Homework 7

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Exercise 7.3

Let the function $f_m(x_1, ..., x_m)$ be 1 iff the m-bit string $x_1, ..., x_m$ contains two consecutive 1s. Prove that $D^{worst}(f_m) = \Theta(m)$.

It's easy to show that for $f_m(x_1, ..., x_m)$, $D^{worst}(f) \leq \frac{m}{2} + 1$. Because for any partition $S : T$, the player with the least number of bits can send them to the other player). Hence we will find a single hard partition to prove lower bounds for the worst-cast communication complexity.

In this case, the hard petition is the odd bits versus the even bits. Computing f_m, according to this partition, is equivalent to computing the function $f_l(x, y)$ ($f_l(x, y)$ will return 1 iff $\exists y_i = 1$ s.t. $x_i = y_i = 1$ or $x_{i+1} = y_i = 1$). In the regular two-party model, we can easily show that the disjointness function, DISJ, on $0, 1^{\frac{m}{2}} \times 0, 1^{\frac{m}{2}}$ can be reduced to the subquestion corresponding to S and T.that at least $d = \Omega(\frac{m}{2})$ bits are required to solve the subquestion. the worst case shows that $D^{worst}(f_M) \geq D(f_l) \geq \Omega(\frac{m}{2})$. Conclude $D^{worst}(f_m) = \Theta(m)$.

Exercise 7.15*

The undirected $s - t$-connectivity problem, USTCON, accepts as input a graph on l vertices (that is, $m = \binom{l}{2}$ input bits representing the edges), and outputs 1 if and only if there exists a path between vertices s and t in the input graph($s \neq t$). Prove that for all n, $D^{n-best}(\text{USTCON}) = \Omega(\frac{n}{l})$. Conclude that $D^{best}(\text{USTCON}) = \Omega(\sqrt{m})$.

Define $G_A(V_A, E_A)$ consist of inputs to Alice and as same relationship with $G_B(V_B, E_B)$ and Bob. Consider there are n edges in each graph. Let $d = \frac{n}{4l}$. Construct the ordered edge set of Alice

$$S_A := \{(u_i, v_i) \in E_A \mid v_i \neq v_j (i \neq j), v_i \neq u_j\}, |S_A| = d$$

Here is the proof of the Existence of S_A. If $\exists p \in G_A$, such that $deg_{G_A}(p) \geq d$ (function $deg_G(x)$ means the degree of x in graph G). we can choose each $u_i = p$. Otherwise, we notice that for each vertex $deg_{G_A}(x) \leq d - 1$. If we choose one edge (u, v) into S_A, we can not choose other edges which endpoint is u or v. Thus the choice would fix at most $2d - 3$ edges and there are at most$d(2d - 3)$ edges we fixed. Because $n \leq m < \frac{l^2}{2}$, $d(2d - 3) < n(\frac{n}{8l^2} - \frac{3}{4l}) < n$, it's enough to choose d times.

Thus consider the graph G_B. Define $G'_B(V'_B, E'_B)$, such that $V'_B := \{x \in V_B \mid x \neq u_i, v_i : \forall (u_i, v_i) \in S_A\}$, $E'_B := \{(u, v) \in E_B \mid u, v \in V'_B\}$. Because $|E'_B| \geq n - 2d \cdot l \geq \frac{n}{2}$, it's enough to construct the similar ordered edge set of Bob

$$S_B := \{(u'_i, v'_i) \in E'_B \mid v'_i \neq v'_j (i \neq j), v'_i \neq u'_j\}, |S_B| = d$$

Now there are four ordered point sets($V_u, V_v, V_{u'}, V_{v'}$), two fixed vertex s and t and two ordered edge setsS_A, S_B. And we formulate other edges to show that the disjointness function DISJ on $0, 1^d \times 0, 1^d$ can be reduced to this problem corresponding to S_A and S_B. For all $u_i \neq u_j (u_i, u_j \in V_u)$, add edge (u_i, u_j). For all $u'_i \neq u'_j (u'_i, u'_j \in V_{u'})$, add edge (u'_i, u'_j). For all $i \in [d]$, ensure edges $(s, u_i), (v_i, v'_i), (u_i, t)$ are all added(If not, we added them). Now there are d feasible paths from s to t, path i is feasible iff (u_i, v_i), (u'_i, v'_i) would be true(both corresponding input bits to 1). Hence that at least $d = \Omega(\frac{n}{l})$ bits are required to solve USTCON by reducing to DISJ. Finally, when $n = \frac{m}{2}$, $D^{best}(f) = D^{\frac{m}{2}-best}(f) = \Omega(l) = \Omega(\sqrt{m})$.