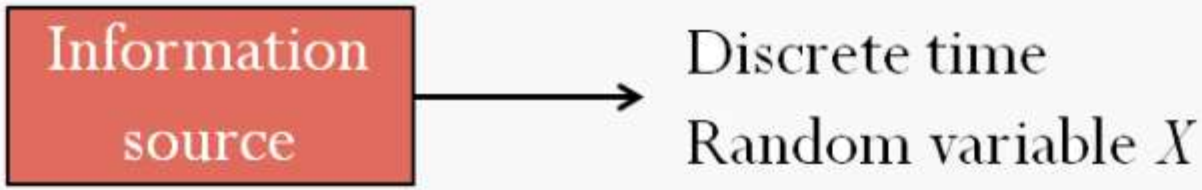
- In communication systems, data compression is done by the **source coding module**.



Source **encoder** side…

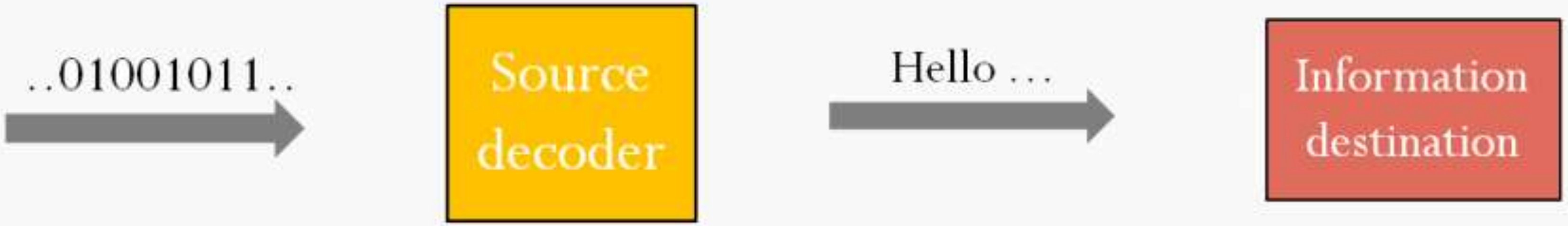


- How to characterize the information source?

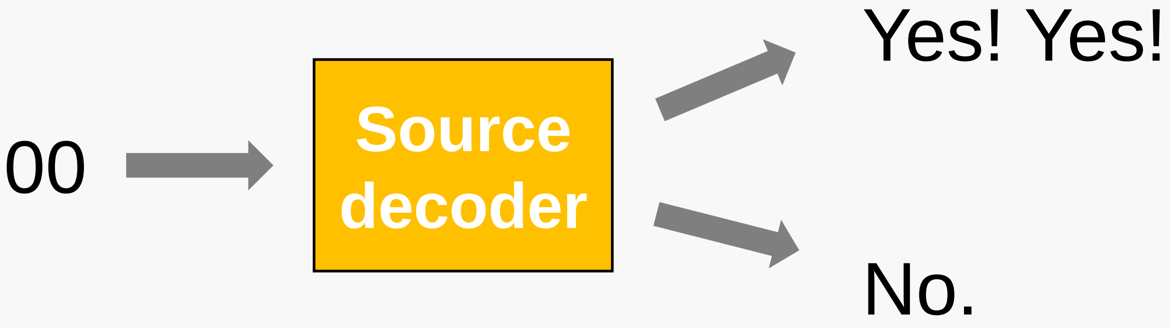


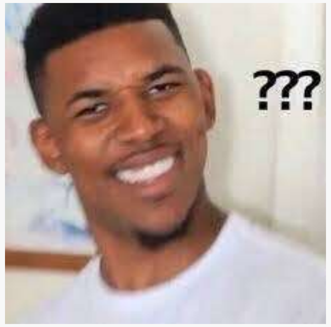
- How to represent the information of the source **efficiently**?

Source **decoder** side…



- **Computational complexity** to recover the original sequence.
- Whether **uniquely** recover the original sequence?
  - e.g. “Yes” is coded as “0”, “No” is coded as “00”.





Source code: definition

- A source code $\boldsymbol{C}$ for a random variable X is a mapping between the space of X to the space of code D.

$$\mathbf{C}:\mathbf{X}\rightarrow\mathbf{D}:\ \mathbf{C}(\mathbf{x}),$$

where $\mathbf{D}$ is the set of finite length strings of symbols from a D-ary alphabet[1].

- Let $C(x)$ denote the codeword corresponding to x.
- Let $l(x)$ denote the length of $C(x)$.

X		**D**
A	⟷	0
B		10
C		111
…		…

How to design an efficient code?

Which code is more efficient?

- Source X={a, b, c}
- Codebook D={0,10,111}
- Output string: aabaacabaa

A B C	⟷	0 10 111

Pr(X=a)=0.7
Pr(X=b)=0.2
Pr(X=c)=0.1

Source Code #1: C(a)=111, C(b)=10, C(c)=0
Bit String #1: 1111111011111101111101111111

Source Code #2: C(a)=0, C(b)=10, C(c)=111
Bit String#2: 00100011101000

- What is the total code length? $l(a) \times n(a) + l(b) \times n(b) + l(c) \times n(c)$
- What is the average code length per symbol?

$$\frac{l(a) \times n(a) + l(b) \times n(b) + l(c) \times n(c)}{n(a) + n(b) + n(c)} = \sum_{x \in \mathcal{X}} p(x) l(x)$$

Coding efficiency: Expected length of a source code

- Definition: The expected length $L(C)$ of a source code $C(x)$ for a random variable X with p.m.f. $p(x)$ is given by

$$\boxed{L(C) = \sum_{x \in \mathcal{X}} p(x) l(x)}$$

where $l(x)$ is the length of the codeword associated with x.

- Shorter average code length ⟶ Higher efficiency ⟶ Better compression

Source code: example #1

- $r.v.X$

$$\begin{array}{l} \Pr(X = a) = 1/2, \\ \Pr(X = b) = 1/4, \\ \Pr(X = c) = 1/8, \\ \Pr(X = d) = 1/8. \end{array}$$

$$\begin{array}{l} C(a) = 0, \\ C(b) = 10, \\ C(c) = 110, \\ C(d) = 111. \end{array}$$

$$H(X) = -\sum_{x \in \mathcal{X}} p(x) \log p(x) = 1.75 \text{ bits}$$

$$L(C) = \sum_{x \in \mathcal{X}} p(x) l(x) = 1.75$$

- Observation:

$$H(X) = L(C)$$

- Example bit string: 0110111100110
- Decoded symbol: acdbac

Source code: example #2

For $r.v.X$,

$$\boxed{\begin{aligned}\Pr(X=a)&=1/3,\\ \Pr(X=b)&=1/3,\\ \Pr(X=c)&=1/3.\end{aligned}}$$

$$\boxed{\begin{aligned}C(a)&=0,\\ C(b)&=10,\\ C(c)&=11.\end{aligned}}$$

$$\boxed{H(X) = -\sum_{x\in\mathcal{X}} p(x)\log p(x) = 1.58 \text{ bits}}$$

$$\boxed{L(C) = \sum_{x\in\mathcal{X}} p(x)l(x) = 1.66}$$

Observation:

$$H(X) = -\sum_{x\in\mathcal{X}} p(x)\log p(x) = 3\cdot\frac{1}{3}\cdot\log_2 3 = 1.58 \text{ bits}.$$

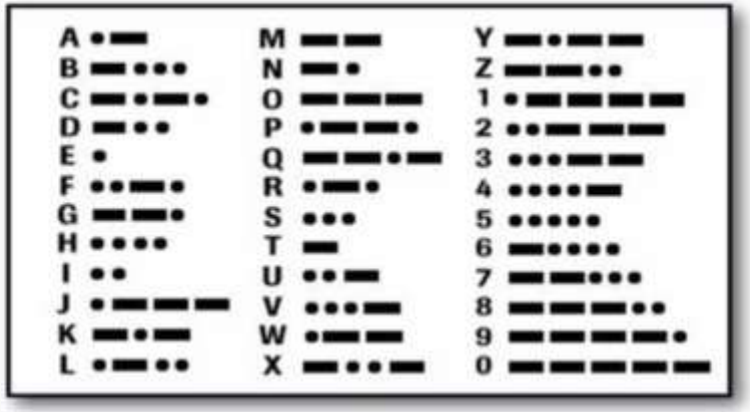
$$L(C) = \sum_{x\in\mathcal{X}} p(x)l(x) = \frac{1}{3}\cdot 1 + \frac{1}{3}\cdot 2 + \frac{1}{3}\cdot 2 = 1.66.$$

$$\therefore H(X) < L(C)$$

Source coding problem: intuitive idea

Information source → Discrete time
Random variable X

- Problem: design a source code to minimize the average codeword length
  - Also referred to as data compression problem
- Intuitive idea
  - Allocate the shortest code words to the most probable outcomes;
  - Allocate the inevitably longer ones to less likely outcomes
- Well-known example: Morse code



04

Get to know some codes!

Non-singular code

$$\mathbf{C} : \mathbf{X} \rightarrow \mathbf{D} \text{: } \mathbf{C}(\mathbf{x})$$

- Non-singular code:
  - Every element of the range of X maps into a different string in $\mathbf{D}$, i.e.

$$\boxed{x_i \neq x_j \Longrightarrow C(x_i) \neq C(x_j)}$$

- Good enough?
  - {A, B, C, D}={0, 1, 10, 110}
  - Received code string: 110
  - Unambiguous for single symbol, for a stream needs comma.

Uniquely decodable code

- Extension $C*$ of a code C
  - Mapping from finite length strings of X to finite length strings of **D**, defined by

$$\boxed{C(x_1x_2\ldots x_n) = C(x_1)\,C(x_2)\ldots C(x_n)}$$

  - Example:
    If $C(x_1) = 00$, $C(x_2) = 11$, then $C(x_1x_2) = 0011$.

- Uniquely Decodable Code
  - A code is called uniquely decodable if its extension is non-singular.
  - Only one possible source string

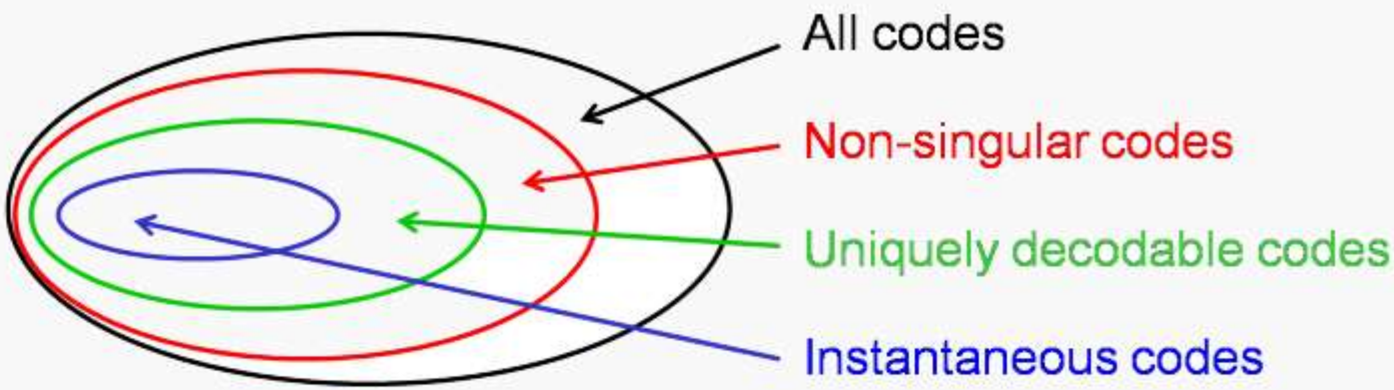
- Good enough?
  - {A, B, C, D}={10, 00, 11, 110}
  - Received code string: 00110001011
  - Need to wait until the entire string is received to decode it.

Prefix/Instantaneous code

- Prefix Code = Instantaneous Code
  - A code is called a prefix code or an instantaneous code if no codeword is a prefix of any other codeword.
  - Can be decoded without reference to the future codewords

- Good enough?
  - {A, B, C, D}={1, 01, 001, 0001}
  - Received code string: 10010100011
  - **Desirable design goal**

Classes of codes



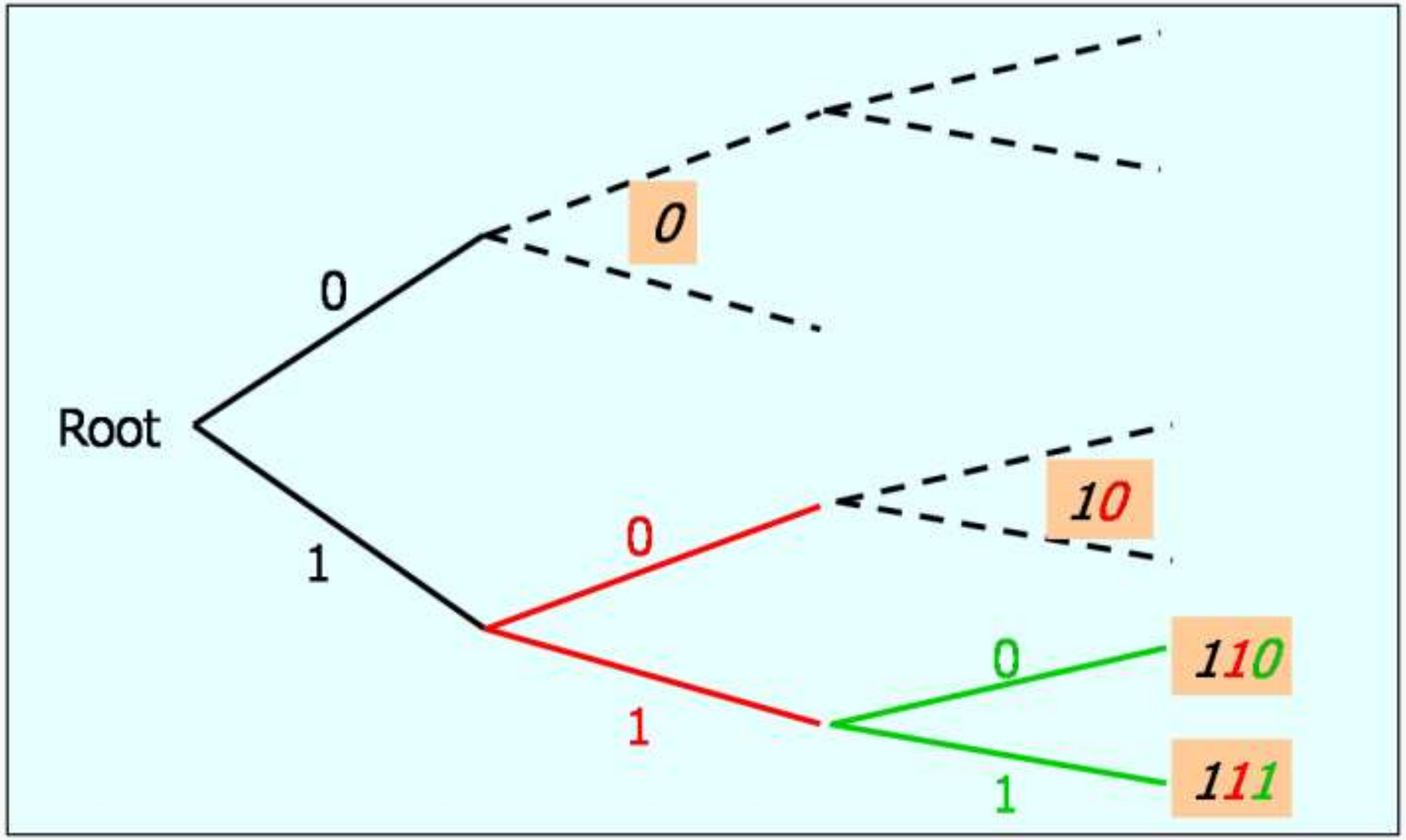
X	Singular	Non-singular	Uniquely decodable	Prefix
A	0	0	10	0
B	1	010	00	10
C	0	01	11	110
D	0	10	110	111

- The shortest codeword cannot be assigned for all symbols in a prefix code.

Code tree

- We can always construct the code tree of a prefix code.

X	Prefix
A	0
B	10
C	110
D	111



- Can you tell what is the signature of a prefix code?

05

What do we want from a source code?

What do we want from a source code?

- **Efficiency**
  - Find codes with the minimum average code length.
- **Reversibility**
  - The code must be uniquely decodable
- **Instantaneous code**
  - Detect where the code for one input symbol ends and the next begins.
- **Easy implementation** of the code
  - From algorithm design's point of view

Compression

Zero-error

Engineering

Which one is the champion code?

For the discrete information source S, the output symbols are A, B, C, D. The probability of each symbol is given as P, $C_1 \sim C_6$ are possible source codes.

Source Symbol	P	C_1	C_2	C_3	C_4	C_5	C_6
A	0.6	00	0	0	0	0	0
B	0.25	01	10	10	01	10	10
C	0.1	10	110	110	011	11	11
D	0.05	11	1110	111	111	01	0

Requirement #1: Efficiency

- Average code length of the other codes

$$
\begin{aligned}
C_1: &\quad 2 \times 0.6 + 2 \times 0.25 + 2 \times 0.1 + 2 \times 0.05 = 2 \\
C_2: &\quad 1 \times 0.6 + 2 \times 0.25 + 3 \times 0.1 + 4 \times 0.05 = 1.60 \\
C_3: &\quad 1 \times 0.6 + 2 \times 0.25 + 3 \times 0.1 + 3 \times 0.05 = 1.55 \\
C_4: &\quad 1 \times 0.6 + 2 \times 0.25 + 3 \times 0.1 + 3 \times 0.05 = 1.55 \\
C_5: &\quad 1 \times 0.6 + 2 \times 0.25 + 2 \times 0.1 + 2 \times 0.05 = 1.40 \\
C_6: &\quad 1 \times 0.6 + 2 \times 0.25 + 2 \times 0.1 + 1 \times 0.05 = 1.35
\end{aligned}
$$

- C_6 is the most efficient code

Requirement #2: Reversibility

- Reversibility
  - Possible to decode the code words.
- Singular codes
  - Multiple source symbols have the same code
  - C_6 is a singular code, as A and D have the same code.
- Not all non-singular codes are reversible
  - C_5 is non-singular, but non-reversible

$$\underset{A}{\underline{0}}\ \underset{C}{\underline{11}}\ \underset{A}{\underline{0}} \qquad \underset{D}{\underline{01}}\ \underset{B}{\underline{10}}$$

- Reversible codes are also called uniquely decodable codes
  - $C_1 \sim C_4$ are uniquely decodable

Requirement #3: Instantaneous Code

- Instantaneous code
  - It is possible to decode each word without reference to succeeding code symbols
  - No need to buffer the source sequence
- Example
  - C_3 and C_4 are uniquely decodable and equally efficient
  - C_4

$0\ 111\ 111\ 0\cdots$	$\rightarrow$	$A\,D\,D\,\cdots$	*OK!*
0111111	$\rightarrow$	$B\,D\cdots$ *or* $C\,D\cdots$	*Which one?*

  - C_3

0111111	$\rightarrow$	$A\,D\,D$

  - C_3 is instantaneous, but C_4 isn't

Which one is the champion code?

Source Symbol	P	C_1	C_2	C_3	C_4	C_5	C_6
A	0.6	00	0	0	0	0	0
B	0.25	01	10	10	01	10	10
C	0.1	10	110	110	011	11	11
D	0.05	11	1110	111	111	01	0
$\bar{L}$		2	1.60	1.55	1.55	1.40	1.35

- C1 and C2 are also instantaneous codes
- Design goal:
  - We want a uniquely decodable, instantaneous code with the shortest average code length.
  - C3 is the best candidate

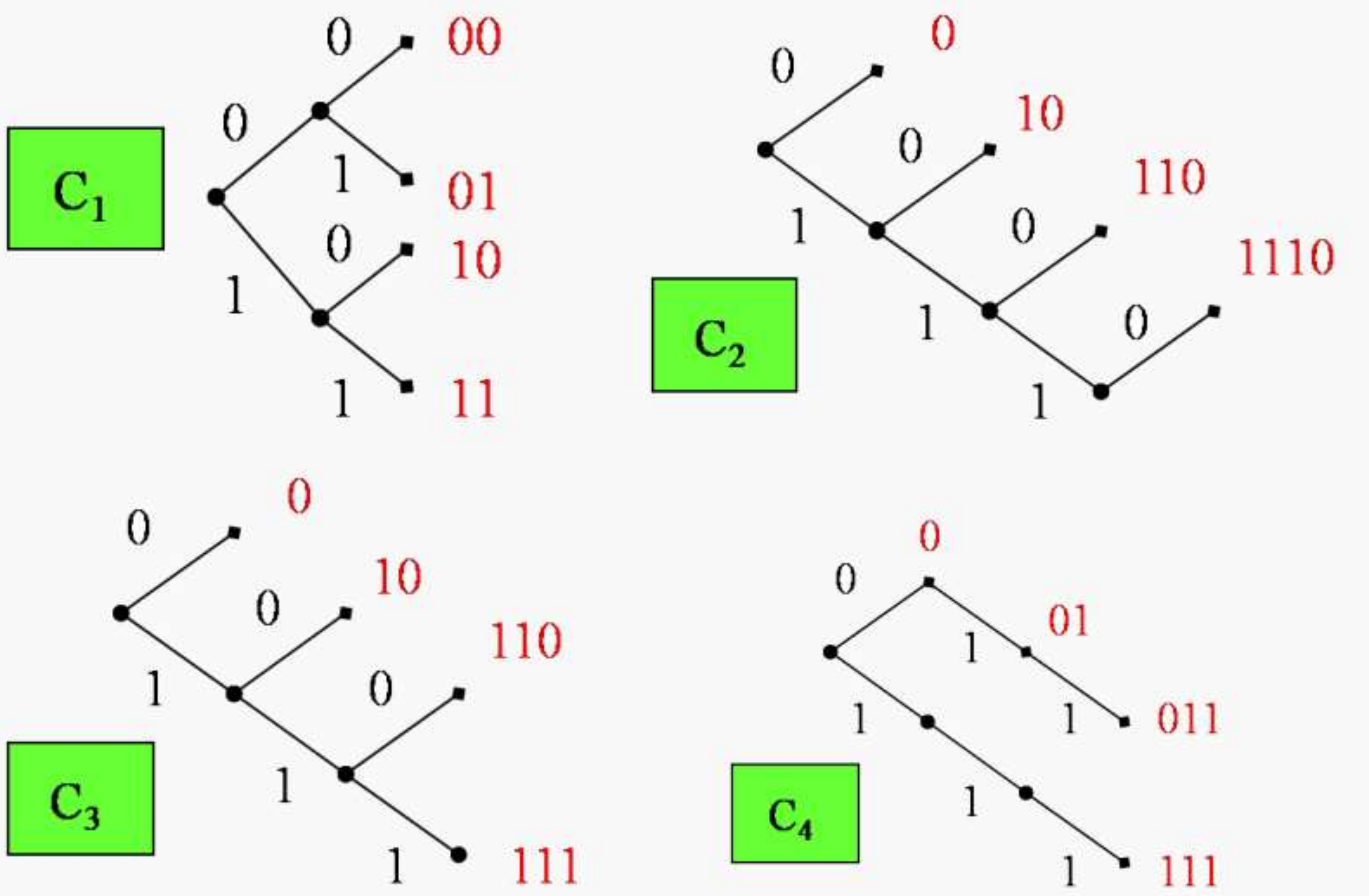
How to recognize an instantaneous code?

- **Prefix condition**
  - A necessary and sufficient condition for a code to be instantaneous is that no complete word of the code is a prefix of some other code word

Source Symbol	C_4
A	0
B	01
C	011
D	111

- Why is C4 not an instantaneous code?
  - In case of a 0 in the sequence, we can not distinguish between the code words 0, 01 or 011.
  - 0 is a prefix of different code words.

How to recognize an instantaneous code? Code tree



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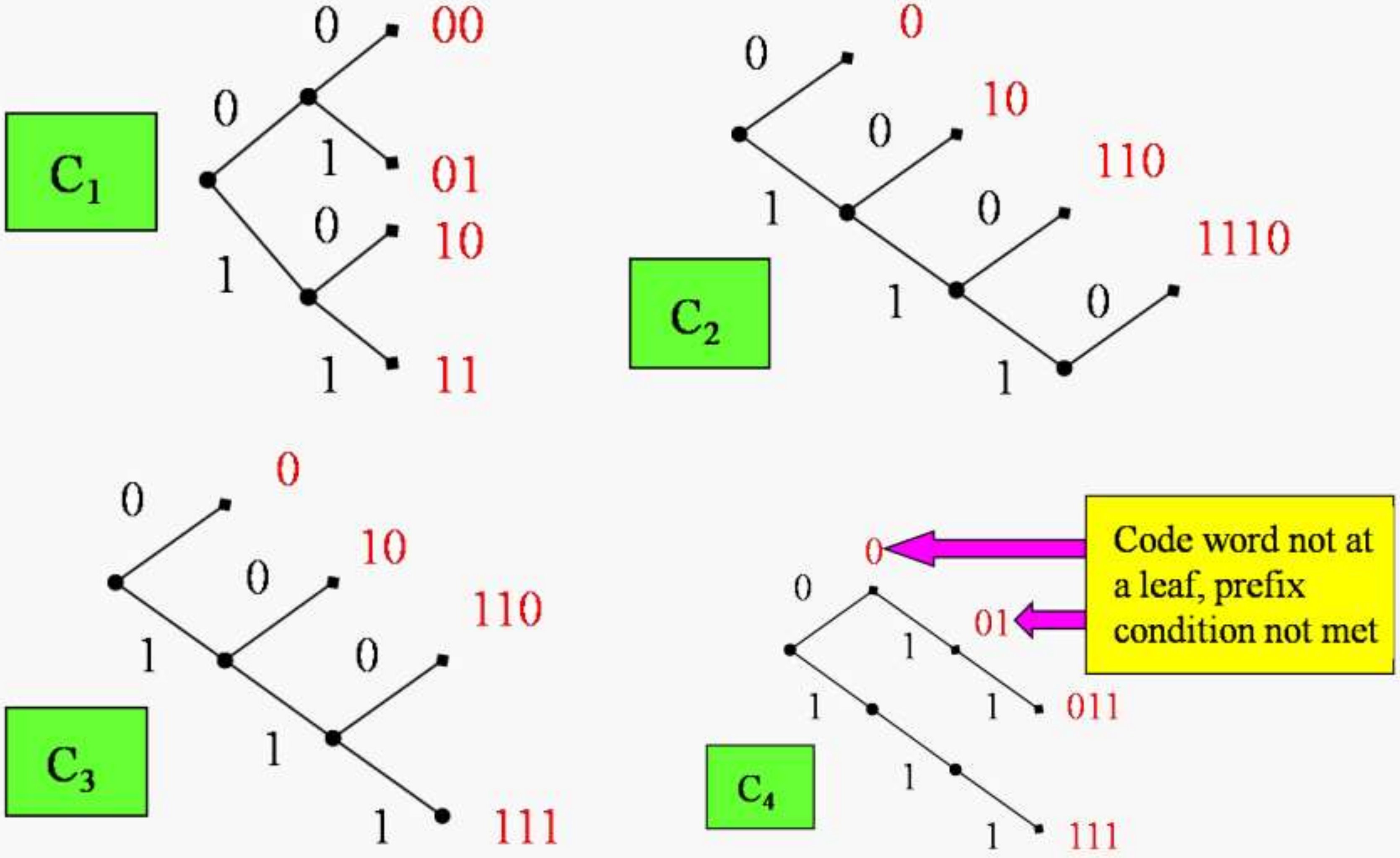
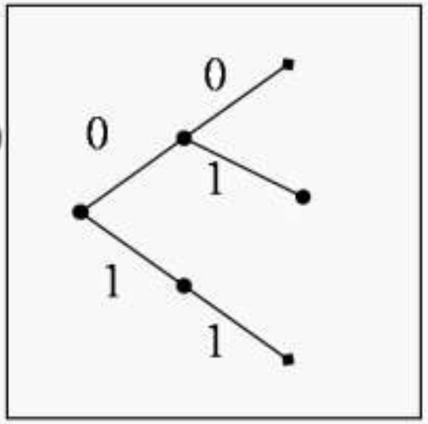


Code tree: example

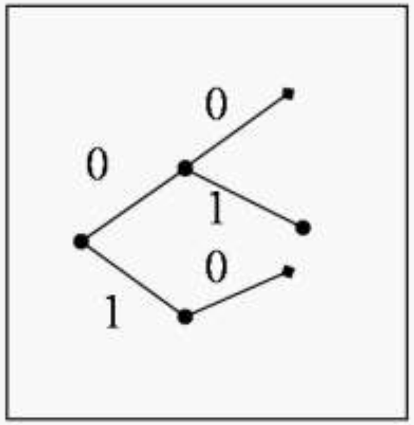
Source Symbol	P	C_1	C_2	C_3	C_4	C_5
S_1	0.5	0	0	1	1	01
S_2	0.1	11	11	01	01	001
S_3	0.02	00	00	00	001	0001
S_4	0.38	11	010	10	0001	00001

- What is the code tree of C_3?

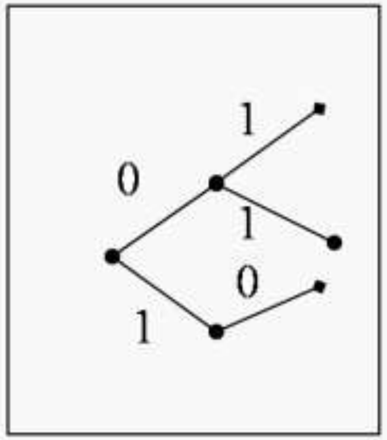
A)



B)



C)



Summary: What do we want from a source code?

- **Efficiency**
  - Find codes with the minimum average code length.
- **Reversibility**
  - The code must be uniquely decodable
- **Instantaneous code**
  - Detect where the code for one input symbol ends and the next begins.
- **Easy implementation** of the code
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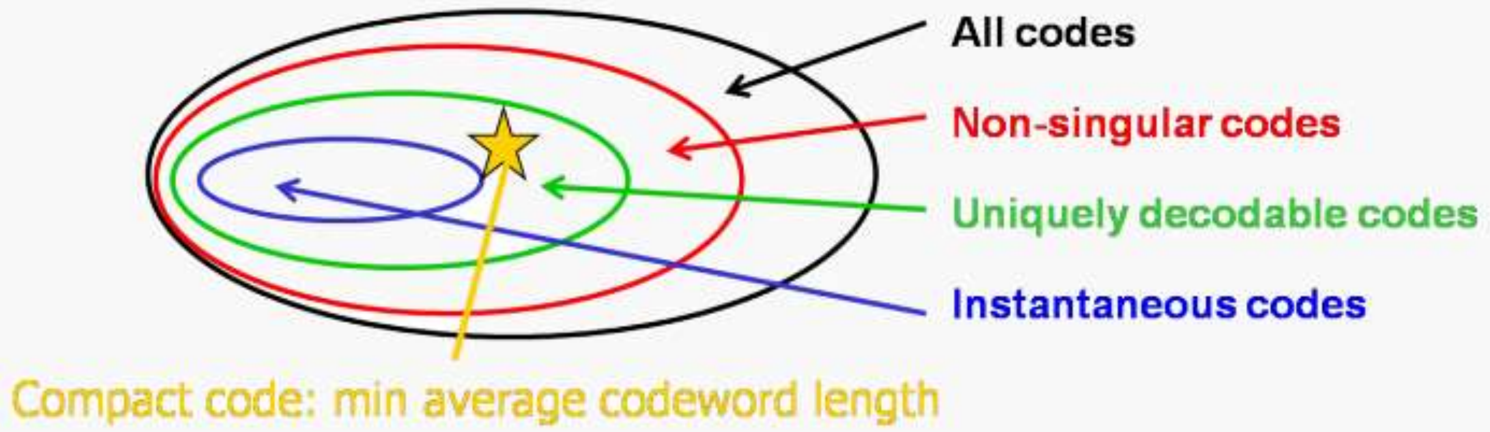
Summary: What do we want from a source code?

- In general, the optimal zero-error source coding problem is equivalent to find the optimal (shortest average length) uniquely decodable codes.
- Such a targeted code is called a compact code.
    - The uniquely decodable code with the smallest average code length for an information source S.
    - **How short can it be?**
        - **Shannon's first theorem**



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My Homepage

Thank you!

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