2)

From section 5.5.1.1 undered Bordered algorithm variant, we know that loop invariant is

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c|c} L/U_{TL} & \hat{A}_{TR} \\ \hline \hat{A}_{BL} & \hat{A}_{BR} \end{array}\right) \wedge L_{TL}U_{TL} = \hat{A}_{TL}$$

Only matrix A_{TL} is being overwritten under this algorithm, the rest remain unchanged. The loop invariant for simple backward analysis is rewritten as

$$\left(\begin{array}{c|c} [\check{L}/\check{U}]_{TL} = [LU(A_{TL})] & \check{U}_{TR} = 0 \\ \hline \check{L}_{BL} = 0 & [\check{L}/\check{U}]_{BR} = 0 \end{array}\right)$$

where ~~[illegible]~~ L and U satisfy $\check{L}\check{U} = A + \delta A$

The error invariant is $\left(\begin{array}{c|c} \check{L}\check{U}_{TL} = (A_{TL} + \Delta A_{TL}) & \check{U}_{TR} = 0 \\ \hline \check{L}_{BL} = 0 & [\check{L}/\check{U}]_{BR} = 0 \end{array}\right)$

a) <u>Base case</u>

For $n=1$, matrix $A \in \mathbb{R}$ such that $A = LU$, where $A = LU$ (L and U are also $\in \mathbb{R}$). Since the matrix is a 1×1 matrix, there will be no L and U factorization that will be suitable in this case. Since these values will be null.

From section 6.2.2, we know that (real number) ~~being stored~~ error is ~~[illegible]~~ ~~real number as floating point~~ ~~matrix~~ being stored as a floating point number as follows:

$$|\Delta A| \leq \epsilon_{mach}|A|$$

In addition, from section 6.3.2.3, we know from the ~~the~~ definition of γ_n, which is:

$$\gamma_n = \frac{n\epsilon_{mach}}{1-n\epsilon_{mach}} \quad \text{for all } n \geq 1 \text{ and } n\epsilon_{mach} < 1$$

~~[illegible]~~ for $n=1$, $\gamma_n = \frac{\epsilon_{mach}}{\underbrace{1-\epsilon_{mach}}_{\approx 10^{-10}}} \approx \epsilon_{mach}$ for a double precision floating point number

Therefore, $|\Delta A| \leq \gamma_n |\check{L}||\check{U}|$ which means $|A| \leq \gamma_n |A|$ is true for $n=1$.

($|LU| = |L||U|$ are all in $\mathbb{R}$)

b) Inductive step:

Assume that for $A_{00} \in \mathbb{R}^{n\times n}$, the bordered LU factorization computes $\check{L}$ and $\check{U}$ where

$$A_{00} + \Delta A_{00} = \check{L}_{00}\check{U}_{00} \text{ with } |\Delta A_{00}| \le \gamma_n |\check{L}_{00}||\check{U}_{00}|$$, which holds for any n.

Now, we apply bordered algorithm to $\left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right)$

From section 5.5.1.1, we know that the above matrix can be rewritten as:

$$A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right), U \rightarrow \left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array}\right), L \rightarrow \left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right)$$

At the top of the loop, after repartioning, A then contains

~~At the~~ At the top of the loop, after updating A it must contain

$$\left(\begin{array}{c|cc} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ A_{20} & a_{21} & A_{22} \end{array}\right)$$ while after updating

$$\begin{pmatrix} A_{00} & a_{01} & A_{02} \\ a_{10}^T & \alpha_{11} & a_{12}^T \\ A_{20} & a_{21} & A_{22} \end{pmatrix} = \left(\begin{array}{cc|c} \check{L}\backslash\check{U}_{00} & \check{U}_{01} & \hat{A}_{02} \\ \check{l}_{10}^T & v_{11} & \hat{a}_{12} \\ \hline \hat{A}_{20} & \hat{a}_{21} & \hat{A}_{22} \end{array}\right)$$

$$\check{L}_{00}\check{U}_{00} = \hat{A}_{00} \quad \check{L}_{00}\check{u}_{01} = \hat{a}_{01}$$

$$\check{l}_{10}^T\check{U}_{00} = \hat{a}_{10}^T \quad \check{l}_{10}^T\check{u}_{01} + v_{11} = \hat{\alpha}_{11}$$

$$\wedge \begin{pmatrix} \check{L}_{00} & 0 \\ \check{l}_{10}^T & 1 \end{pmatrix}\begin{pmatrix} \check{U}_{00} & \check{U}_{01} \\ 0 & \check{U}_{11} \end{pmatrix} = \begin{pmatrix} \hat{A}_{00} & \hat{a}_{01} \\ \hat{a}_{10}^T & \hat{\alpha}_{11} \end{pmatrix}$$

$\hat{a}_{01} = \delta a_{01} + a_{01}$

$\hat{a}_{10}^T = \delta a_{10} + a_{10}^T$

$\hat{\alpha}_{11} = \delta\alpha_{11} + \alpha_{11}$

$$\hat{A}_{00} = A_{00} + \Delta A_{00}$$

~~We know~~ This gives us the equation

$$\left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) + \left(\begin{array}{c|c} \Delta A_{00} & \delta a_{01} \\ \hline \delta a_{10}^T & \delta a_{11} \end{array}\right) = \left(\begin{array}{c|c} \hat{L}_{00} & 0 \\ \hline \hat{l}_{10}^T & 1 \end{array}\right)\left(\begin{array}{c|c} \hat{U}_{00} & \hat{u}_{01} \\ \hline 0 & \hat{v}_{11} \end{array}\right)$$

This is bounded by the error $|\Delta A_{00}| \le \gamma_n |\check{L}_{00}||\check{U}_{00}|$ and the base case which $|\Delta A| \le \gamma_1 |L||U|$

Combining these two boundaries we get

$$\left|\left(\begin{array}{c|c} \Delta A_{00} & \delta a_{01} \\ \hline \delta a_{10}^T & \delta a_{11} \end{array}\right)\right| \le \gamma_{n+1}\left|\left(\begin{array}{c|c} \check{L}_{00} & 0 \\ \hline \check{l}_{10}^T & 1 \end{array}\right)\right|\left|\left(\begin{array}{c|c} \hat{U}_{00} & \hat{u}_{01} \\ \hline 0 & \hat{u}_{11} \end{array}\right)\right|$$