Homework 4

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Exercise 4.8

Thanks for my classmate Di Chen and Yu Gao to help me finish this problem.

Subproblem 1

Express B_*^1 as a linear program. Use the duality theorem of linear programming to write its dual program.

Besides the definition $B_*^1(f) = \max_\mu B_\mu(f)$ and $B_\mu(f) = \frac{1}{\max_R \mu(R)}$. We design a linear program $LP(f)$and to solve $\frac{1}{B_*^1(f)} = \min_\mu \max_R \mu(R)$. such that

- minimize value k
- for all 1-rectangle R, $\sum_{(x,y)\in R} \mu(x,y) \leq k$
- $\sum_{(x,y)} \mu(x,y) = 1$

And write it's dual program $DP(f)$:

- maximize value k
- for all (x,y), $\sum_{R\ni(x,y)} v(R) \geq k$
- $\sum_R v(R) \leq 1$

While $v(R)$ is the value function of 1-rectangle R.

Subproblem 2

Prove that $B_*^1(f \wedge g) = B_*^1(f)B_*^1(g)$(that is, Lemma 4.7 holds with equality)

Here we use $DP(f)$ to prove $B_*^1(f \wedge g) \leq B_*^1(f)B_*^1(g)$ which means we need to construct value function $v_{f\wedge g}(R)$ and maximize value $k_{f\wedge g}$ to satisfy $DP(f)$ and ensure the linearity of the program.

Assume that we have already know $v_f(R)$, k_f, $v_g(R)$, k_g. Hence we construct the operation of logical and.

$$v_{f\wedge g}(R_{f\wedge g}) = v_f R_f \cdot v_g R_g$$

while $R_{f\wedge g}$ is generated by R_f and R_g. Thus we calculate $k_{f\wedge g} = \min_R v_{f\wedge g}(R) = k_f \cdot k_g$. It's also means the combine of minimum rectangle R_f and R_g can be construct to $R_{f\wedge g}$. According to the definition, we can get $\frac{1}{B_*^1(f\wedge g)} \geq \frac{1}{B_*^1(f)} \cdot \frac{1}{B_*^1(g)}$, and multiple -1 to both sides of equation. we found $B_*^1(f \wedge g) \leq B_*^1(f) \cdot B_*^1(g)$

Finally, by using **Lemma 4.7**, $B_*^1(f \wedge g) \geq B_*^1(f) \cdot B_*^1(g)$. we prove it all.

Subproblem 3

Prove that $B_*^1(f) \leq R^{1,pub}(f) + O(1)$

We also use $DP(f)$ to prove $B_*^1(f) \leq R^{1,pub}(f)+O(1)$, which means reconstruct the inequality $\frac{1}{B_*^1} \geq \frac{1}{c} \cdot \frac{1}{2^{R^{1,pub}(f)}}$, and c is a constant. So, we need suitable value function $v_f(R)$and maximize value k_f in the program.

Construct a deterministic protocol $\mathcal{P}$ and define $L(D)$ as the set of all 1-rectangle leaves in D. Assume that

$$v_f(R) = \frac{1}{2^{R^{1,pub}(f)}} \sum_{L(D) \ni R} \pi(D)$$

Notice that there are at most $2^{R^{1,pub}(f)}$ leaves in $\mathcal{P}$, besides $\Pr[R(x,y) = 1 \mid f(x,y) = 1] \geq \epsilon > 0$. For all (x,y), $\sum_{R \ni (x,y)} v_f(R) \geq \frac{1}{2^{R^{1,pub}(f)}} \cdot \epsilon$.

Finally, let the constant equal to $\frac{1}{\epsilon}$, we have $\frac{1}{B_*^1} \geq \frac{1}{c} \cdot \frac{1}{2^{R^{1,pub}(f)}}$.

Exercise 4.21

Show that $D(f) \geq \log_2 D^1(f)$ and also $R(f) \geq \log_2 R^1(f)$

Subproblem 1

Prove $D(f) \geq \log_2 D^1(f)$.

We build a 1-round protocol by the protocol of f with $2^D(f)$ communication complexity. In this protocol, Alice sends 0 or 1 for each leaves. Alice sends 1 for one leaf iff exists an appropriate y of Bob to fetch the leaf. Otherwise, sends 0.

Hence, Bob can analysis these informations and simulate the protocol of f. When he should choose one sub-tree by using Alice's information, he just ensure which sub-tree's leaves can be fetched. Notice that there are at most one sub-tree existing 1 leaves. So, Bob can determine the answer finally. And we say $2^{D(f)} \geq D^1(f)$. Thus $D(f) \geq \log_2 D^1(f)$.

Subproblem 2

Prove $R(f) \geq \log_2 R^1(f)$.

We use an similar way to prove this subproblem. We also build a 1-round protocol by the protocol of $R(f)$. And both Alice and Bob would do same actions. Bob also can finish the protocol correctly. And we say $2^{R(f)} \geq R^1(f)$. Thus $R(f) \geq \log_2 R^1(f)$.

Exercise 4.55

Let $0 \leq d \leq \frac{n}{2}$. Let S be the set of all pairs (x,y) such that $x,y \in \{0,1\}^n$ and the Hamming distance between x and y is at most d. Prove that $D(S)$ and $D^1(S)$ are both $\Theta(\log \binom{n}{d})$.

Define the set $r_y := \{x \mid \text{Hamming}(x, y) \leq d\}$. And we know $\binom{n}{d} \leq |r_y| \leq (d+1)\binom{n}{d}$. So, we should get the colour of each $x \in \{0,1\}^n$ called $c(x)$ satisfied for each $x_1 \in r_t$ and $x_2 \in r_t$, $c(x_1) \neq c(x_2)$.

And we establish a protocol $\mathcal{P}$. Alice send $c(x)$, the colour of x, to Bob. And there are only one $c(x)$ colour in r_y. Bob determine the answer and announce it. Thus the complexity of $\mathcal{P}$ depends on the number of different colours.

Thus we find the number of colours is $|r_y|^2$ bounded by $\binom{n}{d}^2$. Hence, $D(S)$ and $D^1(S)$ are both $\Theta(\log \binom{n}{d})$.