Eigenstructure Assignment

Applied to MAV System

John Morrell

Flight Dynamics
Brigham Young University

April 20, 2022

Introduction: Eigenstructure Assigment

What Eigenstructure assignment is a more advanced version of pole placement.
Why Pole placement gives control over stability and the speed of transient response, but not the shape of transient responses. Eigenstructure assignment lets one control the shape of the output as well as the speed of decay.
How Similar methods to pole placement, using state feedback gain matrices (K matrix).

Eigenstructure

Eigenvectors and Eigenvalues are a useful way to analyze systems.

$$Av = \lambda v$$

We can use eigenvectors to make a useful similarity transform. Define $M = [v_1 \ldots v_n]$ where v_i is an eigenvector of A, and $J = \mathrm{diag}(\lambda_i \ldots \lambda_n)$ where λ_i is an eigenvalue of A. Then:

$$AM = MJ \rightarrow A = MJM^{-1}$$

Recall that the solution to the differential equation $\dot{x} = Ax$ is given by $x(t) = e^{At}x(t_0)$. It can be shown that the eigenvectors of A are also eigenvectors of e^{At} and that the associated eigenvalues are simply $e_i = e^{\lambda_i t}$. Thus we can write our solution as:

$$x(t) = M\,\mathrm{diag}(e^{\lambda_1 t} \ldots e^{\lambda_n t})M^{-1}x_0$$

I assume here that there are no repeated eigenvectors; if there are, then J is not diagonal but Jordan form and things are bit more messy.

Eigenstructure

For convenience, we define $M = [w_1 \dots w_n]^T$. Then we can write:

$$x(t) = \sum_{i=1}^{n} (w_i^T x_0) e^{\lambda_i t} v_i$$

In this equation, v_i is an eigenvector of the system, $(w_i^T x_0)$ is a scalar coefficient, and obviously $e^{\lambda_i t}$ is a scalar.

We can see that the eigenvectors of the system are the modes of the system, and the associated eigenvalues determine how quickly these modes grow / decay.

Modes in Aircraft

In our MAV model, we decoupled the longitudinal and lateral dynamics, giving us two sets of modes. The longitudinal modes included the short period and phugoid modes. The Lateral dynamics include the dutch roll modes, roll dynamics, and spiral divergence mode [1].

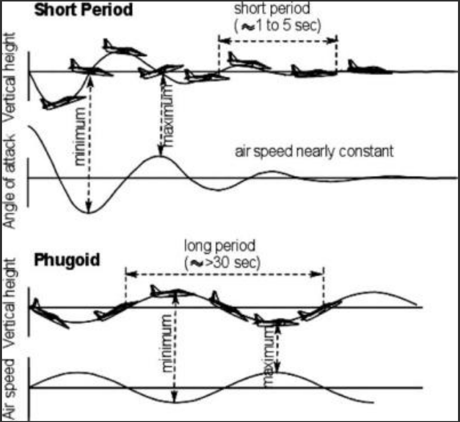

Figure: Illustration of longitudinal modes in aircraft

Modes in Aircraft

Longitudinal States $= [u, w, q, \theta, p_d]$

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Longitudinal Eigenvalues:
[ 0.000+0.000j -4.878+9.869j -4.878-9.869j -0.118+0.492j -0.118-0.492j]
Longitudinal Eigenvectors:
[ 0.000+0.000j  0.000+0.000j  0.000+0.000j  0.000+0.000j  1.000+0.000j]
[-0.034-0.002j  0.925+0.000j -0.016+0.374j  0.031-0.014j -0.022+0.026j]
[-0.034+0.002j  0.925-0.000j -0.016-0.374j  0.031+0.014j -0.022-0.026j]
[-0.334-0.155j -0.006-0.002j -0.009-0.004j -0.004+0.019j  0.930+0.000j]
[-0.334+0.155j -0.006+0.002j -0.009+0.004j -0.004-0.019j  0.930-0.000j]
```

Figure: Longitudinal Eigenvalues and Eigenvectors

Lateral States $= [\dot{p}_e, p, r, \phi, \psi]$

```
Lateral Eigenvalues:
[-22.441+0.000j  -1.140+4.655j  -1.140-4.655j   0.102+0.000j   0.011+0.000j]
Lateral Eigenvectors:
[-0.030+0.000j  0.999+0.000j  0.007+0.000j -0.044+0.000j -0.000+0.000j]
[-0.967+0.000j  0.182+0.045j -0.004+0.167j  0.002-0.039j  0.034-0.008j]
[-0.967-0.000j  0.182-0.045j -0.004-0.167j  0.002+0.039j  0.034+0.008j]
[-0.148+0.000j -0.016+0.000j -0.086+0.000j -0.229+0.000j -0.958+0.000j]
[-0.002+0.000j -0.000+0.000j -0.001+0.000j -0.003+0.000j  1.000+0.000j]
```

Pole Placement by State Feedback

In state feedback, we define a control law: $u = -Kx$ which gives us a new system:

$$\dot{x} = Ax + B(-Kx) = (A - BK)x$$

By picking the elements of K, we can change the $A - BK$ matrix and change the poles and behavior of the system.

For SISO systems, there is a closed form solution for K given a set of desired eigenvalues, and thus there are no additional degrees of freedom. However, for MIMO systems with m inputs, assuming fully observable states, there are additional degrees of freedom, and $n - m$ elements of each eigenvector can be controlled [2].

Eigenstructure Assignment by State Feedback

1. Pick desired eigenvectors v_i^d and associated eigenvalues λ_i
2. Construct the equation [3]:

$$\begin{bmatrix} \lambda_i I - A & B \\ D & 0 \end{bmatrix} \begin{bmatrix} v_i \\ u_i \end{bmatrix} = \begin{bmatrix} 0 \\ v_i^d \end{bmatrix}$$

   v_i are the achievable eigenvectors and u_i are control outputs which will be used to find the gain matrix later. D is a matrix which selects which components of v_i are being chosen for.
3. For each eigenvector, solve the equation for v_i and u_i. v_i are the achievable eigenvectors and u_i are control outputs. This equation may be over-constrained, in which case a least squares solution must be used.
4. Look at the achievable eigenvectors. If they are unacceptable, make new choices about the desired eigenvectors and repeat.
5. Once satisfied, the gain matrix can be found by $K = UV^{-1}$ where $V = [v_1 \dots v_n]$ and $U = [u_1 \dots u_n]$.

Eigenstructure Assignment by State Feedback

- The desired eigenvectors must form a basis of rank n; there can be no repeating eigenvectors or linearly dependent eigenvectors.
- In the case where all states can be observed, only n - m elements of each eigenvector can be strictly controlled.
- It is usually not possible to arbitrarily change modes; we might change the dutch roll mode but probably can't eliminate it entirely

Changing eigenvalues changes the speed of the transient response and ensures stability, while changing eigenvectors changes the shape of the response.

Picking desired eigenvector traits may be more difficult than picking eigenvalues. One possibility is to pick eigenvectors such that certain common inputs do not excite undesirable modes; for example, it may be helpful for the short period mode to be decoupled from the airspeed.

Simple Example: Double Mass System

A double mass spring damper system with force inputs on both blocks. The state is defined as:

$$x = \begin{bmatrix} z_1 \\ z_2 \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix}$$

The natural dynamics obviously have a coupling between the two masses, and even under state space control the motion of the two blocks will be coupled.

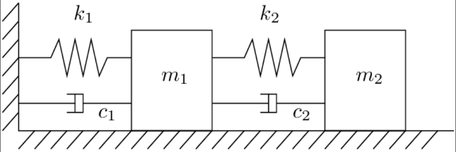

Figure: Schematic of the double mass system

Simple Example: Double Mass System

Open Loop:

```
Eigenvalues:
[-0.0479+2.5819j -0.0479-2.5819j -0.9521+1.4975j -0.9521-1.4975j]
Eigenvectors:
[-0.005 -0.2691j  0.2293-0.0733j  0.6951+0.j       0.1782+0.5956j]
[-0.005 +0.2691j  0.2293+0.0733j  0.6951-0.j       0.1782-0.5956j]
[ 0.1916-0.166j  -0.2256-0.3548j  0.0661+0.445j   0.7461+0.j     ]
[ 0.1916+0.166j  -0.2256+0.3548j  0.0661-0.445j   0.7461-0.j     ]
```

Closed Loop:

```
Eigenvalues:
[-3.+3.j -3.-3.j -2.+2.j -2.-2.j]
Eigenvectors:
[-0.1147-0.1147j  0.1147+0.1147j  0.6882+0.j      -0.6882+0.j     ]
[-0.1147+0.1147j  0.1147-0.1147j  0.6882-0.j      -0.6882-0.j     ]
[-0.1667-0.1667j -0.1667-0.1667j  0.6667+0.j       0.6667-0.j     ]
[-0.1667+0.1667j -0.1667+0.1667j  0.6667-0.j       0.6667+0.j     ]
K:
[[ 8. -3.  4. -0.]
 [-8. 10. -2.  4.]]
```

Simple Example: Double Mass System

Eigenstructure Loop:

```
Eigenstructure Assignment:
Eigenvalues:  [-2.-2.j -2.+2.j -3.+3.j -3.-3.j]
Desired Eigenvectors:
[1. +1.j 0. +0.j 1. +1.j 0.1+0.j]
[1. -1.j 0. +0.j 1. -1.j 0.1+0.j]
[0. +0.j 1. +1.j 0.1+0.j 1. +1.j]
[0. +0.j 1. -1.j 0.1+0.j 1. -1.j]
Achieved Eigenvectors:
[-0.2357+0.2357j  0.     -0.j       0.9428+0.j      -0.     +0.j     ]
[-0.2357-0.2357j  0.     -0.j       0.9428+0.j      -0.     +0.j     ]
[-0.     -0.j     -0.1622-0.1622j -0.     +0.j       0.9733+0.j     ]
[ 0.     -0.j     -0.1622+0.1622j -0.     -0.j       0.9733+0.j     ]
K:
[[ 3.-0.j  2.+0.j  3.-0.j  1.-0.j]
 [-3.+0.j 15.-0.j -1.+0.j  5.-0.j]]
```

Simple Example: Double Mass System

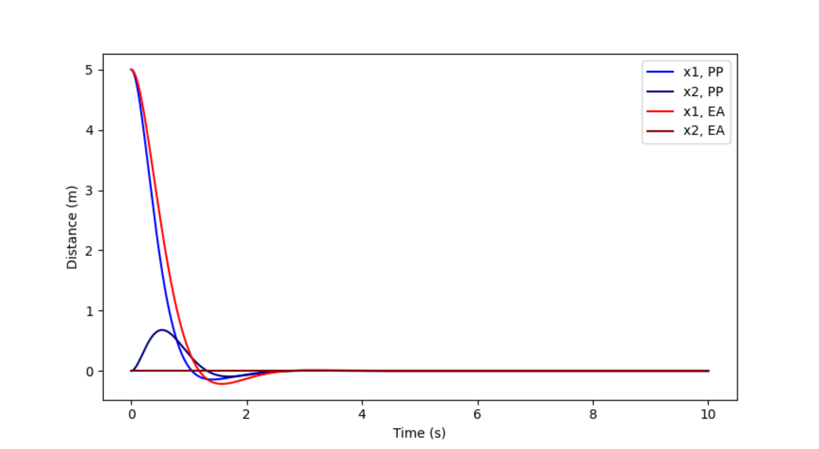

Figure: Simulation of double mass system, showing regular pole placement and full eigenstructure placement. Note that the fully assigned system displays no coupling between the z_1 and z_2 components.

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MAV

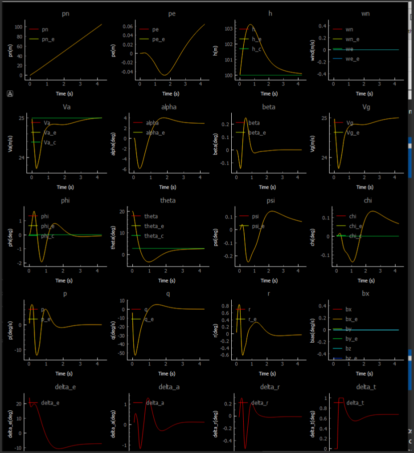

Figure: MAV with traditional pole placement on the longitudinal dynamics

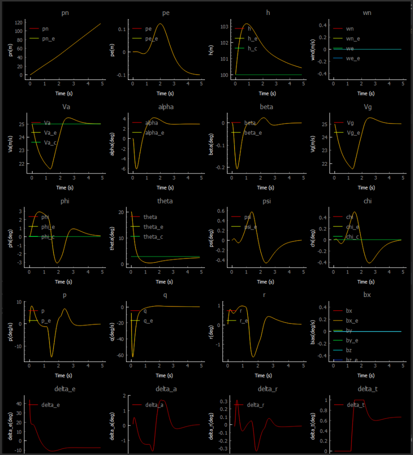

Figure: MAV with eigenstructure assignment on the longitudinal dynamics

Conclusion

- Eigenstructure assignment is, potentially, very powerful and can tailor dynamics of systems.
- It is significantly more difficult to tune.
- Difficult to interpret eigenvectors for complex systems.
- There remains a lack of control for placing the entire eigenvector.

References

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