Information Theory Exam 2024

Yuejin Xie

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1 True/False Problems

Tips: Explain your conclusion.

1. $g(X)$ is a function of the random variable X, $H(g(X)) \geq H(X)$.
2. $I(X;Y|Z) \geq I(X;Y)$.
3. Entropy is non-negative.
4. A prefix code with code lengths $\{1,3,3,3,4,5\}$ can be achieved.
5. Given the Markov transition matrix,is its maximum entropy rate $\log_2 3$?

$$\begin{pmatrix} 1-p & p/2 & p/2 \\ p/2 & 1-p & p/2 \\ p/2 & p/2 & 1-p \end{pmatrix}$$

2 Short Problems

1. State Shannon's first theorem and explain its meaning.
2. A car's license plate ends with a digit from 0 to 9. The probabilities of the last digit being 0, 1, and 2 are $\frac{1}{20}$, $\frac{1}{8}$, and $\frac{1}{8}$ respectively. The probabilities of the other digits are equal. Suppose you ask the car owner if the last digit of their license plate is a specific number, and they will answer "yes" or "no". On average, how many questions do you need to ask to determine the last digit of the license plate?
3. Write down the upper bound of the optimal expected code length, and explain how to decrease the expected code length.
4. Write down the form of the rate-distortion function.
5. Compare channel capacity and rate-distortion function.
6. State why the maximum transmission rate is equal to the channel capacity (Hint: Use AEP).

3 Long Problems

1. X and Y are random variables. Let $Z = X \oplus Y$, where Z donate Modulo 2 additions. The joint distribution of X and Y is as follows:

$X \backslash Y$	0	1
0	$\frac{1}{3}$	$\frac{1}{3}$
1	0	$\frac{1}{3}$

Table 1: Joint Probability Distribution

(a) Find $H(X)$, $H(Y)$, and $H(Z)$.

(b) Find $H(X|Z)$ and $H(Z|X)$.

(c) Find $H(X,Y,Z)$ and $H(X,Y)$, and explain your results.

2. A source emits symbols with the probability distribution {0.45, 0.25, 0.12, 0.05, 0.05, 0.04, 0.03, 0.01}.

(a) Derive the binary Shannon code for this distribution and calculate the coding efficiency.

(b) Derive the binary Huffman code for this distribution, draw the Huffman tree, and calculate the coding efficiency.

(c) Derive the ternary Huffman code for this distribution and calculate the coding efficiency.

3. Prove the following:

(a) $L(C, X) \geq H_D(X)$, and explain when we can get equality.

(b) $H(X) \geq \log |\mathcal{X}|$, and explain when we can get equality.

4. Write down Shannon's formula and explain the meaning of each symbol:

(a) What can you observe from this formula? Please state at least three points.

(b) For an Additive White Gaussian Noise (AWGN) channel with a channel capacity $C = 150$ kbps, bandwidth $B = 30$ kHz, and signal power $S = 0.5$ W, if the bandwidth approaches infinity and you want to maintain the same transmission rate, what will be the required signal power?

5. A military unit uses a flock of pigeons for communication. Each pigeon can carry 8 bits of information, and one pigeon is released every 5 minutes. Each pigeon takes 3 minutes to reach its destination.

(a) Calculate the channel capacity: how many bits per hour?

(b) If a proportion α of the pigeons are shot down by the enemy, and the information receiver can know which pigeons are shot down, calculate the channel capacity.

(c) If the enemy becomes more clever and releases pigeons carrying random information when they shoot down the original pigeons, calculate the channel capacity. (Construct your model and explain it).