ALGO - AMPL

Projet modélisation de nourriture.

Food Manufacture :

A food is manufactured by refining raw oils and blending them together. The raw oils come in two categories :

vegetable oils **VEG 1**
VEG 2

non-vegetable oils **OIL 1**
OIL 2
OIL 3

Each oil may be purchased for immediate delivery (January) or bought on the futures market for delivery in a subsequent month. Prices now and in the futures market are given below in (£/ton) :

	VEG 1	VEG 2	OIL 1	OIL 2	OIL 3
January	110	120	130	110	115
February	130	130	110	90	115
March	110	140	130	100	95
April	120	110	120	120	125
May	100	120	150	110	105
June	90	100	140	80	135

The final product sells at £ 150 per ton.

Vegetable oils and non-vegetable oils require different production lines for refining. In any month it is not possible to refine more than 200 tons of vegetable oils and more than 250 tons of non-vegetable oils. There is no loss of weight in the refining process and the cost of refining may be ignored.

It is possible to store up to 1000 tons of each raw oil for use later. The cost of storage for vegetable and non-vegetable oil is £5 per ton per month. The final product cannot be stored, nor can refined oils be stored.

There is a technological restriction of hardness on the final product. In the units in which hardness is measured this must lie between 3 and 6. It is assumed that hardness blends linearly and that the hardnesses of the raw oils are :

VEG1	**8.8**
VEG2	**6.1**
OIL1	**2.0**
OIL2	**4.1**
OIL3	**5.0**

What buying and manufacturing policy should the company pursue in order to maximize profit ?
At present there are 500 tons of each type of raw oil in storage. It is required that these stocks will also exist at the end of June.

Blending problems are frequently solved using linear programming. Linear programming has been used to find minimum cost blends of fertilizer, metal alloys, clays, and many food products to name only a few. Applications are described in (for example) Fisher and Schruben (1953) and Williams and Redwood (1974).

The problem presented here has two aspects. Firstly it is a series of simple blending problems. Secondly there is a purchasing and storing problem. To understand how this problem may be formulated it is convenient to consider first the blending problem for only one month. This is the single-period problem which has already been presented as the second example in Section 1.2.

The Single-period Problem

If no storage of raw oils were allowed the problem of what to buy and how to blend in January could be formulated as follows:

$$
\begin{array}{ll}
\text{Maximize} & \\
\text{PROFIT} & -110x_1 - 120x_2 - 130x_3 - 110x_4 - 115x_5 + 150y \\
\text{subject to} & \\
\text{VVEG} & x_1 + x_2 \leqslant 200, \\
\text{NVEG} & x_3 + x_4 + x_5 \leqslant 250, \\
\text{UHRD} & 8{\cdot}8x_1 + 6{\cdot}1x_2 + 2x_3 + 4{\cdot}2x_4 + 5x_5 - 6y \leqslant 0, \\
\text{LHRD} & 8{\cdot}8x_1 + 6{\cdot}1x_2 + 2x_3 + 4{\cdot}2x_4 + 5x_5 - 3y \geqslant 0, \\
\text{CONT} & x_1 + x_2 + x_3 + x_4 + x_5 - y = 0.
\end{array}
$$

The variables x_1, x_2, x_3, x_4, x_5 represent the quantities of the raw oils which should be bought respectively, i.e. VEG 1, VEG 2, OIL 1, OIL 2, and OIL 3. y represents the quantity of PROD which should be made.

The objective is to maximize profit which represents the income derived from selling PROD minus the cost of the raw oils.

The first two constraints represent the limited production capacities for refining vegetable and non-vegetable oils.

The second two constraints force the hardness of PROD to lie between its upper limit of 6 and its lower limit of 3. It is important to model these restrictions correctly. A frequent mistake is to model them as

$$8{\cdot}8x_1 + 6{\cdot}1x_2 + 2x_3 + 4{\cdot}2x_4 + 5x_5 \leqslant 6$$

and

$$8{\cdot}8x_1 + 6{\cdot}1x_1 + 2x_3 + 4{\cdot}2x_4 + 5x_5 \geqslant 3$$

Such constraints are clearly dimensionally wrong. The expressions on the left have the dimension of hardness × quantity, whereas the figures on the right have the dimensions of hardness. Instead of the variables x_i in the above two inequalities expressions x/y are needed representing *proportions* of the ingredients rather than the absolute quantities x_i. When such replacements are made the resultant inequalities can easily be re-expressed in a linear form as the constraints UHRD and LHRD.

Finally it is necessary to make sure that the weight of the final product PROD is equal to the weight of the ingredients. This is done by the last constraint CONT which imposes this continuity of weight.

The single-period problems for the other months would be similar to that for January apart from the objective coefficients representing the raw oil costs.

The Multi-period Problem

The decisions of how to buy each month with a view to storing for use later can be incorporated into a linear programming model. To do this a multi-period model is built. It is necessary, each month, to distinguish the quantities of each raw oil bought, used, and stored. These quantities must be represented by different variables. We suppose the quantities of VEG 1 bought, used, and stored in each successive month are represented by variables with the following names:

BVEG11, BVEG12, ...etc
UVEG11, UVEG12, ...etc
SVEG11, SVEG12, ...etc

It is necessary to link these variables together by the relation

$$\text{quantity stored in month } (t-1) + \text{quantity bought in month } t$$
$$= \text{quantity used in month } t + \text{quantity stored in month } t$$

Initially (month 0) and finally (month 6) the quantities in store are constants (500). The relation above involving VEG 1 gives rise to the following constraints:

$$\begin{aligned}
\text{BVEG 11} - \text{UVEG 11} - \text{SVEG 11} &= -500,\\
\text{SVEG 11} + \text{BVEG 12} - \text{UVEG 12} - \text{SVEG 12} &= 0,\\
\text{SVEG 12} + \text{BVEG 13} - \text{UVEG 13} - \text{SVEG 13} &= 0,\\
\text{SVEG 13} + \text{BVEG 14} - \text{UVEG 14} - \text{SVEG 14} &= 0,\\
\text{SVEG 14} + \text{BVEG 15} - \text{UVEG 15} - \text{SVEG 15} &= 0,\\
\text{SVEG 15} + \text{BVEG 16} - \text{UVEG 16} &= 500.
\end{aligned}$$

Similar constraints must be specified for the other five raw oils.

It may be more convenient to introduce variables SVEG 10, etc., and SVEG 16, etc., into the model and *fix* them at the value 500.

In the objective function the buying variables will be given the appropriate raw oil costs in each month. The storage variables will be given the cost of £5 (or profit of -£5). Separate variables PROD 1, PROD 2, etc., must be defined to represent the quantity of PROD to be made in each month. These variables will each have a profit of £150.

The resulting model will have the following dimensions as well as the single objective function:

$6 \times 5 = 30$		buying variables
$6 \times 5 = 30$		using variables
$5 \times 5 = 25$		storing variables
	6	product variables
Total	91	variables
$6 \times 5 = 30$		blending constraints (as in the single-period model)
$6 \times 5 = 30$		storage linking constraints
Total	60	constraints

It is also important to realize the use to which a model such as this might be put for medium term planning. By solving the model in January buying and blending plans could be determined for January together with provisional plans for the succeeding months. In February the model would probably be resolved with revised figures to give firm plans for February together with provisional plans for succeeding months up to and including July. By this means the best use is made of the information for succeeding months to derive an operating policy for the current month.

VARIANTE

It is wished to impose the following extra conditions on the food manufacture problem:

(1) The food may never be made up of more than three oils in any month.

(2) If an oil is used in a month at least 20 tons must be used.

(3) If either of VEG 1 or VEG 2 are used in a month then OIL 3 must also be used.

Extend the food manufacture model to encompass these restrictions and find the new optimal solution.

SOLUTION

The extra restrictions stipulated are quite common in blending problems. It is often desired to: (i) limit the number of ingredients in a blend; (ii) rule out small quantities of one ingredient; and (iii) impose logical conditions on the combinations of ingredients.

These restrictions cannot be modelled by conventional linear programming. Integer programming is the obvious way of imposing the extra restrictions. In order to do this it is necessary to introduce 01 integer variables into the problem as described in Section 9.2. For each using variable in the problem a corresponding 01 variable will also be introduced. This variable will be used as an indicator of whether the corresponding ingredient appears in the blend or not. For example, corresponding to variable UVEG 11 a 01 variable DVEG 11 is introduced. These variables are linked together by two constraints.

Supposing $\boldsymbol{x_1}$ represents UVEG 11 and $\boldsymbol{\delta_1}$ represents DVEG 11 the following extra constraints are added to the model:

$$x_1 - 200\delta_1 \leqslant 0,$$
$$x_1 - \ \ 20\delta_1 \geqslant 0.$$

Since δ_1 is only allowed to take the integer values 0 and 1, x_1 can only be non-zero (i.e. VEG 1 in the blend in month 1) $\delta_1 = 1$ and then it must be at a level of at least 20 tons. The constant 200 in the first of the above inequalities is a known upper limit to the level of UVEG 11 (the combined quantities of vegetable oils used in a month cannot exceed 200). Similar 01 variables and corresponding linkage constraints are introduced for the other ingredients. Suppose x_2, x_3, x_4, represent UVEG 21, UOIL 11; UOIL 21, and UOIL 31, then the following constraints and 01 variables are also introduced:

$$x_2 - 200\delta_2 \leqslant 0, \qquad x_2 - 20\delta_2 \geqslant 0,$$
$$x_3 - 250\delta_3 \leqslant 0, \qquad x_3 - 20\delta_3 \geqslant 0,$$
$$x_4 - 250\delta_4 \leqslant 0, \qquad x_4 - 20\delta_4 \geqslant 0,$$
$$x_5 - 250\delta_5 \leqslant 0, \qquad x_5 - 20\delta_5 \geqslant 0.$$

All these variables and constraints are repeated for all six months.

In this way condition 2 in the statement of the problem is automatically imposed. Condition 1 can be imposed by the constraint

$$\delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 \leqslant 3$$

and the corresponding constraints for the other five months.

Condition 3 can be imposed in two possible ways. Firstly by the following single constraints,

$$\delta_1 + \delta_2 - 2\delta_5 \leqslant 0$$

or by the pairs of constraints

$$\delta_1 - \delta_5 \leqslant 0,$$
$$\delta_2 - \delta_5 \leqslant 0.$$

There is computational advantage to be gained by using the second pair of constraints since they are tighter in the continuous problem (see Section 10.1). Similar constraints are of course imposed for the other five months.

The model has now been augmented in the following way:

$6 \times 5 = \underline{30}$	0–1 variables
Total 30	extra variables (all integer)
$2 \times 6 \times 5 = 60$	linking constraints
6	constraints for condition 1
$2 \times 6 = \underline{12}$	constraints for condition 3
Total 78	extra constraints

It is also necessary to impose upper bounds of 1 on all the 30 integer variables.

There is probably advantage to be gained from the user specifying a priority order for the variables in order to control the tree search in the branch and bound algorithm. The six δ_5 variables (one for each month) were given priority in choosing branching variables.