Approximation and interpolation Exercises with solutions

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This document provides typical exercises related to the course APIN. It contains three parts, the same ones as the course :

1. Approximation. The exercises consist in finding the best approximation of a function f with a polynomial $P \in \mathbb{R}_n[X]$ (typically, $n = 2$). The best approximation is the one which minimizes a quadratic error
$$\int_I w(x)\big[f(x) - P(x)\big]^2\,\mathrm{d}x$$
It is the orthogonal projection of f onto $\mathbb{R}_n[X]$.
2. Interpolation. We have a set of nodes $x_0, x_1, \cdots, x_n$ in $\mathbb{R}$ and we only know from f its values at these nodes. There exists a unique interpolating polynomial P_n of f in $\mathbb{R}_n[X]$ and the exercises consist in providing this polynomial, using Lagrange's or Newton's method.
The error $P_n(x) - f(x)$ needs also to be known. And some specific computations with the polynomial can be required (synthetic division).
3. Numerical integration. The purpose of this part is to provide and study quadrature rules. Such a rule is an estimation of an integral :
$$\int_\alpha^\beta f(x)\,\mathrm{d}x \simeq w_0 f(x_0) + w_1 f(w_1) + \cdot + w_n f(x_n)$$
Typical exercises consist in calculating the w_i, determining the degree of precision of the quadrature rule, its Peano kernel and in providing an upper bound for the integration error.

1 Approximation

1.1 Exercises

Exercise 1

Let $E = C^0([0,1])$ be the vector space of the continuous functions from $[0,1]$ to $\mathbb{R}$. We define on E the inner product $\langle , \rangle$ and the associated norm $\|.\|$ for any $(f,g) \in E^2$ by

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x \qquad \text{and} \qquad \|f\| = \sqrt{\langle f, f\rangle}$$

Let $F = \mathbb{R}_2[X]$ be the linear subspace of E made of the polynomial functions of degree at most 2. It is a linear subspace of dimension 3 : a basis of F is the family $(\varphi_0, \varphi_1, \varphi_2)$ with

$$\varphi_0(x) = 1, \qquad \varphi_1(x) = x, \qquad \varphi_2(x) = x^2$$

Let f be the function defined for on $[0,1]$ by $f(x) = e^x$.

1. Explain why the best approximation of f by a function of F, in the sense of the norm $\|.\|$, is $p_F(f)$, the orthogonal projection of f onto F.
2. Determine $p_F(f)$.

Exercise 2

Let $E = C^0([0,1])$ be the vector space of the continuous functions from $[0,1]$ to $\mathbb{R}$. We define the same inner product and norm as in exercise 1 : for any $(f,g) \in E^2$

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x \qquad \text{and} \qquad \|f\| = \sqrt{\langle f, f\rangle}$$

Let $(\varphi_0, \varphi_1, \varphi_2)$ be the family of functions defined for any $x \in [0,1]$ by

$$\varphi_0(x) = 1 \qquad \varphi_1(x) = \sqrt{3}(2x-1) \qquad \varphi_2(x) = \sqrt{5}(6x^2 - 6x + 1)$$

1. Prove that $(\varphi_0, \varphi_1, \varphi_2)$ is an orthonormal basis of $F = \mathbb{R}_2[X]$, the set of polynomial functions of degree at most 2.
2. Let f and g be the functions defined for any $x \in [0,1]$ by

$$f(x) = \cos(\pi x) \qquad g(x) = \sin(\pi x)$$

   (a) Calculate $\langle f, 1\rangle$ and $\langle g, 1\rangle$.
   (b) Using an integration by parts, deduce $\langle f, x\rangle$ and $\langle g, x\rangle$
   (c) Using an integration by parts, deduce $\langle f, x^2\rangle$ and $\langle g, x^2\rangle$
   (d) Deduce $\langle f, \varphi_i\rangle$ and $\langle g, \varphi_i\rangle$ for all $i \in \{0,1,2\}$.
3. Let P_f and P_g be the orthogonal projections of f and g onto F.
   Since P_f and P_g are in F, there exist $(a_0, a_1, a_2) \in \mathbb{R}^3$ and $(b_0, b_1, b_2) \in \mathbb{R}^3$ such that

$$P_f = a_0\varphi_0 + a_1\varphi_1 + a_2\varphi_2 \qquad \text{and} \qquad P_g = b_0\varphi_0 + b_1\varphi_1 + b_2\varphi_2$$

   (a) Write the system that (a_0, a_1, a_2) satisfies.
   (b) Write the system that (b_0, b_1, b_2) satisfies.
   (c) Deduce P_f and P_g.

1.2 Solutions

Exercise 1

1. The best approximation of f by a function of F is $p_F(f)$, the orthogonal projection of f onto F. This orthogonal projection is defined by the two properties :
$$\begin{cases} p_F(f) \in F \\ f - p_F(f) \in F^\perp \end{cases}$$
Then, for any $P \in F$, we have $p_F(f) - P \in F$ since both $p_F(f)$ and P are in F. It results that
$$\begin{aligned}
\|f-P\|^2 &= \left\|\big(f-p_F(f)\big)+\big(p_F(f)-P\big)\right\|^2 \\
&= \langle f-p_F(f), f-p_F(f)\rangle + 2\left\langle \underbrace{f-p_F(f)}_{\in F^\perp}, \underbrace{p_F(f)-P}_{\in F}\right\rangle + \langle p_F(f)-P, p_F(f)-P\rangle \\
&= \|f-p_F(f)\|^2 + \|p_F(f)-P\|^2 \\
&\geqslant \|f-p_F(f)\|^2
\end{aligned}$$

2. To determine $p_F(f)$, we start with its definition :
$$\begin{aligned}
P_0 = p_F(f) &\iff \begin{cases} P_0 \in F \\ f - P_0 \in F^\perp \end{cases} \\
&\iff \begin{cases} \exists (a_0,a_1,a_2) \in \mathbb{R}^3, P_0 = a_0\varphi_0 + a_1\varphi_1 + a_2\varphi_2 \\ \\ \langle f - P_0, \varphi_0\rangle = 0 \\ \langle f - P_0, \varphi_1\rangle = 0 \\ \langle f - P_0, \varphi_2\rangle = 0 \end{cases}
\end{aligned}$$
We just need to calculate the coefficients a_0, a_1 and a_2. They are solutions of the system :
$$\begin{cases} \langle f - a_0\varphi_0 - a_1\varphi_1 - a_2\varphi_2, \varphi_0\rangle = 0 \\ \langle f - a_0\varphi_0 - a_1\varphi_1 - a_2\varphi_2, \varphi_1\rangle = 0 \\ \langle f - a_0\varphi_0 - a_1\varphi_1 - a_2\varphi_2, \varphi_2\rangle = 0 \end{cases} \iff \begin{cases} a_0\langle\varphi_0,\varphi_0\rangle + a_1\langle\varphi_1,\varphi_0\rangle + a_2\langle\varphi_2,\varphi_0\rangle = \langle f,\varphi_0\rangle \\ a_0\langle\varphi_0,\varphi_1\rangle + a_1\langle\varphi_1,\varphi_1\rangle + a_2\langle\varphi_2,\varphi_1\rangle = \langle f,\varphi_1\rangle \\ a_0\langle\varphi_0,\varphi_2\rangle + a_1\langle\varphi_1,\varphi_2\rangle + a_2\langle\varphi_2,\varphi_2\rangle = \langle f,\varphi_2\rangle \end{cases}$$
But for any (i,j), we have
$$\langle \varphi_i, \varphi_j\rangle = \int_0^1 x^{i+j}\,\mathrm{d}x = \left[\frac{x^{i+j+1}}{i+j+1}\right]_0^1 = \frac{1}{i+j+1}$$
Furthermore,
$$\begin{aligned}
\langle f, \varphi_0\rangle &= \int_0^1 e^x\,\mathrm{d}x = [e^x]_0^1 = e-1 \\
\langle f, \varphi_1\rangle &= \int_0^1 xe^x\,\mathrm{d}x \\
&= [xe^x]_0^1 - \int_0^1 e^x\,\mathrm{d}x \\
&= e - (e-1) = 1 \\
\langle f, \varphi_2\rangle &= \int_0^1 x^2e^x\,\mathrm{d}x \\
&= \left[x^2e^x\right]_0^1 - \int_0^1 2xe^x\,\mathrm{d}x \\
&= e-2
\end{aligned}$$

So the system we must solve is

$$
(S) \quad = \quad \begin{cases} a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 & = \quad e-1 \\ \frac{1}{2}a_0 + \frac{1}{3}a_1 + \frac{1}{4}a_2 & = \quad 1 \\ \frac{1}{3}a_0 + \frac{1}{4}a_1 + \frac{1}{5}a_2 & = \quad e-2 \end{cases}
$$

$$
\iff \begin{cases} a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 & = \quad e-1 \\ \frac{1}{6}a_1 + \frac{1}{6}a_2 & = \quad -e+3 \quad 2\times \text{Eqn.}2 - \text{Eqn.}1 \\ \frac{1}{4}a_1 + \frac{4}{15}a_2 & = \quad 2e-5 \quad 3\times \text{Eqn.}3 - \text{Eqn.}1 \end{cases}
$$

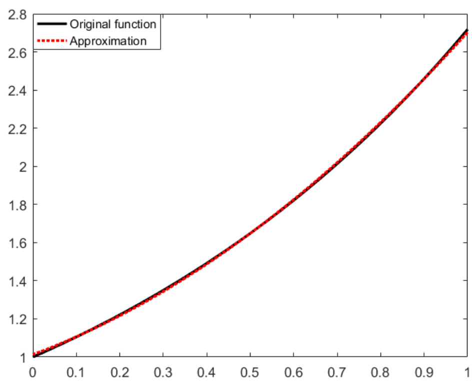
$$
\iff \begin{cases} a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 & = \quad e-1 \\ \frac{1}{6}a_1 + \frac{1}{6}a_2 & = \quad -e+3 \\ \frac{1}{30}a_2 & = \quad 7e-19 \quad 2\times \text{Eqn.}3 - 3\times \text{Eqn.}2 \end{cases}
$$

$$
\iff \begin{cases} a_2 & = \quad 210e-570 \\ a_1 & = \quad -216e+588 \\ a_0 & = \quad 39e-105 \end{cases}
$$

Thus, the orthogonal projection of f onto $F = \mathbb{R}_2[X]$ is

$$
P_0(x) = (210e-570)x^2 + (-216e+588)x + (39e-105)
$$

The following figure shows the graphs of the function and of its approximation.



Exercise 2

1. We have

$$\begin{array}{rcl}\langle \varphi_0, \varphi_1 \rangle & = & \displaystyle\int_0^1 1 \times \sqrt{3}(2x-1)\,\mathrm{d}x \\ & = & \left[\sqrt{3}(x^2-x)\right]_0^1 \\ & = & 0 \\ \langle \varphi_0, \varphi_2 \rangle & = & \displaystyle\int_0^1 1 \times \sqrt{5}(6x^2-6x+1)\,\mathrm{d}x \\ & = & \left[\sqrt{5}(2x^3-3x^2+x)\right]_0^1 \\ & = & 0 \\ \langle \varphi_1, \varphi_2 \rangle & = & \displaystyle\int_0^1 \sqrt{3}(2x-1) \times \sqrt{5}(6x^2-6x+1)\,\mathrm{d}x \\ & = & \displaystyle\int_0^1 (\sqrt{15}(12x^3-18x^2+8x-1)\,\mathrm{d}x \\ & = & \left[\sqrt{15}(3x^4-6x^3+4x^2-x)\right]_0^1 \\ & = & 0\end{array}$$

and the family is orthogonal.

Furthermore,

$$\begin{array}{rcl}\|\varphi_0\|^2 = \langle \varphi_0, \varphi_0 \rangle & = & \displaystyle\int_0^1 1 \times 1\,\mathrm{d}x \\ & = & 1 \\ \|\varphi_1\|^2 = \langle \varphi_1, \varphi_1 \rangle & = & \displaystyle\int_0^1 3(2x-1)^2\,\mathrm{d}x \\ & = & \left[3 \times \frac{(2x-1)^3}{6}\right]_0^1 \\ & = & 1 \\ \|\varphi_2\|^2 = \langle \varphi_2, \varphi_2 \rangle & = & \displaystyle\int_0^1 5(6x^2-6x+1)^2\,\mathrm{d}x \\ & = & 5\displaystyle\int_0^1 (36x^4-72x^3+48x^2-12x+1)\,\mathrm{d}x \\ & = & 5\left[\frac{36}{5}x^5-18x^4+16x^3-6x^2+x\right]_0^1 \\ & = & 5\left(\frac{36}{5}-18+16-6+1\right) \\ & = & 5\left(\frac{36}{5}-7\right) \\ & = & 1\end{array}$$

and the family is orthonormal.
Since the family does not contain the null function 0_E and is orthogonal, it is linearly independant. Since it is made of 3 independant functions of F and since $\dim(F) = 3$, it is a basis of F. Finally, it is an orthonormal basis of F.

2. (a) We have

$$\begin{aligned} \langle f,1\rangle &= \int_0^1 \cos(\pi x)\,\mathrm{d}x \\ &= \left[\frac{\sin(\pi x)}{\pi}\right]_0^1 \\ &= 0 \\ \langle g,1\rangle &= \int_0^1 \sin(\pi x)\,\mathrm{d}x \\ &= \left[\frac{-\cos(\pi x)}{\pi}\right]_0^1 \\ &= \frac{2}{\pi} \end{aligned}$$

(b) Using integrations by parts, we find

$$\begin{aligned} \langle f,x\rangle &= \int_0^1 x\cos(\pi x)\,\mathrm{d}x \\ &= \underbrace{\left[\frac{x\sin(\pi x)}{\pi}\right]_0^1}_{0} - \frac{1}{\pi}\underbrace{\int_0^1 \sin(\pi x)\,\mathrm{d}x}_{\frac{2}{\pi}} \\ &= -\frac{2}{\pi^2} \end{aligned}$$

and

$$\begin{aligned} \langle g,x\rangle &= \int_0^1 x\sin(\pi x)\,\mathrm{d}x \\ &= \underbrace{\left[\frac{-x\cos(\pi x)}{\pi}\right]_0^1}_{\frac{1}{\pi}} + \frac{1}{\pi}\underbrace{\int_0^1 \cos(\pi x)\,\mathrm{d}x}_{0} \\ &= \frac{1}{\pi} \end{aligned}$$

(c) Using integrations by parts, we find

$$\begin{aligned} \left\langle f,x^2\right\rangle &= \int_0^1 x^2\cos(\pi x)\,\mathrm{d}x \\ &= \underbrace{\left[\frac{x^2\sin(\pi x)}{\pi}\right]_0^1}_{0} - \frac{2}{\pi}\underbrace{\int_0^1 x\sin(\pi x)\,\mathrm{d}x}_{\frac{1}{\pi}} \\ &= -\frac{2}{\pi^2} \end{aligned}$$

and

$$\begin{aligned} \left\langle g,x^2\right\rangle &= \int_0^1 x^2\sin(\pi x)\,\mathrm{d}x \\ &= \underbrace{\left[\frac{-x^2\cos(\pi x)}{\pi}\right]_0^1}_{\frac{1}{\pi}} + \frac{2}{\pi}\underbrace{\int_0^1 x\cos(\pi x)\,\mathrm{d}x}_{-\frac{2}{\pi^2}} \\ &= \frac{1}{\pi} - \frac{4}{\pi^3} \end{aligned}$$

(d) We have

$$\begin{aligned}
\langle f,\varphi_0\rangle &= \langle f,1\rangle \\
&= 0 \\
\langle f,\varphi_1\rangle &= \left\langle f,\sqrt{3}(2x-1)\right\rangle \\
&= \sqrt{3}\big(2\langle f,x\rangle - \langle f,1\rangle\big) \\
&= -\frac{4\sqrt{3}}{\pi^2} \\
\langle f,\varphi_2\rangle &= \left\langle f,\sqrt{5}(6x^2-6x+1)\right\rangle \\
&= \sqrt{5}\big(6\left\langle f,x^2\right\rangle - 6\langle f,x\rangle + \langle f,1\rangle\big) \\
&= \sqrt{5}\left[6\left(-\frac{2}{\pi^2}\right) - 6\left(-\frac{2}{\pi^2}\right) + 0\right] \\
&= 0
\end{aligned}
\qquad
\begin{aligned}
\langle g,\varphi_0\rangle &= \langle g,1\rangle \\
&= \frac{2}{\pi} \\
\langle g,\varphi_1\rangle &= \left\langle g,\sqrt{3}(2x-1)\right\rangle \\
&= \sqrt{3}\big(2\langle g,x\rangle - \langle g,1\rangle\big) \\
&= 0 \\
\langle g,\varphi_2\rangle &= \left\langle g,\sqrt{5}(6x^2-6x+1)\right\rangle \\
&= \sqrt{5}\big(6\left\langle g,x^2\right\rangle - 6\langle g,x\rangle + \langle g,1\rangle\big) \\
&= \sqrt{5}\left[6\left(\frac{1}{\pi} - \frac{4}{\pi^3}\right) - \frac{6}{\pi} + \frac{2}{\pi}\right] \\
&= \frac{2\sqrt{5}}{\pi}\left(1 - \frac{12}{\pi^2}\right)
\end{aligned}$$

3. (a) $P_f = a_0\varphi_0 + a_1\varphi_1 + a_2\varphi_2$ is the orthogonal projection of f onto $F = \mathrm{Span}(\{\varphi_0, \varphi_1 n \varphi_2\})$. Thus,

$$\begin{aligned}
P_f = p_F(f) &\iff f - P_f \in F^\perp \\
&\iff \left\{\begin{array}{lcl} \langle f - a_0\varphi_0 - a_1\varphi_1 - a_2\varphi_2, \varphi_0\rangle &=& 0 \\ \langle f - a_0\varphi_0 - a_1\varphi_1 - a_2\varphi_2, \varphi_1\rangle &=& 0 \\ \langle f - a_0\varphi_0 - a_1\varphi_1 - a_2\varphi_2, \varphi_2\rangle &=& 0 \end{array}\right. \\
&\iff \left\{\begin{array}{lcl} a_0\langle\varphi_0,\varphi_0\rangle + a_1\langle\varphi_1,\varphi_0\rangle + a_2\langle\varphi_2,\varphi_0\rangle &=& \langle f,\varphi_0\rangle \\ a_0\langle\varphi_0,\varphi_1\rangle + a_1\langle\varphi_1,\varphi_1\rangle + a_2\langle\varphi_2,\varphi_1\rangle &=& \langle f,\varphi_1\rangle \\ a_0\langle\varphi_0,\varphi_2\rangle + a_1\langle\varphi_1,\varphi_2\rangle + a_2\langle\varphi_2,\varphi_2\rangle &=& \langle f,\varphi_2\rangle \end{array}\right.
\end{aligned}$$

But the family $(\varphi_0, \varphi_1, \varphi_2)$ is orthonormal, which means that

$$\langle\varphi_i,\varphi_j\rangle = \left|\begin{array}{ll} 1 & \text{if } i=j \\ 0 & \text{else} \end{array}\right.$$

So the latter system is

$$\left\{\begin{array}{lcl} a_0 &=& \langle f,\varphi_0\rangle \\ a_1 &=& \langle f,\varphi_1\rangle \\ a_2 &=& \langle f,\varphi_2\rangle \end{array}\right.$$

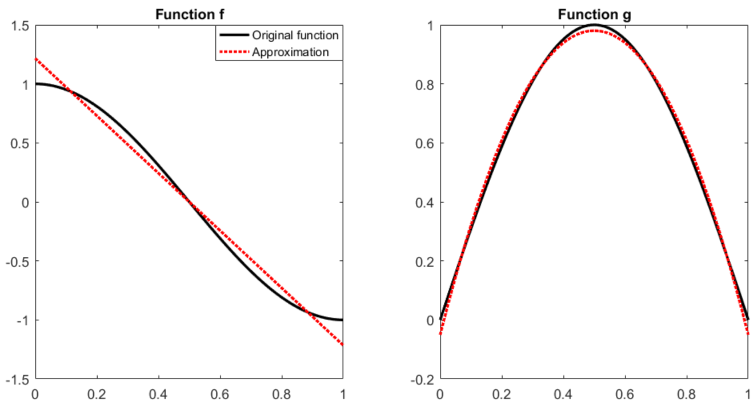
(b) In the same way, for $P_g = b_0\varphi_0 + b_1\varphi_1 + b_2\varphi_2$, we have

$$\left\{\begin{array}{lcl} b_0 &=& \langle g,\varphi_0\rangle \\ b_1 &=& \langle g,\varphi_1\rangle \\ b_2 &=& \langle g,\varphi_2\rangle \end{array}\right.$$

(c) According to the results of previous questions, the polynomials P_f and P_g are

$$\begin{aligned} P_f(x) &= -\frac{4\sqrt{3}}{\pi^2}\varphi_1(x) \\ &= -\frac{12}{\pi^2}(2x-1) \\ P_g(x) &= \frac{2}{\pi}\varphi_0(x) + \frac{2\sqrt{5}}{\pi}\left(1-\frac{12}{\pi^2}\right)\varphi_2(x) \\ &= \frac{60}{\pi}\left(1-\frac{12}{\pi^2}\right)x(x-1) + \frac{12}{\pi}\left(1-\frac{10}{\pi^2}\right) \end{aligned}$$

The following figure shows the graphs of the two functions and of their approximations.



2 Interpolation

2.1 Exercises

Exercise 3

Let f be a function whose values are known at the following nodes :

$$f(-1) = 3, \qquad f(0) = 1, \qquad f(1) = 3 \qquad \text{and} \qquad f(2) = 15$$

1. Using the Lagrange interpolation method, compute an approximation of $f(3)$.
2. Assuming that f is infinitely differentiable over $\mathbb{R}$, provide an expression for the error of this approximation.

Exercise 4

Let g be a function whose values are known at the following nodes :

$$g(-2) = 25, \qquad g(-1) = 3, \qquad g(2) = 9 \qquad \text{and} \qquad g(3) = -5$$

1. Construct the divided difference table for these data.
2. Using the Newton's method, deduce the interpolating polynomial P_g of g.
3. Express P_g in the form
$$P_g = \sum_{k=0}^{n} R_k (X-1)^k$$
4. Deduce the values of $P_g(1)$, $P_g'(1)$, $P_g''(1)$ and $P_g'''(1)$.

2.2 Solutions

Exercise 3

1. The Lagrange polynomials at the nodes are

$$
\begin{array}{lclclcl}
L_{-1} & = & \dfrac{(X-0)(X-1)(X-2)}{(-1-0)(-1-1)(-1-2)} & = & -\dfrac{X(X-1)(X-2)}{6} & \Longrightarrow & L_{-1}(3) = -1 \\
L_0 & = & \dfrac{(X+1)(X-1)(X-2)}{(0+1)(0-1)(0-2)} & = & \dfrac{(X+1)(X-1)(X-2)}{2} & \Longrightarrow & L_0(3) = 4 \\
L_1 & = & \dfrac{(X+1)(X-0)(X-2)}{(1+1)(1-0)(1-2)} & = & -\dfrac{X(X+1)(X-2)}{2} & \Longrightarrow & L_1(3) = -6 \\
L_2 & = & \dfrac{(X+1)(X-0)(X-1)}{(2+1)(2-0)(2-1)} & = & \dfrac{X(X+1)(X-1)}{6} & \Longrightarrow & L_2(3) = 4
\end{array}
$$

The interpolating polynomial of f is

$$P_f = 3L_{-1} + L_0 + 3L_1 + 15L_2$$

Hence, an approximation of $f(3)$ is

$$
\begin{array}{rcl}
P_f(3) & = & 3L_{-1}(3) + L_0(3) + 3L_1(3) + 15L_2(3) \\
& = & 3 \times (-1) + 4 + 3 \times (-6) + 15 \times 4 \\
& = & -3 + 4 - 18 + 60 \\
& = & 43
\end{array}
$$

2. We know that for any $x \in \mathbb{R}$, there exists $c \in \mathbb{R}$ such that

$$f(x) - P_f(x) = (x+1)(x-0)(x-1)(x-2)\frac{f^{(4)}(c)}{4!}$$

For $x = 3$, we deduce that there exists $c \in \mathbb{R}$ such that

$$f(3) - P_f(3) = f^{(4)}(c)$$

Hence,

$$|f(3) - P_f(3)| \leqslant \sup_{\mathbb{R}} \left| f^{(4)}(x) \right|$$

Exercise 4

1. The divided difference table is the following :

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
-2	25			
		-22		
-1	3		6	
		2		-2
2	9		-4	
		-14		
3	-5			

2. Then we use the terms from the upper diagonal of this table to get the interpolating polynomial :

$$\begin{aligned} P_g(X) &= 25 - 22(X+2) + 6(X+2)(X+1) - 2(X+2)(X+1)(X-2) \\ &= -2X^3 + 4X^2 + 4X + 1 \end{aligned}$$

3. Let us do the synthetic division of P_g with $X - 1$:

$$\begin{aligned} P_g(X) &= (X-1)\big[-2X^2 + 2X + 6\big] + 7 \\ &= (X-1)\big[(X-1)(-2X) + 6\big] + 7 \\ &= (X-1)^2(-2X) + 6(X-1) + 7 \\ &= (X-1)^2\big[(X-1)(-2) - 2\big] + 6(X-1) + 7 \\ &= -2(X-1)^3 - 2(X-1)^2 + 6(X-1) + 7 \end{aligned}$$

4. According to the previous question, we have

$$\begin{aligned} P_g(1) &= 7 \times 0! &= 7 \\ P_g'(1) &= 6 \times 1! &= 6 \\ P_g''(1) &= -2 \times 2! &= -4 \\ P_g'''(1) &= -2 \times 3! &= -12 \end{aligned}$$

3 Numerical integration

3.1 Exercises

In these exercises, we construct numerical methods of integration using the Lagrange formula of the interpolating polynomial. We assume that the function f is continuous over an interval $[\alpha, \beta] \subset \mathbb{R}$, and we want to estimate

$$I(f) = \int_{\alpha}^{\beta} f(x) \, \mathrm{d}x$$

The nodes of interpolation are denoted $x_0, x_1, \ldots, x_n$. The resulting quadrature rule will have the form

$$R(f) = \sum_{i=0}^{n} w_i f(x_i)$$

In the solutions of the exercises, there is a property that we widely use in the computations : *when the interval of integration has the form* $[-a, a]$, then :

— for any odd function, that is $f(-x) = -f(x)$, we have

$$\int_{-a}^{a} f(x) \, \mathrm{d}x = 0$$

— for any even function, that is $f(-x) = +f(x)$, we have

$$\int_{-a}^{a} f(x) \, \mathrm{d}x = 2 \int_{0}^{a} f(x) \, \mathrm{d}x$$

For example, when the integration interval is $[-1, 1]$, we use this property when computing the degree of precision of the rules : we then need to compute the integral of $f(x) = x^k$:

$$I(f) = \int_{-1}^{1} x^k \, \mathrm{d}x$$

which is hence 0 when k is odd. *It does not work if the integration interval is* $[0, 1]$ *instead of* $[-1, 1]$.

Exercise 5 : trapezoidal rule

The integration interval is $[-1, 1]$, the nodes of interpolation are $x_0 = -1$ and $x_1 = +1$.
We denote by P_1 the interpolating polynomial of degree 1 for the function f using the nodes x_0 and x_1.

1. Show that

$$P_1(x) = f(-1)L_0(x) + f(1)L_1(x)$$

where

$$L_0(x) = -\frac{1}{2}(x-1) \quad \text{and} \quad L_1(x) = \frac{1}{2}(x+1)$$

2. Deduce the following quadrature rule

$$R(f) = f(-1) + f(1)$$

3. Determine the degree of precision p of this rule R.
4. Determine the Peano kernel K of R and deduce its sign.
5. Evaluate the Peano constant K_c. Assuming that the derivative $f^{(p+1)}$ is defined and continuous over $[-1, 1]$, provide an upper bound for the absolute value of the approximation error with this quadrature rule.

Exercise 6 : Gauss rule with two nodes

The integration interval is $[-1,1]$, the nodes of interpolation are $x_0 = -\frac{1}{\sqrt{3}}$ and $x_1 = +\frac{1}{\sqrt{3}}$.
We denote by P_1 the interpolating polynomial of degree 1 for the function f using the nodes x_0 and x_1.

1. Show that
$$P_1(x) = f(x_0)L_0(x) + f(x_1)L_1(x)$$
where
$$L_0(x) = -\frac{\sqrt{3}}{2}\left(x - \frac{1}{\sqrt{3}}\right) \quad \text{and} \quad L_1(x) = \frac{\sqrt{3}}{2}\left(x + \frac{1}{\sqrt{3}}\right)$$
2. Deduce the following quadrature rule
$$R(f) = f\left(-\tfrac{1}{\sqrt{3}}\right) + f\left(\tfrac{1}{\sqrt{3}}\right)$$
3. Determine the degree of precision p of this rule R.
4. Determine the Peano kernel K of R and study its sign.
5. Evaluate the Peano constant K_c. Assuming that the $(p+1)^{\text{th}}$ derivative of f is defined and continuous over $[-1,1]$, provide an upper bound for the absolute value of the integration error.

Exercise 7 : Gauss rule with three nodes

The integration interval is $[-1,1]$, the nodes of interpolation are

$$x_0 = -\sqrt{\tfrac{3}{5}} \quad x_1 = 0 \quad \text{and } x_2 = \sqrt{\tfrac{3}{5}}$$

We denote by P_2 the interpolating polynomial of degree 2 for the function f using the nodes x_0, x_1 and x_2.

1. Show that
$$P_2(x) = f(x_0)L_0(x) + f(x_1)L_1(x) + f(x_2)L_2(x)$$
where
$$L_0(x) = \frac{5}{6}x\left(x - \sqrt{\tfrac{3}{5}}\right) \quad L_1(x) = -\frac{5}{3}\left(x - \sqrt{\tfrac{3}{5}}\right)\left(x + \sqrt{\tfrac{3}{5}}\right) \quad \text{and} \quad L_2(x) = \frac{5}{6}x\left(x + \sqrt{\tfrac{3}{5}}\right)$$
2. Deduce the following quadrature rule
$$R(f) = \frac{5}{9}f\left(-\sqrt{\tfrac{3}{5}}\right) + \frac{8}{9}f(0) + \frac{5}{9}f\left(\sqrt{\tfrac{3}{5}}\right)$$
3. Determine its degree of precision p.
4. Determine its Peano kernel K and study its sign.
5. Evaluate the Peano constant K_c. Assuming that the derivative $f^{(p+1)}$ is defined and continuous over $[-1,1]$, provide an upper bound for the absolute value of the integration error.

3.2 Solutions

Exercise 5 : trapezoidal rule

1. We know that P_1 is the unique polynomial of degree 1 (or less) that fits f at the nodes $x_0 = -1$ and $x_1 = +1$.
   Now, let Q be the polynomial defined by
   $$Q(x) = f(-1)L_0(x) + f(1)L_1(x)$$
   Since L_0 and L_1 have a degree 1, we know that $d^\circ(Q) \leqslant 1$.
   Furthermore, we have
   $$\left\{\begin{array}{lcl} L_0(-1) & = & -\frac{1}{2}(-1-1) \\ & = & -\frac{1}{2}(-2) \\ & = & 1 \\ L_0(1) & = & -\frac{1}{2}(1-1) \\ & = & 0 \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{lcl} L_1(-1) & = & \frac{1}{2}(-1+1) \\ & = & 0 \\ L_1(1) & = & \frac{1}{2}(1+1) \\ & = & \frac{1}{2}(2) \\ & = & 1 \end{array}\right.$$
   which leads to
   $$\left\{\begin{array}{lcl} Q(-1) & = & f(-1)\times 1 + f(1)\times 0 \\ & = & f(-1) \\ Q(1) & = & f(-1)\times 0 + f(1)\times 1 \\ & = & f(1) \end{array}\right.$$
   Finally, the polynomial Q has degree at most 1 and fits f at the two nodes $x_0 = -1$ and $x_1 = 1$. Since such a polynomial is unique, we have $Q = P_1$.

2. The quadrature rule is
   $$\begin{array}{rcl} R(f) & = & \displaystyle\int_{-1}^{1} P_1(x)\ \mathrm{d}x \\ & = & \displaystyle\int_{-1}^{1} \big[f(-1)L_0(x) + f(1)L_1(x)\big]\ \mathrm{d}x \\ & = & f(-1)\displaystyle\int_{-1}^{1} L_0(x)\ \mathrm{d}x + f(1)\displaystyle\int_{-1}^{1} L_1(x)\ \mathrm{d}x \end{array}$$
   Now, we have
   $$\begin{array}{rcl} \displaystyle\int_{-1}^{1} L_0(x)\ \mathrm{d}x & = & \displaystyle\int_{-1}^{1} -\frac{1}{2}(x-1)\ \mathrm{d}x \\ & = & -\dfrac{1}{2}\left[\displaystyle\int_{-1}^{1} x\ \mathrm{d}x - \int_{-1}^{1} 1\ \mathrm{d}x\right] \\ & = & -\dfrac{1}{2}[0-2] \\ & = & 1 \end{array}$$
   and a similar computation results in
   $$\int_{-1}^{1} L_1(x)\ \mathrm{d}x = 1$$
   So, finally,
   $$R(f) = f(-1) + f(1)$$

3. To determine the degree of precision p of the rule R, we compare $I(f)$ with $R(f)$ for functions $f(x) = x^k, k \in \mathbb{N}$. We know that :
   — When k is odd, the function f is odd. Therefore, $I(f) = \int_{-1}^{1} f(x)\ \mathrm{d}x = 0$ (because the function f is odd), and
   $$I(f) = \int_{-1}^{1} f(x)\ \mathrm{d}x = 0 \qquad \text{and} \qquad \begin{aligned} R(f) &= f(-1) + f(+1) \\ &= (-1)^k + (+1)^k \\ &= -1 + 1 \\ &= 0 \end{aligned}$$
   In this case, we have $I(f) = R(f)$
   — When k is even, the function f is even, so
   $$I(f) = \int_{-1}^{1} x^k\ \mathrm{d}x = 2\int_{0}^{1} x^k\ \mathrm{d}x = \frac{2}{k+1}$$
   and
   $$\begin{aligned} R(f) &= f(-1) + f(+1) = 0 \\ &= (-1)^k + (+1)^k \\ &= 2 \end{aligned}$$

   Now, let's compare $I(x^k)$ and $R(x^k)$ for the successive values of k :
   $k = 0 :\ I(x^0) = 2 = R(x^0)$
   $k = 1 :\ I(x^1) = 0 = R(x^1)$
   $k = 2 :\ I(x^2) = \frac{2}{3}$ and $R(x^2) = (-1)^2 + (1)^2 = 2 \neq I(x^2)$
   So the degree of precision of the rule R is $p = 1$.

   Remark : This result is consistent with the number of nodes.
4. The Peano kernel K is defined over $[-1, 1]$: for each $t \in [-1, 1]$, we define the function $(x - t)_+^p$, of variable x, by
   $$(x-t)_+^p = \left|\begin{array}{ll} 0 & \text{if } x - t \leqslant 0 \\ (x-t)^p & \text{if } x - t \geqslant 0 \end{array}\right.$$
   The exponent p is the degree of precision, so here $p = 1$.
   Then, $K(t)$ is the approximation error of the rule R for the function $\frac{(x-t)_+^p}{p!}$, which results in :
   $$\begin{aligned} K(t) &= I\left((x-t)_+^1\right) - R\left((x-t)_+^1\right) \\ &= \int_{-1}^{1} (x-t)_+^1\ \mathrm{d}x - (-1-t)_+^1 - (1-t)_+^1 \end{aligned}$$
   The integral in this expression is
   $$\begin{aligned} \int_{-1}^{1} (x-t)_+^1\ \mathrm{d}x &= \int_{-1}^{t} (x-t)_+^1\ \mathrm{d}x + \int_{t}^{1} (x-t)_+^1\ \mathrm{d}x \\ &= \int_{-1}^{t} 0\ \mathrm{d}x + \int_{t}^{1} (x-t)^1\ \mathrm{d}x \\ &= 0 + \left[\frac{(x-t)^2}{2}\right]_t^1 \\ &= \frac{(1-t)^2}{2} \end{aligned}$$

Now, for each $t \in [-1,1]$, we have

$$\begin{cases} t \leqslant 1 & \Longrightarrow & (1-t)_+^1 & = & (1-t) \\ t \geqslant -1 & \Longrightarrow & (-1-t)_+^1 & = & 0 \end{cases}$$

Therefore,

$$\begin{aligned} K(t) &= \frac{(1-t)^2}{2} - (1-t) \\ &= \frac{(1-t)}{2}\big((1-t)-2\big) \\ &= \frac{(1-t)(-1-t)}{2} \\ &= \frac{t^2-1}{2} \end{aligned}$$

We can see that, for each $t \in [-1,1]$, the value $K(t)$ is negative. So the Peano kernel has a constant (negative) sign.

5. We have just seen that K has a constant sign. This implies that there exists a Peano constant K_c : for any function $f \in C^2\big([-1,1]\big)$, there is $c \in [-1,1]$ such that

$$I(f) - R(f) = K_c f''(c)$$

Let's choose for f the particular function $f_0(x) = x^2$. Then

$$\left.\begin{array}{l} \forall c \in [-1,1], f_0''(c) = 2 \\ I(f_0) - R(f_0) = \dfrac{2}{3} - 2 = -\dfrac{4}{3} \end{array}\right\} \Longrightarrow K_c = \frac{-\frac{4}{3}}{2} = -\frac{2}{3}$$

Then for any function f whose second derivative is defined and continuous over $[-1,1]$, there exists $c \in [-1,1]$ such that

$$I(f) - R(f) = -\frac{2}{3} \quad f''(c)$$

Therefore,

$$|I(f) - R(f)| \leqslant \frac{2}{3} \quad \sup_{[-1,1]} |f''|$$

6. Additional remark : if we adapt this quadrature rule to evaluate the integral of f on an interval $[a_i, b_i]$, we get

$$\begin{aligned} R_i(f) &= \frac{b_i - a_i}{2}\left[f\left(\frac{a_i+b_i}{2} - \frac{b_i-a_i}{2}\right) + f\left(\frac{a_i+b_i}{2} + \frac{b_i-a_i}{2}\right)\right] \\ &= \frac{b_i-a_i}{2}\big[f(a_i) + f(b_i)\big] \end{aligned}$$

and

$$\big|I_i(f) - R_i(f)\big| \leqslant \frac{2}{3} \times \left(\frac{b_i-a_i}{2}\right)^3 \times \sup_{[a_i,b_i]} |f''|$$

Now, if we subdivise an interval $[A,B]$ into m subintervals $[a_i,b_i]$ with $b_i - a_i = \frac{B-A}{m}$, we get the estimator

$$\int_A^B f(x)\,\mathrm{d}x \simeq \sum_{i=1}^m R_i(f)$$

whose error is bounded by

$$m \times \frac{2}{3} \times \left(\frac{B-A}{2m}\right)^3 \times \sup_{[A,B]} |f''| = \frac{(B-A)^3}{12m^2} \quad \sup_{[A,B]} |f''|$$

Exercise 6 : Gauss rule with two nodes

1. We know that P_1 is the unique polynomial of degree 1 (or less) that fits f at the nodes $x_0 = -\frac{1}{\sqrt{3}}$ and $x_1 = +\frac{1}{\sqrt{3}}$.
   Now, let Q be the polynomial defined by
   $$Q(x) = f\left(-\tfrac{1}{\sqrt{3}}\right) L_0(x) + f\left(\tfrac{1}{\sqrt{3}}\right) L_1(x)$$
   Since L_0 and L_1 have a degree 1, we know that $d^\circ(Q) \leqslant 1$.
   Furthermore, we have
   $$\left\{\begin{array}{lcl} L_0\left(-\frac{1}{\sqrt{3}}\right) & = & -\frac{\sqrt{3}}{2}\left(-\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right) \\ & = & -\frac{\sqrt{3}}{2}\left(-\frac{2}{\sqrt{3}}\right) \\ & = & 1 \\ L_0\left(\frac{1}{\sqrt{3}}\right) & = & -\frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right) \\ & = & 0 \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{lcl} L_1\left(-\frac{1}{\sqrt{3}}\right) & = & \frac{\sqrt{3}}{2}\left(-\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) \\ & = & 0 \\ L_1\left(\frac{1}{\sqrt{3}}\right) & = & \frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) \\ & = & \frac{\sqrt{3}}{2}\left(\frac{2}{\sqrt{3}}\right) \\ & = & 1 \end{array}\right.$$
   which leads to
   $$\left\{\begin{array}{lcl} Q\left(-\frac{1}{\sqrt{3}}\right) & = & f\left(-\frac{1}{\sqrt{3}}\right) \times 1 + f\left(\frac{1}{\sqrt{3}}\right) \times 0 \\ & = & f\left(-\frac{1}{\sqrt{3}}\right) \\ Q\left(\frac{1}{\sqrt{3}}\right) & = & f\left(-\frac{1}{\sqrt{3}}\right) \times 0 + f\left(\frac{1}{\sqrt{3}}\right) \times 1 \\ & = & f\left(\frac{1}{\sqrt{3}}\right) \end{array}\right.$$
   Finally, the polynomial Q has degree at most 1 and fits f at the two nodes x_0 and x_1. Since such a polynomial is unique, we have $Q = P_1$.
2. The quadrature rule is
   $$\begin{array}{lcl} R(f) & = & \displaystyle\int_{-1}^{1} P_1(x)\ \mathrm{d}x \\ & = & \displaystyle\int_{-1}^{1} \left[f\left(-\tfrac{1}{\sqrt{3}}\right) L_0(x) + f\left(\tfrac{1}{\sqrt{3}}\right) L_1(x)\right]\ \mathrm{d}x \\ & = & f\left(-\tfrac{1}{\sqrt{3}}\right) \displaystyle\int_{-1}^{1} L_0(x)\ \mathrm{d}x + f\left(\tfrac{1}{\sqrt{3}}\right) \displaystyle\int_{-1}^{1} L_1(x)\ \mathrm{d}x \end{array}$$
   Now, we have
   $$\begin{array}{lcl} \displaystyle\int_{-1}^{1} L_0(x)\ \mathrm{d}x & = & \displaystyle\int_{-1}^{1} -\tfrac{\sqrt{3}}{2}\left(x - \tfrac{1}{\sqrt{3}}\right)\ \mathrm{d}x \\ & = & -\tfrac{\sqrt{3}}{2}\left[\displaystyle\int_{-1}^{1} x\ \mathrm{d}x - \displaystyle\int_{-1}^{1} \tfrac{1}{\sqrt{3}}\ \mathrm{d}x\right] \\ & = & -\tfrac{\sqrt{3}}{2}\left[0 - \tfrac{2}{\sqrt{3}}\right] \\ & = & 1 \end{array}$$
   and a similar computation results in
   $$\int_{-1}^{1} L_1(x)\ \mathrm{d}x = 1$$
   So we finally have
   $$R(f) = f\left(-\tfrac{1}{\sqrt{3}}\right) + f\left(\tfrac{1}{\sqrt{3}}\right)$$

3. To determine the degree of precision p of the rule R, we compare $I(f)$ with $R(f)$ for functions $f(x) = x^k$ with different values of k. We know that :
— When k is odd, the function f is odd. So

$$I(f) = \int_{-1}^{1} f(x)\ \mathrm{d}x = 0 \qquad \text{and} \qquad R(f) = f\left(-\tfrac{1}{\sqrt{3}}\right) + f\left(\tfrac{1}{\sqrt{3}}\right) = 0$$

In this case, we have $I(f) = R(f)$
— When k is even, the function f is even, so

$$I(f) = \int_{-1}^{1} x^k\ \mathrm{d}x = 2\int_{0}^{1} x^k\ \mathrm{d}x = \frac{2}{k+1}$$

and

$$R(f) = \left(-\frac{1}{\sqrt{3}}\right)^k + \left(\frac{1}{\sqrt{3}}\right)^k = \frac{2}{\left(\sqrt{3}\right)^k}$$

Now, we can compare $I(x^k)$ and $R(x^k)$ for the successive values of k :
$k = 0 :\ I(x^0) = 2 = R(x^0)$
$k = 1 :\ I(x^1) = 0 = R(x^1)$
$k = 2 :\ I(x^2) = \frac{2}{3} = R(x^2)$
$k = 3 :\ I(x^3) = 0 = R(x^3)$
$k = 4 :\ I(x^4) = \frac{2}{5}$ and $R(x^4) = \frac{2}{9} \neq I(x^4)$
So the degree of precision of the rule R is $p = 3$.

4. The Peano kernel K is defined over $[-1, 1]$: for each $t \in [-1, 1]$, $K(t)$ is the approximation error of the rule R for the function $x \mapsto \frac{(x-t)_+^p}{p!} = \frac{(x-t)_+^3}{3!}$, which results in :

$$\begin{array}{rcl} 3!K(t) & = & I\left((x-t)_+^3\right) - R\left((x-t)_+^3\right) \\ & = & \displaystyle\int_{-1}^{1} (x-t)_+^3\ \mathrm{d}x - \left(-\tfrac{1}{\sqrt{3}} - t\right)_+^3 - \left(\tfrac{1}{\sqrt{3}} - t\right)_+^3 \end{array}$$

The integral in this expression is

$$\begin{array}{rcl} \displaystyle\int_{-1}^{1} (x-t)_+^3\ \mathrm{d}x & = & \displaystyle\int_{-1}^{t} (x-t)_+^3\ \mathrm{d}x + \int_{t}^{1} (x-t)_+^3\ \mathrm{d}x \\ & = & \displaystyle\int_{-1}^{t} 0\ \mathrm{d}x + \int_{t}^{1} (x-t)^3\ \mathrm{d}x \\ & = & 0 + \left[\dfrac{(x-t)^4}{4}\right]_t^1 \\ & = & \dfrac{(1-t)^4}{4} \end{array}$$

Now, the expression of $K(t)$ depends on the position of t with respect to the nodes :
— If $t \in \left[\frac{1}{\sqrt{3}}, 1\right]$, then

$$\left.\begin{array}{rcrcl} t \geqslant +\frac{1}{\sqrt{3}} & \Rightarrow & \left(\frac{1}{\sqrt{3}} - t\right)_+^3 & = & 0 \\ t \geqslant -\frac{1}{\sqrt{3}} & \Rightarrow & \left(-\frac{1}{\sqrt{3}} - t\right)_+^3 & = & 0 \end{array}\right\} \Longrightarrow 3!K(t) = \frac{(1-t)^4}{4}$$

— If $t \in \left[-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$, then

$$\left.\begin{array}{lcl} t \leqslant +\frac{1}{\sqrt{3}} & \Rightarrow & \left(\frac{1}{\sqrt{3}} - t\right)_+^3 = \left(\frac{1}{\sqrt{3}} - t\right)^3 \\ t \geqslant -\frac{1}{\sqrt{3}} & \Rightarrow & \left(-\frac{1}{\sqrt{3}} - t\right)_+^3 = 0 \end{array}\right\} \Longrightarrow 3!K(t) = \frac{(1-t)^4}{4} - \left(\frac{1}{\sqrt{3}} - t\right)^3$$

If we develop this expression, using the binomial formula

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

we find

$$3!K(t) = \frac{1}{4}\left[t^4 + \left(6 - \sqrt{3}\right)t^2 + \left(1 - \frac{1}{3\sqrt{3}}\right)\right]$$

— If $t \in \left[-1, -\frac{1}{\sqrt{3}}\right]$, we can use the parity of the kernel K (the three nodes of the rule have symmetric values around 0) : if $t \in \left[-1, -\frac{1}{\sqrt{3}}\right]$, then $-t \in \left[\frac{1}{\sqrt{3}}, 1\right]$ and

$$\begin{array}{rcl} 3!K(t) & = & 3!K(-t) \\ & = & \dfrac{(1+t)^4}{4} \end{array}$$

We can see that, in each subinterval, the function K is positive : the terms $(1-t)^4$, $(1+t)^4$, t^4 and t^2 are all positive and multiplied by positive coefficients. So the Peano kernel has a constant (positive) sign.

5. We have just seen that K has a constant sign. This implies that there exists a Peano constant K_c : for any function f which is 4 times differentiable on $[-1,1]$ with $f^{(4)}$ continuous ($4 = p+1$), there is $c \in [-1,1]$ such that
$$I(f) - R(f) = K_c f^{(4)}(c)$$
Let's choose for f the particular function $f_0(x) = x^4$:
$$\left.\begin{array}{l} \forall c \in [-1,1], f_0^{(4)}(c) = 4! = 24 \\ I(f_0) - R(f_0) = \dfrac{2}{5} - \dfrac{2}{9} = \dfrac{8}{45} \end{array}\right\} \Longrightarrow K_c = \frac{\frac{8}{45}}{24} = \frac{1}{135}$$
Then, for any function f whose fourth derivative is defined and continuous over $[-1,1]$, there exists $c \in [-1,1]$ such that
$$I(f) - R(f) = \frac{1}{135} \quad f^{(4)}(c)$$
Therefore,
$$|I(f) - R(f)| \leqslant \frac{1}{135} \quad \sup_{[-1,1]} \left|f^{(4)}\right|$$

6. Additional remark : if we adapt this quadrature rule to evaluate the integral of f on an interval $[a_i, b_i]$, we get
$$R_i(f) = \frac{b_i - a_i}{2}\left[f\left(\frac{a_i + b_i}{2} - \frac{b_i - a_i}{2\sqrt{3}}\right) + f\left(\frac{a_i + b_i}{2} + \frac{b_i - a_i}{2\sqrt{3}}\right)\right]$$

and

$$\left|I_i(f) - R_i(f)\right| \leqslant \frac{1}{135} \times \left(\frac{b_i - a_i}{2}\right)^5 \times \sup_{[a_i,b_i]} \left|f^{(4)}\right|$$

Now, if we subdivise an interval $[A, B]$ into m subintervals $[a_i, b_i]$ with $b_i - a_i = \frac{B-A}{m}$, we get the estimator

$$\int_A^B f(x)\,\mathrm{d}x \simeq \sum_{i=1}^{m} R_i(f)$$

whose error is bounded by

$$m \times \frac{1}{135} \times \left(\frac{B-A}{2m}\right)^5 \times \sup_{[A,B]} \left|f^{(4)}\right| = \frac{(B-A)^5}{4320m^4} \quad \sup_{[A,B]} \left|f^{(4)}\right|$$

Compare this result with the error bound for the previous rules.

Exercise 7 : Gauss rule with three nodes

1. We know that P_2 is the unique polynomial in $\mathbb{R}_2[X]$ that fits f at the nodes $x_0 = -\sqrt{\frac{3}{5}}$, $x_1 = 0$ and $x_2 = +\sqrt{\frac{3}{5}}$.
Now, let Q be the polynomial defined by

$$Q(x) = f\left(-\sqrt{\tfrac{3}{5}}\right) L_0(x) + f(0)L_1(x) + f\left(\sqrt{\tfrac{3}{5}}\right) L_2(x)$$

Since L_0, L_1 and L_2 have a degree 2, we know that $d^\circ(Q) \leqslant 2$. Furthermore, we have

$$\left\{\begin{array}{lclclcl}
L_0\left(-\sqrt{\frac{3}{5}}\right) & = & \frac{5}{6}\left(-\sqrt{\frac{3}{5}}\right)\left(-\sqrt{\frac{3}{5}}-\sqrt{\frac{3}{5}}\right) & & L_0(0) & = & \frac{5}{6}(0)\left(0-\sqrt{\frac{3}{5}}\right) \\
 & = & \frac{5}{6}\left(-\sqrt{\frac{3}{5}}\right)\left(-2\sqrt{\frac{3}{5}}\right) & & & = & 0 \\
 & = & 1 & & L_0\left(\sqrt{\frac{3}{5}}\right) & = & \frac{5}{6}\left(\sqrt{\frac{3}{5}}\right)\left(\sqrt{\frac{3}{5}}-\sqrt{\frac{3}{5}}\right) \\
 & & & & & = & 0 \\
 & & & & & & \\
L_1\left(-\sqrt{\frac{3}{5}}\right) & = & -\frac{5}{3}\left(-\sqrt{\frac{3}{5}}-\sqrt{\frac{3}{5}}\right)\left(-\sqrt{\frac{3}{5}}+\sqrt{\frac{3}{5}}\right) & & L_1\left(\sqrt{\frac{3}{5}}\right) & = & \frac{5}{3}\left(\sqrt{\frac{3}{5}}-\sqrt{\frac{3}{5}}\right)\left(\sqrt{\frac{3}{5}}+\sqrt{\frac{3}{5}}\right) \\
 & = & 0 & & & = & 0 \\
L_1(0) & = & -\frac{5}{3}\left(0-\sqrt{\frac{3}{5}}\right)\left(0+\sqrt{\frac{3}{5}}\right) & & & & \\
 & = & -\frac{5}{3}\left(-\sqrt{\frac{3}{5}}\right)\left(\sqrt{\frac{3}{5}}\right) & & & & \\
 & = & 1 & & & & \\
 & & & & & & \\
L_2\left(-\sqrt{\frac{3}{5}}\right) & = & \frac{5}{6}\left(-\sqrt{\frac{3}{5}}\right)\left(-\sqrt{\frac{3}{5}}+\sqrt{\frac{3}{5}}\right) & & L_2\left(\sqrt{\frac{3}{5}}\right) & = & \frac{5}{6}\left(\sqrt{\frac{3}{5}}\right)\left(\sqrt{\frac{3}{5}}+\sqrt{\frac{3}{5}}\right) \\
 & = & 0 & & & = & \frac{5}{6}\left(\sqrt{\frac{3}{5}}\right)\left(2\sqrt{\frac{3}{5}}\right) \\
L_2(0) & = & \frac{5}{6}(0)\left(0+\sqrt{\frac{3}{5}}\right) & & & = & 1 \\
 & = & 0 & & & &
\end{array}\right.$$

which leads to

$$\left\{\begin{array}{lcl} Q\left(-\sqrt{\frac{3}{5}}\right) & = & f\left(-\sqrt{\frac{3}{5}}\right)\times 1 + f(0)\times 0 + f\left(\sqrt{\frac{3}{5}}\right)\times 0 \\ & = & f\left(-\sqrt{\frac{3}{5}}\right) \\ Q(0) & = & f\left(-\sqrt{\frac{3}{5}}\right)\times 0 + f(0)\times 1 + f\left(\sqrt{\frac{3}{5}}\right)\times 0 \\ & = & f(0) \\ Q\left(\sqrt{\frac{3}{5}}\right) & = & f\left(-\sqrt{\frac{3}{5}}\right)\times 0 + f(0)\times 0 + f\left(\sqrt{\frac{3}{5}}\right)\times 1 \\ & = & f\left(\sqrt{\frac{3}{5}}\right) \end{array}\right.$$

Finally, the polynomial Q is in $\mathbb{R}_2[X]$ and fits f at the three nodes x_0, x_1 and x_2. Since such a polynomial is unique, we have $Q = P_2$.

2. The quadrature rule is

$$\begin{array}{rcl} R(f) & = & \displaystyle\int_{-1}^{1} P_2(x)\ \mathrm{d}x \\ & = & \displaystyle\int_{-1}^{1}\left[f\left(-\sqrt{\tfrac{3}{5}}\right)L_0(x) + f(0)L_1(x) + f\left(\sqrt{\tfrac{3}{5}}\right)L_2(x)\right]\ \mathrm{d}x \\ & = & \displaystyle f\left(-\sqrt{\tfrac{3}{5}}\right)\int_{-1}^{1} L_0(x)\ \mathrm{d}x + f(0)\int_{-1}^{1} L_1(x)\ \mathrm{d}x + f\left(\sqrt{\tfrac{3}{5}}\right)\int_{-1}^{1} L_2(x)\ \mathrm{d}x \end{array}$$

Now, we have

$$\begin{array}{rcl} \displaystyle\int_{-1}^{1} L_0(x)\ \mathrm{d}x & = & \displaystyle\int_{-1}^{1}\frac{5}{6}x\left(x-\sqrt{\tfrac{3}{5}}\right)\ \mathrm{d}x \\ & = & \displaystyle\frac{5}{6}\left[\int_{-1}^{1} x^2\ \mathrm{d}x - \int_{-1}^{1}\sqrt{\tfrac{3}{5}}x\ \mathrm{d}x\right] \\ & = & \displaystyle\frac{5}{6}\left[\frac{2}{3}-0\right] \\ & = & \displaystyle\frac{5}{9} \end{array}$$

A similar computation results in

$$\int_{-1}^{1} L_2(x)\ \mathrm{d}x = \frac{5}{9}$$

And

$$\begin{array}{rcl} \displaystyle\int_{-1}^{1} L_1(x)\ \mathrm{d}x & = & \displaystyle\int_{-1}^{1} -\frac{5}{3}\left(x-\sqrt{\tfrac{3}{5}}\right)\left(x+\sqrt{\tfrac{3}{5}}\right)\ \mathrm{d}x \\ & = & \displaystyle -\frac{5}{3}\left[\int_{-1}^{1} x^2\ \mathrm{d}x - \int_{-1}^{1}\frac{3}{5}\ \mathrm{d}x\right] \\ & = & \displaystyle -\frac{5}{3}\left[\frac{2}{3}-\frac{6}{5}\right] \\ & = & \displaystyle -\frac{5}{3}\times\frac{10-18}{15} \\ & = & \displaystyle\frac{8}{9} \end{array}$$

So, finally,

$$R(f) = \frac{5}{9}f\left(-\sqrt{\tfrac{3}{5}}\right) + \frac{8}{9}f(0) + \frac{5}{9}f\left(\sqrt{\tfrac{3}{5}}\right)$$

3. To determine the degree of precision p of the rule R, we compare $I(f)$ with $R(f)$ for functions $f(x) = x^k, k \in \mathbb{N}$. We remind that :
   — When k is odd, the function f is odd. So

$$I(f) = \int_{-1}^{1} f(x)\ \mathrm{d}x = 0 \qquad \text{and} \qquad \begin{aligned} R(f) &= \tfrac{5}{9} f\left(-\sqrt{\tfrac{3}{5}}\right) + \tfrac{8}{9} f(0) + \tfrac{5}{9} f\left(\sqrt{\tfrac{3}{5}}\right) \\ &= \tfrac{5}{9}\left(-\sqrt{\tfrac{3}{5}}\right)^k + \tfrac{8}{9} 0^k + \tfrac{5}{9}\left(\sqrt{\tfrac{3}{5}}\right)^k \\ &= 0 \end{aligned}$$

   In this case, we have $I(f) = R(f)$
   — When k is even, the function f is even and

$$I(f) = \int_{-1}^{1} x^k\ \mathrm{d}x = 2\int_{0}^{1} x^k\ \mathrm{d}x = \frac{2}{k+1}$$

   Now, let's compare $I(x^k)$ with $R(x^k)$ for the successive values of k :
   $k = 0 :\ I(x^0) = 2$ and $R(x^0) = \frac{5}{9} + \frac{8}{9} + \frac{5}{9} = I(x^0)$
   $k = 1 :\ I(x^1) = 0 = R(x^1)$
   $k = 2 :\ I(x^2) = \frac{2}{3}$ and $R(x^2) = \frac{5}{9} \times \frac{3}{5} + \frac{8}{9} \times 0 + \frac{5}{9} \times \frac{3}{5} = \frac{1}{3} + \frac{1}{3} = I(x^2)$
   $k = 3 :\ I(x^3) = 0 = R(x^3)$
   $k = 4 :\ I(x^4) = \frac{2}{5}$ and $R(x^4) = \frac{5}{9} \times \frac{9}{25} + \frac{8}{9} \times 0 + \frac{5}{9} \times \frac{9}{25} = \frac{1}{5} + \frac{1}{5} = I(x^4)$
   $k = 5 :\ I(x^5) = 0 = R(x^5)$
   $k = 6 :\ I(x^6) = \frac{2}{7}$ and $R(x^6) = \frac{5}{9} \times \frac{27}{125} + \frac{8}{9} \times 0 + \frac{5}{9} \times \frac{27}{125} = \frac{3}{25} + \frac{3}{25} = \frac{6}{25} \neq I(x^6)$
   So the degree of precision is $p = 5$.

4. The Peano kernel K is defined over $[-1, 1]$: for each $t \in [-1, 1]$, we define the function $(x-t)_+^5$, of variable x, by

$$(x-t)_+^5 = \left| \begin{array}{ll} 0 & \text{if } x - t \leqslant 0 \\ (x-t)^5 & \text{if } x - t \geqslant 0 \end{array} \right.$$

   The exponent 5 is the degree of precision. Then, $K(t)$ is the approximation error of the rule R for the function $x \mapsto \frac{(x-t)_+^5}{5!}$, which results in :

$$\begin{aligned} 5!K(t) &= I\left((x-t)_+^5\right) - R\left((x-t)_+^5\right) \\ &= \int_{-1}^{1} (x-t)_+^5\ \mathrm{d}x - \frac{5}{9}\left(-\sqrt{\tfrac{3}{5}} - t\right)_+^5 - \frac{8}{9}(0-t)_+^5 - \frac{5}{9}\left(\sqrt{\tfrac{3}{5}} - t\right)_+^5 \end{aligned}$$

   The integral in this expression is

$$\begin{aligned} \int_{-1}^{1} (x-t)_+^5\ \mathrm{d}x &= \int_{-1}^{t} (x-t)_+^5\ \mathrm{d}x + \int_{t}^{1} (x-t)_+^5\ \mathrm{d}x \\ &= \int_{-1}^{t} 0\ \mathrm{d}x + \int_{t}^{1} (x-t)^5\ \mathrm{d}x \\ &= 0 + \left[\frac{(x-t)^6}{6}\right]_t^1 \\ &= \frac{(1-t)^6}{6} \end{aligned}$$

   Now, the expression of $K(t)$ depends on the subinterval of $[-1, 1]$ in which t lies :

— if $t \in \left[\sqrt{\frac{3}{5}}, 1\right]$, then

$$\left.\begin{array}{lll} t \geqslant \sqrt{\frac{3}{5}} & \Rightarrow & \left(\sqrt{\frac{3}{5}} - t\right)_+^5 = 0 \\ t \geqslant 0 & \Rightarrow & (0-t)_+^5 = 0 \\ t \geqslant -\sqrt{\frac{3}{5}} & \Rightarrow & \left(-\sqrt{\frac{3}{5}} - t\right)_+^5 = 0 \end{array}\right\} \Longrightarrow 5!K(t) = \frac{(1-t)^6}{6}$$

— if $t \in \left[0, \sqrt{\frac{3}{5}}\right]$, then

$$\left.\begin{array}{lll} t \leqslant \sqrt{\frac{3}{5}} & \Rightarrow & \left(\sqrt{\frac{3}{5}} - t\right)_+^5 = \left(\sqrt{\frac{3}{5}} - t\right)^5 \\ t \geqslant 0 & \Rightarrow & (0-t)_+^5 = 0 \\ t \geqslant -\sqrt{\frac{3}{5}} & \Rightarrow & \left(-\sqrt{\frac{3}{5}} - t\right)_+^5 = 0 \end{array}\right\} \Longrightarrow 5!K(t) = \frac{(1-t)^6}{6} - \frac{5}{9}\left(\sqrt{\frac{3}{5}} - t\right)^5$$

— if $t \in \left[-\sqrt{\frac{3}{5}}, 0\right]$, then

$$\left.\begin{array}{lll} t \leqslant \sqrt{\frac{3}{5}} & \Rightarrow & \left(\sqrt{\frac{3}{5}} - t\right)_+^5 = \left(\sqrt{\frac{3}{5}} - t\right)^5 \\ t \leqslant 0 & \Rightarrow & (0-t)_+^5 = -t^5 \\ t \geqslant -\sqrt{\frac{3}{5}} & \Rightarrow & \left(-\sqrt{\frac{3}{5}} - t\right)_+^5 = 0 \end{array}\right\} \Longrightarrow 5!K(t) = \frac{(1-t)^6}{6} - \frac{5}{9}\left(\sqrt{\frac{3}{5}} - t\right)^5 + \frac{8}{9}t^5$$

But another expression of K is possible in this interval : the nodes being in symmetric position around 0, the kernel K is even. We hence have $K(t) = K(-t)$ and, since $-t \in \left[0, \sqrt{\frac{3}{5}}\right]$,

$$\begin{array}{rcl} 5!K(t) & = & 5!K(-t) \\ & = & \dfrac{(1+t)^6}{6} - \dfrac{5}{9}\left(\sqrt{\frac{3}{5}} + t\right)^5 \end{array}$$

— if $t \in \left[-1, -\sqrt{\frac{3}{5}}\right]$, then the parity of the kernel K and the fact that $-t \in \left[\sqrt{\frac{3}{5}}, 1\right]$ imply that,

$$\begin{array}{rcl} 5!K(t) & = & 5!K(-t) \\ & = & \dfrac{(1+t)^6}{6} \end{array}$$

To study the sign of K, we first note that its parity enables us to focus on the interval $[0, 1]$. Furthermore, the kernel is obviously positive for $t \in \left[\sqrt{\frac{3}{5}}, 1\right]$, so we can focus on the case $t \in \left[0, \sqrt{\frac{3}{5}}\right]$. Let's prove that

$$\forall t \in \left[0, \sqrt{\frac{3}{5}}\right], \frac{(1-t)^6}{6} - \frac{5}{9}\left(\sqrt{\frac{3}{5}} - t\right)^5 \geqslant 0$$

Let us set $T = \sqrt{\frac{3}{5}} - t$ and express K as a function of T : for all $\forall t \in \left[0, \sqrt{\frac{3}{5}}\right]$, we have

$$\left\{\begin{array}{lcl} T \in \left[0, \sqrt{\frac{3}{5}}\right] & & \\ t = \sqrt{\frac{3}{5}} - T & & \\ 5!K(t) & = & \dfrac{\left[1 - \left(\sqrt{\frac{3}{5}} - T\right)\right]^6}{6} - \dfrac{5T^5}{9} \quad = \quad \dfrac{3\left(T + 1 - \sqrt{\frac{3}{5}}\right)^6 - 10T^5}{18} \end{array}\right.$$

If we develop the numerator with the binomial formula, using the notation $\delta = 1 - \sqrt{\frac{3}{5}}$, we get

$$\begin{aligned}
18 \times 5! \times K(t) &= 3(T+\delta)^6 - 10T^5 \\
&= 3(T^6 + 6\delta T^5 + 15\delta^2 T^4 + 20\delta^3 T3 + 15\delta^4 T^2 + 5\delta^5 T + \delta^6) - 10T^5 \\
&= 3T^6 + (18\delta - 10)T^5 + 45\delta^2 T^4 + 60\delta^3 T^3 + 45\delta^4 T^2 + 18\delta^5 T + 3\delta^6 \\
&= a_6 T^6 + a_5 T^5 + a_4 T^4 + a_3 T^3 + a_2 T^2 + a_1 T + a_0
\end{aligned}$$

The coefficients a_k are all positive, except $a_5 = 18\delta - 10 \simeq -5.9$. But we know that, for all $T \in \left[0, \sqrt{\frac{3}{5}}\right]$ and for all $k = 0, 1, \ldots, 4$, we have $T^k \geqslant T^4$. Therefore, for all $T \in \left[0, \sqrt{\frac{3}{5}}\right]$,

$$\begin{aligned}
18 \times 5! \times K(t) &\geqslant a_6 T^6 + a_5 T^5 + a_4 T^4 + a_3 T^4 + a_2 T^4 + a_1 T^4 + a_0 T^4 \\
&\geqslant T^4 \big[a_6 T^2 + a_5 T + (a_4 + a_3 + a_2 + a_1 + a_0)\big]
\end{aligned}$$

The latter expression is the product of T^4, which is positive, with the second degree polynomial $a_6 T^2 + a_5 T + (a_4 + a_3 + a_2 + a_1 + a_0)$. The coefficients of this polynomial are

$$a_6 = 3, \qquad a_5 \simeq -5.9, \qquad \text{and} \qquad a_4 + a_3 + a_2 + a_1 + a_0 \simeq 3.1$$

Its discriminant is $\Delta \simeq -1.9 < 0$ so this second degree polynomial has no real root and has a constant sign over $\mathbb{R}$. This sign can be given by its value for $T = 0$: it is positive. Therefore, for all $t \in \left[0, \sqrt{\frac{3}{5}}\right]$,

$$\begin{aligned}
18 \times 5! \times K(t) &\geqslant T^4 \big[a_6 T^2 + a_5 T + (a_4 + a_3 + a_2 + a_1 + a_0)\big] \\
&\geqslant 0
\end{aligned}$$

5. The Peano kernel K having a constant sign, the quadrature rule has a Peano constant K_c : for any function $f \in C^6\big([-1,1]\big)$, there exists $c \in [-1,1]$ such that

$$I(f) - R(f) = K_c f^{(6)}(c)$$

Let's choose for f the particular function $f_0(x) = x^6$. Then

$$\left.\begin{array}{l}
\forall c \in [-1,1], f_0^{(6)}(c) = 6! \\
I(f_0) - R(f_0) = \dfrac{2}{7} - \dfrac{6}{25} = \dfrac{8}{175}
\end{array}\right\} \Longrightarrow K_c = \frac{\frac{8}{175}}{6!} = \frac{1}{15750}$$

Then, for any function f whose sixth derivative is defined and continuous over $[-1,1]$, there exists $c \in [-1,1]$ such that

$$I(f) - R(f) = \frac{1}{15750} \quad f^{(6)}(c)$$

Therefore,

$$|I(f) - R(f)| \leqslant \frac{1}{15750} \quad \sup_{[-1,1]} \left|f^{(6)}\right|$$