1) ~~[illegible]~~

a) To derive the bordered algorithm for computing the Cholesky factorization of a symmetric ~~matrix~~ positive definite (SPD) ~~matrix (SPD)~~ matrix A, we need to partition A and ~~a~~ L as follows;

$$A = \left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) \quad L = \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)$$

Where A_{00} is a pxp submatrix, a_{01} is a pxq submatrix, a_{10}^T is a qxp submatrix, and α_{11} is a scalar. Also, L_{00} is a pxp lower triangular submatrix, l_{10}^T is a qxp submatrix and λ_{11} is a scalar. Here, we assume that A is partitioned into p and q ~~such that~~ such that $(p+q=n)$ and A_{00} and L_{00} are also SPD matrices,

The Cholesky factorization of A is: $A = LL^T$

Substituting the partioned form of A and L, this results:

$$\left(\begin{array}{c|c} A_{00} & * \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) = \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)\left(\begin{array}{c|c} L_{00}^T & l_{10} \\ \hline 0 & \lambda_{11} \end{array}\right)$$

$*$: part of A that is not updated yet

Multiplying the right hand side, we get: ~~translate this to an algorithm~~

$$A_{00} = L_{00} L_{00}^T$$

~~[illegible]~~

$$a_{10}^T = l_{10}^T L_{00}^T$$

$$\alpha_{11} = l_{10}^T l_{10} + \lambda_{11}^2$$

This gives us:

$$\left(\begin{array}{c|c} L_{00} = Chol(A)_{00} & * \\ \hline l_{10}^T = a_{10}^T L_{00}^{-T} & \lambda_{11} = \sqrt{\alpha_{11} - l_{10}^T l_{10}} \end{array}\right)$$

The equalities above results in this algorithm:

1. Partition $A \rightarrow \left(\begin{array}{c|c} A_{00} & * \\ \hline a_{10}^T & \alpha_{11} \end{array}\right)$

2. Assume that $A_{00} := L_{00} = \mathrm{Chol}(A_{00})$ has already been computed on previous iterations of the ~~loop~~ loop in the algorithm.

3. Overwrite $a_{10}^T := l_{10}^T = a_{10}^T L_{00}^{-T}$

4. Overwrite $\alpha_{11} := \sqrt{\alpha_{11} - l_{10}^T l_{10}}$

Algorithm $A := \mathrm{Chol}(A)$

Partition $A \rightarrow \left(\begin{array}{c|c} A_{TL} & * \\ \hline A_{BL} & A_{BR} \end{array}\right)$

where A_{TL} is 0×0

While $m(A_{TL}) < m(A)$ do:

repartition $\left(\begin{array}{c|c} A_{TL} & * \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & * & * \\ \hline a_{10}^T & \alpha_{11} & * \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)$

where α_{11} is 1×1

(Bordered Algorithm)

$a_{10}^T := a_{10}^T \, \mathrm{TRIL}(A_{00})^{-T}$

$\alpha_{11} := \alpha_{11} - a_{10}^T a_{10}$

$\alpha_{11} := \sqrt{\alpha_{11}}$

Continue with

$\left(\begin{array}{c|c} A_{TL} & * \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & * & * \\ \hline a_{10}^T & \alpha_{11} & * \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)$

end while

1b) We will prove the Cholesky factorization theorem by showing that the Cholesky factorization is well-defined for a matrix A that is SPD. For this proof, we will use some lemmas.

Lemma 1:

Let $A \in \mathbb{R}^{n \times n}$ be an SPD matrix. Then α_{11} is real and positive.

- proof: This is just a special case for lemma 5.4.4.1 in the textbook. Since all SPD matrices are by definition ~~just an~~ also HPD matrices.

Lemma 2:

Let $A \in \mathbb{R}^{m \times m}$ be an SPD matrix and $l_{21} = a_{21}/\sqrt{\alpha_{11}}$. Then, $A_{22} - l_{21} l_{21}^T$ is an SPD matrix.

- proof: Since A is symmetric so are A_{22} and $A_{22} - l_{21} l_{21}^T$. Given $x_1 \neq 0$ be any vector with length $n-1$, let's define x to be $\left(\frac{x_0}{x_1}\right)$ where $x_0 = -a_{21}^T x_1/\alpha_{11}$. Then, because $x \neq 0$,

$$
\begin{aligned}
&= \left(\frac{x_0}{x_1}\right)^T \left(\begin{array}{c|c} \alpha_{11} & a_{21}^T \\ \hline a_{21} & A_{22} \end{array}\right) \left(\frac{x_0}{x_1}\right) = x^T A x > 0 \\
&= \left(\frac{x_0}{x_1}\right)^T \left(\frac{\alpha_{11} x_0 + a_{21}^T x_1}{a_{21} x_0 + A_{22} x_1}\right) \\
&= \alpha_{11} x_0^2 + x_0 a_{21}^T x_1 + x_1^T a_{21} x_0 + x_1^T A_{22} x_1 \\
&= \alpha_{11}\left(\frac{a_{21}^T x_1 x_1^T a_{21}}{\alpha_{11}}\right) - \frac{x_1^T a_{21} a_{21}^T x_1}{\alpha_{11}} - x_1^T a_{21} \frac{a_{21}^T x_1}{\alpha_{11}} + x_1^T A_{22} x_1 \\
&= x_1^T \left(A_{22} - \frac{a_{21} a_{21}^T}{\alpha_{11}}\right) x_1 \\
&= x_1^T \left(A_{22} - l_{21} l_{21}^T\right) x_1
\end{aligned}
$$

Therefore, we conclude that $A_{22} - l_{21} l_{21}^T$ is an SPD matrix.

Proof by induction:

~~Base case~~

Base case:

For $n=1$, the result is trivial for a 1×1 matrix A, which is just equal to α_{11}. In this example, since A is SPD, then α_{11} is real and positive. A cholesky factor is then given as $\lambda_{11} = \sqrt{\alpha_{11}}$. This is unique if we know that λ_{11} is positive.

Inductive step:

Let's assume the result is true for an SPD matrix A, where $A \in \mathbb{R}^{(n-1)\times(n-1)}$. We will also show that this is true as well for SPD matrix A, where $A \in \mathbb{R}^{n\times n}$:

First, given $A \in \mathbb{R}^{n\times n}$ be an SPD matrix, partition A and L such that:

$$A = \left(\begin{array}{c|c} \alpha_{11} & a_{21}^T \\ \hline a_{21} & A_{22} \end{array}\right) \qquad L = \left(\begin{array}{c|c} \lambda_{11} & 0 \\ \hline l_{21} & L_{22} \end{array}\right)$$

and let's set $\lambda_{11} = \sqrt{\alpha_{11}}$ (which has been shown previously), $l_{21} = a_{21}/\lambda_{11}$, and L_{22} will then be equal to $\mathrm{Chol}(A_{22} - l_{21}l_{21}^T)$. Therefore, L is the cholesky factor of a SPD matrix A. By the principle of mathematical induction, this proof holds.