Networks and Flows on Graphs

Optimal Transportation Problems

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May 16, 2017

EPITA

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Optimal transportation problem

Find a bipartite non-negatively (integer-)weighted digraph $G = (V, A)$ linking vertices in O to those of D such that the total cost

$$c(G) = \sum_{a \in A} w(a)c(a)$$

is minimal among all possible digraphs.

Conveniently modeling a transporation problem

The following table/matrix represents the costs of “paths” going from a set of origins $\{I, II, III, IV\}$ to a set of destinations numbered from 1 to 6.

	1	*2*	*3*	*4*	*5*	*6*	**Av.**
I	12	27	61	49	83	35	18
II	23	39	78	28	65	42	32
III	67	56	92	24	53	54	14
IV	71	43	91	67	40	49	9
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The violet colored (i, j) coefficient represents the cost c_{ij} of the path going from i to j. The darker cells correspond to available goods and demand, the available number of goods at a line i is written a_i and the number of needed goods at a column j is b_j. Notice that the total amount of available goods is the same as the number of needed ones.

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Solving the previous transportation problem is about finding a matrix (x_{ij}) such that

$$\sum_{j=1}^{n} x_{ij} = a_i \ , \quad \sum_{i=1}^{m} x_{ij} = b_j \quad \text{and the total cost} \quad c_{tot}\big((x_{ij})\big) = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}c_{ij} \quad \text{is minimal}$$

Needed hypothesis

The algorithm we shall be giving subsequently starts by giving an answer to our transportation problem not taking into account the cost; i.e. we look for a matrix (x_{ij}) only satisfying[1]

$$\sum_{j=1}^{n} x_{ij} = a_i \quad \text{and} \quad \sum_{i=1}^{m} x_{ij} = b_j \tag{$\star$}$$

In order to be able to get a re-usable (optimizable if not optimal) answer we'll need it to be *non-degenerate* ;

Non-degeneracy

A matrix (x_{ij}) satisfying $(\star)$ is called a *basic solution* if it has $nm-(n+m-1)$ zeros.

Remark : It is not always the best idea to look for a basic solution! The point is that, when you have one, you're sure to be able to get a better one, if it's not the best.

[1] In fact, the Balas-Hammer algorithm, studied hereby, "takes into account" the cost function.

First stage : finding a basic solution

Algorithm 1 Balas-Hammer

Input: M a maximal rank matrix of costs having size $m \times n$ and positive integer entries, a positive integer a_i for each line i and one b_j for each column j (sums of which along lines and columns are equal)

Output: A solution for the transportation problem defined by M, (a_i) and (b_j)

for each line and each column, compute the difference between the smallest integer in the line or column and the one just bigger
get the line or column corresponding to the maximum of all differences
get the address (i, j) of the minimum cost of the corresponding line or column
give the highest possible weight x_{ij}
erase the *saturated* line i or column j, obtained previously from M (all corresponding weights are 0 except for x_{ij}) and modify the number of available and needed goods accordingly
start again till M is empty
return the matrix (x_{ij})

Balas-Hammer : Working out an example

Write Δ for the difference of the minimum of any line or column with the number just bigger, in the same line or column.

	1	*2*	*3*	*4*	*5*	*6*	**Av.**	Δ
I	12	27	61	49	83	35	18	15
II	23	39	78	28	65	42	32	5
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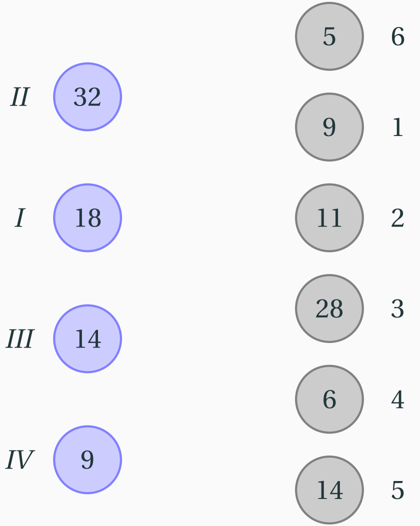
Δ	11	12	17	4	13	7		



Balas-Hammer : Working out an example

The maximum of differences is 29 at row *III*. The minimum of cost of row *III* is 24. It corresponds to the path from origin *III* to destination 4.

	1	*2*	*3*	*4*	*5*	*6*	**Av.**	Δ
I	12	27	61	49	83	35	18	15
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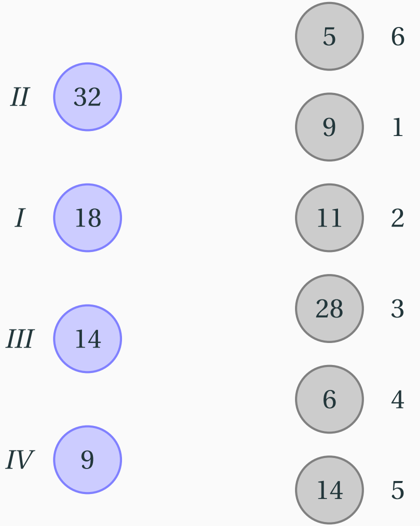

Balas-Hammer : Working out an example

There are 14 available goods at origin *III* and 6 needed at destination 4. We thus choose $x_{III,4} = 6$ and one can forget about column 4.

	1	*2*	*3*	*4*	*5*	*6*	**Av.**	Δ
I	12	27	61	49	83	35	18	15
II	23	39	78	28	65	42	32	5
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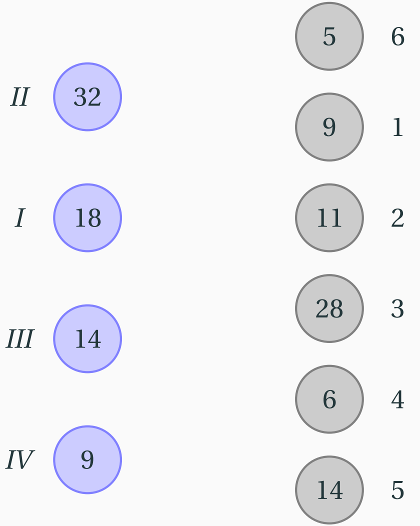

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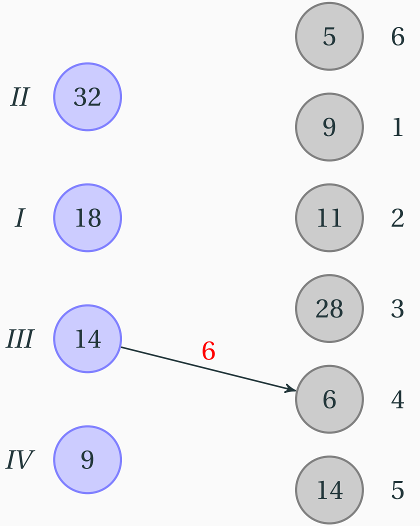

Balas-Hammer : Working out an example

We thus find ourselves with a smaller matrix. In this new matrix cost doesn't change but there are less available goods at origin *III* because 6 went to fill the need at destination 4.

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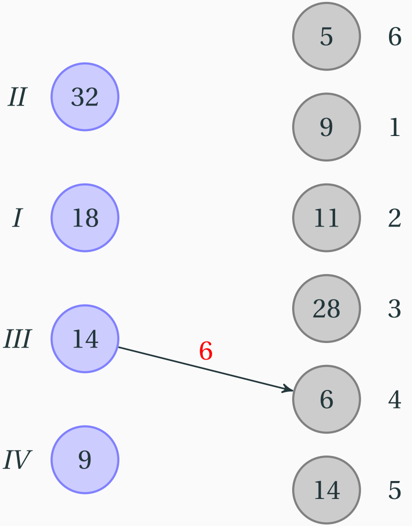
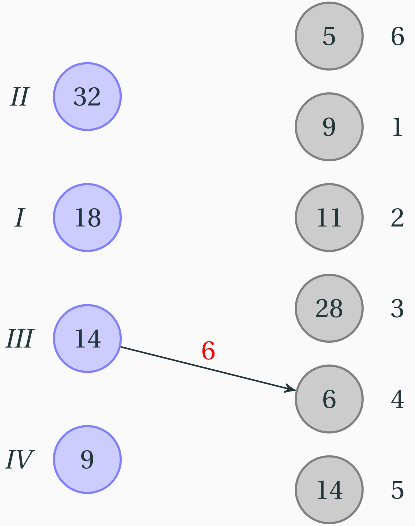
II	23	39	78	65	42	32
III	67	56	92	53	54	8
IV	71	43	91	40	49	9
De.	9	11	28	14	5	67



Balas-Hammer : Working out an example

Let us run one more step towards building a basic solution to our transportation problem.

	1	*2*	*3*	*5*	*6*	**Av.**	Δ
I	12	27	61	83	35	18	15
II	23	39	78	65	42	32	16
III	67	56	92	53	54	8	1
IV	71	43	91	40	49	9	3
De.	9	11	28	14	5	67	
Δ	11	12	17	13	7		



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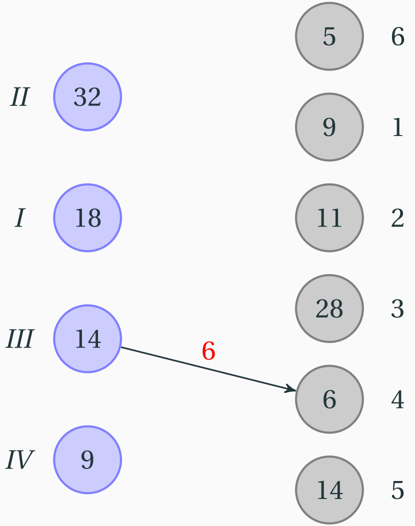

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II	23	39	78	65	42	32	16
III	67	56	92	53	54	8	1
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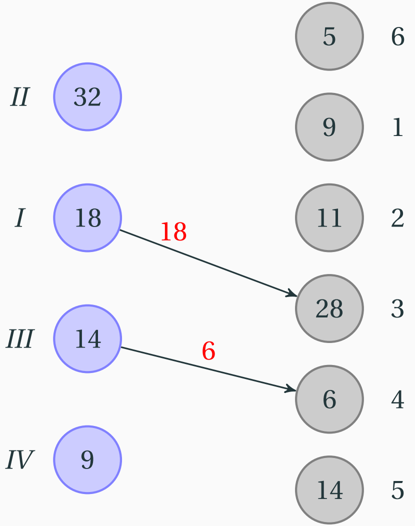

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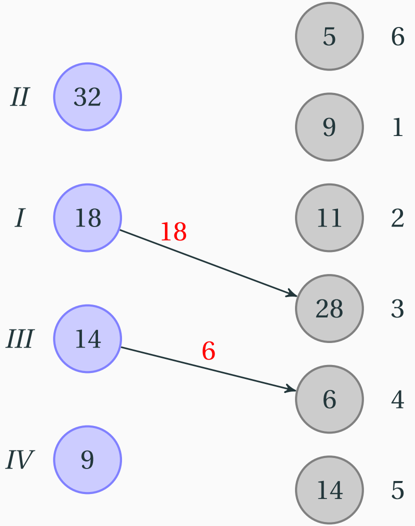
IV	71	43	91	40	49	9
De.	9	11	10	14	5	49

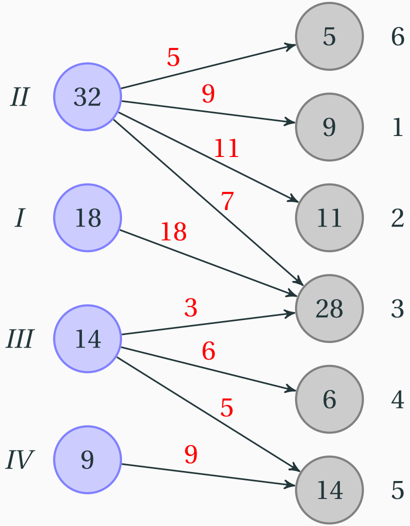


Tree Structure Underlying a Basic Solution

Running Balas-Hammer algorithm till the end we get the solution on the right.

Notice now that the condition to be a *basic* solution means that you get a graph having $m+n$ vertices (m origins and n destinations) with $m+n-1$ arrows; it is thus a tree!
In the case at hand we got $4+6-1=9$ arrows and one can check by looking at the graph on the right that it is a tree.

We are going to use this tree structure for the second step of our algorithm: optimizing the transportation program we have.



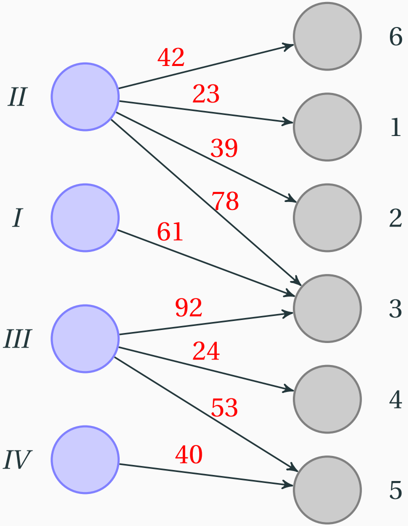
How to look for a better solution ?

We've got a basic solution, i.e. a matrix $(x_{ij})_{i,j}$ satisfying

$$\sum_{j=1}^{n} x_{ij} = a_i \quad \text{and} \quad \sum_{i=1}^{m} x_{ij} = b_j \qquad (\star)$$

where a_i is the number of goods available at i and b_j is the number of ones needed at j.

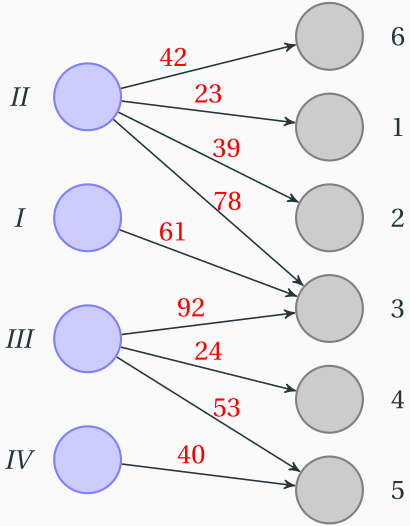
	1	*2*	*3*	*4*	*5*	*6*
I			18			
II	9	11	7			5
III			3	6	5	
IV					9	



How to look for a better solution ?

In order to understand how to look for a better one, let us start by reorganizing our data. Below is the matrix $(x_{ij})_{i,j}$ and on the write is the tree representing this solution with costs along arrows.

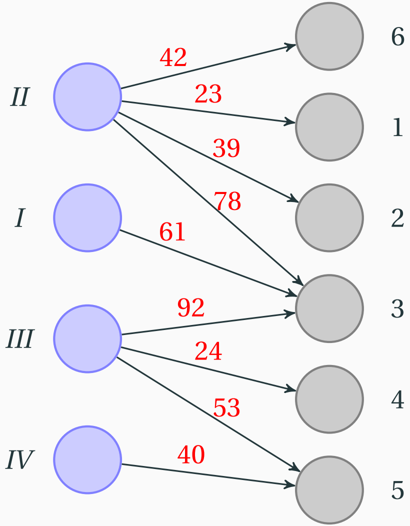
	1	*2*	*3*	*4*	*5*	*6*
I			18			
II	9	11	7			5
III			3	6	5	
IV					9	



How to look for a better solution ?

Assume we want to send a unit of goods along the unused path $(I, 1)$. This means we're adding 1 to the $(I, 1)$ coefficient of our matrix. For the final result to stay a solution to our transportation problem, we're forced to make at least 3 other changes to our matrix.

	1	*2*	*3*	*4*	*5*	*6*
I	+1		18 -1			
II	9 -1	11	7 +1			5
III			3	6	5	
IV					9	



How to look for a better solution ?

These changes correspond to going around the green cycle in the graph on the right. Adding +1 to the path $(I, 1)$, taking 1 out of the route along $(II, 1)$, adding it up to the route along $(II, 3)$ and then taking 1 from the path $(I, 3)$.

	1	*2*	*3*	*4*	*5*	*6*
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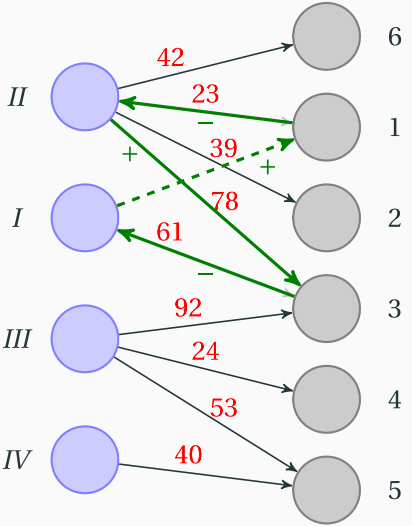

How to look for a better solution ?

Looking for a different solution is about looking for cycles in the tree on the right. For such a solution to be a *better* solution, the *marginal cost* along this cycle has to be negative. In our case $+12-23+78-61=6$, if we tried going through $(I,1)$ it would only get more expensive.

	1	*2*	*3*	*4*	*5*	*6*
I	+1		18 -1			
II	9 -1	11	7 +1			5
III			3	6	5	
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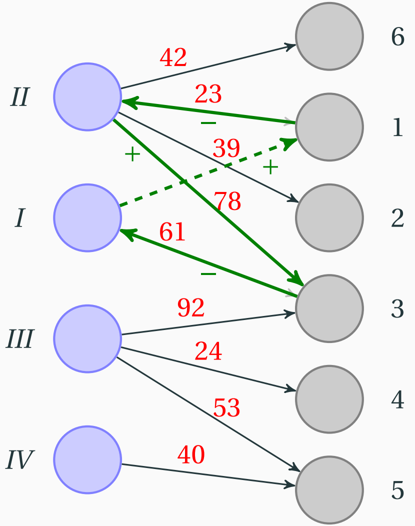
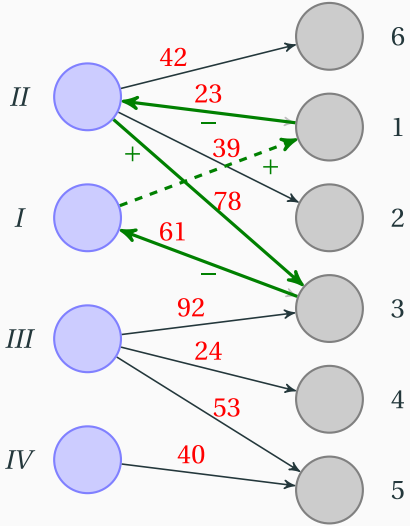


Looking for a better solution : First step conclusion

Let T be the tree given by the Balas-Hammer heuristic. Here is how to proceed in order to get a better solution (if there is any) :

- Let (α, β) be a missing route from an origin to a destination, that is not in T. Since T is a tree, there is a unique *chain* C from β to α.



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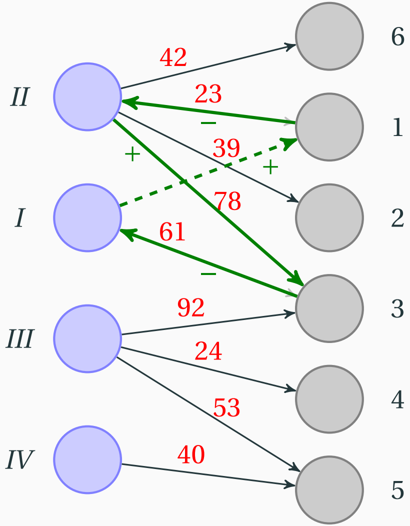
- Let (α, β) be a missing route from an origin to a destination, that is not in T. Since T is a tree, there is a unique *chain* C from β to α.
- Compute the marginal cost of adding a to T along the cycle given by $(\alpha, \beta) \cup C$. If the marginal cost is non-negative do nothing, if it is negative add the route (α, β) with weight 1 and modify T following the cycle $(\alpha, \beta) \cup C$.

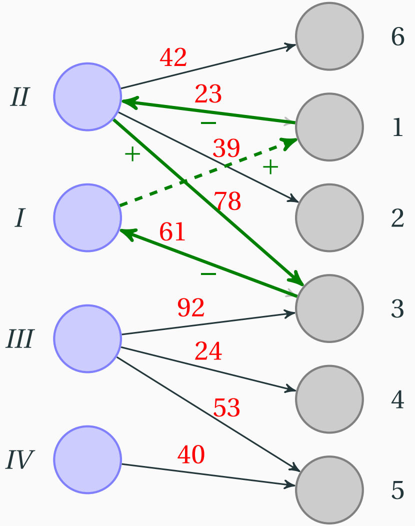


Computing marginal costs : electrical networks

Let us look back at the marginal c_m cost of the green cycle on the right. We have

$$c_m = \underbrace{12}_{(*)} + \underbrace{\left(-23+78-61\right)}_{(**)} = 6.$$

The term $(*)$ is the cost of the route $(I,1)$, there is not much we can do about it. The term $(**)$ can be computed in a way allowing for quicker handwork. The idea is that the $(**)$ should correspond to a difference of potential between 1 and I. This idea takes root in the following result.



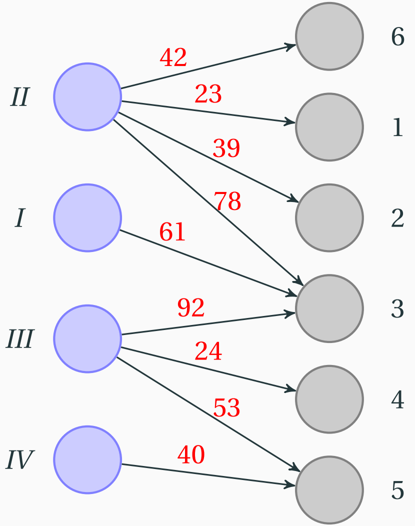
Computing marginal costs : electrical networks

Proposition

Let $T = (V, A)$ be an oriented tree coming with a weight function $\nu : A \to \mathbb{R}$. There is a then a unique function $p : V \to \mathbb{R}$ satisfying

- at a given vertex v, $p(v) = 0$
- for each arrow $a = (x, y)$, $\nu(a) = p(y) - p(x)$.

The function p is called a potential on T.



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Proposition

Let $T = (V, A)$ be an oriented tree coming with a weight function $v : A \to \mathbb{R}$. There is a then a unique function $p : V \to \mathbb{R}$ satisfying

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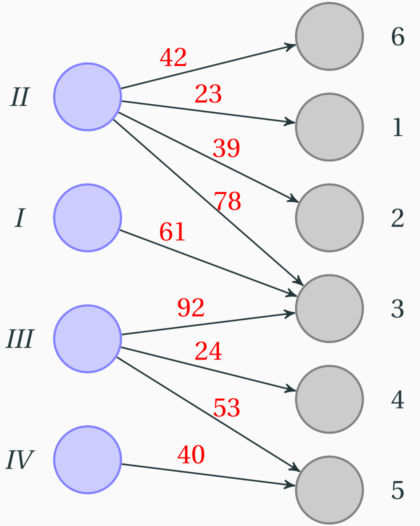
This proposition is proved by building p inductively starting with v, where it is zero, then defining it on its childs. Take v out and start again with each child, with the previously given weight.



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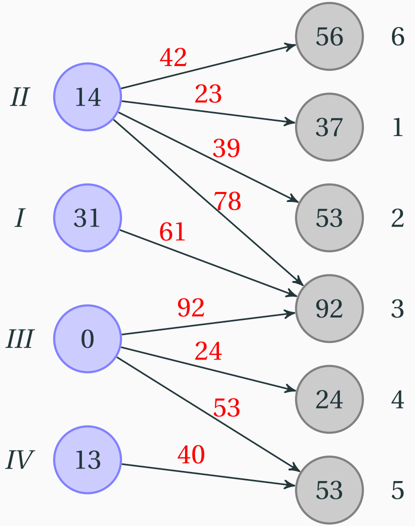
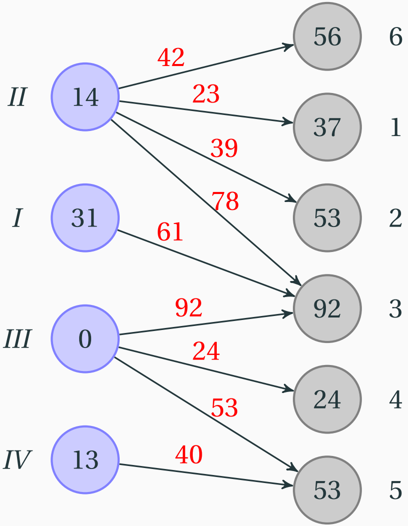


Computing marginal costs : electrical networks

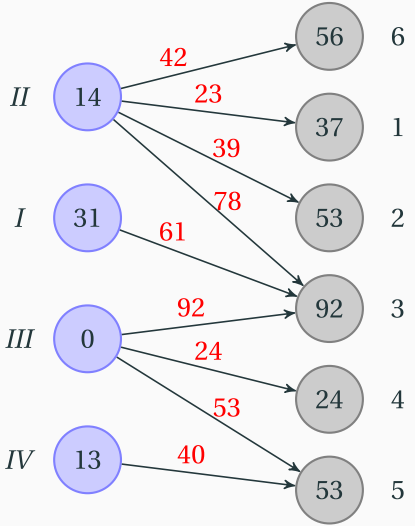
The main property of the potential function p is the following one :



Computing marginal costs : electrical networks

The main property of the potential function p is the following one :

Let C be a chain in T linking two vertices α and β. Give an arrow a in C a $(+1)$ weight if the orientation on C from α to β matches the one of a, otherwise a (-1) weight.



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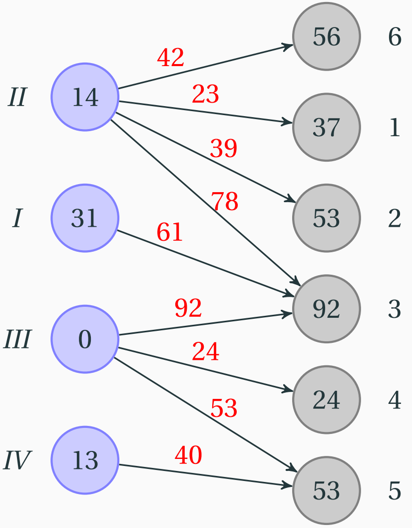
Let C be a chain in T linking two vertices α and β. Give an arrow a in C a $(+1)$ weight if the orientation on C from α to β matches the one of a, otherwise a (-1) weight.

Taking these signs into account the some of weights v_C along C is given by

$$v_C = p(\beta) - p(\alpha).$$

It is the difference between the potential at the target and the one at the origin!



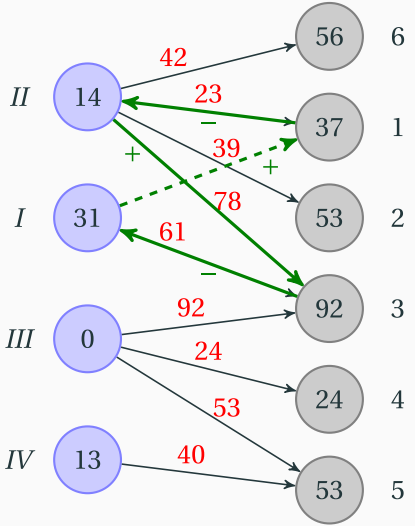
Computing marginal costs : electrical networks

For instance in the case of our previous marginal cost computation along $(I,1)$ we have

$$\begin{aligned} c_m &= 12 + \left(-23 + 78 - 61\right) \\ &= 6 \\ &= 12 - \left(37 - 31\right). \end{aligned}$$

Thus, the marginal cost c_m along a route (α,β) is given by

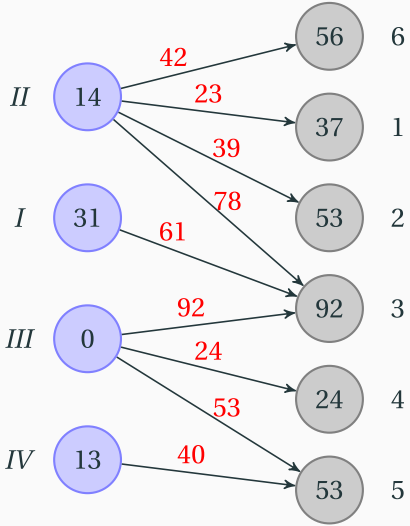
$$c_m = c_{(\alpha,\beta)} - \left(p(\beta) - p(\beta)\right).$$



Looking for a better solution : Final conclusion

Let T be the tree given by the Balas-Hammer heuristic. Here is how to proceed in order to get a better solution (if there is any) :

- Compute the potential function p of T.



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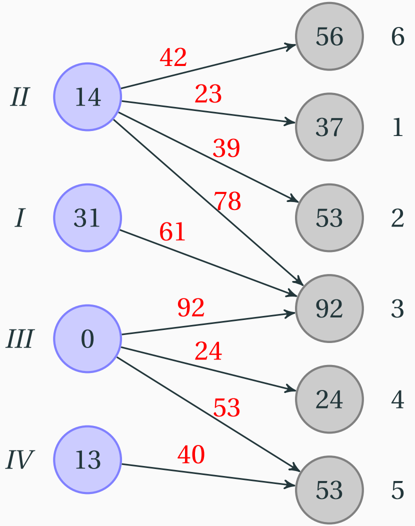
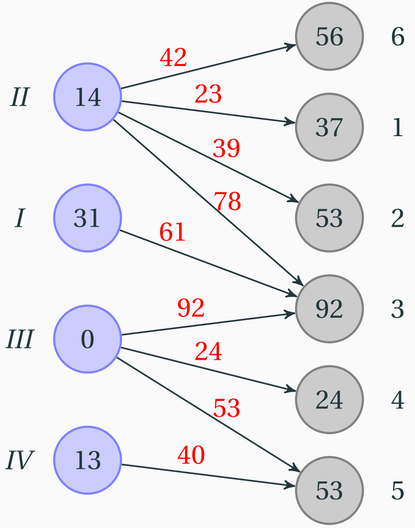
- Compute the potential function p of T.
- For each missing route $a = (\alpha, \beta)$, compute the marginal cost of adding a to T using p. If the marginal cost is non-negative do nothing, else add the route (α, β) with weight 1 and modify T following the unique chain from (β, α).



Looking for a better solution : Final conclusion

Let T be the tree given by the Balas-Hammer heuristic. Here is how to proceed in order to get a better solution (if there is any) :

- Compute the potential function p of T.
- For each missing route $a = (\alpha, \beta)$, compute the marginal cost of adding a to T using p. If the marginal cost is non-negative do nothing, else add the route (α, β) with weight 1 and modify T following the unique chain from (β, α).



Optimization

Is the transportation program we have the one having the least total cost?

This is it for transportation programs but you still need to work out!