Midterm Exam Solutions Curve to 57

M4320–Math Modeling   Fall 2017   Tarleton State   Dr. Scott Cook   Assigned 2017-10-19

1) For the following system, determine (a) linear (L) or non-linear (NL) (b) order (c) autonomous (A) or non-autonomous (NA). Convert to first order, autonomous if it is not already.

10

$$x_t = x_{t-1} + x_{t-2}y_{t-1}$$
$$y_t = y_{t-1} + 5tx_{t-2}$$

2) Consider the Lotka-Volterra model. Describe what each parameter means (ie. what aspect of the predator-prey system it encodes). (Suggestion: Uses phrases like "If we increase ..., then ... will happen").

10

$$x_t = x_{t-1} + rx_{t-1}\left(1 - \frac{x_{t-1}}{K}\right) - \left(1 - \frac{1}{by_{t-1}+1}\right)x_{t-1}$$
$$y_t = y_{t-1} - dy_{t-1} + cx_{t-1}y_{t-1}$$

5 3) Define the terms eigenvector and eigenvalue for a square matrix A.

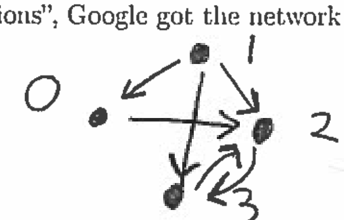
12 4) Find all equilibrium points for the system below and determine stability of at least one of them.

2 per

6 classify

$$x_t = 2x_{t-1} - y_{t-1} - x_{t-1}^2 + x_{t-1}y_{t-1}$$
$$y_t = -x_{t-1}^2$$

13 5) When I searched "good dynamical systems exam questions", Google got the network below.



5.1) What is the adjacency matrix A? 3

5.2) What is the transition matrix T for "naive" PageRank (without random jumps - users can only follow links). 2

5.3) What flaw(s) do you see? 2

5.4) We fix the flaw(s) by introducing a random jump. Let q be the random jump probability in "wise" PageRank. Let $p = 1 - q$. How do you modify ~~the~~ T above? (You don't need to actually compute the new T, just describe what you would ask the computer to do). 3

5.5) np.linalg.eig(T) gives the result below for the T above. In what order will Google display the search results? 3

$$\begin{bmatrix} 1.0+0.j & -0.9+0.j & 0.0+0.j & 0.0-0.j \end{bmatrix}, \begin{bmatrix} 0.03-0.j & -0.00+0.j & 0.71+0.j & 0.71-0.j \\ 0.02-0.j & 0.00+0.j & -0.00+0.j & -0.00-0.j \\ 0.48-0.j & 0.71+0.j & 0.00-0.j & 0.00+0.j \\ 0.46-0.j & -0.71+0.j & -0.71-0.j & -0.71+0.j \end{bmatrix}$$

10 6) Do two of following

6.1) Draw the first 6 steps of the cobweb plot for $x_t = 3 - (x_{t-1} - 2)^2$ $x_0 = 1$

6.2) define chaos (A dynamical system is <u>chaotic</u> if ...)

6.3) define stochastic (A square matrix is <u>stochastic</u> if ...)

6.4) describe variable rescaling (What you DO to rescale and why you would want to do it. You do not need specific details - broad but coherent descriptions are fine)

6.5) State Occam's Razor.

Solutions

1. NL, 2, NA   memory $z_t = x_{t-1}$

clock $w_t = w_{t-1} + 1$, $w_0 = 1$

so that $t = w_{t-1}$ $\forall t$

$$x_t = x_{t-1} + z_{t-1} y_{t-1}$$
$$y_t = y_{t-1} + 5 w_{t-1} z_{t-1}$$
$$z_t = x_{t-1}$$
$$w_t = w_{t-1} + 1$$

2. If we increase...

- ...r, rabbits reproduce more quickly.
- ...b, foxes hunt rabbits more effectively, rabbit death rate
- ...d, foxes die more quickly
- ...c, get more new foxes per rabbit eaten, fox growth rate

K is the carrying capacity of rabbits. It's the # of rabbits there will be if there are no foxes.

3. $\vec{v}$ is an eigenvector for A if
   1. $\vec{v} \neq 0$
   2. $A\vec{v} = \lambda \vec{v}$

The scalar λ is the eigenvalue associated to $\vec{v}$.

4. ① $x = 2x - y - x^2 + xy$

② $y = -x^2$

put ② into ①

$$0 = x - (-x^2) - x^2 + x(-x^2)$$
$$0 = x + x^2 - x^2 - x^3$$
$$0 = x - x^3$$
$$0 = x(1 - x^2)$$

use $y = -x^2$

$x = 0$, $y = 0$ | $x = 1$, $y = -1$ | $x = -1$, $y = -1$

$$J = \begin{bmatrix} 2 - 2x + y & -1 + x \\ -2x & 0 \end{bmatrix}$$

$(0,0)$

$$J = \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\det \begin{bmatrix} 2-\lambda & -1 \\ 0 & 0-\lambda \end{bmatrix} = (2-\lambda)(-\lambda) = 0$$

$\lambda = 0, 2$

<1 >1

semi-stable

saddle

$(1,-1)$

$$J = \begin{bmatrix} -1 & 0 \\ -2 & 0 \end{bmatrix}$$

$$\det \begin{bmatrix} -1-\lambda & 0 \\ -2 & 0-\lambda \end{bmatrix} = (-1-\lambda)(-\lambda) = 0$$

$\lambda = 0, -1$

<1 $=1$??

Dunno

Actually, it's an attractor

$(-1,-1)$

$$J = \begin{bmatrix} 3 & -2 \\ 2 & 0 \end{bmatrix}$$

$$\det \begin{bmatrix} 3-\lambda & -2 \\ 2 & 0-\lambda \end{bmatrix} = (3-\lambda)(-\lambda) + 4$$

$$-3\lambda + \lambda^2 + 4 = 0$$
$$\lambda^2 - 3\lambda + 4 = 0$$
$$\frac{3 \pm \sqrt{9-16}}{2} = \frac{3 \pm \sqrt{7}\,i}{2}$$
$$\left|\frac{3 + \sqrt{7}\,i}{2}\right| > \left|\frac{3+0}{2}\right| > 1$$
$$\left|\frac{3 \pm \sqrt{7}\,i}{2}\right| = \frac{1}{2}\left|3 \pm \sqrt{7}\,i\right|$$
$$= \frac{1}{2}\sqrt{9+7} = \boxed{2}$$

repelling

5. a) $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

3

2 b) $T = \begin{bmatrix} 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1/3 & 0 & 1 \\ 0 & 1/3 & 1 & 0 \end{bmatrix}$

2 c) Flaw - Stuck bouncing $2 \leftrightarrow 3$, nothing left at 0 or 1

3 d) $T' = pT + \frac{q}{4}$ "Multiply every entry by p then add $\frac{q}{4}$ to each entry"

3 e) Dominant eval in slot 1.
It's evect is in COLUMN 1. $\begin{bmatrix} 0.03 \\ 0.02 \\ 0.48 \\ 0.46 \end{bmatrix}$

Page Rank
2
3
0
1

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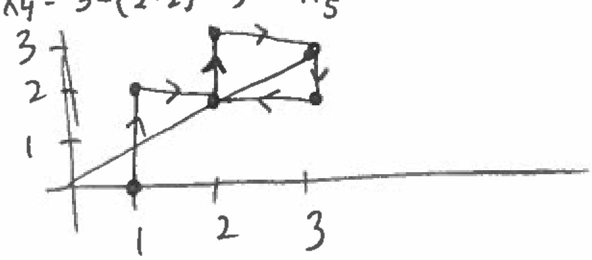
6) $x_0 = 1$ $\quad x_1 = 3-(1-2)^2 = 2$ $\quad x_2 = 3-(2-2)^2 = 3$ $\quad x_3 = 3-(3-2)^2 = 2$

1) $x_4 = 3-(2-2)^2 = 3$ $\quad x_5 = 3-(3-2)^2 = 2$ $\quad$ 3 $\quad$ 2 $\quad$ 3 $\quad$ 2 ...



6.2) Small change to intial condition $\longrightarrow$ giant difference later

6.3) Sum entries in any column, you'll always get 1.

6.4) What - define new variables & parameters from old ones

why - Reduce # of free parameters. Less stuff to vary when you study it.

6.5) Given 2 models that are equally accurate & robust, pick the simpler one.