Networks and Flows on Graphs

Exercise Sheet

1 Basics on Graphs

Exercise 1. Adjacency and Incidence Matrices

Let M be the matrix

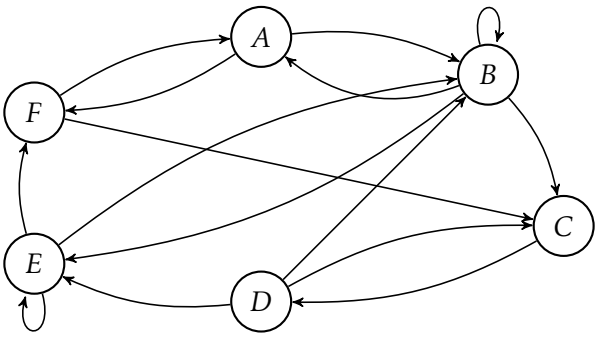
$$M = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

representing a graph G having vertices A, B, C, D, E.

1. Draw the graph having adacency matrix M.
2. What is the incidency matrix of G?
3. Compute M^2, M^3 and M^4. Can you tell what non-zero coefficients correspond to?
4. Knowing that $M^{[k]}$ is the k-th boolean power of M, compute $M^{[2]}$, $M^{[3]}$ and $M^{[4]}$. Can you give a meaning to these matrices?
5. Compute $A = I \oplus M \oplus M^{[2]} \oplus M^{[3]} \oplus M^{[4]}$, where $\oplus$ is the boolean sum. What does A stand for?

Exercise 2. Degrees and Cycles

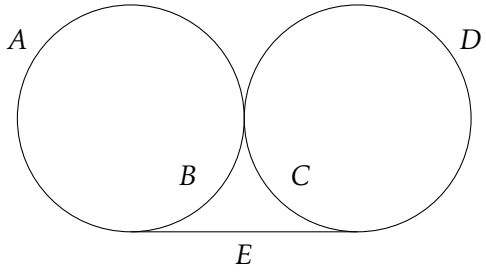
Consider the following graph, subsequently named G:



1. Compute inner and outer degrees of each one of the vertices.
2. Give an example of a simple path which is not elementary.
3. Is there a hamiltonian circuit in G?
4. Draw the oriented graph corresponding to G.
5. Is G connected? strongly connected?

Exercise 3. Railways

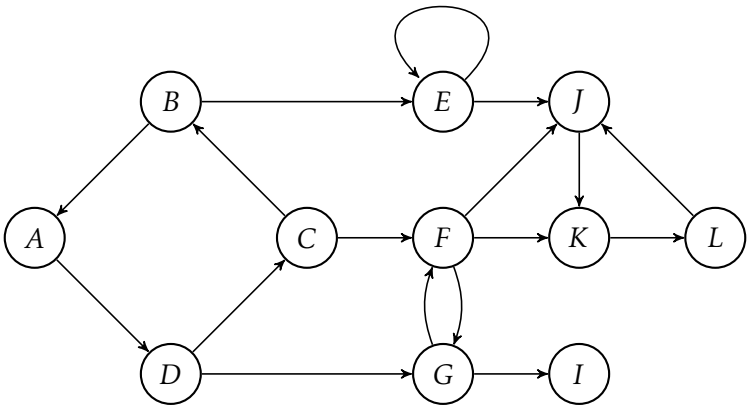
Consider the following railway track :



Each cross-point on the railway track has two available positions. Could you explain why trains will eventually never go through section E? What would happen if we added a section F symmetric to E?

Exercise 4. Strongly Connected Components

Give the strongly connected components of the following graph :



2 Network Flows

Exercise 5. Flows

Ports C, D and E are respectively in need of 9, 12 and 7 containers of a given good X. They are connected to ports A and B, which have each 10 containers of X that are ready to go. There are currently 4 shipping routes :

- from A to C and A to D having respective maximal shipping capacities of 7 and 4 containers ;
- from B to C and B to E having respective maximal capacities of 5 and 5.

1. Is it possible to satisfy all demands?
2. How can one organise the traffic in order to ensure a maximum total number of containers? You're expected to start with initial solution sending 7 containers through route AC. You are to update initial solution using Ford-Fulkerson algorithm. What is the mimimum cut corresponding to max flow solution?

Exercise 6. Pairings

During a bal, 5 men and 5 women are looking forward to dance. We're given the hard task of mathcing up the highest number of couples (gender assymetric ones) taking the below compatibility table

	Marie	Claire	Suzanne	Anne	Jeanne
Joseph	♠	♡	♠	♠	♠
Paul	♡	♠	♠	♠	♠
Luc	♡	♡	♠	♠	♠
Bernard	♡	♡	♠	♠	♡
Francois	♠	♠	♡	♡	♡

1. What is the graph best representing this matching problem?
2. How can network flows formalism help you solve this matching problem?

3 Optimal Transportation Programs

Exercise 7. Transportation Program

A road transport company has to ship a number of goods out of three docks O_1, O_2 and O_3 to 4 destinations D_1, D_2, D_3 and D_4. Each dock has 10 available shipments and the needed shipments at each destination are 7, 4, 14 and 5 respectively at D_1, D_2, D_3 and D_4. The following table sums up the costs of each route between a given dock and any destination

	D_1	D_2	D_3	D_4
O_1	5	10	11	21
O_2	6	8	11	2
O_3	21	12	3	5

The company is looking for the least cost transportation program.

1. Give a basic solution to this transportation problem using the Balas-Hammer heuristic. What is the cost of this solution?
2. What is the optimal transportation program? Is the optimum unique?
3. Following improvements on the connection between O_2 and D_1, cost to go from O_2 to D_1 has gone down from 4 to 6. Is the previous solution still optimal? If not find the new optimal solution.

4 Paths Optimisation

Exercise 8. Along GR 58

At Cervières one can find the map below, it sums the level of difficulty going through any of the pictured paths. You goal is to plan the hike to the Queyras where a *fondue* is waiting and a bottle of *VEP*.

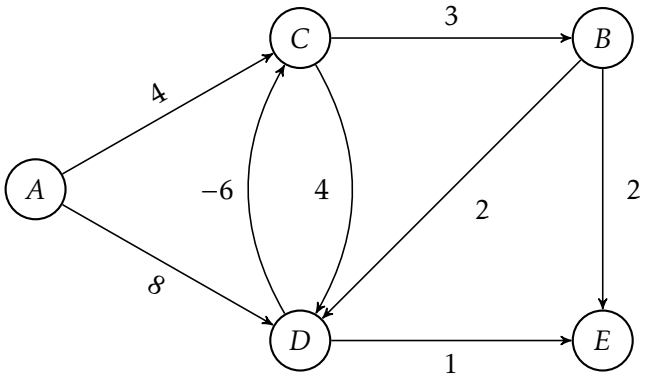
The total difficulty of a given hike is the sum of difficulties of each elememtary route.

1. Using the Ford algorithm compute the easiest hike from Cervières to any other stop. What is the easiest hike from Cervières to Queyras?
2. Is the Ford algorithm the best choice one can make to solve the previous issue, of going from Cervières to Queyras on the easiest hike. How would you do otherwise?

3. If your aim was to break the record of the toughest hike, what route would you take?

Exercise 9. Shortest Path Problem

Using the Floyd-Warshall algorithm give the longest paths in between any two vertices of the following grahps



Does looking for shortest paths in between any two vertices of this graph make sense?

5 Scheduling

Exercise 10. Project Scheduling

A project is split into 8 elementary tasks. In the table below, we sum up the needed time (in days) to accomplish a given task and the tasks one has to go through beforehand.

Tasks	Pre-tasks	Durations
a	–	2
b	–	3
c	–	7
d	b, h	4
e	a	10
f	a, d	6
g	b, c	5
h	–	1

Our aim is to find a project scheduling that minimizes the total duration of the project.

1. Use the potential method to solve previous problem. Point out the earliest starting date for each task, the critical path and its duration. Draw the corresponding potential graph.

2. Figure out the latest possible start of each task.

3. Give up a PERT graph modeling the previous scheduling problem without introducing any external constraints (you'll need a number of virtual tasks). Recover back previous cirtical paths using relevant algorithms.

Exercise 11. Assisting the Director

A movie producer is looking forward to plan his next movie to be. Here are the list of tasks to be done for the movie to get out on the screens.

Task Code	Task detail	Duration	Precedency
A	Writing the scenario	30	
B	Casting	12	Can only start 15 days after A starts
C	Choosing shooting place	8	Can only start 20 days after A starts
D	Technical cut	4	A and C have to be finished
E	Preparing set	7	C and D have to be finished
F	Exterior shooting	10	A, B, C and D have to be finished
G	Interior shooting	12	D, E and F have to be finished
H	Synchronisation	3	F and G have to be finished
I	Editing	14	H has to be finished
J	Sound background	7	Can only start 3 days after the start of I and after H is finished
K	Mixing	6	I and J have to be finished
L	Answer print	1	Can only start 2 days after end of K

1. Using the MPM model, give the earliest dates at which each task can be done. What is the critical path?
2. Delete superfluous precedency constraints obtained by transitivity.
3. Divide tasks A and I into subtasks and add an extra virtual task K' of duration 2. The goal is to have a graphs where all arrows going out from a given task have same valuation, A will be divided into 4 tasks for instance. The extra task K' is there to make sure the arrow going out from K has valuation corresponding to the duration of K.
4. Draw a PERT graph adapted from previous work. What is the main disadvantage of the PERT approach in comparison with the MPM one?
5. Is the set of extra arrows you get in your PERT graph a minimal set? Why do you need each of these arrows?