Eigenstructure Assignment applied to the MAV System

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Abstract

Eigenstructure assignment is a technique which expands on the more common state-feedback pole placement control method by exerting some control over the eigenvectors of the closed loop system. While choosing the eigenvalues of a system determines the speed and stability of the response, the eigenvectors give some control over the shape of the response. This can improve controller performance for systems where the desired eigenstructure can be determined. This paper outlines the equations and methods needed to perform eigenstructure assignment and shows how applying this method to the MAV system can improve longitudinal loop control by decoupling the airspeed and altitude components of the response. Source code can be found at https://github.com/Tarnarmour/FlightDynamics_Eigenstructure.

I. Introduction

One of the most common forms of control is state feedback control, where the output of the controller is defined as a linear combination of the current state of the system. When certain properties of the system are fulfilled, a condition referred to as having a controllable system, a gain matrix can be chosen to change an unstable open-loop system into a stable and precisely tuned closed loop system. More specifically, the choice of gain matrix can place the poles of the closed loop system in arbitrary locations on the imaginary plane, ensuring stability, a desired speed of system response, and desirable damping. Because the desired poles are the input to the design process when using state feedback control, the design of this type of controller or the process for choosing a gain matrix for this type of controller is often called pole placement.

Control over the poles of a system may seem totally sufficient for designing controllers, since the poles determine not only the stability but the speed of the system response. However, multi-input multi-ouput (MIMO) systems often have different design requirements for different states of the system that go beyond what simple pole placement can achieve. For example, when designing an autopilot for a fixed wing aircraft the relative speed of the altitude and course control loops may be very different. Similarly it is essential that the roll loop be fast, and that the yaw loop be relatively slow, or the aircraft may not be stable to course inputs or disturbances. Additionally, there may be desirable control properties, such as a decoupling of specific states, that make higher-level control more effective. An example of this would be a decoupling of altitude and airspeed loops; if the aircraft can be given altitude commands without significantly changing airspeed, it will be easier for a human operator to direct the aircraft.

All of these goals are unreachable with pole placement; traditional pole placement gives the control engineer no influence over anything but the poles of the system, which may correspond to arbitrary combinations of system states. To effect these more advanced design specifications requires some control over the general eigenstructure of the closed loop system. This is the goal of eigenstructure assignment. This paper will explore the meaning of the eigenstructure of a system, the methods and limitations of eigenstructure assignment, and finally demonstrate two examples of systems controlled using eigenstructure assignment.

II. Derivation

I will now give a derivation of eigenstructure assignment, beginning with a mathematical explanation of the meaning of the eigenstructure of a system. I will show that the response of a system can be described as a combination of the eigenvectors of the system. I will give an equation for determining the gain matrix to enforce a desired eigenstructure, then explain the limitations of the method.

i. Interpretation of Eigenstructure

We will consider the canonical linear time invariant system; $\dot{x} = Ax + Bu$. The homogeneous or unforced response of this system is expressed as:

$$x(t) = e^{At}x_0$$

where x_0 is the initial condition of the system. To gain insight into the structure of this response, consider the eigenvalue decomposition of a matrix A. For a square matrix A with n distinct non-zero eigenvalues, we can write:

$$A = MJM^{-1}$$

where $J = \text{diag}(\lambda_1 \dots \lambda_n)$ with λ_i the ith eigenvalue of A and $M = [v_1 \dots v_n]$ with v_i the ith eigenvector of A. It can be shown that taking the matrix exponential of a matrix A does not change the eigenvectors of the matrix, and that the eigenvalues of the exponentiated matrix are given by $\Lambda = \text{diag}(e^{\lambda_1} \dots e^{\lambda_n})$. Thus we can conveniently rewrite the time response of the original system using the eigenvalue decomposition as:

$$x(t) = M\Lambda M^{-1}x_0 = \sum_{i=1}^{n} v_i e^{\lambda_i t}(w_i^T x_0)$$

Here we have let $M^{-1} = [w_1^T \dots w_n^T]^T$ for clarity. This expression shows us that the time response of the system can be thought of a weighted combination of the eigenvectors v_i of the system. Each eigenvector is weighted by $w_i^T x_0$, which indicates the magnitude of the initial conditions in the direction of w_i. These weighted eigenvectors each decay or grow exponentially with their corresponding eigenvalues as time constants. It should here be noted that for any real system, the eigenvectors and eigenvalues must either be real or come in complex conjugate pairs.

When the system is simple enough, this perspective can give a very rich understanding of the behavior. For example, in the case of fixed wing aircraft several distinct modes of behavior have been identified which correspond to specific eigenvectors of the system. Traditionally the longitudinal dynamics are described by two modes; the short period and phugoid modes. The short period mode consists of a coupling between the angle of attack and the elevation, and decays relatively quickly. The phugoid mode consists of a coupling between the altitude and the airspeed and is relatively slow [1]. These modes can be seen in the eigenvectors; the short period mode corresponds to a pair of eigenvectors with significant elements in the attack angle and elevation states.

ii. Eigenstructure Assignment Algorithm

Recall the steps for pole placement. When one considers a system controlled by a state feedback law $u = -Kx$, the resulting closed loop system can be written as:

$$\dot{x} = (A - BK)x$$

Pole placement algorithms seek to choose the gain matrix K to place the eigenvalues of the $A - BK$ matrix at desired values. For eigenstructure assignment, we additionally seek to constrain the eigenvectors of $A - BK$. This is only possible if we have enough degrees of freedom in our feedback law. It can be shown that, for fully observable and fully controllable systems with n states and m inputs, there are additional degrees of freedom in the choice of K and a total of m elements of m eigenvectors can be controlled [2].

To enforce the desired eigenvectors, the following steps are followed:

1. Pick the desired eigenvectors and associated eigenvalues, v_i^d and λ_i.
2. Determine which elements of the eigenvectors are to be determined by picking a matrix D_i such that $D_i v_i^d$ returns the elements of v_i^d which are to be determined.
3. Construct the following equation [3]:

$$\begin{bmatrix} \lambda_i I - A & B \\ D & 0 \end{bmatrix} \begin{bmatrix} v_i \\ u_i \end{bmatrix} = \begin{bmatrix} 0 \\ v_i^d \end{bmatrix} \tag{1}$$

   Here v_i is the achieved eigenvector which solves or minimizes this equation.

4. Solve equation 1 for each eigenvector. Depending on the choice of D, this equation may be over or under constrained, and thus may require a least squares or minimum norm solution.
5. Examine the achieved eigenvectors and determine if they are satisfactory. If not, make new choices and repeat.
6. Once satisfied the gain matrix can be found by setting $U = [u_1 \dots u_n]$ and $V = [v_1 \dots v_n]$ and solving:

$$K = UV^{-1} \tag{2}$$

iii. Limitations

As mentioned above, for a system with less inputs than states only a subset of eigenvectors and eigenvector elements can be determined. To be precise, for a system with n states, r of which are observable, and m inputs, only $\max(r, m)$ eigenvectors and $\min(r, m)$ elements of each vector can be set this way. Minimum norm solutions may give approximate solutions but cannot guarantee arbitrary eigenstructure. Another limitation is that the above equations do not enforce any size on the achieved eigenvectors. That means that while one could set an element of the eigenvector to 1, the scale of the achieved eigenvector is not controlled and this controlled element may still be small in comparison to other elements.

In general, the dynamic behavior of a system can be changed or improved but not completely replaced. Undesirable coupling between states may be limited but the fundamental behavior of a system may not be totally changed. Despite this, the following examples will show that eigenstructure assignment can radically improve the performance of some systems.

III. Examples

In this section two examples of eigenstructure assignment will be shown. First, a simple multi-input double mass system will be demonstrated, where eigenstructure assignment is used to decouple the dynamics of the system. Then a small fixed wing aircraft system will be shown where eigenstructure assignment improves the longitudinal control beyond what regular pole placement could achieve. These results, with all source code, are available at the github repository linked in the abstract.

i. Double Mass

The double mass system contains two masses linked together by a spring and damper with input forces on each mass. The state vector for the system is defined as $x = [z_1 z_2 \dot{z_1} \dot{z_2}]^T$. The natural dynamics of the system couple the motion of one mass to the other, and notably even under state feedback control

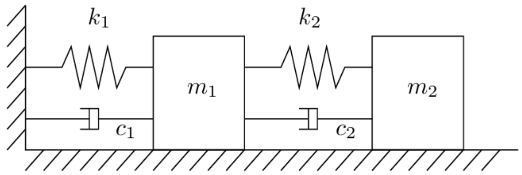

Figure 1: *Schematic of the two-mass system*

with gains found using pole placement this coupling will still exist. Neither of the two pairs of eigenvectors of the closed loop system are orthogonal to either the $[1000]^T$ or $[0100]$ directions, meaning that both modes will be excited by displacement of a single mass. In other words, movement of one mass will excite movement in the other mass, even under state feedback control. Pole placement can determine the eigenvalues of the system, determining the speed of the system response. However, since both pairs of eigenvectors contain non-zeros elements in the z_1 and z_2 directions, any picked eigenvalues will apply unpredictably to both masses, making it impossible to determine the rise times of each block independently.

This is a prime example of the advantages of eigenstructure assignment. Two controllers were designed for the double mass system, one using traditional pole placement (the Python control library place function, which uses the algorithm designed by Tits and Yang [4]) and the second using the above described eigenstructure assignment algorithm. Both controller gains were set using desired poles of $\lambda_d = [-3+3j, -3-3j, -2+2j, -2-2j]$. The eigenstructure controller was given the following desired eigenvectors:

$$V_d = \begin{bmatrix} 1+1j & 0.0 & 1+1j & 0.0 \\ 1-1j & 0.0 & 1-1j & 0.0 \\ 0.0 & 1+1j & 0.0 & 1+1j \\ 0.0 & 1-1j & 0.0 & 1-1j \end{bmatrix}^T$$

The first and second elements of each eigenvector were picked as the enforced elements. Figure 2 shows the results from the two controllers when the system was excited with a

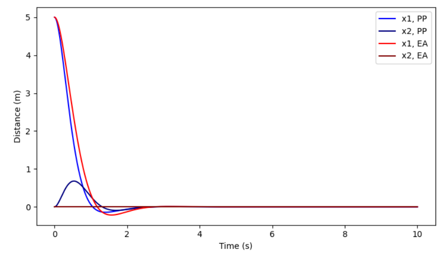

Figure 2: *Results of the two controllers to a non-origin initial condition. Results in red are from the eigenstructure assignment controller, while results in blue are from the regular pole placement controller. Note that while the eigenstructure controller shows no coupling between mass 1 and mass 2, the regular pole placement controller demonstrates that movement of mass 2 perturbed mass.*

non-zero initial condition. The eigenstructure assignment controller clearly managed to decouple the two masses and enforced the eigenvalue assignment to a specific mass.

ii. Micro Air Vehicle

The micro aircraft, or MAV, simulation was developed using the resources described in [5] and code found on the github repository referenced in the abstract of this paper. Details on the parameters and dynamics of the aircraft can be found there. A PID based lateral controller was developed to stabilize the lateral dynamcis. The longitudinal dynamics were controlled by state feedback controllers using either eigenstructure assignment or regular pole placement. Both controllers used the same desired poles but the eigenstructure assignment controller was designed, as detailed above, to decouple the airspeed from the altitude dynamics. Results from both airspeed and altitude commands are shown in figures 4 and figure 3. The eigenstructure assignment method resulted in a controller with clearly superior performance, showing the ability to change airspeed without noticable changes to altitude. This result could probably be further improved with more tuning, and similar im-

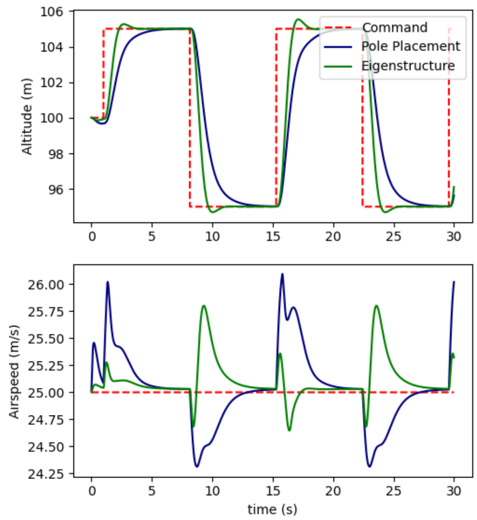

Figure 3: *Results for control under altitude commands. The eigenstructure controller is both faster and exhibits smaller changes in airspeed when altitude commands are given.*

provements could be made to lateral control for the MAV.

IV. Conclusion

Eigenstructure assignment allows more fine control over the performance of state feedback controllers and can make a dramatic difference in the final capabilities of the system. This additional capability comes with additional difficulties, however. Eigenstructure assignment requires the selection eigenvectors as well as eigenvalues, and tuning can be significantly more difficult. I believe that this is the main reason why other methods, such as LQR controllers, are more popular in modern control applications.

Regardless, a knowledge of eigenstructure analysis and assignment can give insight into system performance and guide development when certain system properties are required.

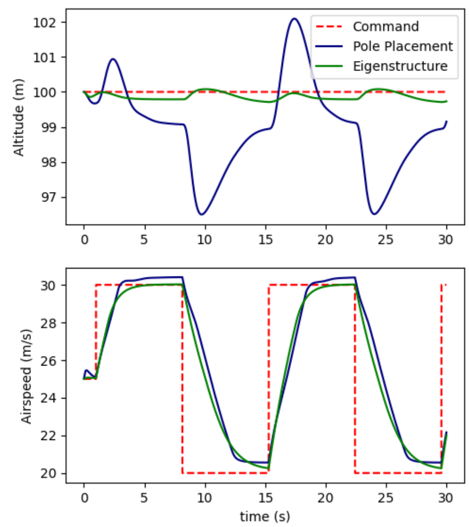

Figure 4: *Results for control under airspeed commands. While both controllers have similar speeds the eigenstructure controller exhibits better steady state performance and shows almost no effect on the altitude of the aircraft while airspeed is changing.*

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