Introduction to Algebraic Information Theory for Quantitative Finance
Homework 1

August 30, 2025

Timothy Tarter
James Madison University
Department of Mathematics

1. Explain why we said that $U_p(\mathbb{Z})$ is only a group if p is prime. Show that all four group axioms hold for $U_p(\mathbb{Z})$ and $\mathbb{Z}_p$.

2. Prove that the G = $\{e^{i\theta} : i = \sqrt{-1}, \theta \in \mathbb{R}\}$ is a group under multiplication.

3. Show that $U_p(\mathbb{Z})$ is an Abelian group.

4. Compute $\sigma_1\tau_2^2\sigma_3$, $\tau_2\sigma_3\tau_1$, and $\sigma_2\sigma_3\tau_2$ in S_3. Remember - we compute from right to left!

5. In class, we said that if G is a finite group and we want to show that H is a subgroup of G, we only had to check closure. Explain in your own words why this is true.

6. Show that $\{e, \tau_1\}$ is not a subgroup of S_3. Find what you need to add to it to make it a subgroup, then prove that that list is a subgroup.

7. Prove that if $M \triangleleft G$ and $N \triangleleft G$, then $M \cap N \triangleleft G$. (Note, $\cap$ means "intersection of sets.")

8. Find all normal subgroups of S_3, recalling that $H \triangleleft G$ iff $aH = Ha$ (or equivalently $H = a^{-1}Ha$, $\forall a \in G$). For each such normal subgroup, compute the elements of S_3/H. Then use the fundamental homomorphism theorem to find an isomorphic group to S_3/H that is easier to work with. Draw a commutative diagram to show the relationship between groups (it may help to use different greek letters for each homomorphism).

9. Find all the normal subgroups of $U_{13}(\mathbb{Z})$, recalling that zero is not in the set. Compute each quotient group like in problem six, then use FHT to find an isomorphic group easily. Draw a commutative diagram to show the relationship between groups (it may help to use different greek letters for each homomorphism).

10. Find an isomorphism from $G =$ the group of all real numbers under addition to $G' =$ the group of all positive real numbers under multiplication (note, G' does not include zero). Prove that this is an isomorphism.

11. Show that $\varphi : G \to G'$, a homomorphism, is a surjective iff $\ker\varphi = \{e\}$.

12. Show that if G is any group, $H \triangleleft G$, and $\varphi : G \to G'$ is a surjective homomorphism, that $\varphi(H) \triangleleft G'$.

13. If G is a group and $N \triangleleft G$, prove that G/N is Abelian iff $aba^{-1}b^{-1} \in N$ for all $a, b \in G$. (Note, iff means you have to prove that G/N is Abelian if $aba^{-1}b^{-1} \in N$ for all $a, b \in G$ AND $aba^{-1}b^{-1} \in N$ for all $a, b \in G$ if G/N is Abelian.)

14. Determine, using the fundamental theorem of finite Abelian groups (FToFAG), all of the $\mathbb{Z} \times \mathbb{Z} \times ... \times \mathbb{Z}$ groups isomorphic to $\mathbb{Z}_{284}$. Note that these are guaranteed to be the ONLY groups isomorphic to $\mathbb{Z}_{284}$ - cool, right? Then go through an note all of the groups which are not isomorphic to $\mathbb{Z}_{284}$ (all of the groups $\mathbb{Z}_i \times ... \times \mathbb{Z}_n$ such that gcd(i, ..., n) $\neq 1$ but $i * \cdots * n = 284$). **Feel free to use a GCD calculator online for this problem if you aren't familiar with calculating the GCD via the Euclidean algorithm.**

15. Determine all of the groups isomorphic to $\mathbb{Z}_{984}$. Determine all of the groups that aren't isomorphic, as above in (14).

16. Determine all of the groups isomorphic to $\mathbb{Z}_{725}$. Determine all of the groups that aren't isomorphic, as above in (14).

17. Determine all of the groups isomorphic to $\mathbb{Z}_{796}$. Determine all of the groups that aren't isomorphic, as above in (14).